THE AREA OF A RANDOM CONVEX POLYGON IN A TRIANGLE.

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ABSTRACT. We deduced the probability distribution function for the area of the convex hull of three, and of four random points in a triangle. We also give some results about the number of vertices of these convex hulls.

1. INTRODUCTION

Either three or four points are generated at random inside a triangle A. Let T be the triangle or quadrangle spanned by the random points. We shall consider the random variable $X = area(T)/area(A)$. It is well known that any affine transformation will preserve the ratio X. This follows from the fact that the area scaling is constant for an affine transformation. The scale equals the determinant of the homogeneous part of the transformation. This means that our results hold for any shape of A.

Various aspects of our problem have been considered in the field of geometric probability, see e.g. [9]. J. J. Sylvester considered the problem of a random triangle T in an arbitrary convex set K and posed the following problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. It has been shown, [2] that $\kappa = \frac{11}{144}$ for K a square.

A. Reńyi and R. Sulanke, [7] and [8], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. R. E. Miles [5] generalizes these asymptotic estimates for K a circle to higher dimensions. C. Buchta and M. Reitzner, [4], has given values of κ (generalized to three dimensions) for $n \geq 4$ points in a tetrahedron.

H. A. Alikoski [2] has given expressions for κ when $n = 3$ and K a regular r-polygon. In a previous paper, [6], we have given the whole probability distribution of X for $n = 3$ and $n = 4$ and K a parallelogram. We also gave some asymptotic estimates in the spirit of Reńyi

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and Sulanke. Here we deduce the whole distribution of X for $n = 3$ and $n = 4$ when K is a triangle. From these distributions we calculate some probabilities for the number of vertices of random convex polygons. All calculated quantities of this paper have been confirmed by Monte Carlo tests.

2. Notation and formulation.

As A, we will chose a triangle having vertices in $(0, 0), (0, 1)$, and $(1, 0)$. We use a constant probability density in A for generating n random points in A. The points will be denoted P_k and have coordinates (x_k, y_k) for $1 \leq k \leq n$. Let T be the convex hull of the n points. We shall determine the probability distribution of the random variable $X = area(T)/area(A)$ when $n = 3$ and 4.

The generated T spans a triangle with sides parallell to the sides of A. We will call this spanned triangle B. The random variable X, that we study will be written as the product of the two independent random variables

$$U = area(T)/area(B) \text{ and } V = area(B)/area(A).$$

Roughly speaking, U describes the shape of T and V its size. It's not obvious that the size V is independent of the shape U. The independence will be shown in section 3.3. We shall determine the distributions for U and V and combine them to get the distribution of $X = UV$.

3. The convex hull of three points.

The convex hull of three points is with probability one a triangle.

3.1. **The two geometrical cases for three points and their probabilities.**

Since B is spanned by T, two or three of the vertices of T sit on the boundary of B. We have two cases: (1) Two vertices sit on the boundary and one is interior and (2) All three vertices sit on the boundary. These two cases are pictured in Figures 1 and 2.

In Figure 1, we have without loss of generality (WLOG) chosen P_1 to be the point that sits in a vertex of B, and this vertex is chosen to be the lower left vertex. The point P_2 sits on the opposite side and P_3 is interior.

When studying the ratio U, we have enlarged B so that its vertices are $(0, 0)$, $(0, 1)$, and $(1, 0)$.

We shall show that the cases 1 and 2 occur with the probabilities $p_1 = \frac{3}{5}$ and $p_2 = \frac{2}{5}$, respectively.

To this end, we shall calculate the probability that P_1, P_2, and P_3 sit as in Figure 1. This is the case if

$$x_1 \leq x_2, \quad x_1 \leq x_3, \quad y_1 \leq y_2, \quad y_1 \leq y_3, \text{ and } x_3 + y_3 \leq x_2 + y_2.$$

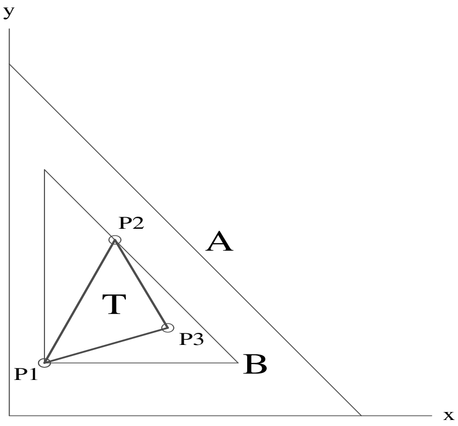

FIGURE 1. Triangle with point 1 in a vertex, point 2 on the opposite side, and point 3 interior in the 'big' subtriangle B that it spanns (Case 1) .

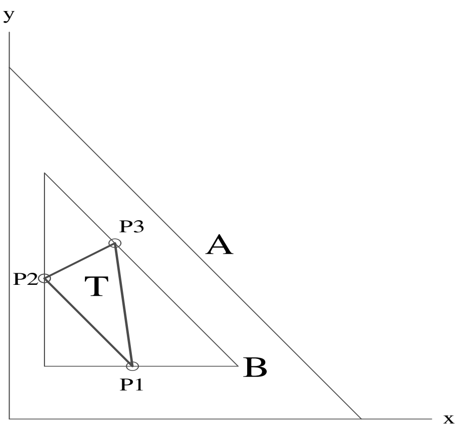

FIGURE 2. Triangle with points 1, 2, and 3 on the boundary of the 'big' subtriangle B that it spanns (Case 2) .

We write the expression first and explain it afterwards.

$$\text{Prob(The } P_k \text{ sit as in Figure 1)} = 8\int_0^1 dx_1 \int_0^{1-x_1} dy_1$$
$$\int_{x_1}^{1-y_1} dx_2 \int_{y_1}^{1-x_2} dy_2 \int_{x_1}^{x_2+y_2-y1} dx_3 \int_{y_1}^{x_2+y_2-x_3} dy_3 = \frac{1}{30}.$$

The factor 8 is 2^3, where the factor 2 is the inverse of the triangle area. The three points can be put in the three positions: lower left corner, interior, and right side in $\frac{3!}{1!1!1!} = 6$ ways. It follows from the affine invariance that all three vertices of B have the same probability of coinciding with one of the generated points, so Case 1 occurs with the probability $p_1 = \frac{6 \cdot 3}{30} = \frac{3}{5}$. The complementary event, Case 2, has the probability $p_2 = \frac{2}{5}$.

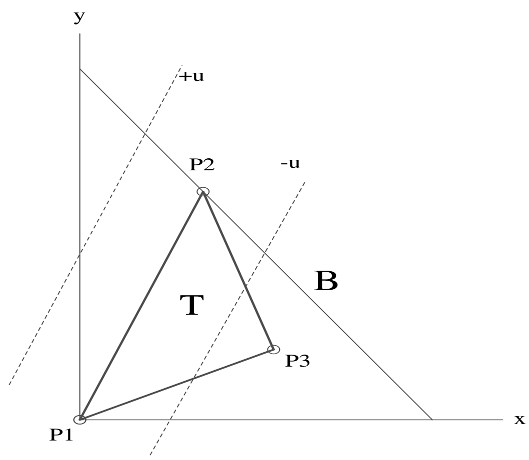

FIGURE 3. The triangle area fraction U is less than u if the point P_3 sits between the lines marked $\pm u$. This doesn't happen in this figure.

3.2. **The distribution of U.** We denote the distribution functions of the fraction U by $H_{3,1}(u)$ and $H_{3,2}(u)$, respectively, in the two cases.

3.2.1. *Calculation of $H_{3,1}(u)$ for Case 1.* Referring on Figure 3, we let P_1 be in the vertex, P_2 be the point on the side opposite to P_1 and P_3 be interior. Once P_2 is known to sit on the side of B, its conditional density is the restriction to the side of the density in the triangle, meaning that the density is constant along the side. The side P_1P_2 of T has length $d = \sqrt{x_2^2 + (1-x_2)^2}$. We have $area(T)/area(B) \le u$ if the distance from P_1P_2 to P_3 is smaller than u/d. This happens if P_3 sits between the lines

$$-\frac{\xi}{x_2} + \frac{\eta}{1-x_2} = \pm\frac{u}{x_2(1-x_2)}.$$

For fixed u and x_2, the area between the lines takes up the following fraction of B when $0 \le u \le 1/2$.

$$f(u|x_2) = \begin{cases} 1 - \frac{(1-u-x_2)^2}{1-x_2}, & 0 \le x_2 \le u \\ 1 - \frac{(1-u-x_2)^2}{1-x_2} - \frac{(x_2-u)^2}{x_2}, & u \le x_2 \le 1-u \\ 1 - \frac{(x_2-u)^2}{x_2}, & 1-u \le x_2 \le 1. \end{cases} \tag{1}$$

For $1/2 \le u \le 1$, we have instead.

$$f(u|x_2) = \begin{cases} 1 - \frac{(1-u-x_2)^2}{1-x_2}, & 0 \le x_2 \le 1-u \\ 1, & 1-u \le x_2 \le u \\ 1 - \frac{(x_2-u)^2}{x_2}, & u \le x_2 \le 1. \end{cases} \tag{2}$$

We get the wanted distribution function by integrating the conditional probability $f(u|x_2)$ over x_2:

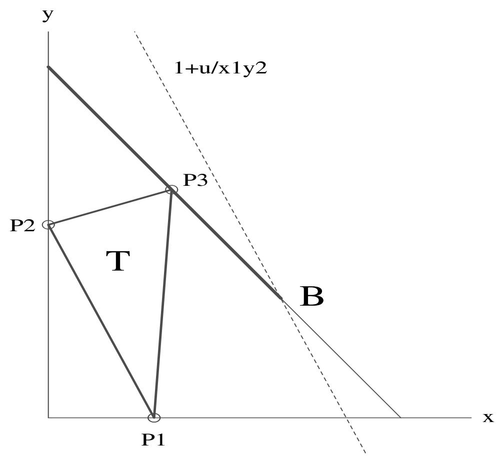

FIGURE 4. The triangle area fraction U is less than u if the point $P3$ sits to the left of the line marked $1+u/x_1y_2$, as it does in this figure.

$$\text{Prob}(U \leq u) = H_{3,1}(u) = \int_0^1 f(u|x_2)\,dx_2. \tag{3}$$

Even though the expressions (1) and (2) are different, we get one analytic expression for $H_{3,1}(u)$ valid for all u. We will not carry out the integrations in detail, but just present the results [1]

$$H_{3,1}(u) = 4u - 3u^2 + 2u^2\log(u), \quad 0 \leq u \leq 1. \tag{4}$$

3.2.2. *Calculation of $H_{3,2}(u)$ for Case 2.* In Case 2, we have the situation depicted in Figure 4 with one point on each side of B. Let P_1 be the point on the x-axis, P_2 on the y-axis and let P_3 sit on the slanted side. For fixed x_1 and y_2, one side of T has the length $d = \sqrt{x_1^2 + y_2^2}$. If the orthogonal distance from this side to P_3 is less than u/d, we have an area fraction less than u. This happens if P_3 is to the left of the line

$$\frac{\xi}{x_1} + \frac{\eta}{y_2} = 1 + \frac{u}{x_1 y_2}.$$

This line intersects the P_3-line $x_3 + y_3 = 1$ in the point

$$\xi_3 = \frac{u - x_1(1 - y_2)}{y_2 - x_1}. \tag{5}$$

When handling this expression, WLOG, we can assume that $y_2 > x_1$.

We have x_3 evenly distributed in $(0, 1)$, so the area fraction $f(u|x_1, y_2)$ will be less than u when $x_3 \leq \xi_3$ provided $0 \leq \xi_3 \leq 1$, otherwise zero or one. We get

[1] We are indebted to Maple for helping us with the integrations of this paper.

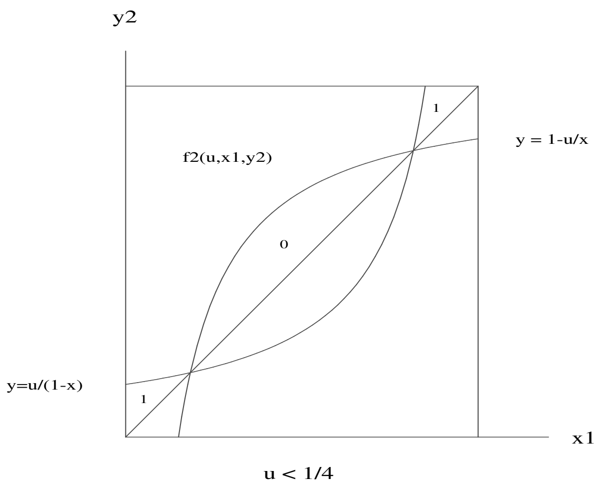

FIGURE 5. Area to integrate $f(u|\, x_1, y_2)$ over when $u \le 1/4$.

$$f(u|x_1, y_2) = \begin{cases} 0, & u \le x_1(1-y_2), \\ \frac{u - x_1(1-y_2)}{y_2 - x_1}, & x_1(1-y_2) \le u \le y_2(1-x_1), \\ 1, & y_2(1-x_1) \le u. \end{cases} \tag{6}$$

Like in Case 1, we get

$$\text{Prob}(U \le u) = H_{3,2}(u) = 2\int_0^1 dx_1 \int_{x_1}^1 f(u|x_1, y_2)\, dy_2. \tag{7}$$

The factor 2 accounts for the restriction $y_2 > x_1$. The nonlinear boundaries of the domain of definition of $f(u|x_1, y_2)$, requires two different ways of calculating this integral, one for $u \le 1/4$ and one for $u > 1/4$. The areas to integrate over in the two cases are depicted in Figures 5 and 6. For $u \le 1/4$, we use the substitution $t = x_1 - \frac{1}{2} + \sqrt{1/4 - u}$ when carrying out the integration in 7.

We get

$$H_{3,2}(u) = \begin{cases} \frac{1}{3}(\log \frac{1-3u+(1-u)\sqrt{1-4u}}{2u^{3/2}})(1-4u)^{3/2} \\ +\frac{1}{2}(1-6u)\log u + u, \\ & 0 \le u \le 1/4, \\ \frac{1}{3}(\arccos \frac{1-3u}{2u^{3/2}} - \pi)(4u-1)^{3/2} \\ +\frac{1}{2}(1-6u)\log u + u, \\ & 1/4 < u \le 1. \end{cases} \tag{8}$$

The density functions for Case 1 and 2 are shown in Figure 7.

3.3. **The distribution of V for three generated points.** We shall calulate the distribution $G_3(v)$ of the ratio $V = area(B)/area(A)$. One could suspect that the distribution of V might be different in the Cases

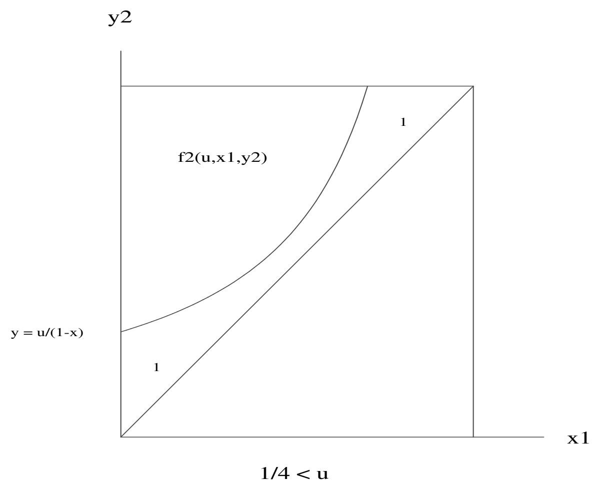

FIGURE 6. Area to integrate $f(u|\,x_1, y_2)$ over when $u > 1/4$.

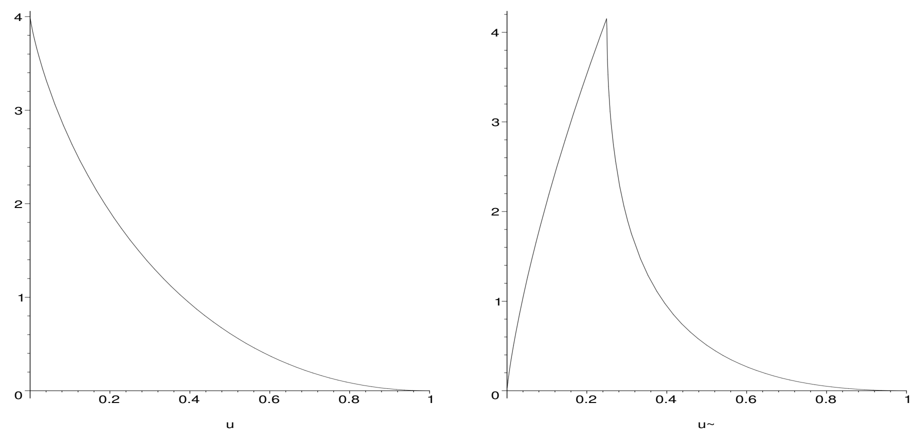

FIGURE 7. Density functions $dH_{3,1}/du$ and $dH_{3,2}/du$ for cases 1 and 2.

1 and 2, so that V would be dependent of U. This is not the case, but we have found no simple way of proving it. Therefore, we shall calculate the distribution function of V for both cases and we will find that they are identical. We do Case 1 here and Case 2 in Appendix B.

Case 1 is characterized by the following set of inequalities

$$x_1 \le x_2 \quad x_1 \le x_3 \quad y_1 \le y_2 \quad y_1 \le y_3.$$

WLOG, we can add the inequality $x_3 + y_3 \le x_2 + y_2$. We start by calculating the probability that this situation occurs .

(9)

$$p_{c1} = \text{Prob(Case 1)} =$$

$$\int_0^1 dx_1 \int_0^{1-x_1} dy_1 \int_{x_1}^{1-y_1} dx_3 \int_{y_1}^{1-x_3} dy_3 \int_{x_1}^{1-y_1} dx_2 \int_{y_1}^{1-x_2} 1\, dy_2 = \frac{1}{120}.$$

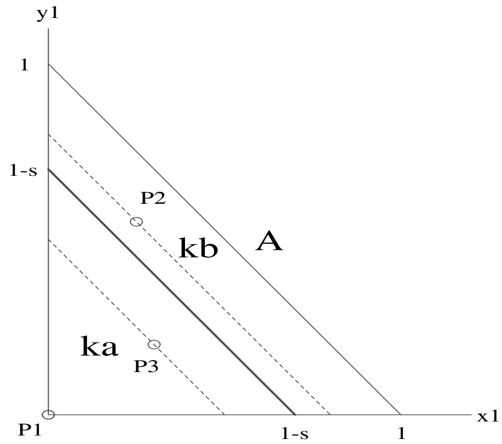

FIGURE 8. Area to integrate k_a and k_b over in the x_1, y_1-plane

We continue by calculating the probability that the horisontal side length of B, which is $S = x_2 + y_2 - x_1 - y_1$, is less than s. We have two different integrations over x_2, y_2, x_3, and y_3 depending on whether $s + x_1 + y_1 - 1$ is negative or positive. Calling the two cases a and b, we have the following conditional probabilities

$$k_{1a}(s|x_1, y_1; s + x_1 + y_1 \leq 1) = \int_{x_1}^{s+x_1} dx_3 \int_{y_1}^{s+x_1+y_1-x_3} dy_3 \int_{x_1}^{s+x_1} dx_2 \int_{y_1}^{s+x_1+y_1-x_2} dy_2 = \frac{s^4}{4} \tag{10}$$

and

$$k_{1b}(s|x_1, y_1; s + x_1 + y_1 > 1) = \int_{x_1}^{1-y_1} dx_3 \int_{y_1}^{1-x_3} dy_3 \int_{x_1}^{1-y_1} dx_2 \int_{y_1}^{1-x_2} dy_2 = \frac{(1 - x_1 - y_1)^4}{4}. \tag{11}$$

The areas to integrate over in the $x_1 y_1$-plane are shown in Figure 8. We get

$$\begin{aligned} K_{3,1}(s) = \text{Prob}(S \leq s) &= \frac{1}{p_{c1}} \left(\int_0^{1-s} dx_1 \int_0^{1-s-x_1} k_{1a} \, dy_1 \right. \\ &\left. + \int_0^{1-s} dx_1 \int_{1-s-x_1}^{1-x_1} k_{1b} \, dy_1 + \int_{1-s}^{1} dx_1 \int_0^{1-x_1} k_{1b} \, dy_1 \right) \\ &= 15s^4 - 24s^5 + 10s^6. \end{aligned} \tag{12}$$

The corresponding calculation in Case 2 (compare Figure 2) is more complicated and deferred to appendix B. It results in the same distribution function as in (12).

Since the area ratio $V = \frac{area(B)}{area(A)} = \frac{S^2/2}{1/2} = S^2$, we get the distribution function for V directly

$$G_3(v) = 15v^2 - 24v^{5/2} + 10v^3, \qquad 0 \le v \le 1\,. \tag{13}$$

3.4. **The distribution of X for three generated points.** Having the distribution functions $H_{3,1}(u)$, $H_{3,2}(u)$ and $G_3(v)$ of U and V, we can start calculating the distribution function $F_3(x)$ of $X = UV$. We have

$$\begin{aligned}&\text{Prob}(X \le x) = F_3(x) = \\ &\text{Prob(Case 1)}\, F_3(x|\text{Case 1}) + \text{Prob(Case 2)}\, F_3(x|\text{Case 2}).\end{aligned}$$

For $i = 1$ or 2, we have

$$F_3(x|\text{Case i}) = \text{Prob}(X = U_iV_i \le x) = \int_0^1 G_{3,i}(x/u)\, dH_{3,i}(u).$$

Since G is the same in both cases, we form

$$H_3(u) = p_{c1}H_{3,1}(u) + p_{c2}H_{3,2}(u) = \frac{3}{5}H_{3,1}(u) + \frac{2}{5}H_{3,2}(u),$$

and can write

$$\begin{aligned}F_3(x) &= \int_0^1 G_3(x/u)\, dH_3(u) = \\ &= [G_3(x/u)H_3(u)]_0^1 - \int_x^1 H_3(u)\frac{d}{du}G_3(x/u)\, du = \\ &= G_3(x) - \int_x^1 H_3(u)\frac{d}{du}G_3(x/u)\, du, \qquad 0 \le x \le 1.\end{aligned} \tag{14}$$

The partial intergration in (14) is used to avoid integrating to the lower bound $u = 0$.

We will not carry out the integration (14) in detail, but will just give the result

(15)

$$F_3(x) = \begin{cases} 8(2x^3+3x^2)(3\log\frac{1+\sqrt{1-4x}}{2}\log\frac{1+\sqrt{1-4x}}{2x} - \pi^2/3) \\ +\frac{2}{5}(324x^2+28x-1)(\log x/2 - \log\frac{1+\sqrt{1-4x}}{2})\sqrt{1-4x} \\ +12x^3(\log x)^2 - (54x^2+6x-1/5)\log x - 57x^2/5 + 62x/5, \\ \hfill 0 \le x \le 1/4, \\ 8(2x^3+3x^2)((2\pi - 3\arccos\frac{1}{2\sqrt{x}})\arccos\frac{1}{2\sqrt{x}} - \pi^2/3) \\ +\frac{2}{5}(324x^2+28x-1)(\arccos\frac{1}{2\sqrt{x}} - \pi/3)\sqrt{4x-1} \\ -18x^2(\log x)^2 - (54x^2+6x-1/5)\log x - 57x^2/5 + 62x/5, \\ \hfill 1/4 < x \le 1. \end{cases}$$

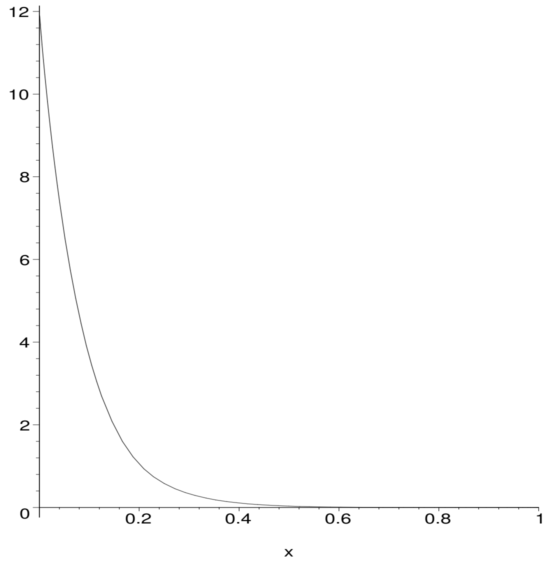

FIGURE 9. Density function $dF_3(x)/dx$ for the area fraction of the convex hull of three random points inside a triangle.

The density $dF_3(x)/dx$ is shown in Figure 9.

The first moments and the standard deviation of X are

$$\alpha_1 = \int_0^1 x\,dF_3(x) = \frac{1}{12}, \tag{16}$$

$$\alpha_2 = \int_0^1 x^2\,dF_3(x) = \frac{1}{72}, \tag{17}$$

$$\sigma = \sqrt{\alpha_2 - {\alpha_1}^2} = \frac{1}{12}, \tag{18}$$

$$\alpha_3 = \int_0^1 x^3\,dF_3(x) = \frac{31}{9000} \approx .00344. \tag{19}$$

4. THE CONVEX HULL OF FOUR POINTS.

With four generated points, the convex hull can be either a triangle or a quadrangle.

4.1. **The five geometrical cases for four points and their probabilities.** We have the same cases as with three points characterized by the number of generated points that span the 'big' subtriangle B, but these cases split up into subcases depending on the position of the fourth point. The cases and their probabilities are described in table I. The points will be denoted P_1, P_2, P_3, and P_4. The indices have nothing to do with the order in which they were generated or anything else.

It can be read from the total probability row of table 1 that the probability that four points generate a triangle is $\frac{1}{3}$. This is in accordance

case	a	b	c	d	e
char.	P_1 in vertex of B			P_1, P_2, and P_3 on the sides of B	
Prob	$\frac{3}{7}$			$\frac{4}{7}$	
char.	P_3 and P_4 on same side		P_3 and P_4 on opposite sides	P_4 interior	P_4 exterior
Prob	$\frac{2}{3}$		$\frac{1}{3}$	$\frac{1}{4}$	$\frac{3}{4}$
			quadrangle	triangle	quadrangle
char.	triangle	quadrangle			
Prob	$\frac{2}{3}$	$\frac{1}{3}$			
total Prob	$\frac{4}{21}$	$\frac{2}{21}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{3}{7}$
E_U	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{10}{27}$

TABLE 1. The 5 geometrical cases for four points, their characterizations and their probabilities. The last row holds the mathematical expectation of U in each case.

with the following arguement: The probability that the fourth point shall sit in the triangle generated by the first three points equals the average area fraction for three points which is $\frac{1}{12}$. The fourth point can be chosen in 4 ways, so the probability for triangle is $\frac{4}{12} = \frac{1}{3}$.

4.2. **Case probabilities for four points.** We shall calculate the probabilities listed in table 1 and the $U-$distributions for the five cases. We start with the probabilities $\frac{3}{7}$ and $\frac{4}{7}$, for the main cases which correspond to to cases 1 and 2 for three points. Case 1 encompasses cases a, b, and c, Case 2 cases d, and e. The calculations are analogous to those in section 3.1.

To this end, we shall calculate the probability that P_1 with coordinates x_1 and y_1 is the lower left vertex of B and P_2 sits on the opposite side of B. This is the case if

$$x_1 \leq x_2, \quad x_1 \leq x_3, \quad x_1 \leq x_4, \quad y_1 \leq y_2, \quad y_1 \leq y_3 \quad y_1 \leq y_4,$$

$$x_3 + y_3 \leq x_2 + y_2 \text{ and } x_4 + y_4 \leq x_2 + y_2.$$

We write the expression first and explain it afterwards.

Prob(the P_i sit as described above) =

$$16\int_0^1 dx_1\int_0^{1-x_1} dy_1\int_{x_1}^{1-y_1} dx_2\int_{y_1}^{1-x_2} dy_2\int_{x_1}^{x_2+y_2-y1} dx_3$$
$$\int_{y_1}^{x_2+y_2-x_3} dy_3\int_{x_1}^{x_2+y_2-y_1} dx_4\int_{y_1}^{x_2+y_2-x_4} dy_4 = \frac{1}{84}.$$

The factor 16 is 2^4, where factor 2 is the inverse of the triangle area. The four points can be put in the three positions: lower left corner, two interior, and one on the right side in $\frac{4!}{1!2!1!} = 12$ ways. It follows from the affine invariance that all three vertices of B have the same probability of coinciding with one of the generated points, so Case 1 occurs with the probability $p_1 = \frac{12\cdot 3}{84} = \frac{3}{7}$. The complementary event, Case 2, has the probability $p_2 = \frac{4}{7}$.

Moving to the second level in table 1, we start by considering the splitting of case 1 into cases a+b and case c. Case c is characterized by that the points P_3 and P_4 sit on opposite sides of the line P_1P_2. When calculating the probability for this to happen, we scale up B so that it fills the whole A. For fixed x_2, the probability is x_2 that a point sits to the left of P_1P_2. We get

$$\text{Prob(one on each side)} = \binom{2}{1}\int_0^1 x_2(1-x_2)dx_2 = \frac{1}{3}.$$

Case d, which is a subcase of 2 occurs when P_4 sits in the triangle $T = P_1P_2P_3$, cf. Figure 2. The area of T is $\frac{1}{2}(x_1(1-y_2)+x_3(y_2-x_1))$. We get

$$\text{Prob}(P_4 \text{ in } T) = \int_0^1 dx_1\int_0^1 dy_2\int_0^1 (x_1(1-y_2)+x_3(y_2-x_1))dx_2 = \frac{1}{4}$$

In studying the discrimination between cases a and b on the third level of table 1, we refer to Figure 10. This Figure shows an affine transformation of the triangle with vertices: top vertex of B, P_1, and P_2 on the vertices $(0,1)$, $(0,0)$, and $(1,0)$, respectively. The point P_3 is assumed to sit in this triangle and is mapped on (x,y). The point (x,y) determines the four triangles T_1 - T_4 in Figure 10. Let $q_i = \text{Prob}(\, P_4 \in T_i\,)$. We have q_i is proportional to the area of T_i:

$$q_1 = \frac{xy}{1-x},\quad q_2 = \frac{x}{x+y} - \frac{xy}{1-x},\quad q_3 = \frac{y}{x+y} - y,\quad q_4 = y.$$

The quadrangle case b occurs if P_4 sits in T_1 or T_3 and by the affine invariance, the probabilities are the same for these two triangles. We get

$$\text{Prob(quadrangle)} = 2\binom{2}{1}\int_0^1 dx\int_0^{1-x} q_1 dy = \frac{1}{3}.$$

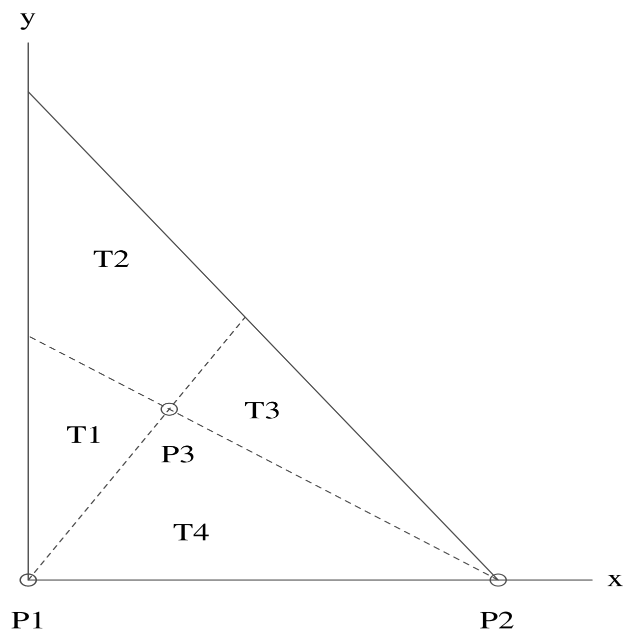

FIGURE 10. The areas relevant for calculation of cases a and b.

We get the total probabilities for each case in table 1 by multiplying together the case splitting probabilities above them.

4.3. **Calculation of the U-distributions for the five cases.** Each case requires its own calculation. As it turns out, the complexity of the calculations increases from case a through b, c, and d to the most complicated case e. The cases a, b, and c are similar and we will do cases a and b here and case c in Appendix C. Case d will be done here and case e in Appendix D.

4.3.1. *Calculation of $H_{4a}(u)$.* In cases a and b, we have P_1, P_2, and P_3 as in Figure 3 and P_4 falling on the same side of P_1P_2 as P_3. Case a occurs when P_4 sits in T_2 or T_4 in Figure 10. By the affine invariance, the distributions for these two sets are the same and we will do the calculations for T_4. We have

$$\text{Prob}(U < z) = \text{Prob}(2T_4 < z) = \text{Prob}(y < z) = \begin{cases} 1, & y < z \\ 0, & y \geq z. \end{cases}$$

We integrate this conditional probability times the probability that it occurs $= q_4 = y$ and get

(20)
$$K_a(z) = \left(\frac{1}{2}\right)^{-1} \left(\frac{1}{3}\right)^{-1} \int_0^z dy \int_0^{1-y} y\,dx = 3z^2 - 2z^3, \quad 0 \leq z \leq 1.$$

The whole triangle in Figure 10 is the affine image of the triangle to the left of P_1P_2 of size $x_2/2$. This implies that the conditional distribution for U is $f(u \mid x_2) = K_a(u/x_2)$, when $x_2 \geq u$ else 1. We get the distribution function for U by integrating $f(u \mid x_2)$ times the probability $= 3x_2^2$ that P_3 and P_4 sit to the left of P_1P_2

(21)
$$H_{4a}(u) = \int_0^u 3x_2^2\,dx_2 + \int_u^1 3x_2^2 K_a(u/x_2)\,dx_2 = u^2(9 - 8u + 6u\log u), \quad 0 \leq u \leq 1.$$

4.3.2. *Calculation of $H_{4b}(u)$.* In case b, we have again the situation depicted in Figure 3 with P_3 and P_4 falling on the same side of P_1P_2. Case b occurs when P_4 sits in T_1 or T_3 in Figure 10. By the affine invariance, the distributions for these two sets are the same and we will do the calculations for $P_4 \in T_1$. Let w be the fraction of T_1 that together with T_4 forms the quadrangle. We get the quadrangle area $Z = wq_1 + q_4$. The distribution function for the fraction that the area nearest to the base takes up of a triangle is

(22) $\phi(w) = 1 - (1-w)^2 = 2w - w^2$ when $0 \leq w \leq 1$, else 0 or 1.

We get

$$f(z \mid x, y) = \phi(w) = \phi(\frac{z - q_4}{q_1}) = \phi(\frac{(z-y)(1-x)}{xy}),$$

when $y < z \leq y/(1-x)$, else 0 or 1. We get the Z-distribution by integrating $f(z \mid x, y)$ times the probability density $(= q_1)$ for (x, y)

$$K_b(z) = \left(\frac{1}{6}\right)^{-1} \int_0^z dy\,(\int_0^{1-y/z} q_1 dx + \int_{1-y/z}^{1-y} q_1\phi(\frac{(z-y)(1-x)}{xy})\,dx)$$
$$= 6(1-3z)(1-z)\log(1-z) - 8z^3 + 3z^2 + 6z - 12z^2\nu(z),$$

where $\nu(z)$ is defined in Appendix A. With the same argument as in case a, we get (compare (21))

(23)
$$\begin{aligned} H_{4b}(u) &= \int_0^u 3x_2^2\,dx_2 + \int_u^1 3x_2^2 K_b(u/x_2)\,dx_2 \\ &= u(-44u^2 + 39u + 6) + 12u^3\log u \\ &\quad - 6(2u^2 + 5u - 1)(1-u)\log(1-u) - 36u^2\nu(u), \\ &\qquad 0 \leq u \leq 1. \end{aligned}$$

4.3.3. *Calculation of $H_{4d}(u)$.* In case d, we have the situation depicted in Figure 4 with P_4 falling inside T. The calculations are similar to those in section 3.2.2. There, we got $f(u \mid x_1, y_2)$ by integrating 1 from 0 to ξ_3. Here, we shall not integrate 1, but the probability that P_4 sits in T over the same interval. This probability is proportional to the size of $T = x_1(1-y_2) + x_3(y_2 - x_1)$. Integrating over x_3, we get (compare (6))

(24)

$$f(u|x_1, y_2) = \begin{cases} 0, & u \le x_1(1-y_2), \\ x_1(1-y_2)\,\xi_3 + (y_2 - x_1)\,{\xi_3}^2/2, & \\ & x_1(1-y_2) \le u \le y_2(1-x_1), \\ (x_1+y_2)/2 - x_1 y_2, & y_2(1-x_1) \le u, \end{cases}$$

where ξ_3 is given in (5). Like in (3), we get

$$\text{(25)}\quad \text{Prob}(U \le u) = H_{4d}(u) = 2\left(\frac{1}{4}\right)^{-1} \int_0^1 dx_1 \int_{x_1}^1 f(u|x_1, y_2)\, dy_2.$$

The areas to integrate over are the same as in Figures 5 and 6, but the value of $f(u|x_1, y_2$ is not 1 in the corners as given in the Figures but the value given above.

We get

(26)

$$H_{4d}(u) = \begin{cases} \frac{2}{5}(6u+1)(1-4u)^{3/2}(\log\frac{1+\sqrt{1-4u}}{2} - \log(u)/2) & \\ +\frac{1}{5}(1-30u^2)\log(u) + \frac{1}{5}u(3u+2), & \\ & 0 \le u \le 1/4, \\ \frac{2}{5}(6u+1)(4u-1)^{3/2}(\arccos\frac{1}{2\sqrt{u}} - \pi/3) & \\ +\frac{1}{5}(1-30u^2)\log(u) + \frac{1}{5}u(3u+2), & \\ & 1/4 \le u \le 1. \end{cases}$$

4.4. **The distribution of U for four generated points.** We get the total fraction distribution $H_4(u)$ by weighting together the five case distributions with the weights given on the last line but one in table 1.

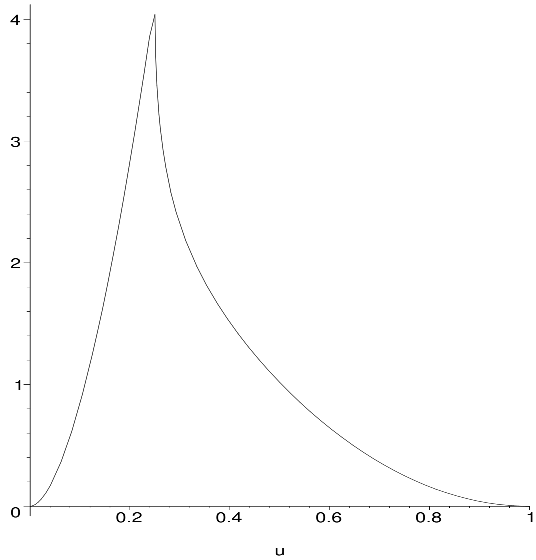

FIGURE 11. Density function $dH_{4d}(u)/du$ for U in case d. The peak value is 6 log(2).

(27)

$$H_4(u) = \frac{4}{21}H_{4a}(u) + \frac{2}{21}H_{4b}(u) + \frac{1}{7}H_{4c}(u) + \frac{1}{7}H_{4d}(u) + \frac{3}{7}H_{4e}(u) =$$

$$= \begin{cases} +\frac{16}{35}(9u^2+13u-1)\sqrt{1-4u}\,(\log(\frac{1+\sqrt{1-4u}}{2}) - \log(u)/2) \\ -\frac{72}{7}u^2\log(\frac{1+\sqrt{1-4u}}{2})\,(\log(\frac{1+\sqrt{1-4u}}{2}) - \log(u)) \\ -\frac{2}{35}(5u^4-20u^3+60u^2-60u+4)\log(u) \\ -\frac{2}{7}(u^3+9u^2+33u-7)(1-u)\log(1-u) \\ -\frac{1}{35}(230u^3-211u^2-54u) + \frac{8}{7}u^2(\pi^2-9\nu(u)), \\ \qquad 0 \le u \le 1/4. \\ +\frac{16}{35}(9u^2+13u-1)\sqrt{4u-1}(\pi/3 - \arccos\frac{1}{2\sqrt{u}}) \\ -\frac{72}{7}u^2(2\pi/3 - \arccos\frac{1}{2\sqrt{u}})\arccos\frac{1}{2\sqrt{u}} + \frac{18}{7}u^2(\log(u))^2 \\ -\frac{2}{35}(5u^4-20u^3+60u^2-60u+4)\log(u) \\ -\frac{2}{7}(u^3+9u^2+33u-7)(1-u)\log(1-u) \\ -\frac{1}{35}(230u^3-211u^2-54u) + \frac{8}{7}u^2(\pi^2-9\nu(u)), \\ \qquad 1/4 \le u \le 1. \end{cases}$$

The expected value of U is $E(U) = \frac{5}{14}$. Calculating the expectation for the triangle cases a and d separately gives $E(U_{triangle}) = \frac{5}{14}$ so $E(U_{quadrangle})$ must also have this value.

4.5. **The distribution of V for four generated points.** We shall calulate the distribution $G_4(v)$ of the ratio $V = area(B)/area(A)$.. The calculations are similar to those for $G_3(v)$. Like for $G_3(v)$, we

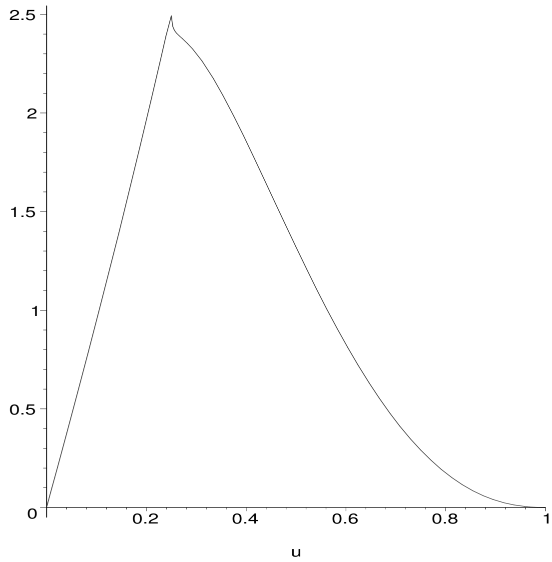

FIGURE 12. Density function $dH_4(u)/du$ for the area fraction of the 'big' subtriangle for four random points in a triangle.

shall do the calculations for the main Case 1, which is characterized by the following set of inequalities

$$x_1 \le x_2 \quad x_1 \le x_3 \quad x_1 \le x_4 \quad y_1 \le y_2 \quad y_1 \le y_3 \quad y_1 \le y_4 .$$

WLOG, we can add the inequality $x_4 + y_4 \le x_3 + y_3 \le x_2 + y_2$. The probability that this situation occurs is the one in (9) extended with integration over x_4 and y_4. The result is $p_{c1} = \frac{1}{448}$.

The side length of B is $S = x_2 + y_2 - x_1 - y_1$. The calculation of its distribution goes as in (10) extended with integration over x_4 and y_4. We get

$$k_{1a}(s|x_1, y_1; s + x_1 + y_1 \le 1) = \frac{s^6}{8}$$

and

$$k_{1b}(s|x_1, y_1; s + x_1 + y_1 > 1) = \frac{(1 - x_1 - y_1)^6}{8}.$$

The areas to integrate over in the x_1y_1-plane are shown in Figure 8 and the integration is exactly the same as in (11). We get

$$K_4(s) = \text{Prob}(S \le s) = 28s^6 - 48s^7 + 21s^8,$$

which gives

$$G_4(v) = 28v^3 - 48v^{7/2} + 21v^4, \qquad 0 \le v \le 1\,. \tag{28}$$

The expectation of V is $E(V) = \frac{7}{15}$. Combining this with $E(U) = \frac{5}{14}$, we get $E(X) = \frac{5}{14} \cdot \frac{7}{15} = \frac{1}{6}$.

4.6. **The distribution of X for four generated points.** The combination of the distributions of U and V to get the distribution of $X = UV$ is done exactly in the same way as for three points. We shall calculate (compare (14))

$$F_4(x) = \int_0^1 G_4(x/u)\,dH_4(u) = \tag{29}$$
$$= G_4(x) - \int_x^1 H_4(u)\frac{d}{du}G_4(x/u)\,du, \qquad 0 \le x \le 1.$$

We get

(30)

$$F_4(x) =$$
$$= \begin{cases} 48x^2(6x^2+4x-3)\log(\frac{1+\sqrt{(1-4x)}}{2})\ \log(\frac{1+\sqrt{1-4x}}{2x}) \\ -\frac{8}{35}(2826x^3-1101x^2-80x+2)\sqrt{1-4x}\ \log(\frac{1+\sqrt{1-4x}}{2\sqrt{x}}) \\ +84x^4(\log x)^2 \\ -\frac{2}{35}(1505x^4+7840x^3-1890x^2-168x+4)\log x \\ -2(43x^3+123x^2+15x-1)(1-x)\log(1-x) \\ -\frac{16}{3}(6x^2+4x-3)x^2\pi^2 \\ -\frac{1}{35}(4622x^2-4603x-54)x - 24x^2(x^2+8x+6)\nu(x), \\ \qquad\qquad 0 \le x \le 1/4, \\ 48x^2(6x^2+4x-3)(2\pi/3-\arccos\frac{1}{2\sqrt{x}})\arccos\frac{1}{2\sqrt{x}} \\ -\frac{8}{35}(2826x^3-1101x^2-80x+2)\sqrt{4x-1}(\pi/3-\arccos\frac{1}{2\sqrt{x}}) \\ +12x^2(x-1)(x-3)(\log x)^2 \\ -\frac{2}{35}(1505x^4+7840x^3-1890x^2-168x+4)\log x \\ -2(43x^3+123x^2+15x-1)(1-x)\log(1-x) \\ -\frac{16}{3}(6x^2+4x-3)x^2\pi^2 \\ -\frac{1}{35}(4622x^2-4603x-54)x - 24x^2(x^2+8x+6)\nu(x), \\ \qquad\qquad 1/4 \le x \le 1. \end{cases}$$

The density $dF_4(x)/dx$ is shown in Figure 13.

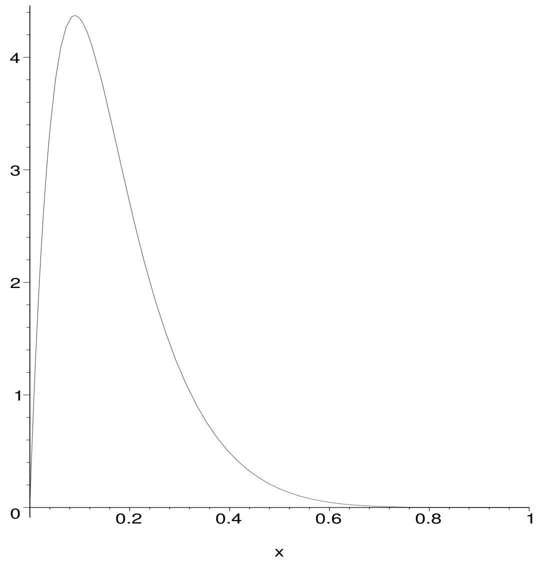

FIGURE 13. Density function $dF_4(x)/dx$ for the area fraction of the convex hull of four random points inside a triangle.

The first moments and the standard deviation of X are

$$\begin{aligned} \alpha_1 &= \int_0^1 x\,dF_4(x) = \frac{1}{6}, \\ \alpha_2 &= \int_0^1 x^2\,dF_4(x) = \frac{181}{4500}, \\ \sigma &= \sqrt{\alpha_2 - {\alpha_1}^2} = \frac{1}{75}\sqrt{70} \approx .1116, \\ \alpha_3 &= \int_0^1 x^3\,dF_4(x) = \frac{14}{1125}. \end{aligned} \tag{31}$$

5. THE NUMBER OF VERTICES OF THE CONVEX HULL.

Like in [6], we define for $k \le n$

$$q_n(k) = \text{ Prob}(\ n \text{ points generate a convex polygon with } k \text{ vertices}).$$

Of course, $q_3(3) = 1$. We noted in the beginning of section 4.1 that $q_4(3) = \frac{1}{3}$ implying $q_4(4) = \frac{2}{3}$. From [6], we have for all $n \ge 3$

$$q_n(3) = \binom{n}{3} \int_0^1 x^{n-3}\,dF_3(x). \tag{32}$$

Some values are

$$q_5(3) = \frac{5}{36} \approx .1389, \quad q_6(3) = \frac{31}{450} \approx .0689, \quad q_7(3) = \frac{7}{180} \approx .0389,$$

$$q_8(3) = \frac{1063}{44100} \approx .0241, \quad q_9(3) = \frac{403}{25200} \approx .0160.$$

For $k = 4$, we need the conditional probability that points 5 through n are generated inside the area generated by the first four points, provided these four points span a quadrangle. Cases b, c, and e are quadrangles, so summing the H_4 for these cases multiplied by their weights will give us the wanted conditional distribution function $H_{4q}(u)$. Combining H_{4q} with G_4 will give the conditional distribution function $F_{4q}(x)$. Including the probability $\frac{2}{3}$ of getting a quadrangle, we get

$$q_n(4) = \frac{2}{3}\binom{n}{4}\int_0^1 x^{n-4}\, dF_{4q}(x). \tag{33}$$

Some values are

$$q_5(4) = \frac{5}{9}, \quad q_6(4) = \frac{119}{300} \approx .3966, \quad q_7(4) = \frac{7}{25}.$$

From the above, we can deduce $q_5(5) = 1 - \frac{5}{36} - \frac{5}{9} = \frac{11}{36} \approx .3056$

6. Asymptotic estimates.

For $n \geq 5$, we cite two asymptotic estimates from [7], [8], and [6] . The average area of the convex hull of n points in a triangle with area $= 1$ is

$$E(\text{area}) = 1 - 2\frac{\log(n) + \gamma}{n} + \mathrm{O}\left(\frac{1}{n^2}\right),$$

where $\gamma = .5772$ is Euler's constant. The average number of vertices of the convex hull of n points in a triangle is

$$E(\#\text{ vertices}) = 2(\log(n/2) + \gamma) + \mathrm{o}(1), \quad n \to \infty.$$

The latter estimate is very good even for small n.

7. Concluding comment.

We have not shown any integral calculations in detail. In principle, they are elementary, which doesn't mean that they don't require a substantial effort. As indicated, the calculations have been done in Maple. The calculations would not have been possible without some tool for handling the huge number of terms that come out of the integrations, sometimes more than a hundred. This doesn't mean that Maple performs the integrations automatically. Generally, we had to split up the integrands in parts and use a particular substitution for each part. Often, we had to do partial integrations manually. Many integrals were improper, calling for a limiting process. We will supply any interested reader with Maple files describing the calculations.

Appendix A

The function $\nu(x)$ is defined

$$(34)\qquad \begin{aligned}\nu(x) &= \operatorname{dilog}(x) + \log(x)\log|1-x| = \\ &= \int_x^1 \frac{\log(t)}{t-1}\,dt + \log(x)\log|1-x| = \int_1^x \frac{\log|1-s|}{s}\,ds.\end{aligned}$$

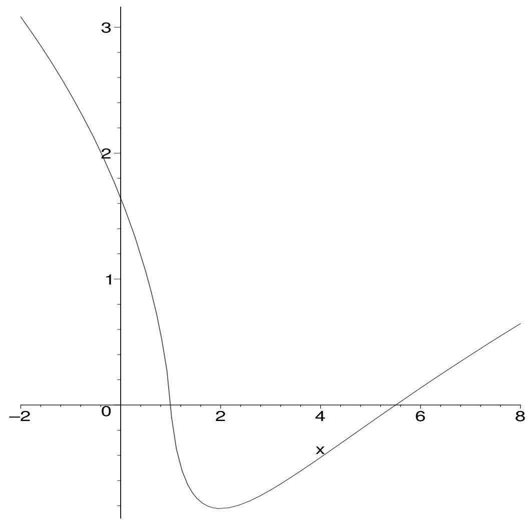

Figure 14. The function $\nu(x)$.

The dilog function and some of its properties are described in [1], page 1004. The dilog function has a series expansion in a unit disk centered at $x = 1$. It follows from the last integral in (34) that the ν function is defined and is real on the whole real axis. We use this function for $0 \le x \le 1$.

$\nu(x)$ is decreasing from $\nu(0) = \pi^2/6$ via $\nu(1) = 0$ to $\nu(2) = -\pi^2/12$.

The integrals involving $\nu(x)$ needed for calculating the moments of various distributions take rational values like

$$\int_0^1 x\,d\nu(x) = -1, \quad \int_0^1 x^2\,d\nu(x) = -\frac{3}{4}, \quad \int_0^1 x^3\,d\nu(x) = -\frac{11}{18}.$$

Appendix B

We shall calculate the distribution function $K_{3.2}(s)$ for the side length in Case 2 for three generated points. This case is depicted in Figure 2 and is characterized by

$$x_2 \le x_1,\ x_2 \le x_3,\ y_1 \le y_2,\ y_1 \le y_3\,.$$

WLOG, we can use

$$x_1 + y_1 \le x_2 + y_2 \le x_3 + y_3.$$

We start by calculating the probability that this situation occurs The integration over x_3 and y_3 gives

$$t(x_2, y_1, y_2) = \frac{(1 - x_2 - y_1)^2}{2} - \frac{(y_2 - y_1)^2}{2}.$$

We get the probability

$$p_{c2} = \int_0^1 dy_1 \int_{y_1}^1 dy_2 \int_0^{1-y_2} dx_2 \int_{x_2}^{x_2+y_2-y_1} t(x_2, y_1, y_2)\, dx_1 = \frac{1}{240}.$$

Notice the integration order dx_1, dx_2, dy_2, dy_1. Any other order makes the integration much more complicated.

Here, the side length of B is $S = x_3 + y_3 - x_2 - y_1$. Define

$$ts(s, y_1, y_2) = \frac{s^2}{2} - \frac{(y_2 - y_1)^2}{2}.$$

The integration over x_3 and y_3 and $S \le s$ gives

$$k_2(s, x_2, y_1, y_2) = \begin{cases} 0, & s < y_2 - y_1, \\ ts(s, y_1, y_2), & y_2 - y_1 \le s < 1 - x_2 - y_1, \\ t(x_2, y_1, y_2), & 1 - x_2 - y_1 \le s. \end{cases}$$

The integration of k_2 over x_1 and x_2 splits into two cases (a) when $y_1 \le 1 - s$ and (b) when $y_1 > 1 - s$. We get

$$\begin{aligned} k_{2a}(s, y_1, y_2) &= \int_0^{1-s-y_1} dx_2 \int_{x_2}^{x_2+y_2-y_1} ts(s, y_1, y_2)\, dx_1 \\ &\quad + \int_{1-s-y_1}^{1-y_2} dx_2 \int_{x_2}^{x_2+y_2-y_1} t(x_2, y_1, y_2)\, dx_1 \\ &= (y_1 - y_2)(y_2 - y_1 - s)({y_1}^2 + y_1 y_2 - 2{y_2}^2 \\ &\quad - (3 + s)y_1 + (3 + 2s)y_2 - 2s^2 + 3s)/6, \end{aligned}$$

and

$$\begin{aligned} k_{2b}(s, y_1, y_2) &= \int_0^{1-y_2} dx_2 \int_{x_2}^{x_2+y_2-y_1} t(x_2, y_1, y_2)\, dx_1 \\ &= (1 - y_2)^2 (y_2 - y_1)(1 + 2y_2 - 3y_1)/6. \end{aligned}$$

The final integration over y_1 and y_2 gives

$$\begin{aligned} K_{3,2}(s) &= \mathrm{Prob}(S \le s) \\ &= \frac{1}{p_{c2}} \left(\int_0^{1-s} dy_1 \int_{y_1}^{y_1+s} k_{2a}\, dy_2 + \int_{1-s}^1 dy_1 \int_{y_1}^1 k_{2b}\, dy_2 \right) \qquad (35) \\ &= 15s^4 - 24s^5 + 10s^6. \end{aligned}$$

This implies that $K_{3,1}(s) = K_{3,2}(s)$.

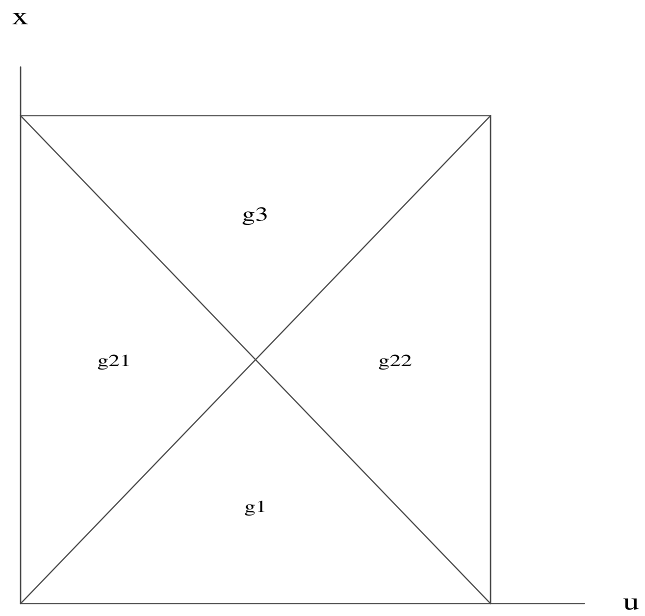

FIGURE 15. Areas to integrate $f(u \mid x)$ over in case c.

APPENDIX C. CALCULATION OF $H_{4c}(u)$.

In case c, we have P_1, P_2, and P_3 as in Figure 3 and P_4 falling on opposite sides of P_1P_2 from P_3. The polygon is the sum of the triangles $T_1 = P_1P_2P_3$ and $T_2 = P_1P_2P_4$. We shall convolve the distributions of these two triangles to get the distribution for $U = T_1 + T_2$. The distribution function for the fraction of a triangle with area = 1, is given in (22). For the two triangles, we get

$$f_1(z \mid x_2) = \phi(z/x_2) \text{ and } f_2(v \mid x_2) = \phi(v/(1-x_2)).$$

Because the expressions for f_1 and f_2 vary over the intervals of z and v, the expression for the conditional distribution $f(u \mid x_2)$ varies in the (u, x_2)-plane. Let $\psi(u,v) = f_2(v)f_1'(u-v)$. In the sequel, we shall skip the index 2 and replace x_2 by x. We get

(36)

$$f(u|x) = \begin{cases} \int_{u-x}^{u} \psi(u,v)dv, & 0 \le x \le u \le \min(u, 1-u), \\ \int_0^u \psi(u,v)dv, & u \le x \le 1-u, \\ \int_{u-x}^{1-x} \psi(u,v)dv + \int_{1-x}^{u} f_1'(u-v)dv, & 1-u \le x \le u, \\ \int_0^{1-x} \psi(u,v)dv + \int_{1-x}^{u} f_1'(u-v)dv, & \max(u, 1-u) \le x \le 1. \end{cases}$$

Performing the integrations in (36), we get the following expressions valid in the regions depicted in Figure 15

(37)

$$f(u|x) = \begin{cases} g_1 = \frac{3x^2-8ux-4x-6u^2+12u}{6(1-x)^2}, & 0 \le x \le u \le \min(u, 1-u), \\ g_{21} = \frac{2u^2}{x(1-x)} - \frac{(4-u)u^3}{6x^2(1-x)^2}, & u \le x \le 1-u, \\ g_{22} = 1 - \frac{(1-u)^4}{6x^2(1-x)^2}, & 1-u \le x \le u, \\ g_3 = \frac{3x^2+8ux-2x-6u^2+4u-1}{6x^2}, & \max(u, 1-u) \le x \le 1. \end{cases}$$

We get

(38)
$$\begin{aligned} \mathrm{Prob}(U \le u) = H_{4c}(u) &= \binom{2}{1}\binom{1}{3}^{-1} \int_0^1 x(1-x) f(u \mid x)\,dx \\ &= 2(4-u)u^3 \log(u) + 2(1-u)^4 \log(1-u) - u(6u^2 - 5u - 2), \\ &\qquad 0 \le u \le 1. \end{aligned}$$

Appendix D. Calculation of $H_{4e}(u)$.

The geometry is described in Figure 4. The x-coordinate of P_1 will be denoted x and the y-coordinate of P_2 will be denoted y. We will have a quadrangle if P_4 sits in one of the three corner triangles. These are affinely equivalent and we do the calculations for P_4 in the lower left triangle. The probability for this to happen is xy. The quadrangle area $U = T_1 + T_2$. For fixed x and y, we get the conditional distribution of U by convolving the conditional distributions $f_1(v \mid x, y)$ and $f_2(v \mid x, y)$ of T_1 and T_2.

(39)
$$f(u \mid x, y) = \int_0^1 f_2(v \mid x, y)\, \frac{df_1}{dv}(u - v \mid x, y)\, dv$$

Like in (6), we assume $x \le y$ and get for fixed x and y, $\frac{df_1}{dv}(v \mid x, y) = \frac{1}{y-x}$ when $x(1-y) \le v \le y(1-x)$, otherwise zero. We have

(40)
$$f_2(v \mid x, y) = \begin{cases} 1 - \left(1 - \frac{v}{xy}\right)^2, & 0 \le v \le xy, \\ 1, & xy \le v \le 1. \end{cases}$$

Because the expressions for f_1 and f_2 vary from interval to interval, we get different expressions for $f(u \mid x, y)$ valid in different sets of u, x, and y. These sets are depicted in Figure 16 for $0 \le u \le 1/4$, and Figure 17 for $1/2 \le u \le 1$. The figure for $1/4 \le u \le 1/2$ is not shown. In fact, integrating for u in this interval gives the same result as for $1/2 \le u \le 1$. We have

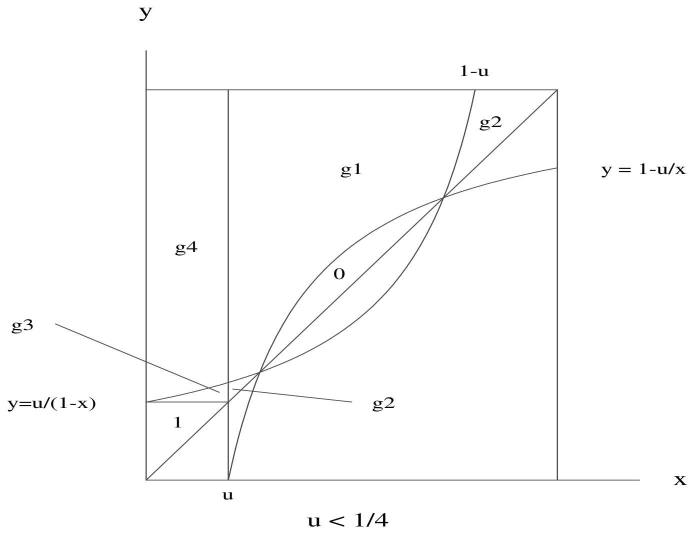

FIGURE 16. Areas to integrate (39) over for $0 \leq u \leq 1/4$.

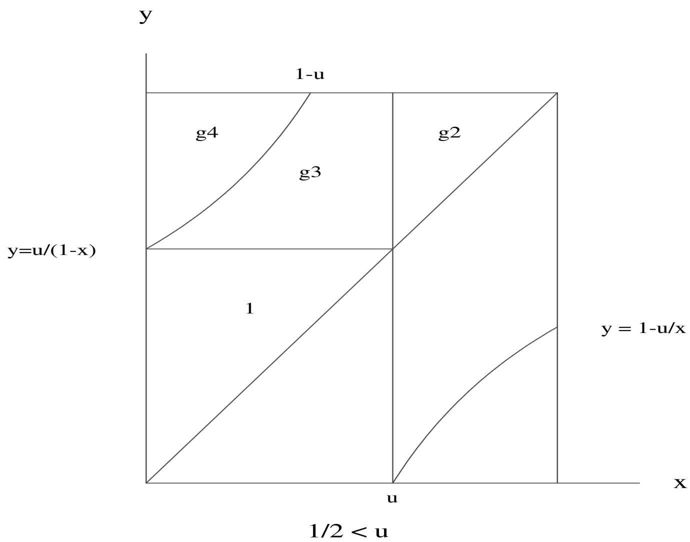

FIGURE 17. Areas to integrate (39) over for $1/2 \leq u \leq 1$.

$$f(u \mid x, y) = \begin{cases} 0, & 0 \leq u \leq x(1-y), \\ \frac{1}{y-x}\int_0^{u-x+xy} f_2(v)dv, & x(1-y) \leq u \leq y(1-x), \\ \frac{1}{y-x}\int_{u-x+xy}^{u-y+xy} f_2(v)dv, & y(1-x) \leq u \leq x, \\ \frac{u-x}{y-x} + \frac{1}{y-x}\int_{u-x+xy}^{xy} f_2(v)dv, & y(1-x) \leq u \leq y \text{ and } x \leq u, \\ \frac{u-x}{y-x} + \frac{1}{y-x}\int_0^{xy} f_2(v)dv, & x \leq u \leq y(1-x), \\ 1, & y \leq u \leq 1. \end{cases} \tag{41}$$

Performing the integrations in (41), we get

(42)

$$f(u \mid x,y) = \begin{cases} g_0 = 0, & 0 \le u \le x(1-y), \\ g_1 = \frac{(u-x+xy)(x-u+2xy)}{3x^2y^2(y-x)}, & x(1-y) \le u \le y(1-x), \\ g_2 = \frac{3x^2y^2-x^2-y^2-xy+3u(x+y-u)}{3x^2y^2}, & y(1-x) \le u \le x, \\ g_3 = \frac{3y(uy-u^2+x^2y^2-x^3y)+u^3-y^3}{3x^2y^2(y-x)}, & y(1-x) \le u \le y \text{ and } x \le u, \\ g_4 = \frac{3u-3x+2xy}{3(y-x)}, & x \le u \le y(1-x), \\ g_5 = 1, & y \le u \le 1. \end{cases}$$

We get H_{4e} by integrating $f(u \mid x,y)$ multiplied by the probability $= xy$ that this case occurs

$$H_{4e}(u) = 2\binom{3}{1}\left(\frac{3}{4}\right)^{-1} \int_0^1 dx \int_x^1 x\, y\, f(u \mid x,y)\, dy. \tag{43}$$

We get

(44)

$$H_{4e}(u) = \begin{cases} \frac{2}{5}(32u^2+34u-3)\sqrt{1-4u}\ (\log(\frac{1+\sqrt{1-4u}}{2}) - \log(u)/2) \\ -24u^2\log(\frac{1+\sqrt{1-4u}}{2})\,(\log(\frac{1+\sqrt{1-4u}}{2}) - \log(u)) \\ -\frac{1}{15}(80u^3+90u^2-120u+9)\log(u) \\ -\frac{8}{3}(2u^2+5u-1)(1-u)\log(1-u) \\ -\frac{7}{15}u^2+\frac{22}{15}u+\frac{8}{3}\pi^2u^2-16u^2\nu(u), \\ \qquad\qquad 0 \le u \le 1/4, \\ \frac{2}{5}(32u^2+34u-3)\sqrt{4u-1}(\pi/3 - \arccos\frac{1}{2\sqrt{u}}) \\ -24u^2(2\pi/3 - \arccos\frac{1}{2\sqrt{u}})\arccos\frac{1}{2\sqrt{u}} \\ -\frac{1}{15}(80u^3+90u^2-120u+9)\log(u) \\ -\frac{8}{3}(2u^2+5u-1)(1-u)\log(1-u) \\ -\frac{7}{15}u^2+\frac{22}{15}u+\frac{8}{3}\pi^2u^2-16u^2\nu(u), \\ \qquad\qquad 1/4 \le u \le 1. \end{cases}$$

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