THE AREA AND PERIMETER OF A RANDOM SPHERICAL TRIANGLE.

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ABSTRACT. We determine the distribution functions for the area and the perimeter of a random spherical triangle. We consider two natural 'polar' ways of generating the random triangle. It turns out that the distribution for the area in the first way is closely related to the distribution of the perimeter in the second way, and vice versa.

1. INTRODUCTION

Various problem in the field of geometric probability have been considered over the years, see e.g. [19]. J. J. Sylvester considered the problem of a random triangle T in an arbitrary convex set K. Assuming that the density for the vertices of the triangle is constant in K and defining $X = area(T)/area(K)$, he posed the problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. H. A. Alikoski [2] has given expressions for κ when T is a triangle and K a regular r-polygon. A. Reńyi and R. Sulanke, [17] and [18], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. R. E. Miles [10] generalizes these asymptotic estimates for K a circle to higher dimensions. C. Buchta and M. Reitzner, [4], have given values of κ (generalized to three dimensions) for $n \geq 4$ points in a tetrahedron.

The distribution function for X when K is a triangle is given by V. S. Alagar [1] and J. Philip[14] , when K is a square by N. Henze [6] and J.Philip [12]. We also have given the distributions for a regular pentagon [16], and hexagon [15]. A. M. Mathai [9] used a method entirely different from that for polygons to find the distribution function for X when K is a circle. The distribution function when K is the whole plane equipped with the normal density has also been studied.

Date: May 7, 2015.
1991 *Mathematics Subject Classification.* Primary: 60D05: Secondary: 52A22.
Key words and phrases. Area and perimeter of spherical triangle.

2. Notation and formulation.

Our space is the unit sphere in three dimensions centered at the origin and equipped with a constant probability density equal to $\frac{1}{4\pi}$. We use three random points on the sphere to construct a spherical triangle T. This can be done in two ways, which we shall call Case 1 and Case 2. We discuss the generation of random points on the sphere in section 10.

Case 1: The random points are used to construct the three angles of the triangle. Let the three random points give the tips of three vectors from the origin and at let these vectors define three half spaces These half spaces cut out the spherical triangle T from the sphere.

Case 2: The random points are taken as the vertices of the spherical triangle T. Then, the sides are parts of great circles.

We rule out the events, with probability zero, that two of the random points are opposite to each other and that all three sit on a great circle.

Three great circles divide the sphere into eight triangles. Of them, two opposite triangles have all three angles less than π. These are called Euler triangles and our T is one of them. Compare Figure 1.

We shall determine the probability distributions of the random variables $X = area(T)$ and $Y = perimeter(T)$ in the two cases.

We will denote the three sides of T by a, b, and c and the three angles by α, β, and γ. All six elements are anglemeasured and less than π.

From any text book on spherical trigonometry e.g. [20] , we take the following equations

$$0 < a + b + c < 2\,\pi \text{ and } \pi < \alpha + \beta + \gamma < 3\,\pi \tag{1}$$

$$\frac{\sin(\alpha)}{\sin(a)} = \frac{\sin(\beta)}{\sin(b)} = \frac{\sin(\gamma)}{\sin(c)}. \tag{2}$$

and further the following equalities and inequalities and their cyclic permutations

$$a + b > c, \qquad \alpha + \beta < \gamma + \pi. \tag{3}$$

$$\cos(a) = \cos(b)\,\cos(c) + \sin(b)\,\sin(c)\,\cos(\alpha). \tag{4}$$

3. The three-dimensional distribution of α, β, and γ in Case 1.

The three angles α, β, and γ all lie between 0 and π, so that their common probability density sits in a cube with side π. The inequalities (1) and (3) restrict their volume of definition to the tetrahedron shown in Figure 2.

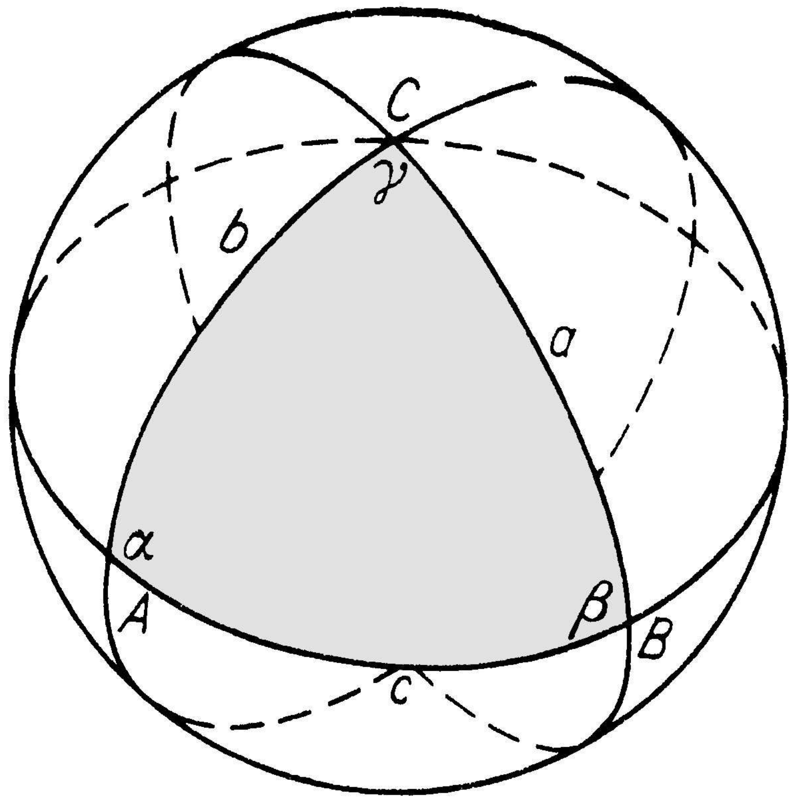

FIGURE 1. A spherical triangle with angles α, β, and γ all less than π.

Let us study the bounding plane $\alpha+\beta=\gamma+\pi$ in Figure 2 . We get

$$\cos(\alpha+\beta)=\cos(\gamma+\pi)=-\cos(\gamma)$$

or

$$\cos(\alpha)\,\cos(\beta)-\sqrt{1-\cos(\alpha)^2}\,\sqrt{1-\cos(\beta)^2}=-\cos(\gamma).$$

Moving the square roots to one side and squaring, we get

$$1-\cos(\alpha)^2-\cos(\beta)^2-\cos(\gamma)^2-2\,\cos(\alpha)\,\cos(\beta)\,\cos(\gamma)=0. \tag{5}$$

Since cosine is an even function and because we used squaring in deducing equation (5), it turns out that this equation is also the boundary resulting from the other three boundaries in Figure 2.

Turning to the random points, we let ϕ be the angle between the random unit vectors n_1 and n_2. Without loss of generality, assume for the moment that n_1 is the north pole $(0,0,1)$. We have

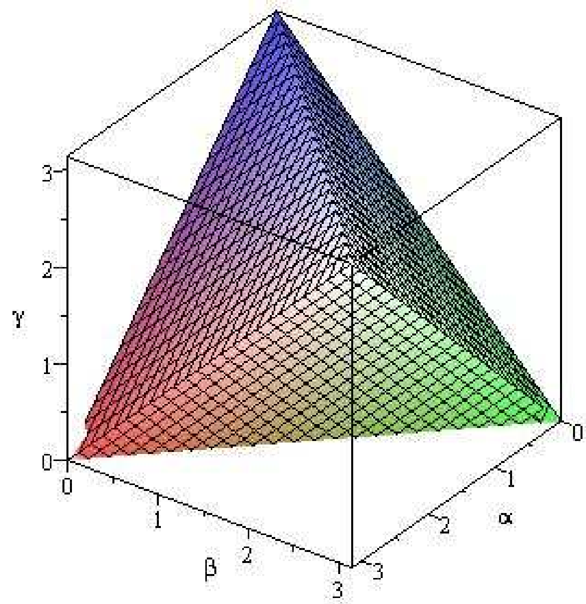

FIGURE 2. The tetrahedron in which α, β, and γ take their values.

(6)

$$\begin{aligned}
&\mathrm{Prob}(\cos(\phi) < t) = 1 - \mathrm{Prob}(\phi < \arccos(t)) = \\
&1 - \text{ the fraction of the globe area above the meridian} \arccos(t) = \\
&1 - \frac{2\pi(1-t)}{4\pi} = \frac{1+t}{2}, \quad -1 \le t \le 1.
\end{aligned}$$

We have found that it is not ϕ but $\cos(\phi)$ that is evenly distributed with a constant density equal to $\frac{1}{2}$. The angle between the planes orthogonal to n_1 and n_2 will be $\pi - \phi$. The angles of the random spherical triangle are such angles. Let us change to the variables

$$\begin{aligned}
t_1 &= \cos(\pi - \alpha) = -\cos(\alpha) \\
t_2 &= \cos(\pi - \beta) = -\cos(\beta) \qquad (7) \\
t_3 &= \cos(\pi - \gamma) = -\cos(\gamma).
\end{aligned}$$

Equation (5) then reads

$$1 - {t_1}^2 - {t_2}^2 - {t_3}^2 + 2\, t_1\, t_2\, t_3 = 0. \qquad (8)$$

The volume bounded by (8) in t-space is the “blown up tetrahedron” shown in Figure 3.

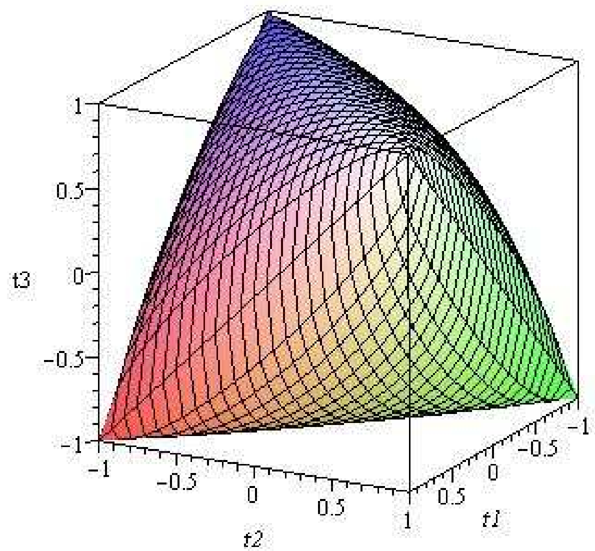

FIGURE 3. The domain of definition in t-space.

We are looking for a probability density $\rho_t(t_1, t_2, t_3)$ in the volume of Figure 3, which then can be transformed to α-, β-, γ-space. The density ρ_t shall be invariant under all permutations of t_1, t_2, and t_3.

All four inequalities in (1) and (3) bounding the tetrahedron in Figure 2 involve all three angles and there is no other inequality with only two variables. This implies that the t_i are pairwise independent in the sense that the two-dimensional marginal distribution in say t_1 and t_2 is of the form $\rho_1(t_1) \cdot \rho_2(t_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, i.e. constant. This implies that

$$\int \rho_t(t_1, t_2, t_3)\, dt_3 = \frac{1}{4} \quad \text{for } -1 < t_1 < 1,\ -1 < t_2 < 1. \tag{9}$$

For this to hold all the way to the boundary of the bulging domain, ρ_t must tend to infinity at the boundary. Since ρ_t also shall be invariant under permutation of the variables, we try

$$\rho_t = C \cdot (1 - {t_1}^2 - {t_2}^2 - {t_3}^2 + 2\, t_1\, t_2\, t_3)^p,$$

where C is a normalizing function of t_1 and t_2. The power p shall be negative so that ρ_t tends to infinity at the boundary and bigger than -1 so that the integral exists. We shall show that the choice $p = -\frac{1}{2}$ satisfies the conditions on ρ_t.

For the integration, we substitute t_3 by $v = t_3 - t_1\, t_2$. Then $\rho_t = C \cdot k^{2p}\, (1 - \frac{v^2}{k^2})^p$, where $k = \sqrt{(1-{t_1}^2)(1-{t_2}^2)}$. The boundaries for v are $-k < v < k$. This ρ_t can be integrated and gives

$$\int \rho_t\, dv = C \cdot v \cdot \mathrm{Hypergeom}([-p, \frac{1}{2}], [\frac{3}{2}], \frac{v^2}{k^2}). \tag{10}$$

Insertion of the boundaries gives

$$\int_{-k}^{k} \rho_t\, dv = C \cdot k^{2p+1} \frac{\Gamma(p+1)\sqrt{\pi}}{\Gamma(p+\frac{3}{2})}. \tag{11}$$

This is constant if $C = \text{const.} \cdot k^{-(2p+1)}$ so that

$$\rho_t = \text{const.} \cdot \frac{1}{k} \cdot (1 - \frac{v^2}{k^2})^p = \text{const.} \cdot \frac{(1-\frac{v^2}{k^2})^p}{\sqrt{(1-t_1^2)(1-t_2^2)}}.$$

This ρ_t tends to infinity at the same rate when t_1 or t_2 tends to 1 as when v tends to k, only if $p = -\frac{1}{2}$.

For $p = -\frac{1}{2}$, the integral in (10) takes the more familiar form

$$\int \rho_t\, dv = C \cdot \mathrm{arcsine}(\frac{v}{k}). \tag{12}$$

With the boundaries inserted, we get $\int \rho_t\, dv = C \cdot \pi$. Integrating also over t_1 and t_2 gives $4\, C\, \pi$. The ρ_t having total mass $= 1$ is

$$\rho_t = \frac{1}{4\,\pi\, \sqrt{1 - {t_1}^2 - {t_2}^2 - {t_3}^2 + 2\, t_1\, t_2\, t_3}} \tag{13}$$

One could have imagined ρ_t to be a linear combination of terms with different p, but this would not have given the right infinity rate in all three directions. We shall show that (13) gives the correct first and second moments and that it is in accordance with Monte Carlo tests.

Using (7) to transform back to the angles and including the Jacobian, we get

(14)
$$\rho(\alpha, \beta, \gamma) = \frac{\sin\alpha\, \sin\beta\, \sin\gamma}{4\,\pi\, \sqrt{1 - \cos\alpha^2 - \cos\beta^2 - \cos\gamma^2 - 2\, \cos\alpha\, \cos\beta\, \cos\gamma}}$$

4. The first and second moments in Case 1.

The area X of the spherical triangle is

$$X = \alpha + \beta + \gamma - \pi, \quad 0 \le X \le 2\pi. \tag{15}$$

For the first moment, we have

(16) $E(X) = E(\alpha+\beta+\gamma-\pi) = E(\alpha)+E(\beta)+E(\gamma)-\pi = 3\cdot E(\alpha)-\pi.$

To get $E(\alpha)$, we shall integrate over the tetrahedron in Figure 2. We see in the Figure that the boundaries in the γ-direction depend on α and β. The upper bound changes along the diagonal $\alpha = \beta$ while the lower bound changes along the diagonal $\alpha+\beta = \pi$. This splits the (α, β)-region into four triangles with different bounds. Integrating ρ over γ , we get the arcsine function in (12) and insertion of the boundaries gives the same result in all four regions, namely

$$\int \rho\, d\gamma = \frac{1}{4}\,\sin\alpha\,\sin\beta.$$

This is the density for two independent variables and we can continue with

$$E(\alpha) = \frac{1}{4}\int_0^\pi \int_0^\pi \alpha\,\sin\alpha\,\sin\beta\,d\beta\,d\alpha = \frac{\pi}{2}\ . \tag{17}$$

Invoking (16), we have

$$E(X) = \frac{\pi}{2}\,. \tag{18}$$

In the same way as in (17), we have

$$E(\alpha^2) = \frac{1}{4}\int_0^\pi \int_0^\pi \alpha^2\,\sin\alpha\,\sin\beta\,d\beta\,d\alpha = \frac{\pi^2}{2} - 2\ . \tag{19}$$

Using $E(\alpha\,\beta) = E(\alpha)\cdot E(\beta) = E(\alpha)^2$, we have

(20)
$$\begin{aligned} E(X^2) &= E((\alpha+\beta+\gamma-\pi)^2) = 3\,E(\alpha^2) + \pi^2 + 6\,E(\alpha\,\beta) - 6\,\pi\,E(\alpha) \\ &= 3\,\left(\frac{\pi^2}{2} - 2\right) + \pi^2 + 6\,\left(\frac{\pi}{2}\right)^2 - 3\,\pi^2 = \pi^2 - 6\ \approx 3.8696. \end{aligned}$$

and

$$\sigma(X) = \sqrt{E(X^2) - E(X)^2} = \frac{1}{2}\,\sqrt{3\,\pi^2 - 24} \approx 1.1841\,. \tag{21}$$

We cannot compute higher moments of X in this way because the independence argument doesn't hold for three variables.

5. The probability density of the area in Case 1.

The distribution function of the area X of the spherical triangle is

$$\begin{aligned} F(x) =& \mathrm{Prob}(X \leq x) = \mathrm{Prob}(\alpha + \beta + \gamma \leq \pi + x) \\ =& \int \int_B \int \rho \, d\alpha \, d\beta \, d\gamma, \end{aligned} \tag{22}$$

where B is the intersection of the bounding tetrahedron in Figure 2. and the halfspace $\alpha + \beta + \gamma \leq \pi + x$.

We will not be able to calculate this integral because we encounter integrals of the form

$$\int \int \sin\alpha \, \sin\beta \, \arcsin\left(\frac{\cos\alpha \, \cos\beta - \cos(x - \alpha - \beta)}{\sin\alpha \, \sin\beta} \right) d\alpha \, d\beta$$

Resorting to the density $f(x) = \frac{dF(x)}{dx}$, we shall integrate ρ over the two-dimensional intersection of the bounding tetrahedron with the plane $\alpha + \beta + \gamma = \pi + x$. Inserting $\gamma = \pi + x - \alpha - \beta$ in ρ in (14) we have

$$\rho(\alpha, \beta, \pi + x - \alpha - \beta) \tag{23}$$

to be integrated over the projection of the intersection on the $\alpha\beta$-plane. Also this integral for $f(x)$ defies our efforts to integrate analytically, so we must resort to numerical integration. Knowing that ρ tends to infinity on the boundary, we make the following substitution to move a boundary point to the origin,

$$\alpha = v + \frac{x}{2}, \quad \beta = w + \frac{x}{2}.$$

The integrand changes from (23) to the following

$$\rho_{vw}(v + \frac{x}{2}, w + \frac{x}{2}, \pi - v - w). \tag{24}$$

We shall take advantage of the invariance of the integrand under permutation of the variables. Using also that sine is odd and cosine is even around both 0 and π gives a sixfold invariance so that we only have to integrate over one sixth of the domain. The details are given in the following equations and in Figure 4, where small circles mark six points having the same ρ_{vw}-value.

We have three lines of symmetry for three different actions:

$$\begin{aligned} v = w & \quad \text{for swapping } v \text{ and } w\,, \\ 2\,v + w = \pi - \frac{x}{2} & \quad \text{for } v \text{ while keeping } w \text{ fixed}, \\ v + 2\,w = \pi - \frac{x}{2} & \quad \text{for } w \text{ while keeping } v \text{ fixed}. \end{aligned} \tag{25}$$

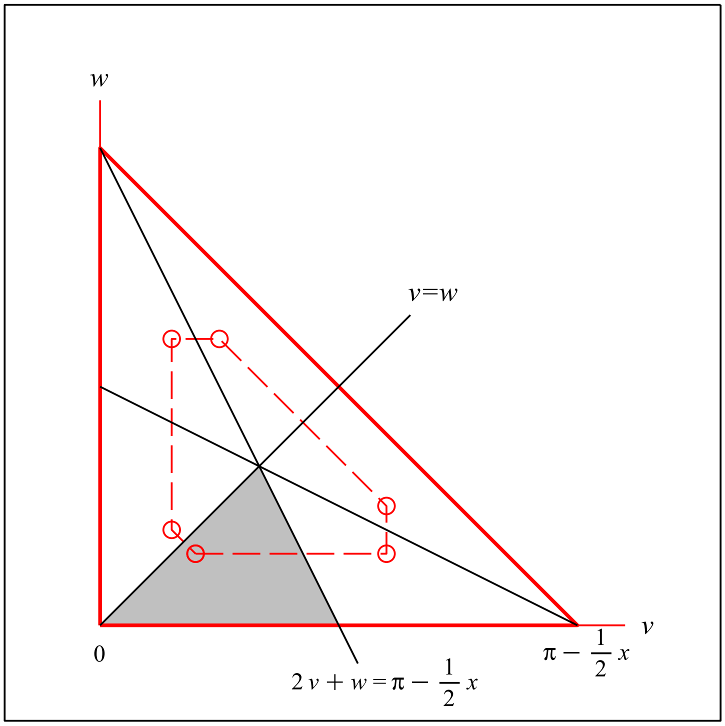

FIGURE 4. The triangle to integrate over. The thin black lines are the symmetry lines. The integrand takes the same value in the six marked points.

We shall integrate numerically over the gray triangle marked in Figure 4 and multiply the result by 6:

$$f(x) = 6 \int_0^{\frac{\pi}{3}-\frac{x}{6}} \int_w^{\frac{\pi}{2}-\frac{x}{4}-\frac{w}{2}} \rho_{vw}\, dv\ dw. \tag{26}$$

The integrand tends to infinity when w tends to zero, calling for approximating it with an integrable function for small w. We get an approximation for small w by first series expanding ρ_{vw} in the v-direction around the midpoint $\frac{\pi}{4} - \frac{x}{8}$ and just use the first constant term. Then, this term is integrated in the v-direction from $v = w$ to $v = \frac{\pi}{2} - \frac{x}{4} - \frac{w}{2}$. Series expanding the result in the w direction around $w = 0$ gives a first term proportional to $1/\sqrt{w}$. We integrate this term from $w = 0$ to $w = d$, where d is a small number, typically $d = .001$. The integral over this narrow strip along the v-axis has the value:

$$\frac{3\,(2\,\pi - x)\,\sqrt{d}}{8\,\pi\,\sqrt{2}} \sqrt{\sin\left(\frac{x}{2}\right)\,\left(\cos\frac{x}{2} + \sin\frac{x}{4}\right)}. \tag{27}$$

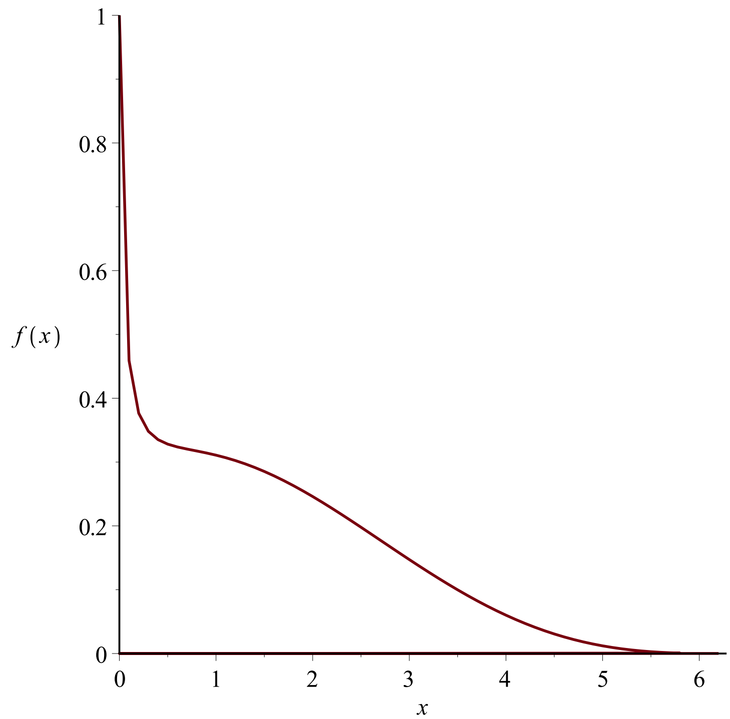

Figure 5. The probability density function $f(x)$, $(0 < x < 2\pi)$, for the area of a random spherical triangle in Case 1. The density tends to infinity like $x^{-1/2}$ at $x = 0$.

Alternatively, one can series expand the numerator and denominator of ρ_{vw} separately around $w = 0$, and integrate v numerically, leading to evaluating

$$\frac{3\sqrt{d}}{4\pi}\int_d^{\frac{\pi}{2}-\frac{x}{4}-\frac{d}{2}} \sqrt{\sin(2v)+\sin(x)-\sin(2v+x)}\,dv. \tag{28}$$

The two methods give essentiallly the same result. The integral in (27) over the triangle for $w > d$ is done numerically. Together, the two parts give the density for the area of a random triangle shown in Figure 5. The contribution from (27) or (28) is small. Its maximal contribution amounts to 3 percent occurring around $x = \frac{\pi}{2}$.

6. The three-dimensional distribution of a, b, and c in Case 1.

A spherical triangle is defined by its three angles α, β, and γ, or by its three sides a, b, and c or any three of these six entities. For the area, we used the angles because the area formula $X = \alpha + \beta + \gamma - \pi$ is so simple. For the same reason, here we want to use the sides because

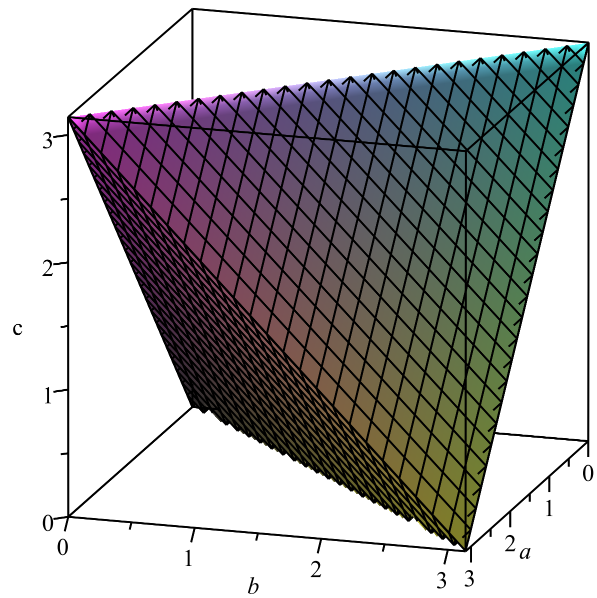

FIGURE 6. The tetrahedron in which a, b, and c take their values.

it makes the formula for the perimeter Y so simple

$$Y = a + b + c.$$

The relations between angles and sides is given in (2) and in (4) with its cyclic permutations.

We shall express the volume density $\rho(\alpha, \beta, \gamma)$ valid in Case 1 in the variables a, b, and c . This is done by first solving $\cos(\alpha)$ from (4) and do the same thing for $\cos(\beta)$ and $\cos(\gamma)$ from the cyclic permutations of (4) and then inserting them in (14). Including also the Jacobian for the change of coordinates, we arrive after simplification at the density for a, b, and c:

(29)

$$\mu(a, b, c) = \frac{1 - \cos(a)^2 - \cos(b)^2 - \cos(c)^2 + 2\cos(a)\cos(b)\cos(c)}{4\pi\sin(a)^2\sin(b)^2\sin(c)^2}.$$

Notice that this is not ρ^{-2} because the sign of the last term in numerator has changed. The domain of definition is the tetrahedron given in (1) and (3) and shown in Figure 6. The present tetrahedron has one vertex at the origin while the one in Figure 2 has a vertex in (π, π, π).

The density μ can be integrated analytically in the c-direction. Like in (α, β)-space, the (a,b)-square is divided by its diagonals into four

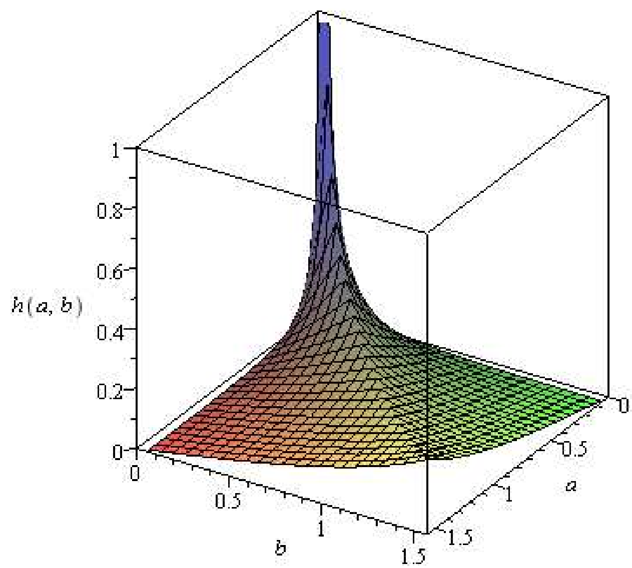

FIGURE 7. The two-dimensional marginal density function $h(a,b)$ given over the quarter of its domain $0 < a < \frac{\pi}{2}$, $0 < b < \frac{\pi}{2}$. The density is symmetric around $a = \frac{\pi}{2}$ and $b = \frac{\pi}{2}$.

triangles with different c-boundaries, compare Figure 6. The two-dimensional marginal density $h(a,b)$ is by symmetry the same in all four triangles and is given here for the triangle nearest to the a-axis:

$$h(a,b) = \int_{a-b}^{a+b} \mu \, dc = \frac{1}{2\,\pi} \, \frac{b - \sin{(b)}\, \cos{(b)}}{\sin{(a)}^2 \sin{(b)}^2}, \; b < a < \pi - b. \tag{30}$$

By symmetry, we have $h(a,b) = h(b,a)$ for $a < b$ and $h(a,b) = h(\pi - b, \pi - a)$ for $a > \pi - b$. This results in the unexpected density shown in Figures 7 and 8.

We calculate the one-dimensional marginal distribution $k(b)$ by integrating $2\,h(a,b)$ over a, $0 < a < \frac{\pi}{2}$, for a fixed b as in Figure 8 , getting:

$$k(b) = 2 \left(\int_0^b h(b,a)\, da + \int_b^{\frac{\pi}{2}} h(a,b)\, da \right) = \frac{1}{\pi} \, , \; 0 < b < \pi. \tag{31}$$

Somewhat unexpectedly, we find that $k(b)$ is constant.

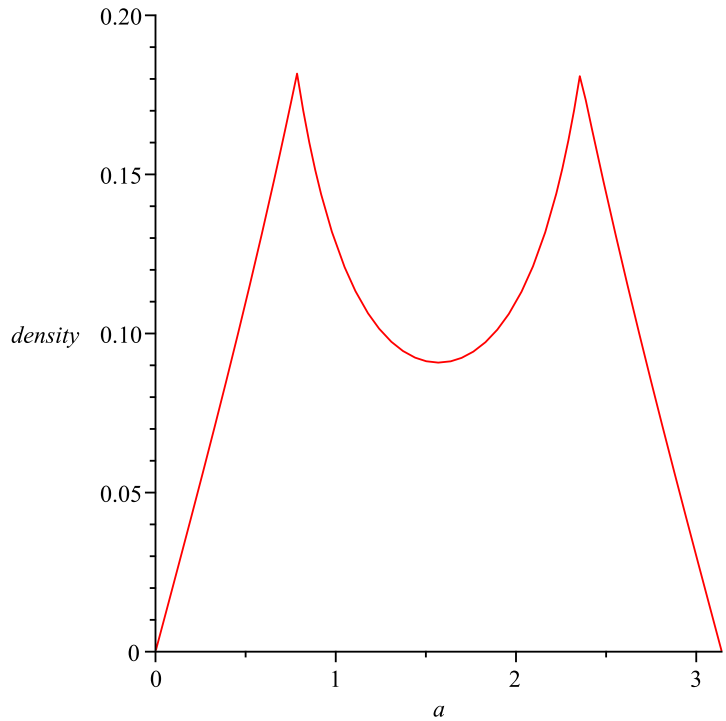

FIGURE 8. A cut though the density $h(a,b)$ for $b = \frac{\pi}{4}$. The peaks of the density occur at $a = b$ and $a = \pi - b$ and amount to $h(b,b) \approx \frac{1}{3\pi}\, b^{-1} + \frac{7}{45\pi}\, b$.

7. First and second moments of the perimeter in Case 1

First, since $k(b)$ is constant over its domain$(0,\pi)$, we have $E(b) = \frac{\pi}{2}$ giving

$$E(Y) = E(a) + E(b) + E(c) = 3\,E(b) = \frac{3}{2}\,\pi.$$

For the second moment, we have

$$E(Y^2) = E((a+b+c)^2) = 3\,E(a^2) + 6\,E(a\,b). \tag{32}$$

Again from (31), we have

$$E(a^2) = E(b^2) = \int_0^\pi b^2\,\frac{1}{\pi}\,db = \frac{\pi^2}{3}.$$

We need

$$E(ab) = \int_0^\pi \int_0^\pi b\,a\,h(a,b)\,da\,db,$$

where the two-dimensional marginal distribution $h(a,b)$ looks like in Figure 8. Using the dilog formula

$$Li_2(e^{ix}) + Li_2(e^{-ix}) = \frac{x^2}{2} - \pi\,|x| + \frac{\pi^2}{3}$$

which is valid for real x, $|x| < 2\pi$ and a lengthy simplification of *Maple's* result, we find the inner integral to be

$$\int_0^\pi a\, h(a,b)\, da = \frac{1}{2}.$$

Then, the outer integral becomes

$$E(ab) = \int_0^\pi \frac{1}{2}\, b\, db = \frac{\pi^2}{4},$$

giving

$$E(Y^2) = \pi^2 + 6\, E(ab) = \frac{5}{2}\,\pi^2$$

and

$$\sigma(Y) = \sqrt{E(Y^2) - E(Y)^2} = \frac{\pi}{2}.$$

8. The probability density of the perimeter in Case 1.

We obtain the probability density $g(y)$ of the perimeter in the same way as we did for the area, i.e. by integrating μ over the two-dimensional intersection of the bounding tetrahedron in Figure 6 with the plane $a+b+c=y$. Inserting $c = y-a-b$ in μ in (30), we have

$$\mu(a, b, y-a-b)$$

to be integrated over the projection of the intersection on the ab-plane. By symmetry, we can take two times the integral over half the domain. The integration domain is marked in Figure 9.

$$g(y) = 2\int_{\frac{y}{4}}^{\frac{y}{2}} \int_{\frac{y}{2}-a}^{a} \mu(a, b, y-a-b)\; db\; da \quad . \tag{33}$$

The first integration of $\mu(a, b, y-a-b)$ with respect to b can be done analytically. To give an impression of what this integral looks like, we give the expression for the indefinite integral.

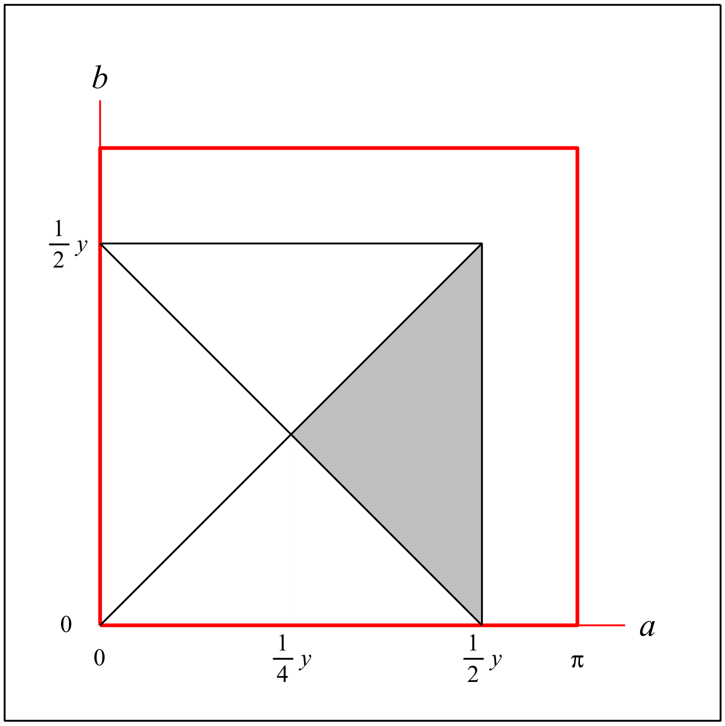

FIGURE 9. The area to integrate over in (33) is marked gray.

$$
\begin{aligned}
&\phi(y,a,b) = \int \mu(a,b,y-a-b)\,db = \\
&= \frac{\cos(y-a-b)}{\sin(a)^2\,\sin(y-a-b)} + \frac{2\,\cos(a)\,\sin(b)}{\sin(a)^2\sin(b)^2\,\sin(y-a-b)} \\
&+ \frac{(\sin(2y-2a-b)+3\sin(b))(1+\cos(a)^2)}{2\sin(y-a)^2\,\sin(y-a-b)\sin(a)^2\,\tan(b)\,\cos(y-a)} \\
&- \frac{\cos(b)}{\sin(a)^2\,\sin(b)} - \frac{\cos(a)\,(4\,\cos(y-a)\,\tan(b)+2\,\sin(y-a))}{\sin(a)^2\sin(y-a)^2\tan(b)^2} \\
&+ \frac{\ln(\sin(y-a-b)) - \ln(\sin(b))}{\sin(a)^2\sin(y-a)^3}\cdot \\
&\cdot (2\,\cos(y-a)\,(1+\cos(a)^2) - \cos(a)\,\cos(2y-2a) - 3\,\cos(a))
\end{aligned} \tag{34}
$$

Inserting the boundaries, we have

$$
\psi(y,a) = 2\left(\phi(y,a,a) - \phi(y,a,\frac{y}{2}-a)\right)
$$

to integrate over a. Some but not all of the obtained 12 terms can be integrated analytically, so we must use numerical integration. We show $\psi(y,a)$ for $y=6$ in Figure 10.

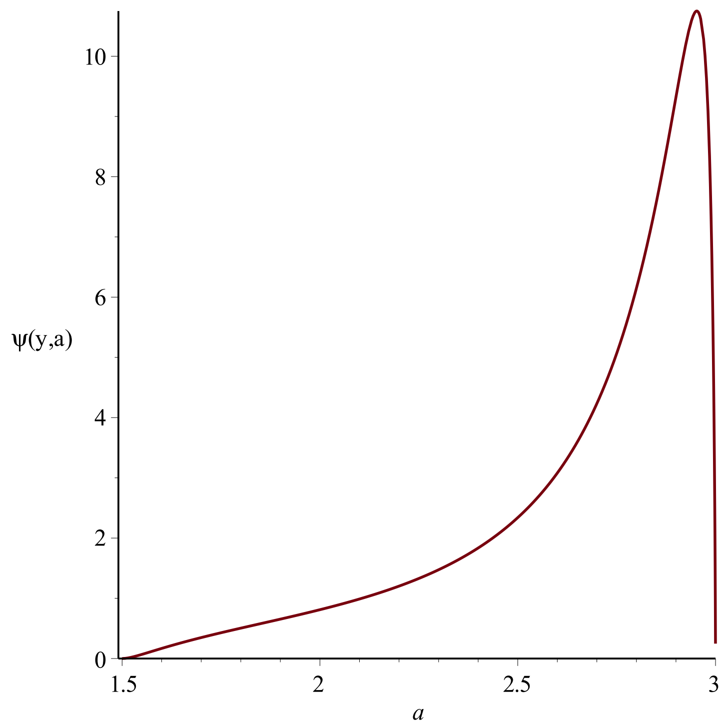

FIGURE 10. The function $\psi(y,a)$ for $y=6$, to be integrated from $\frac{y}{4}$ to $\frac{y}{2}$.

When y approaches $2\,\pi$, the peak of ψ moves towards $\frac{y}{2}$ and increases towards infinity. In the numerical integration, we use the estimate $pp = .66\,y - .32\,\pi$ of the peak position valid for y close to 2π by using the following statement in the code

$$\text{if } y < 5 \text{ then } pp = \frac{y}{2} - .01 \text{ else } pp = .66y - .32\,\pi \text{ end if}$$

and divide the integration into two parts

$$g(y) = g_1(y) + g_2(y) \ \text{, where}$$
$$g_1(y) = \int_{\frac{y}{4}}^{pp} \psi(y,a)da \quad \text{and} \quad g_2(y) = \int_{pp}^{\frac{y}{2}-.001} \psi(y,a)\,da. \tag{35}$$

Without this split of g, the numerical integration stalls when y is close to 2π and the integrator decreases the steplength at the peak.

The obtained probability density for the perimeter in Case 1 is shown in Figure 11.

9. Case 2.

This case is the alternative way of using the random points on the sphere to construct a triangle. Here, we let the generated points be the vertices of the triangle and connect them by great circles that form the

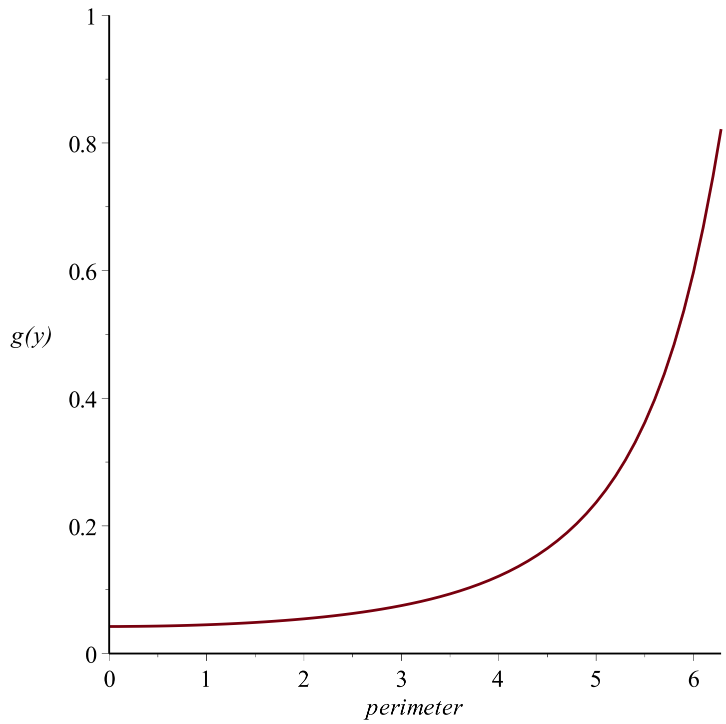

FIGURE 11. The probability density function $g(y)$, $(0 < y < 2\pi)$, for the perimeter in Case 1.

triangle sides. The random vectors n_1, n_2, and n_3 point out the points A, B, and C in Figure 1. The angle between n_1 and n_2 is the angle-measured side c of the triangle and similarly for the sides a and b. The so obtained spherical triangle is said to be polar to the one studied in the earlier sections. The angle ϕ between n_1 and n_2 that gave $\gamma = \pi - \phi$ in Case 1 now gives side c. Denoting case with subindices, we have the following relation between the two cases

$$\begin{aligned} a_2 + \alpha_1 &= \pi \quad b_2 + \beta_1 = \pi \quad c_2 + \gamma_1 = \pi \\ \alpha_2 + a_1 &= \pi \quad \beta_2 + b_1 = \pi \quad \gamma_2 + c_1 = \pi. \end{aligned} \tag{36}$$

The area X_2 in Case 2 becomes

$$X_2 = \alpha_2 + \beta_2 + \gamma_2 - \pi = 2\pi - a_1 - b_1 - c_1 = 2\pi - Y_1,$$

meaning that the density $f_2(x)$ of the area in Case 2 equals the reversed density of the perimeter in Case 1 and the perimeter Y_2 in Case 2 becomes

$$Y_2 = a_2 + b_2 + c_2 = 3\pi - \alpha_1 - \beta_1 - \gamma_1 = 2\pi - X_1.$$

We get

$$f_2(x) = g_1(2\pi - x) \quad \text{and} \quad g_2(y) = f_1(2\pi - y)\,.$$

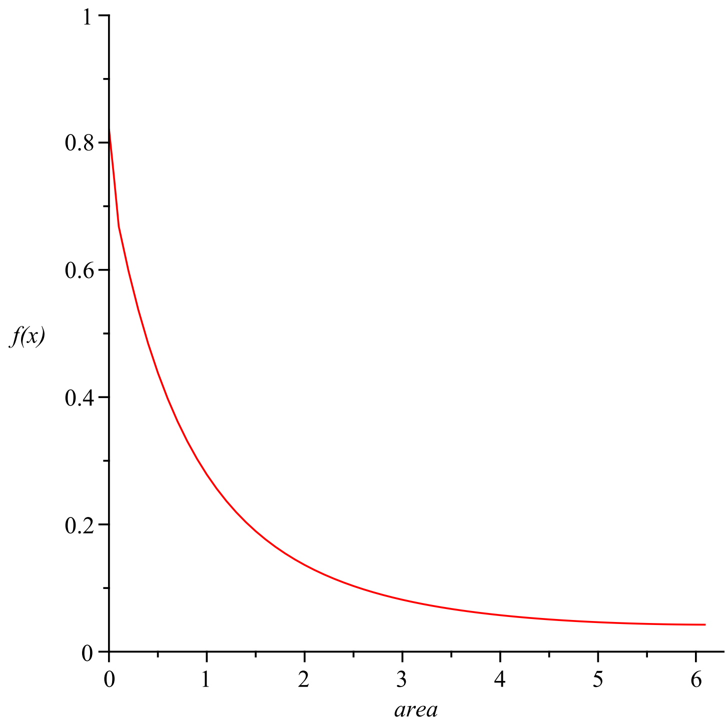

FIGURE 12. The probability density for the area in Case 2, $(0 < x < 2\pi)$.

This implies that

$$E(X_2) = 2\pi - E(Y_1) = \frac{\pi}{2}\,, E(X_2^2) = 4\pi^2 - 4\pi\, E(Y_1) + E({Y_1}^2) = \frac{\pi^2}{2}$$

and

$$\sigma(X_2) = \sqrt{E({X_2}^2) - E(X_2)^2} = \frac{\pi}{2}.$$

We also have that

$$E(Y_2) = 2\pi - E(X_1) = \frac{3\pi}{2}\,, E(Y_2^2) = 4\pi^2 - 4\pi\, E(X_1) + E({X_1}^2) = 3\,\pi^2 - 6$$

and

$$\sigma(Y_2) = \sqrt{E({Y_2}^2) - E(Y_2)^2} = \frac{1}{2}\sqrt{3\,\pi^2 - 24}.$$

We give the two probability densities in Figures 12 and 13.

R. E. Miles [11] mentions that Mr. J. N. Boots (Mt. Stromlo Observatory) has determined

(37)

$$f_2(x) =$$
$$-\frac{(x^2 - 4\pi x + 3\pi^2 - 6)\ \cos(x) - 6(x - 2\pi)\ \sin(x) - 2(x^2 - 4\pi x + 3\pi^2 + 3)}{16\,\pi\cos(x/2)^4}.$$

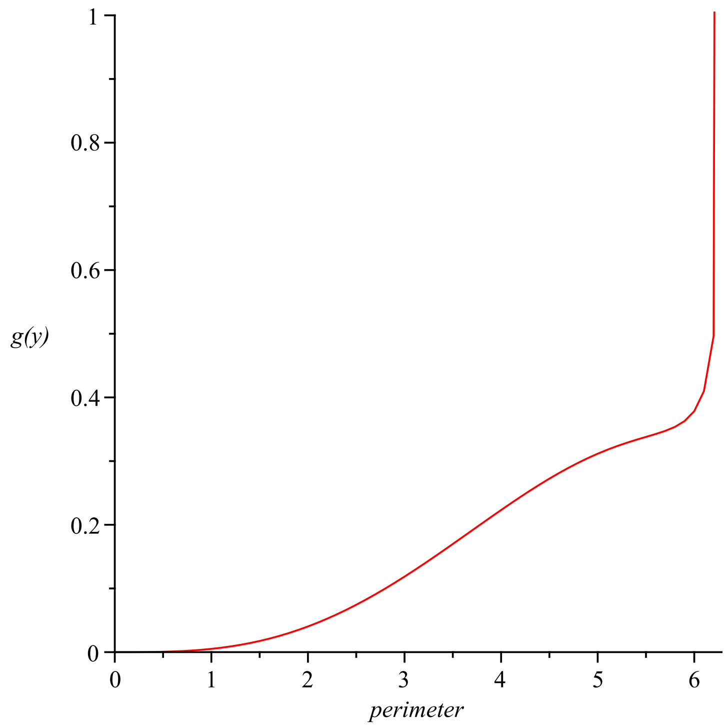

FIGURE 13. The probability density for the perimeter in Case 2, $(0 < y < 2\pi)$.

We cannot see how integration of $\psi(y,a)$ can become this expression for $f_2(x)$, but the values coincide with the numerical calculation given in Figure 12. However, Finch and Jones [5] have deduced (37) starting from the following formula for the area x:

$$\tan(\frac{x}{4}) = \sqrt{\tan(\frac{s}{2})\tan(\frac{s-a}{2})\tan(\frac{s-b}{2})\tan(\frac{s-c}{2})},$$

where $s = \frac{1}{2}(a+b+c)$.

10. Monte Carlo tests.

All distributions and values that we have presented have been checked by Monte Carlo tests. We generate a random point $n = (n_1, n_2, n_3)$ on the unit sphere by the following program using the function RAN which produces a random number between 0 and 1.

DO
x=2*RAN-1 ; y=2*RAN-1 ; z=2*RAN-1;
$r2 = x^2 + y^2 + z^2$;
LOOP WHILE $r2 > 1$;
r=sqrt(r2) ;

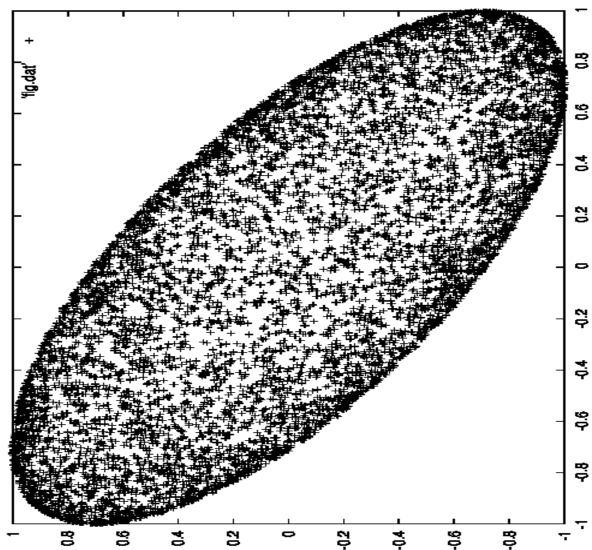

FIGURE 14. A cut through the 'blown up' tetrahedron in Figure 3 showing that the density is great at the boundary. The figure also shows that the cut is an ellips.

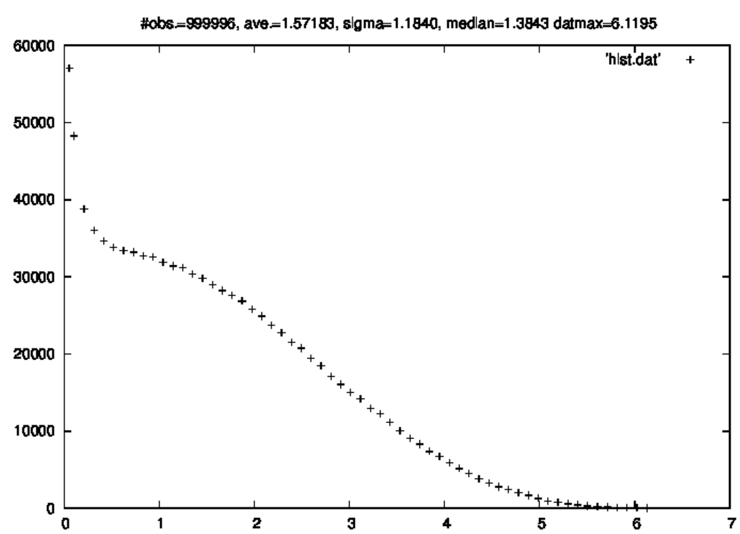

FIGURE 15. The probability density in Figure 5.

n1=x/r ; n2=y/r ; n3=z/r ;

We give some of the printouts in Figures 14 - 16.

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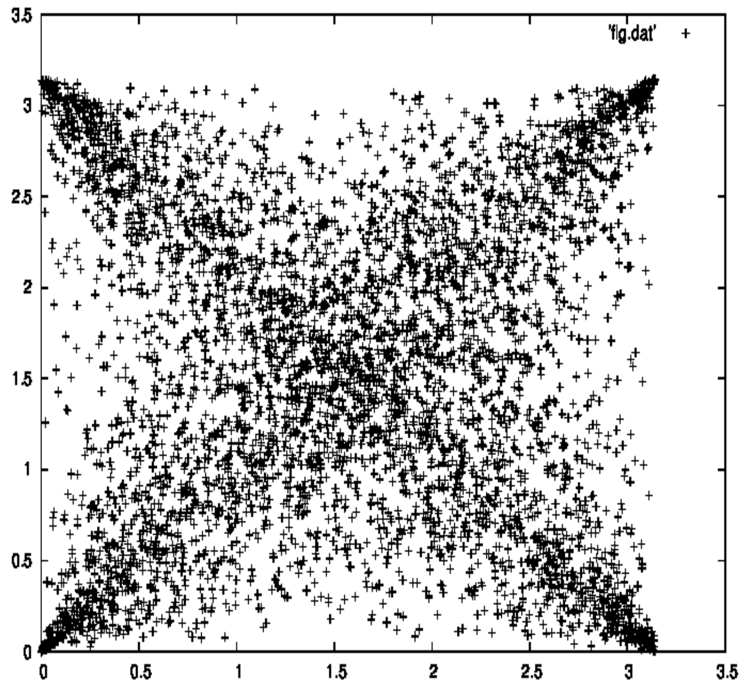

Figure 16. The density shown in Figures 7 and 8.

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