



# Globally Convergent Optimization Methods Based on Conservative Convex Approximations

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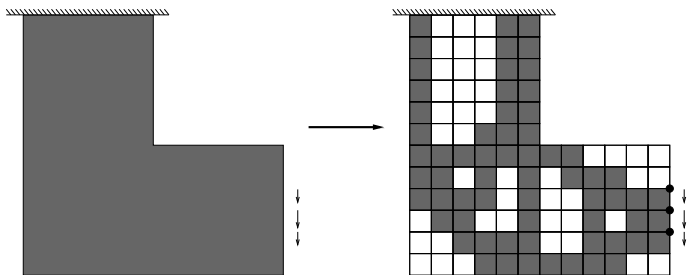
# Outline

- 1 Application background
- 2 Conservative convex approximations – main ideas
- 3 Global convergence
- 4 Extended/restricted problem formulation

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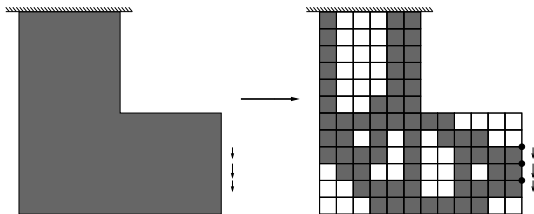
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# Ground structure approach for topology optimization



- The discretized structure has  $n$  finite elements.
- The design is represented by  $\mathbf{x} \in \{0, 1\}^n$ .
- Elements filled with material have  $x_j = 1$  (black).
- Elements without material have  $x_j = 0$  (white).

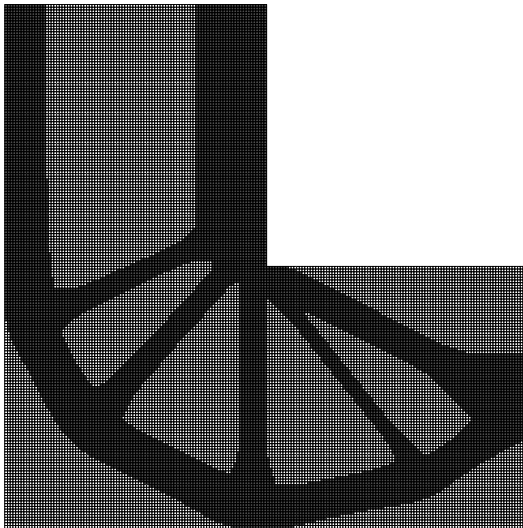
# Possible topology optimization problem



- $\mathbf{f}$  = vector of (given) external nodal forces,
- $\mathbf{d}$  = vector of nodal displacements,
- $\mathbf{K}(\mathbf{x}) = \sum_{j=1}^n x_j \mathbf{K}_j$  = structural stiffness matrix.
- State equations:  $\mathbf{K}(\mathbf{x})\mathbf{d} = \mathbf{f}$ .

**Possible problem:** minimize  $\sum_{j=1}^n x_j$  subject to  $\mathbf{f}^T \mathbf{d}(\mathbf{x}) \leq S$ ,  
where  $\mathbf{d}(\mathbf{x})$  is a solution to the state equations.

# Possible solution, 27648 variables



# Naive but possible heuristic approach

Replace the integer programming problem by a relaxed and penalized continuous **nonlinear programming** problem

$$\begin{aligned} & \text{minimize} && \sum_{j=1}^n x_j + C \sum_{j=1}^n x_j(1-x_j) \\ & \text{subject to} && \mathbf{f}^T \mathbf{d}(\mathbf{x}) \leq S, \\ & && 0 \leq x_j \leq 1, \quad j = 1, \dots, n. \end{aligned}$$

where  $\mathbf{d}(\mathbf{x})$  is the solution to  $(\sum_{j=1}^n x_j \mathbf{K}_j) \mathbf{d} = \mathbf{f}$ .

The penalty parameter  $C$  is gradually increased from zero, until (almost) all  $x_j \in \{0, 1\}$ .

# Characteristics of topology optimization problems

Some typical properties of the (relaxed and penalized) nonlinear programming problems in topology optimization:

- Nonconvex problem with a large number of variables.
- Given lower and upper bounds on the variables.
- Relatively few general nonlinear constraints (inequalities).
- Expensive function evaluations...
- ...but still possible to calculate derivatives.
- Not always important with high accuracy optimality...
- ...but the algorithms should be reliable.

One frequently used solution method is the *method of moving asymptotes* (MMA), which is based on the concept of **conservative convex approximations**.

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# Conservative convex approximations

Consider a problem of the following type:

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m, \end{array}$$

where  $\mathbf{x} \in R^n$  and  $f_0, f_1, \dots, f_m$  are given continuously differentiable functions.

# Approximating subproblem

At iteration  $k$ , the feasible iteration point  $\mathbf{x}^{(k)} \in R^n$  is given. A corresponding convex first order approximating subproblem is defined by

$$\begin{aligned} & \text{minimize} && \tilde{f}_0^{(k)}(\mathbf{x}) \\ & \text{subject to} && \tilde{f}_i^{(k)}(\mathbf{x}) \leq 0, \quad i = 1, \dots, m, \end{aligned}$$

where  $\tilde{f}_0^{(k)}, \tilde{f}_1^{(k)}, \dots, \tilde{f}_m^{(k)}$  are certain **convex** functions which satisfy  $\tilde{f}_i^{(k)}(\mathbf{x}^{(k)}) = f_i(\mathbf{x}^{(k)})$  and  $\nabla \tilde{f}_i^{(k)}(\mathbf{x}^{(k)}) = \nabla f_i(\mathbf{x}^{(k)})$ . In particular,  $\tilde{f}_0^{(k)}$  should be **strictly** convex.

The optimal solution  $\hat{\mathbf{x}}^{(k)}$  of this subproblem **possibly** becomes the next iteration point  $\mathbf{x}^{(k+1)}$ .

# Linear + separable quadratic approximations

A simple example of a convex first order approximating subproblem is

$$\begin{aligned} & \text{minimize} && f_0(\mathbf{x}^{(k)}) + \nabla f_0(\mathbf{x}^{(k)})(\mathbf{x} - \mathbf{x}^{(k)}) + \rho_0^{(k)} \|\mathbf{x} - \mathbf{x}^{(k)}\|^2 \\ & \text{subject to} && f_i(\mathbf{x}^{(k)}) + \nabla f_i(\mathbf{x}^{(k)})(\mathbf{x} - \mathbf{x}^{(k)}) + \rho_i^{(k)} \|\mathbf{x} - \mathbf{x}^{(k)}\|^2 \leq 0, \\ & && i = 1, \dots, m, \end{aligned}$$

where  $\rho_0^{(k)} > 0$  and  $\rho_i^{(k)} \geq 0$  for  $i = 1, \dots, m$ .

Easy to solve, e.g. by a **dual** method, due to the convexity and separability.

# Conservative approximating subproblem

The convex first order approximating subproblem

$$\begin{aligned} & \text{minimize} && \tilde{f}_0^{(k)}(\mathbf{x}) \\ & \text{subject to} && \tilde{f}_i^{(k)}(\mathbf{x}) \leq 0, \quad i = 1, \dots, m, \end{aligned}$$

with optimal solution  $\hat{\mathbf{x}}^{(k)}$ , is a **conservative** approximating subproblem if  $\tilde{f}_i^{(k)}(\hat{\mathbf{x}}^{(k)}) \geq f_i(\hat{\mathbf{x}}^{(k)})$  for  $i = 0, 1, \dots, m$ .

In that case,

$$f_i(\hat{\mathbf{x}}^{(k)}) \leq \tilde{f}_i^{(k)}(\hat{\mathbf{x}}^{(k)}) \leq 0, \quad \text{for } i = 1, \dots, m,$$

$$f_0(\hat{\mathbf{x}}^{(k)}) \leq \tilde{f}_0^{(k)}(\hat{\mathbf{x}}^{(k)}) < \tilde{f}_0^{(k)}(\mathbf{x}^{(k)}) = f_0(\mathbf{x}^{(k)}), \quad \text{unless } \hat{\mathbf{x}}^{(k)} = \mathbf{x}^{(k)}.$$

Reasonable to let  $\hat{\mathbf{x}}^{(k)}$  become the next iteration point  $\mathbf{x}^{(k+1)}$ .

# Inner iterations to obtain conservatism

Let  $(k, \ell)$  denote the  $\ell$ :th **inner** iteration within the  $k$ :th **outer** iteration. The corresponding approximating functions are

$$\tilde{f}_i^{(k,\ell)}(\mathbf{x}) = f_i(\mathbf{x}^{(k)}) + \nabla f_i(\mathbf{x}^{(k)})(\mathbf{x} - \mathbf{x}^{(k)}) + \rho_i^{(k,\ell)} \|\mathbf{x} - \mathbf{x}^{(k)}\|^2,$$

and the corresponding approximating subproblem is

$$\begin{aligned} & \text{minimize} && \tilde{f}_0^{(k,\ell)}(\mathbf{x}) \\ & \text{subject to} && \tilde{f}_i^{(k,\ell)}(\mathbf{x}) \leq 0, \quad i = 1, \dots, m, \end{aligned}$$

with a unique optimal solution denoted  $\hat{\mathbf{x}}^{(k,\ell)}$ .

If  $\tilde{f}_i^{(k,\ell)}(\hat{\mathbf{x}}^{(k,\ell)}) \geq f_i(\hat{\mathbf{x}}^{(k,\ell)})$  for all  $i$ , let  $\mathbf{x}^{(k+1)} = \hat{\mathbf{x}}^{(k,\ell)}$ .

# Repeated inner iterations

Otherwise, let

$$\rho_i^{(k,\ell+1)} = \begin{cases} 1.1 \tilde{\rho}_i^{(k,\ell)} & \text{if } \tilde{f}_i^{(k,\ell)}(\hat{\mathbf{x}}^{(k,\ell)}) < f_i(\hat{\mathbf{x}}^{(k,\ell)}), \\ \rho_i^{(k,\ell)} & \text{if } \tilde{f}_i^{(k,\ell)}(\hat{\mathbf{x}}^{(k,\ell)}) \geq f_i(\hat{\mathbf{x}}^{(k,\ell)}), \end{cases}$$

where  $\tilde{\rho}_i^{(k,\ell)}$  is chosen such that

$$f_i(\mathbf{x}^{(k)}) + \nabla f_i(\mathbf{x}^{(k)}) (\hat{\mathbf{x}}^{(k,\ell)} - \mathbf{x}^{(k)}) + \tilde{\rho}_i^{(k,\ell)} \|\hat{\mathbf{x}}^{(k,\ell)} - \mathbf{x}^{(k)}\|^2 = f_i(\hat{\mathbf{x}}^{(k,\ell)}),$$

$$\text{i.e., } \tilde{\rho}_i^{(k,\ell)} = \rho_i^{(k,\ell)} + \frac{f_i(\hat{\mathbf{x}}^{(k,\ell)}) - \tilde{f}_i^{(k,\ell)}(\hat{\mathbf{x}}^{(k,\ell)})}{\|\hat{\mathbf{x}}^{(k,\ell)} - \mathbf{x}^{(k)}\|^2} > \rho_i^{(k,\ell)}.$$

Then solve the new subproblem to obtain  $\hat{\mathbf{x}}^{(k,\ell+1)}$ , etc.

# Summary of the $k$ :th outer iteration

Starting from the (feasible) iteration point  $\mathbf{x}^{(k)}$ , the next (feasible) iteration point  $\mathbf{x}^{(k+1)}$  is obtained as the optimal solution to the **final** of one or several subproblems of the type

$$\begin{aligned} & \text{minimize} && f_0(\mathbf{x}^{(k)}) + \nabla f_0(\mathbf{x}^{(k)})(\mathbf{x} - \mathbf{x}^{(k)}) + \rho_0^{(k,\ell)} \|\mathbf{x} - \mathbf{x}^{(k)}\|^2 \\ & \text{subject to} && f_i(\mathbf{x}^{(k)}) + \nabla f_i(\mathbf{x}^{(k)})(\mathbf{x} - \mathbf{x}^{(k)}) + \rho_i^{(k,\ell)} \|\mathbf{x} - \mathbf{x}^{(k)}\|^2 \leq 0, \\ & && i = 1, \dots, m, \end{aligned}$$

where the only difference between two consecutive subproblems is that some parameters  $\rho_i^{(k,\ell)}$  have increased.

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# Sufficient assumptions to ensure convergence

Let  $\mathcal{F} = \{ \mathbf{x} \in \mathbb{R}^n \mid f_i(\mathbf{x}) \leq 0, i = 1, \dots, m \}$ .

For  $\mathbf{x} \in \mathcal{F}$ , let  $\mathcal{A}(\mathbf{x}) = \{ i \in \{1, \dots, m\} \mid f_i(\mathbf{x}) = 0 \}$ .

## Assumptions:

- The feasible set  $\mathcal{F}$  is nonempty and bounded.
- A feasible point  $\mathbf{x}^{(1)} \in \mathcal{F}$  is known (or can be calculated).
- $f_0, f_1, \dots, f_m$  have continuous second derivatives on  $\mathcal{F}$ .
- Each feasible point  $\mathbf{x} \in \mathcal{F}$  is a **regular point**, i.e. the gradients of the active constraints at  $\mathbf{x}$  are positive-linearly independent, i.e. there are **no** numbers  $\lambda_i$  such that

$$\sum_{i \in \mathcal{A}(\mathbf{x})} \lambda_i \nabla f_i(\mathbf{x}) = \mathbf{0}^T, \quad \sum_{i \in \mathcal{A}(\mathbf{x})} \lambda_i > 0, \quad \text{and} \quad \lambda_i \geq 0, \quad i \in \mathcal{A}(\mathbf{x}).$$

## Propositions:

- The considered problem (to minimize  $f_0(\mathbf{x})$  s.t.  $\mathbf{x} \in \mathcal{F}$ ) has at least one optimal solution, and each optimal solution is a KKT point. In particular, the set of KKT points is non-empty.
- Within each outer iteration, a **finite** number of inner iterations is needed until the subproblem becomes conservative.
- Each iteration point  $\mathbf{x}^{(k)}$  is a feasible solution, i.e.  $\mathbf{x}^{(k)} \in \mathcal{F}$ .
- Unless  $\mathbf{x}^{(k)}$  is already a KKT point of the original problem, the objective value at  $\mathbf{x}^{(k+1)}$  is strictly better than the objective value at  $\mathbf{x}^{(k)}$ , i.e.  $f_0(\mathbf{x}^{(k+1)}) < f_0(\mathbf{x}^{(k)})$ .

# Global convergence theorem

Let  $\Omega$  denote the set of KKT points to the original problem.  
(Note that  $\Omega$  is non-empty according to above.)

Further, let  $\|\Omega - \mathbf{x}^{(k)}\|$  denote the *distance* from the set  $\Omega$  to the iteration point  $\mathbf{x}^{(k)}$ , i.e.  $\|\Omega - \mathbf{x}^{(k)}\| = \inf_{\mathbf{x} \in \Omega} \{\|\mathbf{x} - \mathbf{x}^{(k)}\|\}$ .

**Theorem:**  $\|\Omega - \mathbf{x}^{(k)}\| \longrightarrow 0$  as  $k \longrightarrow \infty$ .

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# Extended/restricted problem formulation

From now on, the following problem formulation is considered

$$\begin{aligned} & \text{minimize} && f_0(\mathbf{x}) + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\ & \text{subject to} && f_i(\mathbf{x}) - y_i \leq 0, \quad i = 1, \dots, m \\ & && \mathbf{x} \in X, \quad \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

where

- $X = \{ \mathbf{x} \in \mathbb{R}^n \mid x_j^{\min} \leq x_j \leq x_j^{\max}, j = 1, \dots, n \}$ ,
- $c_i \geq 0, d_i > 0, x_j^{\min} < x_j^{\max}$ ,
- $f_0, \dots, f_m$  have continuous second derivatives on  $X$ .

# Some properties of the considered problem

Some theoretical properties of the considered problem

$$\begin{aligned} & \text{minimize} && f_0(\mathbf{x}) + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\ & \text{subject to} && f_i(\mathbf{x}) - y_i \leq 0, \quad i = 1, \dots, m \\ & && \mathbf{x} \in X, \quad \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

- There are always feasible solutions to this problem.
- There is always at least one globally optimal solution.
- Each feasible solution is a regular point.
- Each optimal solution (local or global) is a KKT point.

# Inner and outer iterations

As before,  $(k, \ell)$  denotes the  $\ell$ :th **inner** iteration within the  $k$ :th **outer** iteration. The corresponding approximating subproblem is

$$\begin{aligned} \text{minimize} \quad & \tilde{f}_0^{(k,\ell)}(\mathbf{x}) + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\ \text{subject to} \quad & \tilde{f}_i^{(k,\ell)}(\mathbf{x}) - y_i \leq 0, \quad i = 1, \dots, m \\ & \mathbf{x} \in X, \quad \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

where, as before, the approximating functions **could** be

$$\tilde{f}_i^{(k,\ell)}(\mathbf{x}) = f_i(\mathbf{x}^{(k)}) + \nabla f_i(\mathbf{x}^{(k)}) (\mathbf{x} - \mathbf{x}^{(k)}) + \rho_i^{(k,\ell)} \|\mathbf{x} - \mathbf{x}^{(k)}\|^2.$$

But other choices might be better for some classes of problems!

# Approximating functions in MMA

In the method of moving asymptotes, **MMA**, the convex approximating functions are chosen as

$$\tilde{f}_i^{(k,\ell)}(\mathbf{x}) = \sum_{j=1}^n \left( \frac{p_{ij}^{(k,\ell)}}{u_j^{(k)} - x_j} + \frac{q_{ij}^{(k,\ell)}}{x_j - v_j^{(k)}} \right) + r_i^{(k,\ell)},$$

where  $u_j^{(k)}$  and  $v_j^{(k)}$  are parameters satisfying  $v_j^{(k)} < x_j^{(k)} < u_j^{(k)}$ , defining vertical **asymptotes**.

The coefficients  $p_{ij}^{(k,\ell)} > 0$ ,  $q_{ij}^{(k,\ell)} > 0$  and  $r_i^{(k,\ell)}$  are chosen such that  $\tilde{f}_i^{(k,\ell)}(\mathbf{x}^{(k)}) = f_i(\mathbf{x}^{(k)})$  and  $\nabla \tilde{f}_i^{(k,\ell)}(\mathbf{x}^{(k)}) = \nabla f_i(\mathbf{x}^{(k)})$ .

# Approximating subproblem in MMA

The MMA subproblem thus looks as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^n \left( \frac{p_{0j}^{(k,\ell)}}{u_j^{(k)} - x_j} + \frac{q_{0j}^{(k,\ell)}}{x_j - v_j^{(k)}} \right) + r_0^{(k,\ell)} + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\ \text{subject to} \quad & \sum_{j=1}^n \left( \frac{p_{ij}^{(k,\ell)}}{u_j^{(k)} - x_j} + \frac{q_{ij}^{(k,\ell)}}{x_j - v_j^{(k)}} \right) + r_i^{(k,\ell)} - y_i \leq 0, \quad i = 1, \dots, m \\ & x_j^{\min} \leq x_j \leq x_j^{\max} \quad \text{and} \quad v_j^{(k)} < x_j < u_j^{(k)}, \quad j = 1, \dots, n. \end{aligned}$$

Easy to solve, e.g. by a **dual** method, due to the convexity and separability.

# Default rules for updating the asymptotes

The first two outer iterations, when  $k = 1$  and  $k = 2$ ,

$$\begin{aligned}u_j^{(k)} + v_j^{(k)} &= 2x_j^{(k)}, \\u_j^{(k)} - v_j^{(k)} &= x_j^{\max} - x_j^{\min}.\end{aligned}$$

In later outer iterations, when  $k \geq 3$ ,

$$\begin{aligned}u_j^{(k)} + v_j^{(k)} &= 2x_j^{(k)}, \\u_j^{(k)} - v_j^{(k)} &= \gamma_j^{(k)}(u_j^{(k-1)} - v_j^{(k-1)}),\end{aligned}$$

where

$$\gamma_j^{(k)} = \begin{cases} 0.7 & \text{if } (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) < 0, \\ 1.2 & \text{if } (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) > 0, \\ 1 & \text{if } (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) = 0, \end{cases}$$

# The coefficients $p_{ij}^{(k,\ell)}$ , $q_{ij}^{(k,\ell)}$ and $r_i^{(k,\ell)}$

$$\tilde{f}_i^{(k,\ell)}(\mathbf{x}) = \sum_{j=1}^n \left( \frac{p_{ij}^{(k,\ell)}}{u_j^{(k)} - x_j} + \frac{q_{ij}^{(k,\ell)}}{x_j - v_j^{(k)}} \right) + r_i^{(k,\ell)}, \text{ where}$$

$$p_{ij}^{(k,\ell)} = (u_j^{(k)} - x_j^{(k)})^2 \left( \left( \frac{\partial f_i}{\partial x_j}(\mathbf{x}^{(k)}) \right)^+ + \frac{\rho_i^{(k,\ell)}}{u_j^{(k)} - v_j^{(k)}} \right),$$

$$q_{ij}^{(k,\ell)} = (x_j^{(k)} - v_j^{(k)})^2 \left( \left( \frac{\partial f_i}{\partial x_j}(\mathbf{x}^{(k)}) \right)^- + \frac{\rho_i^{(k,\ell)}}{u_j^{(k)} - v_j^{(k)}} \right),$$

and  $r_i^{(k,\ell)}$  is chosen such that  $\tilde{f}_i^{(k,\ell)}(\mathbf{x}^{(k)}) = f_i(\mathbf{x}^{(k)})$ .

Here,  $\rho_0^{(k,\ell)} > 0$  and  $\rho_i^{(k,\ell)} \geq 0$  for  $i = 1, \dots, m$ .

# Summary of the $k$ :th outer iteration in MMA

Given the iteration point  $(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$ , the next iteration point  $(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)})$  is obtained as the optimal solution to the **final** of one or several subproblems of the type

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^n \left( \frac{p_{0j}^{(k,\ell)}}{u_j^{(k)} - x_j} + \frac{q_{0j}^{(k,\ell)}}{x_j - v_j^{(k)}} \right) + r_0^{(k,\ell)} + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\ \text{subject to} \quad & \sum_{j=1}^n \left( \frac{p_{ij}^{(k,\ell)}}{u_j^{(k)} - x_j} + \frac{q_{ij}^{(k,\ell)}}{x_j - v_j^{(k)}} \right) + r_i^{(k,\ell)} - y_i \leq 0, \quad i = 1, \dots, m \\ & x_j^{\min} \leq x_j \leq x_j^{\max} \quad \text{and} \quad v_j^{(k)} < x_j < u_j^{(k)}, \quad j = 1, \dots, n. \end{aligned}$$

Before each outer iteration,  $f_i(\mathbf{x}^{(k)})$  and  $\nabla f_i(\mathbf{x}^{(k)})$  are needed.  
After each inner iteration,  $f_i(\hat{\mathbf{x}}^{(k,\ell)})$  is needed, but **no gradients**.

# Global convergence

Let  $\Omega$  denote the set of KKT points  $(\mathbf{x}, \mathbf{y})$  to the considered problem

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\ \text{subject to} \quad & f_i(\mathbf{x}) - y_i \leq 0, \quad i = 1, \dots, m \\ & \mathbf{x} \in X, \quad \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

**Theorem:**  $\| \Omega - (\mathbf{x}^{(k)}, \mathbf{y}^{(k)}) \| \rightarrow 0$  as  $k \rightarrow \infty$ .

# Implementations in Matlab and Fortran

In our *Matlab* implementation, the convex subproblems are solved by a *primal-dual interior-point method*.

In our *Fortran* implementation, the convex subproblems are solved by an active-set Newton method applied to the *dual* problem.

M-files in Matlab and a source code in Fortran are available for academic research and education. Just send an e-mail.

Thank You!