Mathematical Foundation for Compressed Sensing

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An outline for today

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  Bi-linear RIP and Polarisation

We were looking for *s-sparse* solutions $\mathbf{x} = \mathbf{x}_{\text{sparse}}$ of the equation

$$\mathbf{y} = \mathbf{A}\mathbf{x},$$

where the $m \times N$-matrix $\mathbf{A} \in \mathbb{C}^m \times \mathbb{C}^N$ and the column vector $\mathbf{y} \in \mathbb{C}^m$ is given, with $m << N$.

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Solution by minimising the l_1 - norm:

$$\mathbf{x}_{\text{sparse}} := \begin{cases} \text{argmin}_{\mathbf{x}} \ \sum_i |x_i|, \\ \mathbf{A}\mathbf{x} = \mathbf{y}. \end{cases} \qquad (P_1)$$

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$MP_1(s) = MP_1(s; m, N)$: The set of Matrices A s.t. every s-sparse vector **x** is: the unique solution of (P_1) for some **y**.

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The LEIP and RIP properties of matrices

We were looking at some properties for $m \times N$ - matrices ($m << N$):

- Definition: Matrix $\mathbf{A}$ satisfies Low Entropy Isometry Property (LEIP) with constant $\tilde{\delta}_e$ if

$$\left| \|\mathbf{Ax}\|_2^2 - \|\mathbf{x}\|_2^2 \right| \leq \tilde{\delta}_e \|\mathbf{x}\|_2^2,$$

for all $\mathbf{x}$ with $\mathrm{Ent}_1(\mathbf{x}) \leq e$.

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- Definition: Matrix $\mathbf{A}$ satisfies Restricted Isometry Property (RIP) with constant δ_s if

$$|\|\mathbf{Ax}\|_2^2 - \|\mathbf{x}\|_2^2| \leq \delta_s \|\mathbf{x}\|_2^2,$$

  for all s- sparse vectors $\mathbf{x}$.

- Easy claim:
  If the matrix **A** satisfies LEIP with constant $\tilde{\delta}_e$ and $\sqrt{s} \leq e$, then **A** satisfies RIP with a constant $\delta_s \leq \tilde{\delta}_e$.

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  If the matrix $\mathbf{A}$ satisfies LEIP with constant $\tilde{\delta}_e$ and $\sqrt{s} \leq e$, then $\mathbf{A}$ satisfies RIP with a constant $\delta_s \leq \tilde{\delta}_e$.
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- In the other direction:
  Unproved statement -proof later on:

$$(\mathbf{A} \text{ has the RIP-property with constant } \delta_s)$$

implies

$$(\mathbf{A} \text{ has the LEIP-property with constant } \tilde{\delta}_e)$$

with $\tilde{\delta}_e \leq 4\delta_s$ provided $s \geq 2e^2$

Easy Claim: If $\tilde{\delta}_e < 1$ then $\mathbf{A} \in NE(e)$, i.e each non-zero $\mathbf{x}$ with $\mathbf{Ax} = 0$ has entropy $\mathrm{Ent}_1(x) > e$.

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Proof: For any $\mathbf{x} \neq 0$ with $\text{Ent}_1(\mathbf{x}) \leq e$ and $\mathbf{Ax} = 0$ then LEIP:

$$\left| \|\mathbf{Ax}\|_2^2 - \|\mathbf{x}\|_2^2 \right| \leq \tilde{\delta}_e \|\mathbf{x}\|_2^2$$

with $\tilde{\delta}_e < 1$ would imply $\|\mathbf{x}\|_2^2 < \|\mathbf{x}\|_2^2$.

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- ($\mathbf{A} \in \mathrm{NS}(s)$ i.e each s-sparse vector $\mathbf{x}$ supported in S with $|S| = s$ and each vector $\mathbf{z} = \mathbf{z}_S + \mathbf{z}_{^cS}$ with $\mathbf{Az} = \mathbf{Ax}$ we have $\|\mathbf{z}_{^cS}\|_1 > \|\mathbf{x} - \mathbf{z}_S\|_1$)
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Using the steps with triangle inequality and definition of entropy in the proof above (without assuming $\mathbf{Ax} = \mathbf{Az}$) we would get

Lemma 3.2: Assume $\mathbf{A}$ $\in$LEIP with constant $\tilde{\delta}_e$ If $\mathbf{x}$ is s-sparse and $2\sqrt{s} \leq e$, then for any vector $\mathbf{z}$ with $\|\mathbf{z}\|_1 \leq \|\mathbf{x}\|_1$ holds

$$\left| \|\mathbf{A}(\mathbf{x} - \mathbf{z})\|_2^2 - \|\mathbf{x} - \mathbf{z}\|_2^2 \right| \leq \tilde{\delta}_e \|\mathbf{x} - \mathbf{z}\|_2^2.$$

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Observe: Theorem 3.1 follows easily from Lemma 3.2:
If we assume that $\mathbf{A}(\mathbf{x}-\mathbf{z}) = 0$ and $\tilde{\delta}_e < 1$ in Lemma 3.2 we get $\mathbf{x}-\mathbf{z} = 0$, i.e we conclude that $\mathbf{x}$ is the unique solution of the minimal problem (P_1).

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Corollary If $\mathbf{x}$ is s-sparse , $\mathbf{y} = \mathbf{Ax}$ and $\mathbf{Az} = \mathbf{y} + \mathbf{n}$ then $\|\mathbf{z}\|_1 \leq \|\mathbf{x}\|_1$ and $\tilde{\delta}_e < 1$ would imply $\|\mathbf{z}-\mathbf{x}\|_2 \leq (1-\tilde{\delta}_e)^{-1}\|\mathbf{n}\|_2$.

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Preparing the proof:

It is enough to show that f $\mathbf{A}$ fulfils the RIP property with constant $\delta_{2s} \leq \sqrt{2} - 1$ then $\mathbf{A} \in \mathrm{NS}(s)$. The proof is based on two technical lemmas:

Proof of Theorem 3.3. (cont.)

- Let $\mathbf{x}$ be a vector that is ordered decreasing in magnitude, i.e $|x_1| \geq |x_2| \geq |x_3| \geq \ldots$.

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- Split $\{1, \dots, N\}$ into disjoint sets $S_k, k = 1, \dots, K$ $\cup S_k = \{1, \dots, N\}$ with $|S_k| = s$ for $k = 1, \dots, K-1$. Denote $S = S_1$ and ${}^cS$ its complement.

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- Write $\mathbf{x} = \sum_k \mathbf{x}_{S_k}$

Proof of Theorem 3.3. (cont.)

- Lemma 3.4: Assume is $\mathbf{x}$ is a vector with its components decreasing in magnitude and $\mathbf{x} = \sum_k \mathbf{x}_{S_k}$ and above $S = S_1$ as above. Then

$$\sum_{k=2} \|\mathbf{x}_{S_k}\|_2 \leq \frac{\alpha}{2\sqrt{s}}\|\mathbf{x}_S\|_1 + \frac{1}{2\sqrt{s}}(\alpha + \frac{1}{\alpha})\|\mathbf{x}_{^cS}\|_1,$$

for $\alpha > 0$, to be optimised later on.

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Proof of Lemma 3.4: For $k > 1$:

$$\|\mathbf{x}_{S_k}\|_2^2 \leq \|\mathbf{x}_{S_k}\|_1 \|\mathbf{x}_{S_k}\|_\infty \leq \|\mathbf{x}_{S_k}\|_1 \|\mathbf{x}_{S_{k-1}}\|_1 / s$$

using $ab \leq \frac{1}{2\alpha} a^2 + \frac{\alpha}{2} b^2$ for any $a, b \geq 0$ and $\alpha > 0$ we get

$$\|\mathbf{x}_{S_k}\|_2 \leq \frac{\alpha}{2\sqrt{s}} \|\mathbf{x}_{S_{k-}}\|_1 + \frac{1}{2\alpha\sqrt{s}} \|\mathbf{x}_{S_k}\|_1.$$

Summation over $k \geq 2$ gives the desired result.

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Proof of Lemma 3.5 uses two lemmas:

Proof of Theorem 3.3. (cont.)

- Lemma 3.6: If $\delta_s < 1$ then

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Proof of Lemma 3.6: RIP gives

$$\|\mathbf{x}_S\|_2^2 - \|\mathbf{A}\mathbf{x}_S\|_2^2 \leq \delta_s\|\mathbf{x}_S\|_2^2,$$

which gives the result.

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- Lemma 3.7: Let the index subset S and S' be disjoint $|S| = |S'| = s$. Then

$$|\langle \mathbf{A}\mathbf{x}_S, \mathbf{A}\mathbf{x}_{S'}\rangle| \leq \delta_{2s}\|\mathbf{x}_S\|_2\|\mathbf{x}_{S'}\|_2.$$

Next: proof of Lemma 3.7.

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Proof of Lemma 3.7: We may assume $\|\mathbf{x}_S\|_2 = \|\mathbf{x}_{S'}\|_2 = 1$ and we will use RIP with so called polarisation. Note that $\mathbf{x}_S \pm \mathbf{x}_{S'}$ are $2s$-sparse. The absolute values of

$$\|\mathbf{A}(\mathbf{x}_S \pm \mathbf{x}_{S'})\|_2^2 - \|\mathbf{x}_S \pm \mathbf{x}_{S'}\|_2^2$$

is bounded by

$$\delta_s \|\mathbf{x}_S \pm \mathbf{x}_{S'}\|_2^2$$

Taking the difference of plus and minus versions we get

$$4|\langle \mathbf{A}\mathbf{x}_S, \mathbf{A}\mathbf{x}_{S'} \rangle - \langle \mathbf{x}_S, \mathbf{x}_{S'} \rangle| \leq \delta_{2s} 2(\|\mathbf{x}_S\|_2^2 + \|\mathbf{x}_{S'}\|_2^2) = 4\delta_{2s}$$

Since $\langle \mathbf{x}_S, \mathbf{x}_{S'} \rangle = 0$ we get the desired result.

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Proof completed!

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Lemma 3.5: If $\mathbf{A}$ satisfies RIP properties and $\mathbf{Ax} = 0$ then

$$\|\mathbf{x}_{S_1}\|_2 \leq \frac{\delta_{2s}}{1-\delta_s} \sum_{k=2} \|\mathbf{x}_{S_k}\|_2.$$

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Proof of Lemma 3.5: We will use $\mathbf{Ax}_s = -\mathbf{Ax}_{^cS}$. By Lemma 3.6

$$\begin{aligned} \|\mathbf{x}\|_2^2 &\le \tfrac{1}{1-\delta_s}\langle \mathbf{Ax}_S, \mathbf{Ax}_S\rangle \\ &= -\tfrac{1}{1-\delta_s}\langle \mathbf{Ax}_S, \mathbf{Ax}_{^cS}\rangle = -\tfrac{1}{1-\delta_s}\textstyle\sum_{k\ge 2}\langle \mathbf{Ax}_S, \mathbf{Ax}_{S_k}\rangle. \end{aligned}$$

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$$\|\mathbf{x}_S\|_2^2 \leq \frac{\delta_{2s}}{1-\delta_s}\|\mathbf{x}_S\|_2 \sum_{k\geq 2}\|\mathbf{x}_{S_k}\|_2,$$

which gives the desired estimate. Proof of Lemma 3.5 completed!

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Theorem 3.3: If **A** fulfils the RIP property with constant $\delta_{2s} \leq \sqrt{2} - 1$ then $\mathbf{A} \in \mathrm{MP}_1(s)$

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For your convenience: the equations in Lemma 3.4 and Lemma 3.5

$$\sum_{k=2} \|\mathbf{x}_{S_k}\|_2 \leq \frac{\alpha}{2\sqrt{s}} \|\mathbf{x}_S\|_1 + \frac{1}{2\sqrt{s}} (\alpha + \frac{1}{\alpha}) \|\mathbf{x}_{^cS}\|_1,$$

$$\|\mathbf{x}_{S_1}\|_2 \leq \frac{\delta_{2s}}{1 - \delta_s} \sum_{k=2} \|\mathbf{x}_{S_k}\|_2.$$

By Lemma 3.5 and Lemma 3.4 we have

$$\|\mathbf{x}_{S_1}\|_2 \leq \frac{\delta_{2s}}{(1-\delta_s)} \sum_{k=2} \|\mathbf{x}_{S_k}\|_2$$

Proof of Theorem 3.3. (cont.)

Back to proof of:

Theorem 3.3: If $\mathbf{A}$ fulfils the RIP property with constant $\delta_{2s} \leq \sqrt{2} - 1$ then $\mathbf{A} \in \mathrm{MP}_1(s)$

Proof of Theorem 3.3: Since $\mathbf{A} \in \mathrm{MP}_1(s)$ is equivalent to $\mathbf{A}\mathrm{NS}(s)$ it is enough to show that if S is an index subset with $|S| = s$, and $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_{^cS}$ is a non-zero vector with $\mathbf{Ax} = 0$ then $\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{^cS}\|_1$.
By Lemma 3.5 and Lemma 3.4 we have

$$\begin{aligned} \|\mathbf{x}_{S_1}\|_2 &\leq \tfrac{\delta_{2s}}{(1-\delta_s)} \textstyle\sum_{k=2} \|\mathbf{x}_{S_k}\|_2 \\ &\leq \tfrac{\delta_{2s}}{(1-\delta_s)} \left(\tfrac{\alpha}{2\sqrt{s}} \|\mathbf{x}_S\|_1 + \tfrac{1}{2\sqrt{s}} (\alpha + \tfrac{1}{\alpha}) \|\mathbf{x}_{^cS}\|_1 \right) \end{aligned}$$

We may replace $\|\mathbf{x}_s\|_2$ by $\frac{1}{\sqrt{s}}\|\mathbf{x}_s\|_1$ on the left hand side of this inequality and move all $\|\mathbf{x}_s\|_1$ terms to the left hand side and let the $\|\mathbf{x}_{^cS}\|_1$ term stay on the right hand side.

Proof of Theorem 3.3. (cont.)

Thus we get

$$\left(\frac{1}{\sqrt{s}} - \frac{\delta_{2s}}{(1-\delta_s)}\frac{\alpha}{2\sqrt{s}}\right)\|\mathbf{x}_s\|_1 \leq \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2\sqrt{s}}\left(\alpha + \frac{1}{\alpha}\right)\|\mathbf{x}_{{}^cS}\|_1.$$

Proof of Theorem 3.3. (cont.)

Thus we get

$$(\frac{1}{\sqrt{s}} - \frac{\delta_{2s}}{(1-\delta_s)}\frac{\alpha}{2\sqrt{s}})\|\mathbf{x}_s\|_1 \leq \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2\sqrt{s}}(\alpha + \frac{1}{\alpha})\|\mathbf{x}_{^cS}\|_1.$$

We will conclude that $\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{^cS}\|_1$ provided

Proof of Theorem 3.3. (cont.)

Thus we get

$$(\frac{1}{\sqrt{s}} - \frac{\delta_{2s}}{(1-\delta_s)}\frac{\alpha}{2\sqrt{s}})\|\mathbf{x}_s\|_1 \leq \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2\sqrt{s}}(\alpha + \frac{1}{\alpha})\|\mathbf{x}_{^cS}\|_1.$$

We will conclude that $\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{^cS}\|_1$ provided

$$(\frac{1}{\sqrt{s}} - \frac{\delta_{2s}}{(1-\delta_s)}\frac{\alpha}{2\sqrt{s}}) > \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2\sqrt{s}}(\alpha + \frac{1}{\alpha}),$$

Proof of Theorem 3.3. (cont.)

Thus we get

$$(\frac{1}{\sqrt{s}} - \frac{\delta_{2s}}{(1-\delta_s)}\frac{\alpha}{2\sqrt{s}})\|\mathbf{x}_s\|_1 \leq \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2\sqrt{s}}(\alpha + \frac{1}{\alpha})\|\mathbf{x}_{{}^cS}\|_1.$$

We will conclude that $\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{{}^cS}\|_1$ provided

$$(\frac{1}{\sqrt{s}} - \frac{\delta_{2s}}{(1-\delta_s)}\frac{\alpha}{2\sqrt{s}}) > \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2\sqrt{s}}(\alpha + \frac{1}{\alpha}),$$

which is same as

$$1 > \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2}(2\alpha + \frac{1}{\alpha}).$$

Proof of Theorem 3.3. (cont.)

Remains to analyse:

$$1 > \frac{\delta_{2s}}{(1-\delta_s)} \frac{1}{2} (2\alpha + \frac{1}{\alpha}).$$

We minimise $(2\alpha + \frac{1}{\alpha})$, $\alpha > 0$:

$$(2\alpha + \frac{1}{\alpha})' = 2 - \alpha^{-2} = 0$$

has root for $\alpha = \frac{1}{\sqrt{2}}$ and minimum value $2\sqrt{2}$.

Proof of Theorem 3.3. (cont.)

Remains to analyse:

$$1 > \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2}(2\alpha + \frac{1}{\alpha}).$$

We minimise $(2\alpha + \frac{1}{\alpha})$, $\alpha > 0$:

$$(2\alpha + \frac{1}{\alpha})' = 2 - \alpha^{-2} = 0$$

has root for $\alpha = \frac{1}{\sqrt{2}}$ and minimum value $2\sqrt{2}$.
Thus with $\alpha = \frac{1}{\sqrt{2}}$ above we need

$$1 > \frac{\delta_{2s}}{(1-\delta_s)}\frac{1}{2} \cdot 2\sqrt{2}.$$

We conclude that $\|\mathbf{x}_S\|_1 < \|\mathbf{x}_{^cS}\|_1$ provided

$$\frac{\delta_{2s}}{1-\delta_s} < \frac{1}{\sqrt{2}}.$$

By using $\delta_s \leq \delta_{2s}$ in

$$\frac{\delta_{2s}}{1-\delta_s} < \frac{1}{\sqrt{2}}.$$

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Thus we have completed the proof of
Theorem 3.3 : If **A** fulfils the RIP property with constant $\delta_{2s} \leq \sqrt{2} - 1$ then $\mathbf{A} \in \mathrm{MP}_1(s)$

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