Mathematical Foundations for Compressed Sensing
Exercise Set 1

Please hand in individual handwritten solutions. Theorems, lemmas, definitions and propositions numbered in the format "Theorem 3.3, Lemma 3.5" etc. refers to the lecture notes, single numbering such as "Lemma 1" is only used for this exercise paper.

1. Prove

   Lemma 1. *Suppose that* $\{n_k\}_{k=1}^L$ *is a sequence of integers such that*

   $$\frac{n_{k+1}}{n_k} = v > 1,$$

   and $\{f_k\}_{k=1}^L$ *is a sequence of non-negative functions on a probability space such that for* $k = 1, 2, \ldots, L$,

   $$\|f_k\|_{n_k} \le B,$$

   for some $B > 0$*, where* $\|\cdot\|_p$ *denotes the usual* L^p*-norm. Then there is a constant* $A \ge A_v > 1$ *so that*

   $$\|\max_{1\le k\le L} f_k\|_{n_1}^{n_1} \le AB^{n_1}.$$

   Hint: Use induction on $J \le L$ with the assumption that

   $$\|\max_{J\le k\le L} f_k\|_{n_J}^{n_J} \le AB^{n_J}.$$

2. Prove

   Lemma 2. *Assume that an* $m \times N$*-matrix* A *satisfies the RIP estimate with constants* $\delta_s \le \delta_{2s}$ *and that* $\mathbf{x}$ *are* $\mathbf{y}$ *are* s*-sparse vectors with* $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ *. Let* $-\frac{\delta_s}{\delta_{2s}} \le t \le \frac{\delta_s}{\delta_{2s}}$*, so that* $|\|A\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2| = t\delta_{2s}\|\mathbf{x}\|_2^2$*. Then*

   $$|\langle A\mathbf{x}, A\mathbf{y}\rangle| \le \delta_{2s}\sqrt{1-t^2}\|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$

   Hint: Use polarization arguments on vectors $\alpha^2\mathbf{x} + \gamma\mathbf{y}$ and $\beta^2\mathbf{x} - \gamma\mathbf{y}$ where $\mathbf{x}, \mathbf{y}$ are s-sparse, $\gamma = \pm 1, \alpha \ge 0, \beta \ge 0$. Using RIP-estimates, obtain an estimate $|\langle A\mathbf{x}, A\mathbf{y}\rangle - \langle \mathbf{x}, \mathbf{y}\rangle| \le f(\alpha,\beta)\delta_{2s}\|\mathbf{x}\|_2\|\mathbf{y}\|_2$ and optimize f properly (consider cases $\alpha\beta > 0, \alpha = 0$ or $\beta = 0$).

3. Improve Lemma 3.3 in the notes of Lecture 3 to

   Lemma 3. *If* A *is RIP and* $A\mathbf{x} = \mathbf{0}$*, then*

   $$\|\mathbf{x}_{S_1}\|_2 \le \frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2}}\sum_{k\ge 2}\|\mathbf{x}_{S_k}\|_2,$$

where the notation is analogous as in the notes. *Hint:* Use Lemma 2 and optimize with respect to the introduced parameter t.

4. Prove that if $\delta_{2s} < \frac{1}{\sqrt{3}}$ for an $m \times N$-matrix A, then $A \in MP_1(s)$, i.e. the solution to the optimization problem

$$\begin{cases} \min \|\mathbf{z}\|_1 \\ A\mathbf{z} = \mathbf{w} \end{cases}$$

is s-sparse. *Hint:* Proceed as in the proof of Theorem 3.3 in Lecture 3 using the improved Lemma 3 above.

5. Suppose $\|\mathbf{x}\|_2 \le 1, \|\mathbf{x}\|_1 \le \sqrt{s}$. Denote by

$$X(s) = \bigcup_{\substack{S \subset [N] \\ |S| = s}} X_S$$

the set of all s-sparse vectors (compare with Lecture 4) where $[N] = \{1, 2, \dots, N\}$ and X_S denotes the set of all $\mathbf{x} \in \mathbf{C}^N$ where $\operatorname{supp} \mathbf{x} = S$ for some indexset $S \subset [N]$ of cardinality s.
Let $X_B(s) = X(s) \cap \{\mathbf{x} : \|\mathbf{x}\|_2 \le 1\}$.
Prove that

$$\frac{\mathbf{x}}{2} \in \operatorname{ch} X_B(s),$$

where ch denotes the *convex hull*. Remember that the convex hull of a set U is defined as

$$\left\{ \sum_{i=1}^{n} a_i u_i : u_i \in U, a_i \ge 0, \sum_{i=1}^{n} a_i = 1, n = 1, 2, \dots \right\}.$$

Hint: Divide $\mathbf{x}$ into decreasing blocks as in Lemma 3 and use estimates of the type

$$\|\mathbf{x}_{S_k}\|_2 \le \frac{\|\mathbf{x}_{S_{k-1}}\|_1 + \|\mathbf{x}_{S_k}\|_1}{2\sqrt{s}}$$

(why is this true?).