HJEMMEOPPGAVER (utgave av 25-5-2004):

Oppgave 15 til 31. Mai:

For non negative integers $h_1, h_2, \dots, h_n$ we define *Schur functions* $S_{h_1 h_2 \dots h_n}$ by:

$$V_{h_1+n-1\, h_2+n-2\, \dots\, h_n} = s_{h_1 h_2 \dots h_n} V_{n-1\, n-2\, \dots\, 0},$$

where $V_{h_1,h_2,\dots,h_n} = \det(x_i^{h_j})_{ij}$ are the basis elements for the alternating polynomials. Show that the Schur functions for $h_1 \geq h_2 \geq \dots \geq h_n$ form a basis for the symmetric polynomials.

Oppgave 1' til 9. Februar:

Let G be a set with a multiplication such that

(1) There is an element 1 such that $1x = x$ for all x in G.
(2) For all x in G there is an element y such that $xy = 1$.
(3) For all x, y, z in G we have $(xy)z = x(yz)$.

Is it true that G with the given multiplication is a group?

Oppgave 1 til 9. februar:

Let G be the group of symmetries of a square, and let σ be rotation $\pi/2$ around the origin and τ the "flip" around the y-axis. Write $x_1 = 1, x_2 = \sigma, x_3 = \sigma^2, x_4 = \sigma^3, x_5 = \tau, x_6 = \sigma\tau, x_7 = \sigma^2\tau, x_8 = \sigma^3\tau$. Give the permutations that constitute the image by the representation of G on $\mathfrak{S}_8$ given by *Cayleys Theorem.*

Oppgave 2 til 16. februar:

Let $N : \mathfrak{S}_3 \to Gl_3(V)$ be the permutation representation of $\mathfrak{S}_3$. The representation N induces a representation on the *alternating* subgroup $\mathfrak{A} = \{1, c = (123), f = (132)\}$.

(1) Find a chain of submodules $0 = U_0 \subset U_1 \subset \dots \subset U_l = V$ such that U_i/U_{i-1} for $i = 1, 2, \dots, l$ is irreducible.
(2) Give the matrices that determine the corresponding representation on the modules U_i/U_{i-1} in some convenient basis.

Oppgave 3 til 23. Februar

Find an example of an irreducible representation $A : G \to Gl_m(\mathbf{R})$ that becomes reducible when it is considered as a representation $G \to Gl_m(\mathbf{C})$.

Oppgave 4 til 1. Mars:

Give an example showing that the Corollary to Schurs Lemma does not hold when the field is not algebraically closed.

Oppgave 5 til 8. Mars:

Prove that the center of $\mathcal{M}_n$ consists of diagonal matrices with one on the diagonal.

Oppgave 5' til 8. Mars:

Let U and V be vector spaces.

(1) Show that the canonical bilinear map $\beta : U \otimes_K V$ has the universal property:
For every bilinear map $\alpha : U \times V \to W$ there is a unique linear map $\varphi : U \otimes_K V \to W$ such that $\alpha = \varphi\beta$.

(2) Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be bases for U respectively V. Show that $u_i \otimes_K v_j$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ is a basis for $U \otimes_K V$.

(3) Let $\alpha : U \to U$ and $\beta : V \to V$ be linear maps and A and B the corresponding matrices in the bases above. Show that the matrix corresponding to the linear map $\alpha \otimes_K \beta$ in the above basis for $U \otimes_K V$ is equal to $A \otimes_K B$.

Oppgave 6 til 15. Mars:

Show that the maps $G \to K$ with *pointwise addition* and the *convolution product* is a K-algebra. (Show at least the *most important* properties).

Oppgave 6' til 15. Mars:

Give the orthogonality relations for the characteristic functions in the group algebra.

Oppgave 7 til 22. Mars

Find all irreducible representations of $\{\pm 1\} \times \{\pm 1\}$ and determine their characters.

Oppgave 8 til 29. mars

We say that a sequence of numbers $n_1, n_2, \ldots, n_r$ is a *cycle type* of permutations on the integers $1, 2, \ldots, n$ if $n_1 + 2n_2 + \cdots + rn_r = n$. Show that there is a one-to-one correspondence between cycle types and conjugacy classes of the symmetric group $\mathfrak{S}_n$.

Oppgave 9' til 5. april:

Is there a *natural* isomorphism $\varphi_V : V \to V^*$ from vector spaces to their dual, in the sence that, for any linear map $\alpha : V \to W$ of vector spaces the diagram

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ {\scriptstyle \varphi_V}\downarrow & & \downarrow{\scriptstyle \varphi_W} \\ V^* & \xleftarrow[\alpha^*]{} & W^* \end{array}$$

commutes?

Oppgave 9" til 5. april:

Over which fields is it true that all *periodic* matrices can be diagonalized?

Oppgave 10 til 30. april:

Decompose into irreducible representations the group ring of the cyclic group with n elements.

Oppgave 10' til 30. april:

Let G be a finite group and let U and V be G-modules with characters φ and ψ.

(1) Show that the G-module homomorphisms $\mathrm{Hom}_{\mathbf{C}}(U,V)$ have a natural structure of a G-module.
(2) Show that $\dim_{\mathbf{C}}(\mathrm{Hom}_G(U,V)) =< \varphi, \psi >$.

Oppgave 11 til 3. Mai:

Let $H \subseteq G$ be a subgroup of the group G, and let V be a G-module and W a H-module.

(1) Show that G_K, in a natural way, is a left H_K-module, and that it is free with any system of representatives $t_1, \dots, t_n$ of right cosets as representatives.
(2) Show that $W^G = W \otimes_{H_K} G_K$ can be considered as a G-module via the multiplication $(w \otimes x)y = w \otimes xy$.
(3) What is the dimension of W^G as a vector space over K?
(4) Show that there is a canonical isomorphism of vector spaces

$$\mathrm{Hom}_H(W, V_H) \xrightarrow{\sim} \mathrm{Hom}_G(W^G, V).$$

Oppgave 12' til 10. Mai:

Let

$$0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$$

be an exact sequence of G-modules (that is, φ is an injective G-module homomorphism, ψ is a surjective G-module homomorphism and $\ker\psi = \mathrm{im}\,\varphi$.)

(1) Show that there are G-module maps $\sigma : W \to V$ and $\tau : V \to U$ such that $\psi\sigma = \mathrm{id}_W$ and $\tau\varphi = \mathrm{id}_U$.
(2) Show that there is a G-module isomorphism $V \xrightarrow{\sim} U \oplus W$.

Oppgave 12" til 10. Mai:

Let $e_1, \dots, e_n$ be a basis for the vector space V. Show that $e_i \circ e_j$ for $i \leq j$ is a basis for $V \circ V$.

Oppgave 13' til 17 Mai:

Let V be a G-module.

(1) Characterize the n'th exterior product by its universal property.
(2) Show that the sequence

$$0 \to V \wedge V \to V \otimes V \to V \circ V \to 0$$

is exact.
(3) Is the sequence corresponding to the one in (2) exact for all A-modules M?

Oppgave 13" til 17 Mai:

Show that the two subgroups

$$\begin{cases} A : 1, (12)(34), (13)(24), (14)(23) \\ B : 1, (12), (34), (12)(34) \end{cases}$$

of $\mathfrak{S}_4$ are isomorphic.

Oppgave 14 til 24. Mai:

Show that the number of elements in a conjugation class C_α corresponding to the cycle structure $\alpha : \alpha_1, \alpha_2, \ldots, \alpha_n$ with $n = 1\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$ is equal to

$$h_\alpha = \frac{n!}{1^{\alpha_1}\alpha_1! 2^{\alpha_2}\alpha_2! \cdots n^{\alpha_n}\alpha_n!}.$$