KTH Engineering Sciences

Mathematical Foundation for Compressed Sensing

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An outline for today

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- The Restricted Isometry Property (RIP()) and LEIP().
- B-RIP(): Bilinear version of RIP

We were looking for *sparse* solutions $\mathbf{x} = \mathbf{x}_{\text{sparse}}$ of the equation

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

where the $m \times N$-matrix $\mathbf{A} \in \mathbb{C}^m \times \mathbb{C}^N$ and the column vector $\mathbf{y} \in \mathbb{C}^m$ is given, with $m << N$.

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Two algorithms:

Minimize the number of non-zero elements:

$$\mathbf{x}_{\text{sparse}} := \begin{cases} \text{argmin} \# \text{ non-zero element in } \mathbf{x}, \\ \mathbf{A}\mathbf{x} = \mathbf{y}. \end{cases} \qquad (P_0)$$

and minimize the l_1 - norm:

$$\mathbf{x}_{\text{sparse}} := \begin{cases} \text{argmin}_{\mathbf{x}} \ \sum_i |x_i| \\ \mathbf{A}\mathbf{x} = \mathbf{y} \end{cases} \qquad (P_1)$$

Central result in the course:

(P_0) and (P_1) have equivalent solutions if columns in $\mathbf{A}$ are sufficiently "independent" (Candés–Romberg–Tao, 2004)

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We define sets of matrices.

- $MP_0(s) = MP_0(s; m, N)$: The set of Matrices A s.t. every s-sparse vector $\mathbf{x}$ is: the unique solution of (P_0) for some $\mathbf{y}$.

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Preliminaries

Definition Let $\mathbf{x} = (x_i) \in \mathbb{R}^N$, we define the $\|\mathbf{x}\|_p$-norm, $1 \leq p \leq \infty$ by

$$\|\mathbf{x}\|_p = \begin{cases} \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}, & \text{when } 1 \leq p < \infty \\ \max |x_i|, & \text{when } p = \infty \end{cases} .$$

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We will only use the $\|\cdot\|_0$ -, $\|\cdot\|_1$ -, $\|\cdot\|_2$ - and $\|\cdot\|_\infty$ - norms.

We have the following simple relation between those norms

$$\|\mathbf{x}\|_1 \begin{cases} \leq \|\mathbf{x}\|_0^{\frac{1}{2}} \|\mathbf{x}\|_2, \\ \leq \|\mathbf{x}\|_0 \|\mathbf{x}\|_\infty, \end{cases}$$

and

$$\|\mathbf{x}\|_2 \begin{cases} \leq \|\mathbf{x}\|_0^{\frac{1}{2}} \|\mathbf{x}\|_\infty, \\ \leq \|\mathbf{x}\|_1^{\frac{1}{2}} \|\mathbf{x}\|_\infty^{\frac{1}{2}}, \end{cases}$$

Definition The l_1 - entropy of a non-zero vector $\mathbf{x}$ is defined by

$$\mathrm{Ent}_1(\mathbf{x}) = \frac{\|\mathbf{x}\|_1}{\|\mathbf{x}\|_2}$$

Note that is x is a s - sparse non-zero vector
then $\mathrm{Ent}_1(x) \leq \sqrt{s}$

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- The pair of two orthonormal bases $\{\mathbf{x}_k\}_{k=1}^{N}$ and $\{\mathbf{y}_k\}_{k=1}^{N}$ in $\mathbb{C}^N$ is incoherent if all inner products $< \mathbf{x}_k, \mathbf{y}_j >$ are small. More precizly if

$$< \mathbf{x}_k, \mathbf{y}_j > \leq K/\sqrt{N} \text{ for all } 1 \leq j, k \leq N,$$

for some constant $K > 0$. we say the bases are K-incoherent

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  7. The construction is finished!

Construction of matrices in $MP_0(s)$ with more controll 2s -tiple of unitvectors A_N

This page indicates what we may do later on

The construction of preceeding frame gives a matrix $\mathbf{A} \in \mathrm{MP}_0()s$ with very little controll how close the vectors $\mathbf{A}_n$ are to each other. If m is much larger than $2s$ one can controll the distance from each new colonmvector $\mathbf{A}_n$ to all preceeding columnvectors $\mathbf{A}_k, k \leq n$, (which might be useful):

- If $m > 2s$ Any $2s - 1$ - tiple of chosen column vectors span a subspace of codimension $m - 2s + 1$.

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- If $m > 2s$ Any $2s - 1$ - tiple of chosen column vectors span a subspace of codimension $m - 2s + 1$.
- Build an m-dimensional plate by intersecting each such $2s - 1$ - dimensional subspace with the unit ball and take direct sum with the perpendicular ball of radius r and dimension $m - 2s + 1$.

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- Build an m-dimensional plate by intersecting each such $2s - 1$ - dimensional subspace with the unit ball and take direct sum with the perpendicular ball of radius r and dimension $m - 2s + 1$.
- The volume of such a plate can be estimated from above..
- Choose the new column A_n in the unite sphere, but outside each such plate.

Example of controls of the columnvector in **A**

- The repeating may stop will go on as long or $n < N$ or it will stop when the plates cover the m-dimensional unit ball . in this case

  (total number plates)$\times$(plate volume)$\geq$ (volume of the m- dimensio

  This gives a relation between r, s, m, and maximal value of N before the algorithm stops.

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Some examples of control

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We will come back to this later on in the course.

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- If $0 < 1 - r < c/s$ for some small constant $c > 0$ and $m > Cs^2 \log N$ for some constant $C > 0$ large enough the this construction will give a matrix in $MP_1(s, ; m, N)$

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Sparse measurements

- Assume $\mathcal{C}$ class of signals $\mathbf{x}$ of length N with s-sparse representation in:
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- Assume the forward operator the signal x will will is able to provide the a complete set measured data $\mathbf{y}$ of length N such that
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  where $N \times N$ matrix $\mathbf{A}$ is invertible.

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- Assume forward operator can be modified by a $m \times N$ -matrix $\mathbf{E}$, to give a measurement vector $\mathbf{z}$ by
  $$\mathbf{z} = \mathbf{E}\mathbf{y},$$
  where $\mathbf{z}$ is a vector of dimension m.

Sparse measurements

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  Let $\mathbf{A}'$ be a $2s \times N$ - matrix in $\mathrm{MP}_0(s)$ and set $\mathbf{E} = \mathbf{A}'\mathbf{A}^{-1}\mathbf{D}^{-1}$ and $\mathbf{z} = \mathbf{E}\mathbf{z}$. Then the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$ is transformed into

$$\mathbf{A}'z = \eta$$

- Note that it is sufficient with $m = 2s$ independent of how large N is.

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- Definition: Matrix $\mathbf{A}$ satisfies Null Space condition $NS(s)$ if for any non-zero vector $\mathbf{x}$ with $\mathbf{A}\mathbf{x} = 0$ and any index subset $S, |S| = s$ we have

$$\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{^cS}\|_1,$$

where $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_{^cS}$.

Let S be any index subset of $\{1, \dots, N\}$ with $|S| = s$.
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- Definition: Matrix $\mathbf{A}$ satisfies Null Space condition $NS(s)$ if for any non-zero vector $\mathbf{x}$ with $\mathbf{Ax} = 0$ and any index subset $S, |S| = s$ we have

$$\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{^cS}\|_1,$$

where $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_{^cS}$.

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- Definition: Matrix $\mathbf{A}$ satisfies Null Entropy condition $NE(e)$ if for any non-zero vector $\mathbf{x}$ with $\mathbf{Ax} = 0$ we have

$$\mathrm{Ent}_1(x) \geq e$$

Easy claim

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- If $e \geq 2\sqrt{s}$ then

$$(\mathbf{A} \in NE(e)) \text{ implies } (\mathbf{A} \in NS(s))$$

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Proof: $(\mathbf{A} \in NE(e))$ and $\|\mathbf{x}_S\|_1 \geq \|\mathbf{x}_{{}^cS}\|_1$, implies

$$e \leq \text{Ent}_1(\mathbf{x}_S + \mathbf{x}_{{}^cS}) < \frac{\|\mathbf{x}_S\|_1 + \|\mathbf{x}_{{}^cS}\|_1}{\|\mathbf{x}_S\|_2} \leq 2\text{Ent}_1(\mathbf{x}_S) \leq 2\sqrt{s}$$

Easy claim

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Equivalent condtions

For a matrix $\mathbf{A}$ we have

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Proof: if $\mathbf{Ax} = 0$ then $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_{^cS}$ for some index set S with $|S| \leq s$ and $A\mathbf{x}_S = A(-\mathbf{x}_{^cS)})$.
$\mathbf{A} \in \mathrm{MP}_1(s)$ implies x_c is the uniqe solution of a l_1 minimization problem thus $\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{^cS}\|_1$

Equivalent condtions

For a matrix $\mathbf{A}$ we have

$$\mathbf{A} \in \mathrm{MP}_1(s)$$

if and only if

$$\mathbf{A} \in NS(s)$$

Proof: Let $\mathbf{x}$ be s sparse with support in S, $|S| = s$ an $\mathbf{z}$ any vector such that $\mathbf{Az} = \mathbf{Ax}$, Write $\mathbf{z} = \mathbf{z}_S + \mathbf{z}_{^cS}$. Then

$$\|\mathbf{z}\|_1 = \|\mathbf{z}_S\|_1 + \|\mathbf{z}_{^cS}\|_1 \geq \|\mathbf{x}\|_1 - \|\mathbf{z}_S - \mathbf{x}\|_1 + \|\mathbf{z}_{^cS}\|_1$$

Since $\mathbf{x} - \mathbf{z}_S$ is s-sparse, $\mathbf{A}(\mathbf{x} - \mathbf{z}_S) = \mathbf{Az}_{^cS}$ and $\mathbf{A} \in NS(s)$ we have $\|\mathbf{z}_{^cS}\|_1 - \|\mathbf{z}_S - \mathbf{x}\|_1 > 0$. Thus $\|\mathbf{z}\|_1 > \|\mathbf{x}\|_1$

Equivalent condtions

For a matrix $\mathbf{A}$ we have

$$\mathbf{A} \in \mathrm{MP}_1(s)$$

if and only if

$$\mathbf{A} \in NS(s)$$

This is how far we got on lecture 2!

- Definition: Matrix $\mathbf{A}$ satisfies Low Entropy Isometry Property (LEIP) with constant $\tilde{\delta}_e$ if

$$\left| \|\mathbf{Ax}\|_2^2 - \|\mathbf{x}\|_2^2 \right| \leq \tilde{\delta}_e \|\mathbf{x}\|_2^2$$

for all $\mathbf{x}$ with $\mathrm{Ent}_1(\mathbf{x}) \leq e$.

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- Definition Matrix $\mathbf{A}$ satisfies Restricted Isometry Property (RIP) with constant δ_s if

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Easy claims

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Easy claims

- If the matrix $\mathbf{A}$ satisfies LEIP with constant $\tilde{\delta}_e$ and $\sqrt{s} \leq e$, then $\mathbf{A}$ satisfies RIP with a constant $\delta_s \leq \tilde{\delta}_e$.

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Easy claims

- If the matrix $\mathbf{A}$ satisfies LEIP with constant $\tilde{\delta}_e$ and $\sqrt{s} \leq e$, then $\mathbf{A}$ satisfies RIP with a constant $\delta_s \leq \tilde{\delta}_e$.
Proof: if $\mathbf{x}$ is s-sparse then $\mathrm{Ent}_1(\mathbf{x}) \leq \sqrt{s}$

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Easy claims

- If the matrix $\mathbf{A}$ satisfies LEIP with constant $\tilde{\delta}_e$ and $\sqrt{s} \le e$, then $\mathbf{A}$ satisfies RIP with a constant $\delta_s \le \tilde{\delta}_e$.
- If $\tilde{\delta}_e < 1$ then $\mathbf{A} \in NE(e)$

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Easy claims

- If the matrix $\mathbf{A}$ satisfies LEIP with constant $\tilde{\delta}_e$ and $\sqrt{s} \leq e$, then $\mathbf{A}$ satisfies RIP with a constant $\delta_s \leq \tilde{\delta}_e$.
- If $\tilde{\delta}_e < 1$ then $\mathbf{A} \in NE(e)$
Proof: For any $\mathbf{x} \neq 0$ with $\mathrm{Ent}_1(\mathbf{x}) \leq e$ and $\mathbf{Ax} = 0$ then LEIP with $\tilde{\delta}_e < 1$ would imply $\|\mathbf{x}\|_2^2 < \|\mathbf{x}\|_2^2$.

Referencer

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Thats it for today!
- Thank you for your attension!