Mathematical Foundation for Compressed Sensing

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Lecture 4, February 27, 2012

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  - The LEIP and RIP properties which imply MP_1
  - Bi-linear RIP and Polarisation.
- A lower estimate for RIP.
- RIP of Random matrices.

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We were looking for *s-sparse* solutions $\mathbf{x} = \mathbf{x}_{\text{sparse}}$ of the equation

$$\mathbf{y} = \mathbf{A}\mathbf{x},$$

where the $m \times N$-matrix $\mathbf{A} \in \mathbb{C}^m \times \mathbb{C}^N$ and the column vector $\mathbf{y} \in \mathbb{C}^m$ is given, with $m << N$.

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Solution by minimising the l_1 - norm:

$$\mathbf{x}_{\text{sparse}} := \begin{cases} \text{argmin}_{\mathbf{x}} \ \sum_i |x_i|, \\ \mathbf{A}\mathbf{x} = \mathbf{y}. \end{cases} \qquad (P_1)$$

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$MP_1(s) = MP_1(s; m, N)$: The set of Matrices A s.t. every s-sparse vector $\mathbf{x}$ is: the unique solution of (P_1) for some $\mathbf{y}$.

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- Definition: Matrix $\mathbf{A}$ satisfies Low Entropy Isometry Property (LEIP) with constant $\tilde{\delta}_e$ if

$$|\|\mathbf{Ax}\|_2^2 - \|\mathbf{x}\|_2^2| \leq \tilde{\delta}_e \|\mathbf{x}\|_2^2,$$

  for all $\mathbf{x}$ with $\mathrm{Ent}_1(\mathbf{x}) \leq e$..

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We got the easy :
Theorem 3.1: If $e \leq 2\sqrt{s}$ and the matrix $\mathbf{A}$ has the LEIP-property with constant $\tilde{\delta}_e < 1$ then $\mathbf{A} \in \mathrm{MP}_1(s)$.

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With little more effort we got:
Theorem 3.3: If $\mathbf{A}$ fulfils the RIP property with constant $\delta_{2s} < \sqrt{2} - 1 \approx 0.412\ldots$, then $\mathbf{A} \in \mathrm{MP}_1(s)$.

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Best known: enough with $\frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2}} < \frac{1}{\sqrt{2}}$. Thus it is enough with $\delta_{2s} < \frac{1}{\sqrt{3}} \approx 0.577\ldots$.

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Remark: With some effort the constant $\frac{\delta_{2s}}{1-\delta_s}$ in Lemma 3.5 can be replaced by to $\frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2}}$.

A short summary from last lecture:

For s-sparse vetors $\mathbf{x}$ and $\mathbf{z}$ we have the bi-linear version of the RIP:

$$|\langle \mathbf{Ax}, \mathbf{Az} \rangle - \langle \mathbf{x}, \mathbf{z} \rangle| \leq \delta_{2s} \|\mathbf{x}\|_2 \|\mathbf{z}\|_2,$$

by polarization of the RIP estimate with the vectors $\mathbf{x} \pm \mathbf{z}$.

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Theorem 4.1 If $\delta_s < 1$ there is a constant $C > 0$ such that if any $m \times N$ matrix $\mathbf{A}$ har RIP with constant δ_s, then

$$m > Cs \log(Ne/s)$$

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where the union is taken over all subsets S of $[1, \ldots, N]$ with length s. The number of such subsets is $\begin{pmatrix} N \\ s \end{pmatrix}$. By Stirlings formula ($n! \approx (\frac{n}{e})^n \sqrt{2\pi n}$):

$$\begin{pmatrix} N \\ s \end{pmatrix} \sim \frac{N^{N+\frac{1}{2}}}{\sqrt{2\pi}(N-s)^{N-s+\frac{1}{2}}\, s^{s+\frac{1}{2}}},$$

If $s << N$ this is approximately equal to

$$(N\mathrm{e}/s)^s / \sqrt{2\pi s}.$$

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Let $0 < r < 1$, and let $N_m(r)$ be the maximal number of points $\{\mathbf{x}_i\}$ in a set in the unit ball $B_m = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_2 \leq 1$ in $\mathbb{R}^m$ such that

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Let $0 < r < 1$, and $N_{X(s)}(r)$ be the maximal number of points $\{x_i\}$ in a set in $X(s) \cap B_N$ such that

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Recall:

$A \in$RIP with constant δ_s means:

$$\sqrt{1-\delta_s}\|\mathbf{x}\|_2 \leq \|\mathbf{Ax}\|_2 \leq \sqrt{1+\delta_s}\|\mathbf{x}\|_2 \text{ for } \mathbf{x} \in X(s).$$

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This make it possible to estimate $N_{X(s/2)}(r_1)$ by $N_m(r_2)$.

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Proof: Let $N = N_{X(s/2)}(r_1)$ and let $\{\mathbf{x}_i\}_1^N$ in $X(s/2) \cap B_N$ such that $\|\mathbf{x}_i - \mathbf{x}_j\|_2 > r_1$ for all $i \neq j$.

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It follows than $N_m(r_2) \geq N = N_{X(s/2)}(r_1)$.
The proof of Lemma 4.2 is complete.

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Proof:

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- The disjoint balls centered at the points $\mathbf{x}_i$ and with radius $\frac{1}{2}$ are contained in the ball centered at the 0 with radius $1 + \frac{r}{2}$

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- we see that W contains more than

$$2^s \binom{N}{s} \approx (2Ne/s)^s \sqrt{2\pi s}$$

vectors, all with norm $\|\mathbf{x}\|_2 \leq \sqrt{s}$.

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  5. When W' is empty we are finished.

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- Note that for any fixed j and $\|\mathbf{z} - \mathbf{x}_j\|_2 \leq \sqrt{s}/2$ then entries of $\mathbf{z} - \mathbf{x}_j$ are in {-2,-1,0,1,2}} and that $\mathbf{z} - \mathbf{x}_j$ is in $X(s/2)$.

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- For any fixed $\mathbf{x}_j \in X$ the number of points $\mathbf{z}$ such that $\|\mathbf{z}-\mathbf{x}_j\|_2 \leq \sqrt{s}/2$ is estimated by

$$\sum_{k=0}^{s/2} \binom{N}{s} 4^k \approx \sum_{k=0}^{s/2} \frac{(4Ne/k)^k}{\sqrt{2\pi k}} \approx \frac{(8Ne/s)^{s/2}}{\sqrt{4\pi s}}$$

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- We conclude that the process does not end before

$$|X| \ge\approx \frac{(2Ne/s)^s/\sqrt{2\pi s}}{(8Ne/s)^{s/2}/\sqrt{4\pi s}} = \frac{(Ne/(s/2))^{s/2}}{\sqrt{2}}.$$

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$$|X| \geq\approx \frac{(2Ne/s)^s/\sqrt{2\pi s}}{(8Ne/s)^{s/2}/\sqrt{4\pi s}} = \frac{(Ne/(s/2))^{s/2}}{\sqrt{2}}.$$

Thus $N_{X(s)}(s) \geq \frac{1}{2}(Ne/(s/2))^{s/2}$.

Proof of Theorem 4.1

According to Lemma 4.2 following condition must hold:

$$N_{X(s/2)}(r_1) \leq N_m(r_2).$$

when $r_2 \leq r_1 \frac{\sqrt{1-\delta_s}}{\sqrt{1+\delta_s}}$.

Proof of Theorem 4.1

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$$\frac{(Ne/(s/2))^{s/2}}{2} \leq N_{X(s/2)}(r_1) \leq N_m(r_2) \leq (1+\frac{2}{r_2})^m = \left(1 + \frac{4\sqrt{1+\delta_s}}{\sqrt{1-\delta_s}}\right)^m$$

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Taking the logarithm we get

$$m \geq \frac{s\log(2Ne/s) - \log 4}{2(\log(\sqrt{1-\delta_s} + 4\sqrt{1+\delta_s}) - \log(1-\delta_s))}.$$

RIP for Gaussian Matrices

Let random matrix $\mathbf{A} = (A_{ij})_{1 \leq i \leq m, 1 \leq j \leq N}$ where each A_{ij} is an independent Gaussian random variable, i.e each $A_{ij} \in N(0, 1)$ with ditribution function $\phi(t)$ where

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If m is large enough, depending on s, δ and N is will with large probability have the RIP property with constant $\delta_s = \delta$. Theorem 4.5

There is a constant $C > 0$ such for any $\epsilon > 0, \delta > 0$ if we let

$$m > C \frac{s}{\delta^2} \log(Ne/(s\epsilon).$$

Then the Gaussian $m \times N$ matrix $\mathbf{A}$ (as above) satisfy RIP constant $\delta_s = \delta$ with a probability not less than $1 - \epsilon$

RIP for Gaussian Matrices

Proof of Theorem 4.5:

Let $\mathbf{x}$ be a vector in R^N with length $\|\mathbf{x}\|_2 = 1$ and define the random variable $T = \|\mathbf{A}\mathbf{x}\|_2^2/m$.

RIP for Gaussian Matrices

Proof of Theorem 4.5:
Let $\mathbf{x}$ be a vector in R^N with length $\|\mathbf{x}\|_2 = 1$ and define the random variable $T = \|\mathbf{A}\mathbf{x}\|_2^2/m$. The distribution function $\psi(\tau)$ can be determined exactly:

- Write a vector-distriubtion function of $(Z_1, \ldots, \mathbf{z}_m)$

$$\phi((t_1 \ldots, t_m) = \pi^{-m/2} e^{-\sum_i {t_i}^2/2}.$$

- Set $\tau = \sum_i {t_i}^2/m$. It follows that T has distribution function

$$\Psi(\tau) = c_m \tau^{\frac{m}{2}-1} e^{-m\tau/2},$$

for some normalization constant c_m.

RIP for Gaussian Matrices

Proof of Theorem 4.5:

Let $\mathbf{x}$ be a vector in R^N with length $\|\mathbf{x}\|_2 = 1$ and define the random variable $T = \|\mathbf{Ax}\|_2^2/m$. The distribution function $\psi(\tau)$ can be determined exactly:

- Let $Z = \mathbf{Ax}$ be the random columnvector with elements $Z_j, 1 \leq j \leq m$.

- Write a vector-distriubtion function of $(Z_1, \ldots, \mathbf{z}_m)$

$$\phi((t_1 \ldots, t_m) = \pi^{-m/2} e^{-\sum_i {t_i}^2/2}.$$

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- Let $Z = \mathbf{Ax}$ be the random columnvector with elements $Z_j, 1 \leq j \leq m$.
- Observe that Z_j are independent randomvariables with distribution function
$$\phi(t) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-t^2/2}.$$
- Write a vector-distriubtion function of $(Z_1, \ldots, \mathbf{z}_m)$
$$\phi((t_1 \ldots, t_m) = \pi^{-m/2} \mathrm{e}^{-\sum_i {t_i}^2/2}.$$
- Set $\tau = \sum_i {t_i}^2/m$. It follows that T has distribution function
$$\Psi(\tau) = c_m \tau^{\frac{m}{2}-1} \mathrm{e}^{-m\tau/2},$$
for some normalization constant c_m.

RIP for Gaussian Matrices

We take the logarithm of $\Psi(\tau)$ analyse for maximum and estimate the second derivative.

- $$\log \Psi(\tau) = \log c_m + (\frac{m}{2} - 1) \log \tau - \frac{m}{2}\tau$$
- derivative $\Psi(\tau)$
$$(\frac{m}{2} - 1)\frac{1}{\tau} - \frac{m}{2}$$
- second derivative of $\Psi(\tau)$
$$-(\frac{m}{2} - 1)\frac{1}{\tau^2}$$

We want to estimate

$$\int_{|\tau-1|>\delta} \Psi(\tau) d\tau$$

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Thus for one fixed point probability for the RIP estimate does not hold is less than $c_1 e^{-c_2 m \delta^2}$.