Mathematical Foundations for Compressed Sensing

Course at research level. 7.5 hp
Start: February 2012
Extent: The course is planned for 5–6 weekly double-lectures (2x45 min)
Examination: Homework (not yet decided)
Lecturer: Jan-Olov Strömberg

Background Signal compression and reconstruction in the context of incomplete data and(or) highly noisy data is very challenging. During the recent decade there has however been a significant breakthrough with the advent of *sparse signal processing* methods. The underlying basic idea is that almost all real-world signals have an underlying pattern, hence viewed mathematically, such signals also have low information content since these patterns can be utilized for providing a good description of the signal. Sparse signal processing is a general mathematical framework for exploiting such patterns of the signal in e.g. compression and reconstruction. A key insight is that *the ability to represent a signal sparsely is a driving factor behind many important signal and image processing problems*, including signal compression, denoising and model fitting. Repeatedly, it turns out that a better representation technique – one that leads to more sparsity – is the basis for a practically better solution to such problems. Therefore, sparse signal processing has had tremendous impact on compression and reconstruction problems, especially in situations with incomplete data and(or) highly noisy data, cases that previously were considered impossible to handle.

Compressed sensing is a sub-dicipline within sparse signal processing that offers a new sampling/data acquisition theory based on exploiting sparsity (compressibility) already when the signal of interest is experimentally acquired. One can also design non-adaptive sampling techniques that condense the information in the compressible signal into a small amount of data. This offers new approaches to many engineering problems involving signal acquisition, e.g. in analog-to-digital conversion, digital optics, biomedical imaging and seismic exploration. There is also a relation to statistical problems dealing with the recovery of large data matrices from incomplete sets of entries.

Course content The course will focus on the mathematical foundations of compressed sensing. The starting point is to consider a linear system of equations $A \cdot x = y$ where we seek to find the signal x (an n-vector) from data y (an m-vector) given measurements A (an $m \times n$-matrix). The central problem is to solve for x when the system is highly underdetermined (n is much larger than m) using knowledge that the signal x is sparse, or more precisely we know of a $n \times n$ matrix D such that $z = D \cdot x$ has only s non-zero entices with s much smaller that n. Such situations arise e.g. when one knows that a signal x can be represented by a Fourier series with only a few non-zero frequencies (which ones that are non-zero are unknown). Then D is the matrix representing the Fourier basis. The central question is to investigate how many sampling points one needs for recovering such a signal exactly. It turns out that if the elements in the matrix $A \cdot D^{-1}$ are "independent enough" (more technically the matrix satisfies the restricted isometry property), then x can be uniquely recovered from vastly under sampled data. The theory also applies more generally to incoherent pair of bases, of which the pair Fourier basis and standard sampling basis is an example.

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