THE EXPECTED VOLUME OF A RANDOM TETRAHEDRON IN A CUBE.

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Abstract. We calculate the expected size of the volume of a random tetrahedron inside a cube. The result is not new, but the method is different from that of previous calculations.

1. Introduction

Four points are generated at random inside a cube C. Let T be the tetrahedron spanned by the random points. We shall consider the random variable $X = volume(T)/volume(C)$. It is well known that any affine transformation will preserve the ratio X. This follows from the fact that the volume scaling is constant for an affine transformation. The scale equals the determinant of the homogeneous part of the transformation. This means that our results hold for any parallelotope C.

Various aspects of our problem have been considered in the field of geometric probability, see e.g. [12]. J. J. Sylvester considered the plane problem of a random triangle T in an arbitrary convex set K and the variable $X = area(T)/area(K)$. He posed the following problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. It has been shown, [2] that $\kappa = \frac{11}{144}$ for K a square.

A. Reńyi and R. Sulanke, [10] and [11], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. H. A. Alikoski [2] has given expressions for κ when $n = 3$ and K a regular r-polygon. We have given the whole probability distribution of X for $n = 3$ and $n = 4$ and K a parallelogram [7] and for K a triangle [8].

R. E. Miles [6] generalizes the asymptotic estimates for K a circle to higher dimensions. C. Buchta and M. Reitzner, [4], has given values of κ for $n \geq 4$ and K a triangle and the three-dimensional generalization

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to K a tetrahedron. The same value was also calculated by D. Mannion [5] and J. Philip. [9]. The value of κ for a tetrahedron in a cube that we consider in this paper was calculated by A. Zinani [13] to

$$\kappa = \frac{3977}{216000} - \frac{\pi^2}{2160} \approx .013842776.$$

The calculations of this paper are done with the aid of *Maple 10*. The *Maple* worksheet `ETC.mw` is available at `www.math.kth.se/~johanph`.

2. Notation and formulation.

As C, we will chose a unit cube in the positive orthant having one vertex at the origin. We use a constant probability density in C for generating 4 random points in C. The points will be denoted P_k and have coordinates (x_k, y_k, z_k) for $1 \leq k \leq 4$. Let T be the tetrahedron spanned by the 4 points. We shall consider the probability distribution of the random variable $X = volume(T)/volume(C)$.

The generated T spans a parallelotope with sides parallell to the sides of C. We will call this spanned parallelotope B. The random variable X, that we study will be written as the product of the two random variables

$$U = volume(T)/volume(B) \text{ and } V = volume(B)/volume(C).$$

Roughly speaking, U describes the shape of T and V its size. It's not obvious that the size V is independent of the shape U. The independence will be shown in section 6. We shall determine the averages of U and V and combine them to get $E(X) = E(U) \cdot E(V)$.

3. The six geometrical cases for calculating $E(U)$.

The way B is spanned by the four points gives rise to the six cases characterized primarily by the number of faces of B that a point determines. In table 1, the points have been reindexed so that a higher index corresponds to more faces. Cases 5 and 6, are equal combinatorically but differ in that Case 5 occurs when points 1 and 2 determine two opposite faces of B and case 6 when they determine two adjacent faces

In case 1 depicted in Figure 1, we have without loss of generality (WLOG) chosen P_4 to sit in the vertex nearest to the origin and P_3 to sit in the opposite vertex. P_1 and P_2 are interior. In the other cases, the point numbering and their positions have (WLOG) been chosen in a similar way. When studying the ratio U, we have enlarged B to a unit cube. The six cases are depicted in Figure 2, where we show six enlarged B with their point positions.

case	P_1	P_2	P_3	P_4	p_j	$E(U_j)$
1	0	0	3	3	$\frac{1}{36}$	$\frac{11}{432}$
2	0	1	2	3	$\frac{1}{3}$	$\frac{605}{7776} - \frac{\pi^2}{324}$
3	1	1	1	3	$\frac{1}{9}$	$\frac{2281}{3888} - \frac{11\pi^2}{216}$
4	0	2	2	2	$\frac{1}{9}$	$\frac{19}{1944} + \frac{\pi^2}{216}$
5	1	1	2	2	$\frac{1}{12}$	$\frac{23}{1944} + \frac{\pi^2}{324}$
6	1	1	2	2	$\frac{1}{3}$	-$\frac{101}{3888} + \frac{11\pi^2}{972}$

TABLE 1. Giving, for each of the 6 cases, the number of faces determined by each P_k, their probability to occur ($=p_j$), and the expectation of U_j. Cases 5 and 6 have different geometries described in the text.

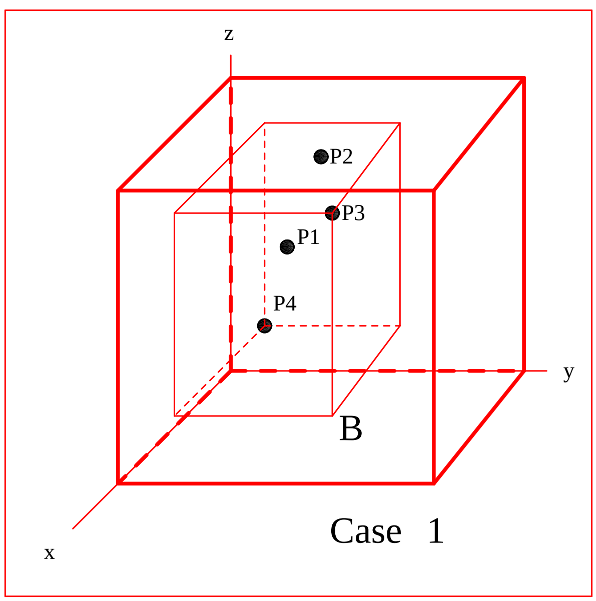

FIGURE 1. The unit cube and the parallelotope B spanned by the vertices of the tetrahedron in Case 1. P_4 sits in one vertex of B, P_3 in the opposite vertex. P_1 and P_2 are interior in B.

3.1. **The probabilities for the six cases.** We shall show that the six cases occur with the probabilities given in table 1.

The Cases 5 and 6 are equal combinatorically but differ in that Case 5 occurs when points 1 and 2 determine two opposite faces of B and case 6 when they determine two adjacent faces. The calculation of each p_j has a geometrical and a combinatorial part.

Case 1 is in the x-direction characterized by

$$x_4 \le x_1 \le x_3 \text{ and } x_4 \le x_2 \le x_3 \ .$$

The probality that this shall occur is

$$p_0 = \int_0^1 dx_4 \int_{x_4}^1 dx_3 \int_{x_4}^{x_3} dx_2 \int_{x_4}^{x_3} dx_1 = \frac{1}{12}.$$

The geometry is the same in the y- and z-directions so the total probability for the configuration in Figure 1 to occur is ${p_0}^3$. Case 1 occurs also when the points are reindexed. With the numbers 0, 0, 3, 3 for case 1, this can be done in $\frac{4!}{2!2!} = 6$ ways. The point P_4 can be put in 8 vertices. Altogether, we get the probability for case 1:

$$p_1 = 6 \cdot 8 \cdot {p_0}^3 = \frac{1}{36}.$$

When doing the other cases, we have the same integrations as above but with other names on the integration variables, so the geometrical probability is the same in all cases.

In case 2, the numbers in the table are all different and we have $\frac{4!}{1!1!1!1!} = 24$ reindexings. Point P_4 can be put in 8 corners and P_3 on 3 edges, giving

$$p_2 = 24 \cdot 8 \cdot 3 \cdot {p_0}^3 = \frac{1}{3}.$$

In case 3, we have $\frac{4!}{3!1!} = 4$ reindexings. Point P_4 can be put in 8 corners, P_3 on 3 edges and P_2 in 2 faces, giving

$$p_3 = 4 \cdot 8 \cdot 3 \cdot 2 \cdot {p_0}^3 = \frac{1}{9}.$$

In case 4, we have $\frac{4!}{1!3!} = 4$ reindexings. Point P_4 can be put on 12 edges and P_3 and on 4 edges, giving

$$p_4 = 4 \cdot 12 \cdot 3 \cdot {p_0}^3 = \frac{1}{9}.$$

In case 5, we have $\frac{4!}{2!2!} = 6$ reindexings. Point P_4 can be put on 12 edges, P_3 on 1 edge and P_2 in 2 faces, giving

$$p_5 = 6 \cdot 12 \cdot 1 \cdot 2 \cdot {p_0}^3 = \frac{1}{12}.$$

In case 6, we have $\frac{4!}{2!2!} = 6$ reindexings. Point P_4 can be put on 12 edges, P_3 on 4 edges and P_2 on 2 faces, giving

$$p_6 = 6 \cdot 12 \cdot 4 \cdot 2 \cdot {p_0}^3 = \frac{1}{3}.$$

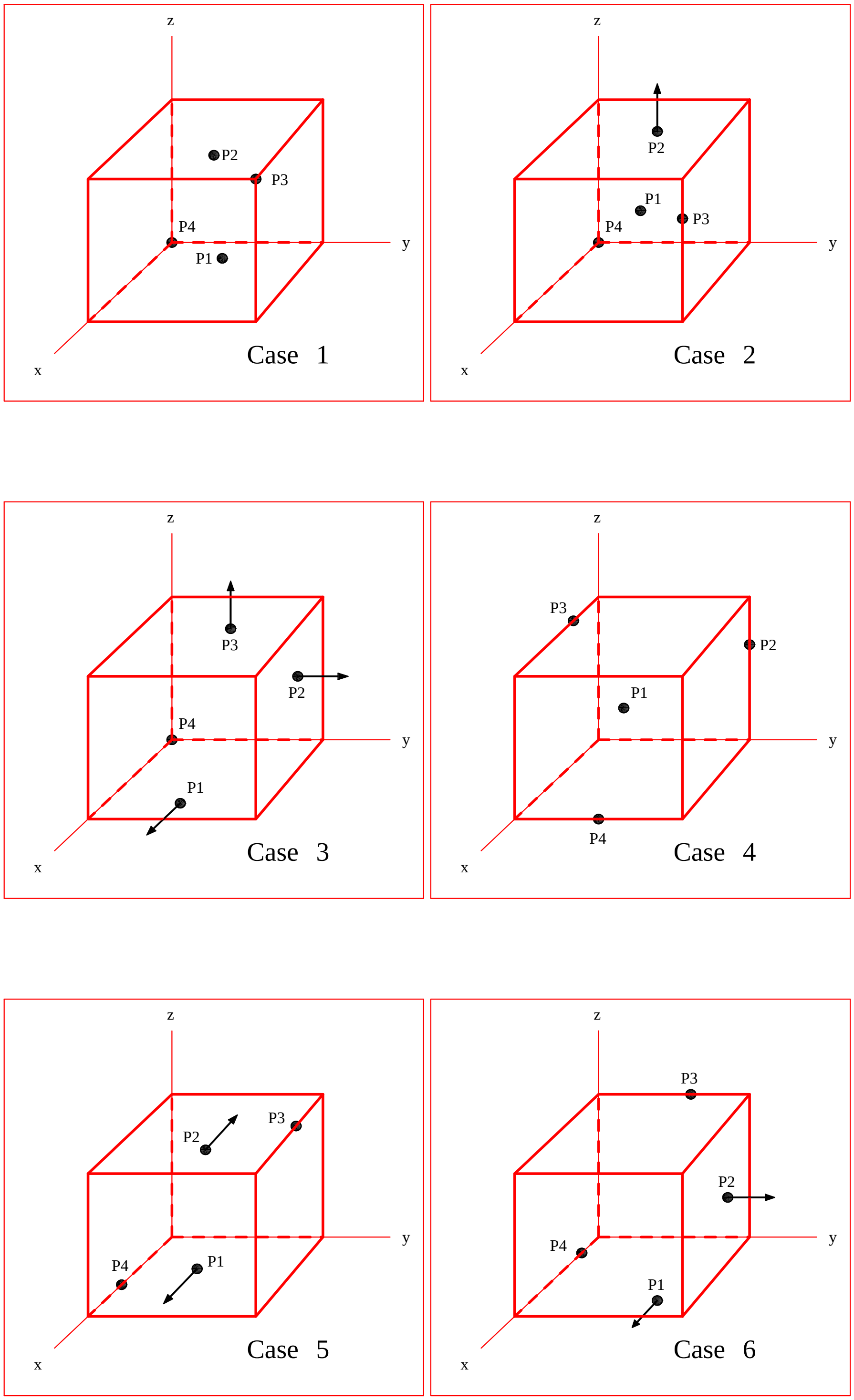

FIGURE 2. The figure shows the enlarged B in the six cases. Points sitting in a face are marked with an arrow orthogonal to th face.

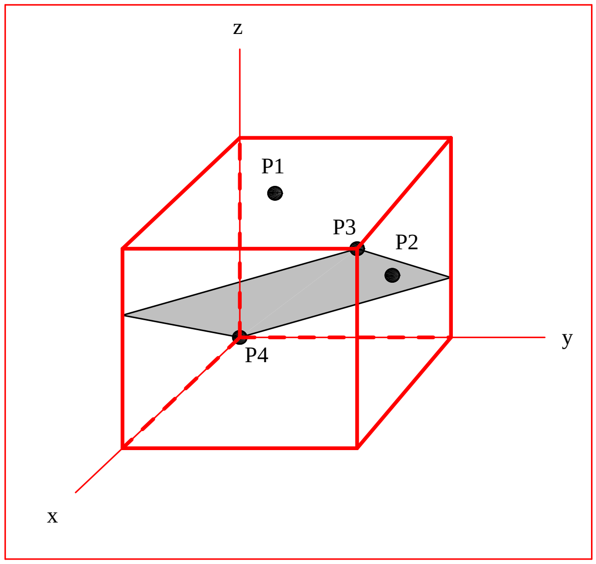

FIGURE 3. The plane $f = 0$ in Case 1.

4. THE EXPECTATION OF U IN EACH OF THE SIX CASES.

When calculating the expectation of U, we enlarge the parallelotope B so that it fills up C. This doesn't affect the ratio U. We will continue to call the points P_k even though the problem has been translated and rescaled. The transformed tetrahedron T is spanned by, e.g., the three vectors

$$P_1 - P_4,\ P_2 - P_4,\ \text{and}\ P_3 - P_4.$$

The side spanned by $P_2 - P_4$ and $P_3 - P_4$ has the normal $n = (P_2 - P_4) \times (P_3 - P_4)$. The volume fraction $U = |f|/6$, where f is the scalar product

$$f = n \cdot (P_1 - P_4). \tag{1}$$

The complexity of the calculation stems from the absolute value. We have to identify the sets where f is positive and negative. Let B_+ and B_- be these subsets of the enlarged B. We have $B = B_+ + B_-$. To get the expectation of U, we are going to integrate f over the whole of B and subtract twice its integral over B_-.

4.1. **Calculation of $E(U_1)$.** In case 1, we use the n and f defined above. With P_4 at the origin, all components of $P_2 - P_4$ and $P_3 - P_4$ are positive, implying that the n_k cannot all have the same sign. To determine where f is negative, we assume WLOG that

$$n_1 \geq n_2 \geq 0.$$

This implies that $n_3 \leq 0$. With these inequalities, we single out one of twelve cases which all have the same probability of occurring and all have the same expectation of $E(U_1)$.

We start by integrating over (x_1, y_1, z_1) for fixed P_2, Cf. Figure 3. With $n_3 \leq 0$, we have $f < 0$ above the plane $f = 0$. Let $z(x_1, y_1)$ be the solution of $f = 0$ for z_1. With the assumed inequalities for the n_k, one can show that $0 \leq z(x_1, y_1) \leq 1$. We get the conditional expectation

$$e_1(x_2, y_2, z_2) = \int_0^1 dx_1 \int_0^1 dy_1 \int_{z(x_1,y_1)}^1 f\, dz_1.$$

The inequalities $n_2 \geq 0$, $n_3 \leq 0$, and $n_1 \geq n_2$ imply respectively $z_2 \geq x_2$, $y_2 \geq x_2$, and $x_2 + y_2 \geq 2z_2$. The whole integral over B_- is

$$e_2 = \int_0^1 dx_2 \int_{x_2}^1 dy_2 \int_{x_2}^{(x_2+y_2)/2} e_1\, dz_2 = -\frac{11}{1728}.$$

By symmetry, the integral of f over the whole of B is zero. We get

$$E(U_1) = -12 \cdot 2 \cdot e_2/6 = \frac{11}{432}.$$

4.2. **Calculation of $E(U_2)$.** In this case we have P_4 at the origin, $P_3 = (1, 1, z_3)$, $P_2 = (x_2, y_2, 1)$, and P_1 interior. Cf. Figure 4. In this case, we define

$$n = (P_3 - P_4) \times (P_2 - P_4)$$

and

$$f = n \cdot (P_1 - P_4) = n_1 x_1 + n_2 y_1 + n_3 z_1.$$

The plane separating positive and negative f goes though P_2, P_3, and P_4. We assume WLOG that $y_2 \geq x_2$, implying $n_1 \geq 0$, $n_2 \leq 0$, and $n_3 \geq 0$. With $n_3 \geq 0$, we have $f \leq 0$ below the plane $z(x_1, y_1)$. In this case, we do not have $0 \leq z(x_1, y_1) \leq 1$ so the bounds of z_1 are those indicated in Figure 4.

We have $y_{10} = -n_1/n_2$, $y_{20} = -n_3/n_2$, and $x_{20} = -(n_2 + n_3)/n_1$ and all three numbers are between 0 and 1. We get

$$e = \int_0^1 dx_2 \int_{x_2}^1 dy_2 \int_0^1 dz_3 \int_0^1 dx_1 \int_0^1 dy_1 \int_{max(0,z)}^{min(1,z)} f\, dz_1 = -\frac{497}{5184} + \frac{\pi^2}{216}. \tag{2}$$

In this case, the integral of f over the part of B where $y_2 \geq x_2$ is not zero but equals $\frac{1}{6}$. We get

$$E(U_2) = 2(\frac{1}{6} - 2e)/6 = \frac{605}{7776} - \frac{\pi^2}{324},$$

where the factor 2 stems from the condition $y_2 \geq x_2$.

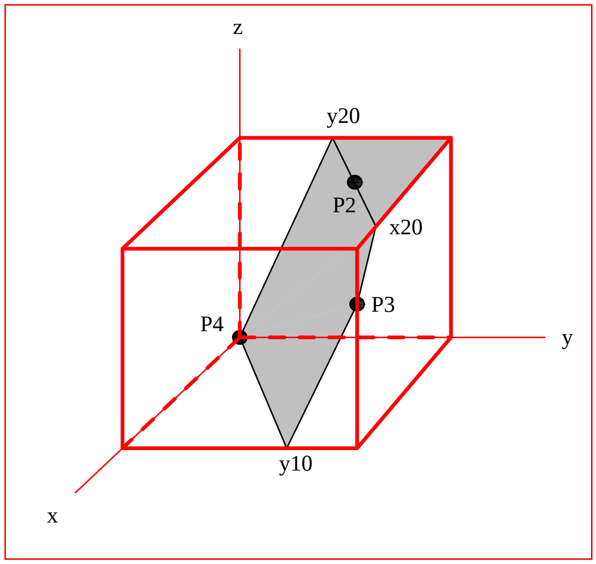

FIGURE 4. The upper bounds for z_1 in the calculation of $E(U_2)$.

4.3. **Calculation of $E(U_3)$.** In this case we have P_4 at the origin and the other points in the non-adjacent faces. Let $P_3 = (x_3, y_3, 1)$, $P_2 = (x_2, 1, z_2)$, and $P_1 = (1, y_1, z_1)$ Cf. Figure 5. Because the points P_1, P_2, and P_3 sit in a similar position, we have a three-fold symmetry. WLOG, we are going to treat one of the three equivalent cases and define

$$n = (P_1 - P_4) \times (P_2 - P_4)$$

and

$$f = n \cdot (P_3 - P_4) = n_1 x_3 + n_2 y_3 + n_3.$$

Typically, $f \le 0$ when z_1, z_2, x_3, and y_3 are big, compare Figure 5. More precisely, assume that $n_1 \le 0$ and $n_2 \le 0$. We always have $n_3 \ge 0$. The area to integrate x_3 and y_3 over is seen in Figure 5. We have $x_{31} = -(n_2 + n_3)/n_1$ and $y_{31} = -(n_1 + n_3)/n_2$. We have $x_{31} \le 1$ and $y_{31} \le 1$ when $n_1 + n_2 + n_3 \le 0$. As can be seen in Figure 6, this condition does also imply that $x_{31} \ge 0$ and $y_{31} \ge 0$.

Let $y(x_3, ...)$ be the solution of $f = 0$ for y_3 and let $z(z_1, ...)$ be the solution of $n_1 + n_2 + n_3 = 0$ for z_2. We get

$$e = \int_0^1 dx_2 \int_0^1 dy_1 \int_{y_1}^1 dz_1 \int_{z(z_1,...)}^1 dz_2 \int_{x_{31}}^1 dx_3 \int_{y(x_3,..)}^1 dy_3 = -\frac{1957}{3888} + \frac{11}{216}\pi^2. \tag{3}$$

In this case, the integral of f over B equals $\frac{1}{2}$. We get

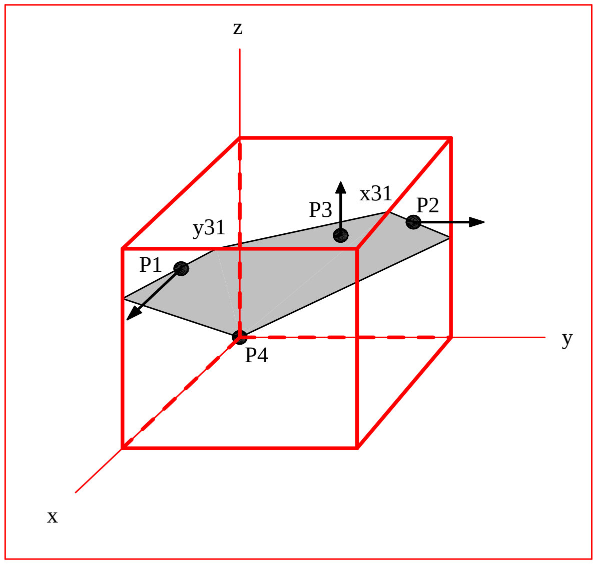

FIGURE 5. The plane $f = 0$ in Case 3. f is negative below the plane and x_3 and y_3 shall be integrated over the triangle in the top front corner.

$$E(U_3) = (\frac{1}{2} - 2 \cdot 3 \cdot e)/6 = \frac{2281}{3888} - \frac{11}{216}\pi^2.$$

4.4. **Calculation of $E(U_4)$.** In this case we have P_2, P_3 and P_4 all sit on edges and P_1 is interior. Let $P_2 = (0, 1, z_2)$, $P_3 = (x_3, 0, 1)$, and $P_4 = (1, y_4, 0)$ Cf. Figure 7.

Define

$$n = (P_2 - P_4) \times (P_3 - P_4)$$

making all n_k positive. We get

$$f = n \cdot (P_1 - P_4) = n_1 x_1 + n_2 y_1 + n_3 z_1 - x_3 y_4 z_2 - 1,$$

and f is negative below the plane $f = 0$ in Figure 7. This plane cuts three edges without points at the coordinates $x_0 = z_2 n_3 / n_1$, $y_0 = x_3 n_1 / n_2$, and $z_0 = y_4 n_2 / n_3$ and these three coordinates are between 0 and 1. Let $z(x_1, y_1, ...)$ be the solution of $f = 0$ for z_1. The integral over negative f is

$$e = \int_0^1 dx_3 \int_0^1 dy_4 \int_0^1 dz_2 \int_0^1 dx_1 \int_0^1 dy_1 \int_{max(z(x_1,y_1,...),0)}^{min(z(x_1,y_1,...),1)} dz_1 = -\frac{19}{648} - \frac{\pi^2}{72}. \tag{4}$$

By symmetry, the integral of f over B equals 0. We get

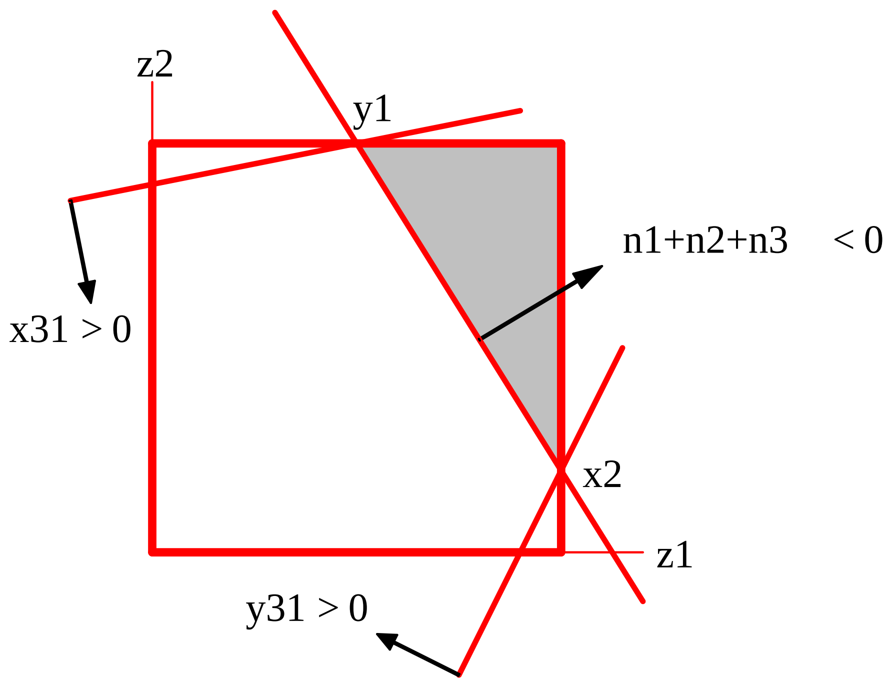

FIGURE 6. The area to integrate z_1 and z_2 over when calculating $E(U_3)$. The figure alse shows that the conditions $x_{31} \geq 0$ and $y_{31} \geq 0$ are fulfilled in this area.

$$E(U_4) = -2e/6 = \frac{19}{1944} + \frac{\pi^2}{216}.$$

4.5. **Calculation of $E(U_5)$.** In this case P_1 and P_2 shall sit in opposite faces. We put P_4 on the x-axis and let $P_3 = (x_3, 1, 1)$, $P_2 = (0, y_2, z_2)$, and $P_1 = (1, y_1, z_1)$ Cf. Figure 8. Define

$$n = (P_3 - P_4) \times (P_2 - P_4)$$

and

$$f = n \cdot (P_1 - P_4) = n_1(1 - x_4) + n_2 y_1 + n_3 z_1.$$

We have $n_2 \leq 0$ and $n_3 \geq 0$. Assuming, WLOG, that $z_2 \geq y_2$, we have $n_1 \geq 0$. We have f negative below the plane $f = 0$ so we shall integrate over P_1 sitting in the triangle in the lower right corner of the front of the cube. Let $z(y_1, ...)$ be the equation of the line through y_{10} and z_{10}. This triangle exists if $y_{10} = -(1-x_4)n_1/n_2 \leq 1$ (and $z_{10} \geq 0$).

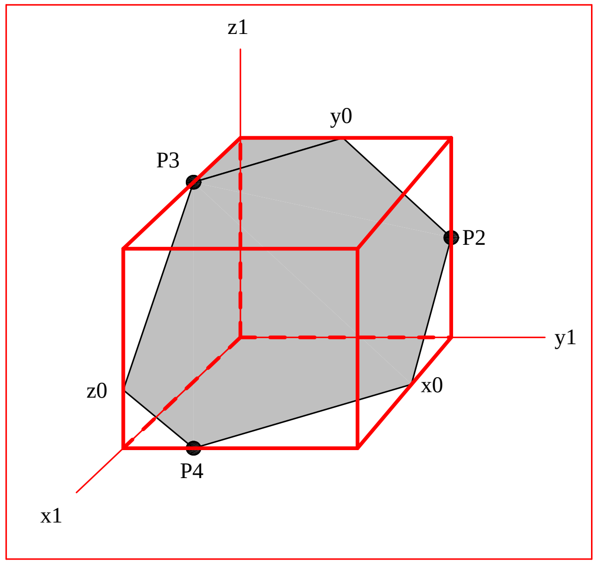

FIGURE 7. The upper bound of z_1 when calculating $E(U_4)$.

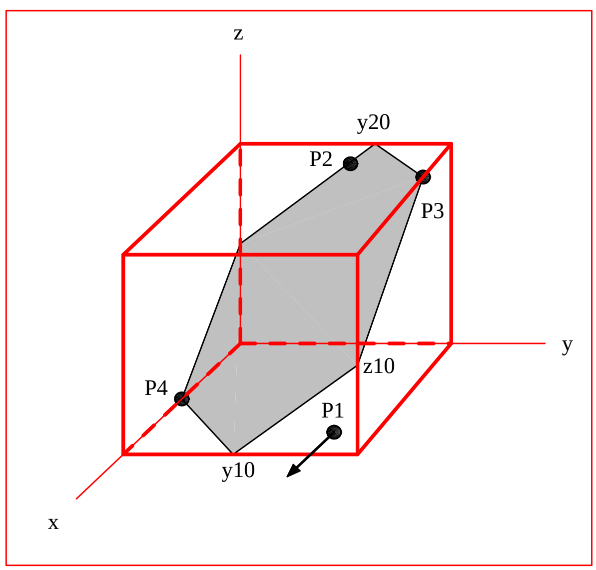

FIGURE 8. The plane $f = 0$ when calculating $E(U_5)$.

Explicitely, this condition reads

$$z_2 x_3 + (1 - z_2) x_4 \geq (1 - x_4)(z_2 - y_2).$$

Let $x(x_3, ...)$ be the solution of this expression for x_4. We get

$$e = \int_0^1 dy_2 \int_{y_2}^1 dz_2 \int_0^1 dx_3 \int_{max(0,x(x_3,...))}^1 dx_4 \int_{y_{10}}^1 dy_1 \int_0^{z(y1,...)} f\, dz_1.$$

This case is not symmetric and the integral of f over the part of B where $z_2 \geq y_2$ equals $\frac{1}{12}$. Taking the condition $z_2 \geq y_2$ into account, we get

$$E(U_5) = 2(\frac{1}{12} - 2e)/6 = \frac{23}{1944} + \frac{\pi^2}{324}.$$

4.6. **Calculation of $E(U_6)$.** This case is similar to case 5, but P_1 and P_2 shall sit in adjacent faces. We put P_4 on the x-axis and let $P_3 = (0, y_3, 1)$, $P_2 = (x_2, 1, z_2)$, and $P_1 = (1, y_1, z_1)$ Cf. Figure 9. Define

$$n = (P_2 - P_4) \times (P_3 - P_4)$$

and

$$f = n \cdot (P_1 - P_4) = n_1(1 - x_4) + n_2 y_1 + n_3 z_1.$$

Here, we have $n_1 \geq 0$, $n_3 \geq 0$, while n_2 can have any sign. For f to be negative for positive y_1, we must have $n_2 \leq 0$. Then, $y_{10} = -(1-x_4)n_1/n_2 \geq 0$. Let $z(y_1, ...)$ be the equation of the line through y_{10} and z_{10}. The triangle to integrate over exists if $y_{10} \leq 1$ (and $z_{10} \geq 0$). Explicitely, this condition reads

$$x_2 + (x_4 + y_3 - x_4 y_3) z_2 \geq 1.$$

Let $x(z_2, ...)$ be the solution of this expression for x_2. The smallest value of $x(z_2, ...)$ is obtained for $z_2 = 1$ and is $x_{20} = (1 - y_3)(1 - x_4) \geq 0$. We get

$$e = \int_0^1 dx_4 \int_0^1 dy_3 \int_0^1 dz_2 \int_{x(z_2,...)}^1 dx_2 \int_{y_{10}}^1 dy_1 \int_0^{z(y1,...)} f\, dz_1.$$

The integral of f over B equals $\frac{1}{2}$. We get

$$E(U_6) = (\frac{1}{2} - 2e)/6 = -\frac{101}{3888} + \frac{11}{972}\pi^2.$$

5. The expectation of U.

We get the expectation of U by weighting together the expectations of the six cases by their probabilities. The required numbers can be

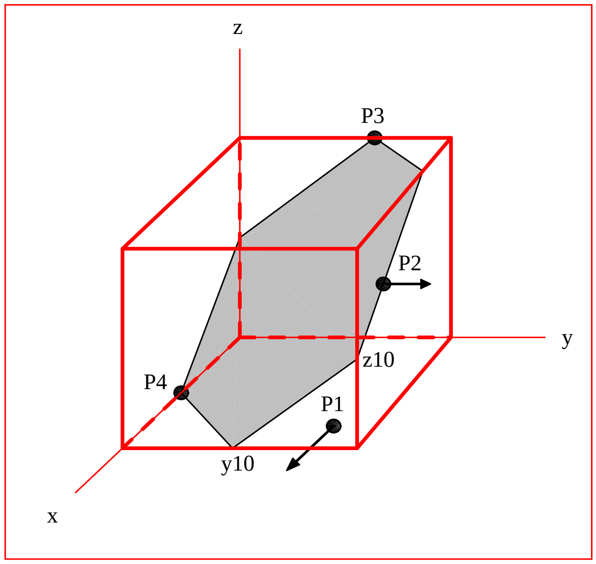

FIGURE 9. The plane $f = 0$ when calculating $E(U_6)$.

read from Table 1. We get

$$E(U) = \sum_{j=1}^{6} p_j \cdot E(U_j)$$

$$= \frac{1}{36} \cdot \frac{11}{432} + \frac{1}{3} \cdot (\frac{605}{7776} - \frac{\pi^2}{324}) + \frac{1}{9} \cdot (\frac{2281}{3888} - \frac{11\pi^2}{216}) + \frac{1}{9} \cdot (\frac{19}{1944} + \frac{\pi^2}{216})$$

$$+ \frac{1}{12} \cdot (\frac{23}{1944} + \frac{\pi^2}{324}) + \frac{1}{3} \cdot (-\frac{101}{3888} + \frac{11\pi^2}{972}) = \frac{3977}{46656} - \frac{25\pi^2}{11664}.$$

6. THE INDEPENDENCE OF U AND V.

U and V are independent because they are functions of separate coordinates of the points P_k and the coordinates are independent. The separation varies from case to case. For instance in case 1, V is a function of the six variables x_3, y_3, z_3, x_4, y_4, and z_4. The remaining six variables are those integrated over when calculating U_1. In case 6, for instance, V is a function of the six variables x_1, x_3, y_4, y_2, z_4, and z_3 while U_6 is a function of the remaining variables.

7. THE EXPECTATION OF V.

We shall calulate the expectation of $V = volume(B)/volume(A)$. The sidelength of B in the x-direction is

$$s_x = \max_{1 \le k \le 4} x_k - \min_{1 \le k \le 4} x_k.$$

In the average, the four x_k divide the unit length of the cube side into five intervals of equal length We get $E(s_x) = \frac{3}{5}$. The average in the y-, and z-directions are the same so we have

$$E(V) = \left(\frac{3}{5}\right)^3.$$

8. The expectation of the volume of a tetrahedron in a cube.

Having calculated the expectations of the independent random variables U and V, we get the expectation of their product X, which is the sought expectation of the volume of a random tetrahedron in a unit cube:

$$\begin{aligned} E(X) =& E(U) \cdot E(V) = (\frac{3977}{46656} - \frac{25\pi^2}{11664}) \cdot \frac{27}{125} \\ =& \frac{3977}{216000} - \frac{\pi^2}{2160} \approx .013842776. \end{aligned}$$

This number coincides with that obtained by Zinani [13] and with Monte Carlo tests.

9. Comments.

Even though the multiple integrals of this paper seem harmless to evaluate, they are not. They involve integrals of unbounded functions and several boundary insertions require a limiting process. Relations between dilogarithmic functions must be employed, see [1]. The order of integration is crucial. In many places the number of terms is of the order one hundred and we rely heavily on the use of a symbolic calculation program, in this case *Maple 10*.

Like us, Zinani [13] considers several geometrical cases. He has five cases and we have six. There is no correspondence between these case.

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