# Divided powers

## 0. Contents.

## 1. Notation.

**(1.1) Notation.** Let !$I$! be a set. For every set !$M$! we write !$M^I$! for the set of all maps $I \to M$, that is the set of all *families* !$(x_\alpha)_{\alpha\in I}$! with !$x_\alpha \in M$! for all !$\alpha \in I$!. When !$G$! is a group we we define the *support* of an element $(g_\alpha)\alpha \in I$ as the subset of $I$ consisting of the $\alpha$ such that $g_\alpha \neq 0$. We write !$G^{(I)}$! for the set of all maps $I \to G$ with *finite support* , that is all !$(g_\alpha)_{\alpha\in I}$! in $G^I$ with only a finite number of the $g_\alpha$ different from 0.

Denote by !$\mathbf{N}$! the natural numbers. For !$(\nu_\alpha)_{\alpha\in I}$! in $\mathbf{N}^{(I)}$ we let !$|\nu| = |(\nu_\alpha)_{\alpha\in I}| = \sum_{\alpha\in I}\nu_\alpha$!. Let !$\mu = (\mu_\alpha)_{\alpha\in I}$! be in $\mathbf{N}^{(I)}$. We write $\mu \leq \nu$ if $\mu_\alpha \leq \nu_\alpha$ for all $\alpha \in I$, and we write $\mu < \nu$ if $\mu \leq \nu$ and $\mu \leq \nu$. Moreover we write $\mu + \nu = (\mu_\alpha + \nu_\alpha)_{\alpha\in I}$.

For every element !$g = (g_\alpha)_{\alpha\in I}$! in $G^I$ we write

$$!g^\nu! = \prod_{\alpha\in I} g_\alpha^{\nu_\alpha} = \prod_{\alpha\in I,\nu_\alpha\neq 0} g_\alpha^{\nu_\alpha}.$$

Let !$u^n : M \to G$! for $n \in \mathbf{N}$ be maps from $M$ to $G$. For every element !$x = (x_\alpha)_{\alpha\in I}$! in $M^I$, and every element $\nu = (\nu_\alpha)_{\alpha\in I}$ in $\mathbf{N}^{(I)}$ we write

$$!u^\nu(x)! = \prod_{\alpha\in I} u^{\nu_\alpha}(x_\alpha) = \prod_{\alpha\in I,\nu_\alpha\neq 0} u^{\nu_\alpha}(x_\alpha).$$

Hence the maps $u^n : M \to G$ for $n \in \mathbf{N}$ give a map

$$u^\nu : M^I \to G$$

for each $\nu \in \mathbf{N}^{(I)}$.

## 2. Exponential sequences.

n **(2.1) Notation.** Let !$A$! be a ring and let !$B$! be an $A$-algebra.

**(2.2) Definition.** (III §7 p. 256, B 87) An *exponential sequence* with values in
n $B$ is a sequence !$e = (e_n)_{n\in\mathbf{N}}$! of elements !$e_n \in B$! such that

(i) $e_0 = 1$.
(ii) $e_m e_n = \binom{m+n}{m} e_{m+n}$.

n The set of exponential sequences with values in $B$ we denote by !$\mathcal{E}(B)$!. We define
n the product of two exponential sequences $e = (e_n)_{n\in\mathbf{N}}$ and !$f = (f_n)_{n\in\mathbf{N}}$! by

$$ef = (\sum_{i+j=n} e_i f_j)_{n\in\mathbf{N}}.$$

**(2.3) Module structure.** The product of two exponential sequences with values in $B$ is again an exponential sequence with values in $B$. This follows from the equality

$$\sum_{i+j=m} \binom{m}{i}\binom{n}{j} = \binom{m+n}{m}$$

that we obtain by considering the coefficient of $t^m$ in the identity $(1+t)^m(1+t)^n = (1+t)^{m+n}$. With this product it is clear that $\mathcal{E}(B)$ is an abelian group with unit $(1, 0, 0, \dots)$. The inverse of the element $f = (f_n)_{n\in\mathbf{N}}$ is $((-1)^n f_n)_{n\in\mathbf{N}}$.

n For all elements !$f \in B$! and all exponential sequences $e = (e_n)_{n\in\mathbf{N}}$ in $\mathcal{E}(B)$ we have that the product

$$fe = (f^n e_n)_{n\in\mathbf{N}}$$

is an exponential sequence. It is clear that $\mathcal{E}(B)$ with this product is a $B$-module.
n We let !$\mathcal{E}(B) = \mathcal{E}_A(B)$! when we want to emphasize that we consider $\mathcal{E}(B)$ as an $A$-module via the $A$-algebra structure on $B$.

n **(2.4) Remark.** Let $B$ be an $A$-algebra. When !$C$! is a $B$-algebra we have that $\mathcal{E}_A(C)$ and $\mathcal{E}_B(C)$ are the same abelian group, and that $\mathcal{E}_A(C)$ is the $B$-module $\mathcal{E}_B(C)$ considered as an $A$-module via the $A$-algebra structure on $B$.

n **(2.5) The exponential functor.** Let $B$ and $C$ be $A$-algebras and let !$\psi : B \to C$! be an $A$-algebra homomorphism. We obtain a map

$$\mathcal{E}(\psi) : \mathcal{E}(B) \to \mathcal{E}(C)$$

defined on all exponential sequences $e = (e_n)_{n\in\mathbf{N}}$ of $\mathcal{E}(B)$ by

$$\mathcal{E}(\psi)(e) = \mathcal{E}(\psi)((e_n)_{n\in\mathbf{N}}) = (\psi(e_n))_{\in\mathbf{N}}.$$

It is clear that $\mathcal{E}(\psi)$ is an $A$-module homomorphism and that $\mathcal{E}$ is a functor from $A$-algebras to $A$-modules.
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**(2.6) Characterization of homomomorphisms to exponential sequences.**
**n** Let !$M$! be an $A$-module. A map

$$!u : M \to \mathcal{E}(B)!$$

is the same as maps

$$!u^n : M \to B! \quad \text{for} \quad n = 0, 1, \ldots$$

where $u$ and $u^0, u^1, \ldots$ are related by

$$u(x) = (u^n(x))_{n \in \mathbf{N}}$$

for all $x$ in $M$. We have that the maps $u^n : M \to B$ for $n = 0, 1, \ldots$ define an $A$-module homomorphism $u : M \to \mathcal{E}(B)$ if and only if, for all $f \in A$ and all $x$ and $y$ in $M$, the following equations hold:

(i) $u^0(x) = 1$.
(ii) $u^n(fx) = f^n u^n(x)$.
(iii) $u^m(x)u^n(x) = \binom{m+n}{m} u^{m+n}(x)$.
(iv) $u^n(x + y) = \sum_{i+j=n} u^i(x)u^j(y)$.

## 3. The algebra of divided powers.

→ **(3.1) Introduction.** Let $M$ be an $A$-module. We shall in (3.19) show that the functor from $A$-algebras to $A$-modules which maps an $A$-algebra $B$ to the $A$-module homomorphisms $!\mathrm{Hom}_A(M, \mathcal{E}(B))!$ from $M$ to $\mathcal{E}(B)$ is representable. That is, there is a $A$-algebra $!\Gamma(M)!$, and for every $A$-algebra $B$ a canonical bijection

n n

$$!\Psi_M(B) : \mathrm{Hom}_{A\text{-alg}}(\Gamma(M), B) \to \mathrm{Hom}_A(M, \mathcal{E}(B))!$$

from the $A$-algebra homomorphisms from $\Gamma(M)$ to $B$, which makes $\Psi_M$ into an isomorphism of functors from $A$-algebras to $A$-modules.

The $A$-algebra $\Gamma(M)$ is called the *algebra of divided powers* of $M$. It is easy to construct $\Gamma(M)$ directly. However, before we construct $\Gamma(M)$ we shall give its properties in order to emphasize that these properties follow since th $A$-algebra $\Gamma(M)$ represents the functor that maps $B$ to $\mathrm{Hom}_A(M, \mathcal{E}(B))$, and not of its construction.

We shall denote the multiplication on $\Gamma(M)$ by $\star$. The reason for introducing a particular notation for the multiplication is that we later shall show that the ring $\Gamma(M)$ is graded, and that, when $M$ is an algebra, each graded piece $\Gamma^n(M)$ is a ring under another multiplication, and it is important to distinguish the two multiplications.

→ **(3.2) Assumption.** We assume until Section (3.19) that for every $A$-module $M$ the functor from $A$-algebras to $A$-modules which maps the $A$-algebra $B$ to the $A$-module $\mathrm{Hom}_A(M, \mathcal{E}(B))$ is representable by an $A$-algebra $\Gamma(M)$.

n When we need to emphasize the $A$-algebra structure we write $!\Gamma(M) = \Gamma_A(M)!$.

**(3.3) Unicity.** Since $\Gamma(M)$ represents the functor which maps the $A$-algebra $B$ to the $A$-module $\mathrm{Hom}_A(M, \mathcal{E}(B))$ it is unique up to an isomorphism of $A$-algebras.

n **(3.4) Functoriality in the modules.** Let $!N!$ be an $A$-module and let $!u : M \to N!$ be an $A$-module homomorphism. We obtain an $A$-algebra homomorphism

$$!\Gamma(u) : \Gamma(M) \to \Gamma(N)!$$

which is the image of the identity map on $\Gamma(N)$ by the composite map

$$\mathrm{Hom}_{A\text{-alg}}(\Gamma(N), \Gamma(N)) \xrightarrow{\Psi_N(\Gamma(N))} \mathrm{Hom}_A(N, \mathcal{E}(\Gamma(N)))$$
$$\xrightarrow{\mathrm{Hom}_A(u,\mathrm{id})} \mathrm{Hom}_A(M, \mathcal{E}(\Gamma(N))) \xrightarrow{\Psi_M(\Gamma(N))^{-1}} \mathrm{Hom}_{A\text{-alg}}(\Gamma(M), \Gamma(N)).$$

It is clear from the definition of $\Gamma(u)$ that $\Gamma$ is a functor from $A$-modules to $A$-algebras.

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**(3.5) The universal map.** The image of the identity map on $\Gamma(M)$ by the bijection $\Psi_M(\Gamma(M)) : \mathrm{Hom}_{A\text{-alg}}(\Gamma(M), \Gamma(M)) \to \mathrm{Hom}_A(M, \mathcal{E}(\Gamma(M))$ is a *universal* $A$-module homomorphism

$$!\gamma_M : M \to \mathcal{E}(\Gamma(M))!,$$

such that there is a bijective correspondence between $A$-module homomorphisms $u : M \to \mathcal{E}(B)$ and $A$-algebra homomorphisms $\varphi : \Gamma(M) \to B$ given by

$$u = \mathcal{E}(\varphi)\gamma_M.$$

The homomorphism $\gamma_M : M \to \mathcal{E}(\Gamma(M))$ is given by homomorphisms

$$\gamma_M^n : M \to \Gamma(M) \text{ for } n = 0, 1, \dots$$

that satisfy the conditions

(i) $\gamma_M^0(x) = 1$.
(ii) $\gamma_M^n(fx) = f^n\gamma_M^n(x)$.
(iii) $\gamma_M^m(x) \star \gamma_M^n(x) = \binom{m+n}{m}\gamma_M^{m+n}(x)$.
(iv) $\gamma_M^n(x+y) = \sum_{i+j=n} \gamma_M^i(x) \star \gamma_M^j(y)$.

n **(3.6) The grading.** For every natural number $n$ we let $!\Gamma^n(M)!$ be the sub-A-module of $\Gamma(M)$ generated by the elements $\gamma_M^\nu(x) = \star_{\alpha\in I}\gamma_M^{\nu_\alpha}(x_\alpha)$ for all $\nu$ in $\mathbf{N}^{(I)}$
n with $|\nu| = n$ and all $x = (x_\alpha)_{\alpha\in I}$ in $M^I$, where we let $!\Gamma^0(M) = A!$. We also let

$$!\gamma_M^n : M \to \Gamma^n(M)!$$

denote the map induced by $\gamma_M^n : M \to \Gamma(M)$. In particular we have the map $\gamma_M^0 : M \to A$ given by $\gamma_M^0(x) = 1$ for all $x$ in $M$.

For every $A$-module homomorphism $u : M \to N$ and every nonnegative integer $n$ it follows from the definitions that we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma_M^n} & \Gamma^n(M) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \Gamma^n(u)} \\ N & \xrightarrow[\gamma_N^n]{} & \Gamma^n(N), \end{array}$$

n where $!\Gamma^n(u) : \Gamma^n(M) \to \Gamma^n(N)!$ is the $A$-module homomorphism on graded pieces induced by $\Gamma(u)$.

Let $f = (f_\alpha)_{\alpha\in I}$ be in $A^{(I)}$ and let $x = (x_\alpha)_{\alpha\in I}$ be in $M^I$. By repeated
→ application of the properties (i)-(iv) of Section (3.5) we obtain the formula

$$\gamma_M^n\Big(\sum_{\alpha\in I} f_\alpha x_\alpha\Big) = \sum_{\nu\in\mathbf{N}^{(I)}, |\nu|=n} f^\nu\gamma_M^\nu(x).$$

**(3.7) Theorem.** *The $A$-algebra $\Gamma(M)$ is a graded and augmented with $\Gamma^n(M)$ as the elements of degree $n$, and $\Gamma$ is a functor from $A$-modules to graded augmented $A$-algebras.*

*Proof.* We first show that the $A$-algebra $\Gamma(M)$ is generated by the elements $\gamma_M^n(x)$ for all $n$ in $\mathbf{N}$ and all $x$ in $M$. In order to see this we let $\Gamma(M)'$ be the sub-$A$-algebra of $\Gamma(M)$ generated by these elements. For each $A$-algebra $B$ we have that $\gamma_M : M \to \mathcal{E}(\Gamma(M))$ factors via the $A$-submodule $\mathcal{E}(\Gamma(M)')$ of $\mathcal{E}(\Gamma(M))$. From the isomorphism

$$\Psi_M(\Gamma(M)') : \operatorname{Hom}_{A\text{-alg}}(\Gamma(M), \Gamma(M)') \to \operatorname{Hom}_A(M, \mathcal{E}(\Gamma(M)'))$$

we consequently obtain an $A$-algebra homomorphism $\Gamma(M) \to \Gamma(M)'$ which composed with the inclusion map $\Gamma(M)' \to \Gamma(M)$ gives the identity. Hence we have that $\Gamma(M)' = \Gamma(M)$

To show that $\Gamma(M)$ is a graded $A$-algebra with $\Gamma^n(M)$ as the elements of degree $n$ we observe that the $A$-algebra structure on $\Gamma(M)$ induces an $A$-algebra structure on $\oplus_{n=0}^\infty \Gamma^n(M)$ such that the map $\varphi : \oplus_{n=0}^\infty \Gamma^n(M) \to \Gamma(M)$, induced by the inclusions of the $A$-modules $\Gamma^n(M)$ in $\Gamma(M)$, is an $A$-algebra homomorphism. The maps $\gamma_M^n : M \to \Gamma^n(M)$ for $n$ in $\mathbf{N}$ induce an $A$-module homomorphism
$\to$ $M \to \mathcal{E}(\oplus_{n=0}^\infty \Gamma^n(M))$ since they satisfy the the relations (i)-(iv) of Section (3.5). Consequently we obtain an $A$-algebra homomorphism $\psi : \Gamma(M) \to \oplus_{n=1}^\infty \Gamma^n(M)$ whose composite with $\varphi : \oplus_{n=0}^\infty \Gamma^n(M) \to \Gamma(M)$ is the identity on $\Gamma(M)$. Hence $\psi$ is injective. It follows from the definition of $\psi$ that the composite of the restriction of $\psi$ to $\Gamma^n(M)$ and the projection $\oplus_{n=0}^\infty \Gamma^n(M) \to \Gamma^n(M)$ is the identity on $\Gamma^n(M)$ for all $n$. Consequently we also have that $\psi$ is surjective, and therefore an isomorphism.

Corresponding to the $A$-linear map $M \to \mathcal{E}(A)$ which maps $x$ to $(1, 0, 0, \dots)$ for all $x$ in $M$ we obtain an $A$-algebra homomorphism

$$!\varepsilon : \Gamma(M) \to A!$$

of graded $A$-algebra $\Gamma(M)$ called the *augmentation map* . Let $u : M \to N$ be a homomorphism of $A$-modules. Then $\Gamma(u) : \Gamma(M) \to \Gamma(N)$ clearly induces a map of graded augmented $A$-algebras.

**(3.8) Remark.** The map

$$\gamma_M^1 : M \to \Gamma^1(M)$$

is an isomorphism of $A$-modules.

**n** To see this we let $!A[M] = A \oplus M!$ be the $A$-algebra of *dual numbers* , that is, the multiplication on $A \oplus M$ is given by $(f + gx)(f' + g'x') = ff' + f'gx + fg'x'$ for all

$f, f', g, g'$ in $A$ and all $x, x'$ in $M$. Then we have an $A$-module homomorphism $M \to \mathcal{E}(A[M])$ which maps $x$ to $1 + x$, and thus an $A$-algebra homomorphism $\Gamma(M) \to A[M]$ which maps $\gamma^1_M(x)$ to $x$. Consequently we have a surjection $\Gamma^1(M) \to M$ whose composite with $\gamma^1_M : M \to \Gamma^1(M)$ is the identity on $M$. Hence $\gamma^1_M$ is an isomorphism.

**(3.9) Functoriality in the rings.** Let $B$ be an $A$-algebra and let $C$ be a $B$-algebra via the homomorphism $\psi : B \to C$. Moreover, let $N$ be a $B$-module and
**n** let !$P$! be a $C$-module, and let !$v : N \to P$! be a $B$-module homomorphism,
→ where we consider $P$ as a $B$-module via $\psi$. It follows from Remark (2.4) that the universal $C$-module homomorphism $\gamma_P : P \to \mathcal{E}_C(\Gamma_C(P))$ gives a $B$-module homomorphism $v : P \to \mathcal{E}_B(\Gamma_C(P))$, where we consider $\Gamma_C(P)$ as a $B$-module via $\psi$. Composition with $v$ gives a $B$-module homomorphism $N \to \mathcal{E}_B(\Gamma_C(P))$ which corresponds to a $B$-algebra homomorphism

$$!\Gamma_\psi(v) : \Gamma_B(N) \to \Gamma_C(P)! \tag{3.9.1}$$

uniquely determined by

$$\Gamma_\psi(v)(\gamma^n_N(y)) = \gamma^n_P(v(y))$$

**n** for all !$y \in M$!. Clearly the diagram

$$\begin{array}{ccc} N & \xrightarrow{\gamma_N} & \mathcal{E}_B(\Gamma_B(N)) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(\Gamma_\psi(v))} \\ P & \xrightarrow[\gamma_P]{} & \mathcal{E}_C(\Gamma_C(P)) \end{array}$$

commutes.

→ Let $M$ be an $A$-module. From the homomorphism (3.9.1) with $N = M \otimes_A B$, $P = M \otimes_A C$, and $v = \mathrm{id} \otimes_A \psi$ we obtain a map

$$!\Gamma^n_\psi : \Gamma^n_B(M \otimes_A B) \to \Gamma^n_C(M \otimes_A C)!$$

such that the diagram

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{\gamma_{M \otimes_A B}} & \Gamma^n_B(M \otimes_A B) \\ {\scriptstyle \mathrm{id} \otimes_A \psi}\downarrow & & \downarrow{\scriptstyle \Gamma^n_\psi} \\ M \otimes_A C & \xrightarrow[\gamma_{M \otimes_A C}]{} & \Gamma^n_C(M \otimes_A C) \end{array} \tag{3.9.2}$$

commutes.

n **(3.10) Tensor products.** Let $B$ be an $A$-algebra via the homomorphism $!\varphi :$
n $A \to B!$, and $N$ an $B$-module. Moreover, let $!u : M \to N!$ be an $A$-module homomorphism, where we consider $N$ as a $A$-module via the $A$-algebra structure
→ on $B$. It follows from (3.9.1) that we have an $A$-algebra homomorphism $\Gamma_\varphi(u) : \Gamma_A(M) \to \Gamma_B(N)$ where $\Gamma_B(N)$ is considered as a $A$-module via the $A$-algebra structure on $B$. Extending scalars we obtain a $B$-algebra homomorphism

$$!\Gamma_\varphi(u)_B : \Gamma_A(M) \otimes_A B \to \Gamma_B(N)! \tag{3.10.1}$$

determined by

$$\Gamma_\varphi(u)_B(\gamma_M^n(x) \otimes_A 1) = \gamma_M^n(u(x))$$

for all $x \in M$. Let $C$ be a $B$-algebra via the homomorphism $\psi : B \to C$. Assume that $N$ is also a $C$-module, and that the $B$-modules structure on $N$ is induced by $\psi$ . Then we clearly have a commutative diagram

$$\begin{array}{ccc} \Gamma_A(M) \otimes_A B & \xrightarrow{\Gamma_\varphi(u)_B} & \Gamma_B(N) \\ {\scriptstyle \mathrm{id} \otimes_A \psi}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_\psi(\mathrm{id})} \\ \Gamma_A(M) \otimes_A C & \xrightarrow[\Gamma_{\psi\varphi}(u)_C]{} & \Gamma_C(N). \end{array} \tag{3.10.2}$$

**(3.11) Theorem.** ([R1], Thm. III, 3 p. 262) *Let $B$ be an $A$-algebra via the homomorphism $\varphi : A \to B$. The map of graded $B$-algebras*

$$\Gamma_\varphi(u)_B : \Gamma_A(M) \otimes_A B \to \Gamma_B(M \otimes_A B) \tag{3.11.1}$$

→ *obtained from* (3.10.1) *applied to the canonical $A$-module homomorphism $u : M \to M \otimes_A B$, is an isomorphism. The inverse is determined by mapping $\gamma_M^n(x \otimes_A f)$ to $\gamma_M^n(x) \otimes_A f^n$ for all $f \in B$ and $x \in M$.*

*Moreover, for every map $\psi : B \to C$ of $A$-algebras we have a commutative diagram*

$$\begin{array}{ccc} \Gamma_A(M) \otimes_A B & \xrightarrow{\Gamma_\varphi(u)_B} & \Gamma_C(M \otimes_A B) \\ {\scriptstyle \mathrm{id} \otimes_A \psi}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma(\mathrm{id} \otimes \psi)} \\ \Gamma_A(M) \otimes_A C & \xrightarrow[\Gamma_{\psi\varphi}(v)_C]{} & \Gamma_C(M \otimes_A C). \end{array} \tag{3.11.2}$$

*where $v : M \to M \otimes_A C$ is the canonical map.*

*Proof.* We shall construct the inverse to $\Gamma_\varphi(u)_B$. Let $\chi : \Gamma_A(M) \to \Gamma_A(M) \otimes_A B$ be the canonical $A$-algebra homomorphism. From the composite of the universal map $\gamma_M : M \to \mathcal{E}_A(\Gamma_A(M))$ with the $A$-module homomorphism $\mathcal{E}_A(\chi) :$

$\mathcal{E}_A(\Gamma_A(M)) \to \mathcal{E}_A(\Gamma_A(M) \otimes_A B)$ we obtain an $A$-module homomorphism $M \to \mathcal{E}_A(\Gamma_A(M) \otimes_A B)$. It follows from from Remark (2.4) that we have a $A$-module isomorphism $\mathcal{E}_A(\Gamma_A(M) \otimes_A B) \to \mathcal{E}_B(\Gamma_A(M) \otimes_A B)$. Consequently we obtain an $A$-module homomorphism $M \to \mathcal{E}_B(\Gamma_A(M) \otimes_A B)$ where the target is considered as a $A$-module via $\varphi$. We extend scalars to $B$ and obtain a $B$-module homomorphism

$$M \otimes_A B \to \mathcal{E}_B(\Gamma_A(M) \otimes_A B).$$

This homomorphism corresponds to a $B$-algebra homomorphism

$$\Gamma_B(M \otimes_A B) \to \Gamma_A(M) \otimes_A B \tag{3.11.3}$$

which is easily checked to be the inverse to the map $\Gamma_\varphi(u)_B$. It is clear that the homomorphism (3.11.3) maps $\gamma_M^n(x \otimes_A f)$ to $\gamma_M^n(x) \otimes_A f^n$ for all $f \in B$.

The last part of the Theorem follows from the commutativity of Diagram (3.10.2).

**(3.12) The extended universal map.** ([R1] IV §1 p.265) Let $B$ be an $A$-algebra. For every natural number $n$ we have a canonical map

$$!\gamma_B^n = (\gamma_M^n)_B : M \otimes_A B \to \Gamma^n(M) \otimes_A B! \tag{3.12.1}$$

determined by

$$\gamma_B^n(\sum_{\alpha \in I} x_\alpha \otimes_A f_\alpha) = \sum_{\nu \in \mathbf{N}^{(I)}, |\nu| = n} \gamma_M^\nu(x) \otimes_A f^\nu$$

for all $f = (f_\alpha)_{\alpha \in I}$ in $B^{(I)}$ and all $x = (x_\alpha)_{\alpha \in I}$ in $M^I$. This map is the composite of the universal homomorphism $\gamma_{M \otimes_A B}^n : M \otimes_A B \to \Gamma_B^n(M \otimes_A B)$ with the inverse of the $B$-algebra isomorphism $\Gamma_\varphi(u)_B : \Gamma_A^n(M) \otimes_A B \to \Gamma_B^n(M \otimes_A B)$ of (3.11.1). That is

$$\gamma_{M \otimes_A B}^n = \Gamma_\varphi(u)_B \gamma_B^n.$$

Let $\psi : B \to C$ be a homomorphism of $A$-algebras. It follows from the commutativity of the Diagrams (3.9.2) and (3.11.2) that we have a commutative diagram

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{\gamma_B^n} & \Gamma^n(M) \otimes_A B \\ {\scriptstyle \mathrm{id} \otimes_A \psi} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \otimes_A \psi} \\ M \otimes_A C & \xrightarrow[\gamma_C^n]{} & \Gamma^n(M) \otimes_A C. \end{array} \tag{3.12.2}$$

**(3.13) Direct limits.** ([R1], Thm. 3 p.277) Let $I$ be a *partially ordered* and *directed* set. Moreover let $!(M_\alpha, u_\alpha^\beta)_{\alpha, \beta \in I, \alpha \le \beta}!$ be a *directed system* of $A$-modules. We denote the direct limit of the system by $!!M = \varinjlim_{\alpha \in I} M_\alpha$ and let $!u_\alpha : M_\alpha \to M!$ for $\alpha$ in $I$ be the *canonical maps* . Because of the functoriality of $\Gamma$ we obtain a directed system $!(\Gamma(M_\alpha), \Gamma(u_\alpha^\beta))_{\alpha, \beta \in I, \alpha \le \beta}!$ of $A$-algebras.

**(3.14) Theorem.** *The A-algebra homomorphism*

$$\varinjlim_{\alpha\in I}\Gamma(M_\alpha) \to \Gamma(\varinjlim_{\alpha\in I} M_\alpha) \qquad (3.14.1)$$

*obtained from the maps* $\Gamma(u_\alpha) : \Gamma(M_\alpha) \to \Gamma(\varinjlim_{\alpha\in I} M_\alpha)$ *for all* $\alpha$ *in* $I$ *is an isomorphism.*

→ *Proof.* We construct an inverse to the homomorphism (3.14.1). The canonical map $\Gamma(M_\alpha) \to \varinjlim_{\alpha\in I}\Gamma(M_\alpha)$ corresponds to an $A$-module homomorphism $v_\alpha : M_\alpha \to \mathcal{E}(\varinjlim_{\alpha\in I}\Gamma(M_\alpha))$. We clearly have that $v_\alpha = v_\beta u_\alpha^\beta$ for all $\alpha$ and $\beta$ in $I$ with $\alpha \leq \beta$. Consequently we obtain an $A$-module homomorphism

$$\varinjlim_{\alpha\in I} M_\alpha \to \mathcal{E}(\varinjlim_{\alpha\in I}\Gamma(M_\alpha)). \qquad (3.14.2)$$

→ The map (3.14.2) corresponds to a map of $A$-algebras

$$\Gamma(\varinjlim_{\alpha\in I} M_\alpha) \to \varinjlim_{\alpha\in I}\Gamma(M_\alpha)$$

→ which clearly is the inverse to the map (3.14.1).

**(3.15) Exact sequences.** ([R1], IV 8 p. 278) A sequence of maps

$$!L \underset{u_2}{\overset{u_1}{\rightrightarrows}} M \xrightarrow{v} N! \qquad (3.15.1)$$

is exact if $v$ is surjective, and for all pairs of elements $x_1, x_2$ of $M$ we have that
n $v(x_1) = v(x_2)$ if and only if there is an element $!w \in L!$ such that $u_1(w) = x_1$ and $u_2(w) = x_2$.

→ In particular, when the sequence (3.15.1) is exact, we have that for all $x$ in $M$ there is a $w$ in $L$ such that $u_1(w) = x = u_2(w)$.

**(3.16) Theorem.** ([R1], Thm. IV 5 p. 232) *Let* $L \underset{u_2}{\overset{u_1}{\rightrightarrows}} M \xrightarrow{v} N$ *be an exact sequence of A-modules. Then the sequence*

$$\Gamma(L) \underset{\Gamma(u_2)}{\overset{\Gamma(u_1)}{\rightrightarrows}} \Gamma(M) \xrightarrow{\Gamma(v)} \Gamma(N) \qquad (3.16.1)$$

*is an exact sequence of A-algebras.*

*Proof.* It is clear that $\Gamma(v)$ is surjective and that $\Gamma(v)\Gamma(u_1) = \Gamma(vu_1) = \Gamma(vu_2) = \Gamma(v)\Gamma(u_2)$. Let $\mathfrak{I}$ be the image of $\Gamma(L)$ by the map $\Gamma(u_1) - \Gamma(u_2)$. Then $\mathfrak{I}$ is contained in the kernel of $\Gamma(v)$.

Observe that every element in $\Gamma(M)$ can be written in the form $\gamma(u_1(w')) = \gamma(u_2(w'))$ for some element $w'$ in $L$. This is because every element in $\Gamma(M)$ is an $A$-linear combination of elements $\gamma_M^\mu(x)$ with $!\mu = (\mu_\alpha)_{\alpha\in I}!$ in $\mathbf{N}^{(I)}$ and $x = (x_\alpha)_{\alpha\in I}$ in $M^I$, and for every $\alpha$ in $I$ we can find an element $w'_\alpha$ in $L$ such that $x_\alpha = u_1(w'_\alpha) = u_2(w'_\alpha)$. Hence

$$\gamma_M^\mu(x) = \gamma_M^\mu(u_1(w')) = \gamma_M^\mu(u_2(w')) = \Gamma(u_1)\gamma_L^\mu(w') = \gamma_L(u_2)\gamma_L^\mu(w')). \qquad (3.16.2)$$

In particular we obtain that $\mathfrak{I}$ is an ideal because it is generated, as an ideal, by the elements $\Gamma(u_1)\gamma_L^\nu(w) - \Gamma(u_2)\gamma_L^\nu(w)$ for all $\nu \in \mathbf{N}^{(I)}$ and $w \in M^I$, and from (3.16.2) we obtain

$$\gamma_L^\mu(x)(\Gamma(u_1)\gamma_L^\nu(w) - \Gamma(u_2)\gamma_L^\nu(w)) = \Gamma(u_1)\gamma_L^\mu(w')\Gamma(u_1)\gamma_L^\nu(w)$$
$$- \Gamma(u_2)\gamma_L^\mu(w')\Gamma(u_2)\gamma_L^\nu(w) = \Gamma(u_1)(\gamma_L^\mu(w')\gamma_L^\nu(w)) - \Gamma(u_2)(\gamma_L^\mu(w')\gamma_L^\nu(w)).$$

We shall show that the kernel of $\Gamma(v)$ is contained in the ideal $\mathfrak{I}$, and hence equal to $\mathfrak{I}$. Let $!u_\mathfrak{I} : \Gamma(M) \to \Gamma(M)/\mathfrak{I}!$ be the residue map. It follows from the definition of $\mathfrak{I}$ that the composite maps of $A$-modules

$$L \underset{u_2}{\overset{u_1}{\rightrightarrows}} M \xrightarrow{\gamma_M} \mathcal{E}(\Gamma(M)) \xrightarrow{\mathcal{E}(u_\mathfrak{I})} \mathcal{E}(\Gamma(M)/\mathfrak{I})$$

are equal. Consequently the homomorphism $\mathcal{E}(u_\mathfrak{I})\gamma_M$ factors via an $A$-module homomorphism $N \to \mathcal{E}(\Gamma(M)/\mathfrak{I})$. Correspondingly there is an $A$-algebra homomorphism $\Gamma(N) \to \Gamma(M)/\mathfrak{I}$ which composed with $\Gamma(v) : \Gamma(M) \to \Gamma(N)$ is equal to $u_\mathfrak{I}$. Consequently the kernel of $\Gamma(v)$ is contained in $\mathfrak{I}$.

We have proved that $\mathfrak{I}$ is the kernel of the map $\Gamma(v)$, and thus that the sequence

$$\Gamma(L) \xrightarrow{\Gamma(u_1)-\Gamma(u_2)} \Gamma(M) \xrightarrow{\Gamma(v)} \Gamma(N) \to 0 \qquad (3.16.3)$$

is exact.

Let $x_1$ and $x_2$ be elements in $\Gamma(M)$ that have the same image by $\Gamma(v)$. Since the sequence (3.16.3) is exact we can find an element $w$ in $\Gamma(L)$ such that

$$x_1 - x_2 = (\Gamma(u_1) - \Gamma(u_2))(w).$$

Let $x = x_1 - \Gamma(u_1)(w) = x_2 - \Gamma(u_2)(w)$. As we observed, every element $x$ in $\Gamma(M)$ can be written on the form $x = \Gamma(u_1)(w') = \Gamma(u_2)(w')$ for some $w'$ in $\Gamma(L)$. We therefore obtain that $x_1 = x + \Gamma(u_1)(w) = \Gamma(u_1)(w + w')$ and that $x_2 = x + \Gamma(u_2)(w) = \Gamma(u_2)(w + w')$. Hence we have proved that the sequence (3.16.1) is exact.

**(3.17) Corollary.** ([R1], Prop. IV.8) *Let $v : M \to N$ be a surjection of A-modules. Then we have an exact sequence*

$$0 \to !\mathfrak{J}! \to \Gamma(M) \xrightarrow{\Gamma(v)} \Gamma(N) \to 0$$

*where $\mathfrak{J}$ is the ideal in $\Gamma(M)$ generated by the elements $\gamma_M^n(x)$ with $n \geq 1$ and with $x$ in the kernel of $v$*

*Proof.* Let $L$ be the kernel of $v$. We have an exact sequence

$$L \times M \underset{u_2}{\overset{u_1}{\rightrightarrows}} M \xrightarrow{v} N$$

where $u_1(w, x) = x$ and $u_2(w, x) = w + x$ for all $w \in L$ and all $x \in M$. It follows from the Theorem that we have an exact sequence of $A$-modules

$$\Gamma(L \times M) \underset{\Gamma(u_2)}{\overset{\Gamma(u_1)}{\rightrightarrows}} \Gamma(M) \xrightarrow{\Gamma(v)} \Gamma(N)$$

and consequently an exact sequence of $A$-modules

$$\Gamma(L \times M) \xrightarrow{\Gamma(u_1)-\Gamma(u_2)} \Gamma(M) \xrightarrow{\Gamma(v)} \Gamma(N) \to 0.$$

Hence we have that the kernel of $\Gamma(v)$ is generated by elements on the form $(\Gamma(u_1) - \Gamma(u_2))\gamma^\nu_{L\times M}(w, x) = \gamma^\nu_M(x) - \gamma^\nu_M(x + w)$ for all $\nu \in \mathbf{N}^{(I)}$, all $w = (w_\alpha)_{\alpha \in I}$ in $L^I$, and all $x = (x_\alpha)_{\alpha\in I}$ in $M^I$. It follows from the formula $\gamma_M^{\nu_\alpha}(w_\alpha + x_\alpha) = \gamma_M^{\nu_\alpha}(x_\alpha) + \sum_{\mu=0}^{\nu_\alpha-1} \gamma^\mu_M(x_\alpha) \star \gamma_M^{\nu_\alpha-\mu}(w_\alpha)$, which holds for all $\alpha$ in $I$, that $\gamma^\nu_M(x) - \gamma^\nu_M(w+x) = \prod_{\alpha\in I} \gamma_M^{\nu_\alpha}(x_\alpha) - \prod_{\alpha\in I} \gamma_M^{\nu_\alpha}(w_\alpha + x_\alpha)$ lies in the ideal of $\Gamma(M)$ generated by the elements $\gamma_M^{\nu_\alpha-\mu}(w_\alpha)$ with $\mu < \nu_\alpha$ and $w_\alpha \in L$. Hence we have proved the Corollary.

**(3.18) Direct sums.** Let $M$ and $N$ be $A$-modules. The natural $A$-algebra homomorphism $\Gamma(M) \to \Gamma(M) \otimes_A \Gamma(N)$ gives a canonical homomorphism

$$M \xrightarrow{\gamma_M} \mathcal{E}(\Gamma(M)) \to \mathcal{E}(\Gamma(M) \otimes_A \Gamma(N))$$

of $A$-modules. Similarly we have a canonical homomorphism of $A$-modules

$$N \xrightarrow{\gamma_N} \mathcal{E}(\Gamma(N)) \to \mathcal{E}(\Gamma(M) \otimes_A \Gamma(N)).$$

We therefore obtain an $A$-module homomorphism

$$M \oplus N \to \mathcal{E}(\Gamma(M) \otimes_A \Gamma(N)). \qquad (3.18.1)$$

**(3.19) Theorem.** ([R1], III §7 p. 256) *The $A$-algebra homomorphism*

$$\Gamma(M \oplus N) \to \Gamma(M) \otimes_A \Gamma(N)) \tag{3.19.1}$$

*corresponding to the map* (3.18.1) *is an isomorphism. It is determined by mapping $\gamma^n_{M\oplus N}(x, y)$ to $\sum_{i+j=n} \gamma^i_M(x) \otimes_A \gamma^j_N(y)$ for all $x$ in $M$ and $y$ in $N$.*

*The inverse is determined by mapping $\gamma^n_M(x) \otimes_A 1$ to $\gamma^n_{M\oplus N}(x)$, and $1 \otimes_A \gamma^n_N(y)$ to $\gamma^n_{M\oplus N}(y)$ for all $x \in M$ and $y \in N$.*

*Proof.* We shall construct the inverse to the map (3.19.1). Observe that the natural $A$-module homomorphisms $M \to M \oplus N$ and $N \to M \oplus N$ give homomorphisms $\Gamma(M) \to \Gamma(M \oplus N)$ respectively $\Gamma(N) \to \Gamma(M \oplus N)$ of $A$-algebras. Consequently we obtain an $A$-algebra homomorphism

$$\Gamma(M) \otimes_A \Gamma(N) \to \Gamma(M \oplus N) \tag{3.19.2}$$

which is determined by mapping the elements $\gamma^n_M(x) \otimes_A 1$ to $\gamma^n_{M\oplus N}(x)$, and the elements $1 \otimes_A \gamma^n_N(y)$ to $\gamma^n_{M\oplus N}(y)$. Consequently (3.19.2) is the inverse to the map (3.19.1).

**(3.20) Co-product structure.** We have a homomorphism of graded augmented $A$-algebras

$$!\Delta : \Gamma(M) \to \Gamma(M) \otimes_A \Gamma(M)!$$

determined by

$$\Delta\gamma^n_M(x) = \sum_{i+j=n} \gamma^i_M(x) \otimes_A \gamma^j_M(x) \tag{3.20.1}$$

for all $x$ in $M$ and $n$ in $\mathbf{N}$.

This homomorphism is obtained by composing the $A$-algebra homomorphism $\Gamma(M) \to \Gamma(M \oplus M)$, corresponding to the *diagonal map* $M \to M \oplus M$ with the isomorphism $\Gamma(M \oplus M) \to \Gamma(M) \otimes_A \Gamma(M)$ of (3.19.1).

It is clear that the augmentation $\varepsilon : \Gamma(M) \to A$ is a co-unit for the multiplication defined by $\Delta$, and it follows from formula (3.20.1) that the co-multiplication is associative and commutative.

**(3.21) Construction of divided powers.** It remains to prove that the $A$-algebra $\Gamma(M)$ exists. We want the $A$-algebra $\Gamma(M)$ to give a canonical isomorphism of $A$-modules

$$\mathrm{Hom}_{A\text{-alg}}(\Gamma(M), B) \to \mathrm{Hom}_A(M, \mathcal{E}(B)),$$

for all $A$-algebras $B$, that is functorial in $B$. It follows from the conditions (i)-(iv) of Section (2.6) for $A$-module homomorphisms $M \to \mathcal{E}(B)$ that these requirements are fulfilled by the residue ring $\Gamma(M)$ of the polynomial ring over $A$ in the independent variables $X(n,x)$ for all $(n,x) \in \mathbf{N} \times M$, modulo the ideal generated by the elements

(i) $X(0,x) - 1$.
(ii) $X(n,fx) - f^n X(n,x)$.
(iii) $X(m,x)X(n,x) - \binom{m+n}{m} X(m+n,x)$.
(iv) $X(n,x+y) - \sum_{i+j=n} X(i,x)X(j,y)$.

for all $x$ and $y$ in $M$ and all $f$ in $A$. The universal homomorphism

$$\gamma_M^n : M \to \Gamma^n(M)$$

maps $x \in M$ to the residue class $\gamma_M^n(x)$ of $X(n,x)$ in $\Gamma(M)$.

## 4. Symmetric tensors.

**(4.1) Notation.** Let $M$ be an $A$-module. Denote by $!T(M) = T_A^n(M)!$ the tensor algebra of the module $M$ over $A$. We denote by $!T^n(M) = T_A(M)!$ the tensor product of $M$ with itself $n$ times over $A$. The symmetric group $!\mathfrak{S}_n!$ operates on $T^n(M)$ by!!

$$\sigma(x_1 \otimes_A x_2 \otimes_A \cdots \otimes_A x_n) = x_{\sigma^{-1}(1)} \otimes_A x_{\sigma^{-1}(2)} \otimes_A \cdots \otimes_A x_{\sigma^{-1}(n)}$$

for all $x_1, x_2, \dots, x_n$ in $M$ and $!\sigma!$ in $\mathfrak{S}_n$. The elements $x$ in $T^n(M)$ such that $\sigma(x) = x$ for all $\sigma$ in $\mathfrak{S}_n$ we call *symmetric tensors* of order $n$. The symmetric tensors form an $A$-submodule of $T^n(M)$ that we denote by $!TS^n(M)!$. Let

$$!TS(M) = \oplus_{n=0}^{\infty} TS^n(M)!.$$

We have that $TS^0(M) = A$ and $TS^1(M) = M$.

For each element $x$ in $M$ the tensor product $!x^{\otimes_A n}!$ of $x$ with itself $n$ times is in $TS^n(M)$.

Let $m_1, m_2, \dots, m_n$ be natural numbers, and let!!

$$m(i) = m_1 + m_2 + \cdots + m_i \quad \text{for} \quad i = 1, 2, \dots, n$$

with $m(0) = 0$. The subgroup of $!\mathfrak{S}_{m(n)}!$ of elements that map the interval!!

$$[m(i-1) + 1, m(i)]$$

to itself for $i = 1, 2, \dots, n$, we denote by $!!\mathfrak{S}_{m_1|m_2|\cdots|m_n}$. Let $!\mathfrak{S}_{m_1,m_2,\dots,m_n}!$ be the elements in $\mathfrak{S}_{m(n)}$ such that

$$\sigma(m(i-1)+1) < \sigma(m(i-1)+2) < \cdots < \sigma(m(i)) \quad \text{for} \quad i = 1, 2, \dots, n.$$

It is clear that the elements in $\mathfrak{S}_{m_1,m_2,\dots,m_n}$ form a full set of representatives for the classes of $\mathfrak{S}_{m(n)}/\mathfrak{S}_{m_1|m_2|\cdots|m_n}$.

**(4.2) The shuffle product.** For $x$ in $TS^m(M)$ and $y$ in $TS^n(M)$ we have that $x \otimes_A y$ is invariant under the group $\mathfrak{S}_{m|n}$. We define the *product* of $x$ and $y$ by

$$xy = \sum_{\sigma \in \mathfrak{S}_{m+n}/\mathfrak{S}_{m|n}} !!\sigma(x \otimes_A y) = \sum_{\sigma \in \mathfrak{S}_{m,n}} \sigma(x \otimes_A y).$$

The product gives an $A$-linear homomorphism

$$TS^m(M) \otimes_A TS^n(M) \to TS^{m+n}(M)$$

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that extends, by linearity, to an $A$-linear product

$$TS(M) \otimes_A TS(M) \to TS(M).$$

With this product, often called the *shuffle product* , we have that $TS(M)$ becomes a commutative graded $A$-algebra with unit 1 in $TS^0(M)$. To prove this the only difficult part is to verify that the product is commutative and associative.

We first show associativity. Let $x, y, z$ be elements in $T^m(M)$, $T^n(M)$ and $T^p(M)$ respectively. We shall consider $\mathfrak{S}_{m+n}$ as the subgroup of $\mathfrak{S}_{m+n+p}$ which fixes the elements in the interval $[m+n+1, m+n+p]$. Since we have a sequence of subgroups $\mathfrak{S}_{m+n+p} \supseteq \mathfrak{S}_{m+n|p} \supseteq \mathfrak{S}_{m|n|p}$ the product in $\mathfrak{S}_{m+n+p}$ of the representatives $\mathfrak{S}_{m+n,p}$ for $\mathfrak{S}_{m+n+p}/\mathfrak{S}_{m+n|p}$, and the representative $\mathfrak{S}_{m,n}$ for $\mathfrak{S}_{m+n|p}/\mathfrak{S}_{m|n|p}$ are representatives for $\mathfrak{S}_{m+n+p}/\mathfrak{S}_{m|n|p}$. Consequently we obtain that

$$\begin{aligned}(x \star y) \star z &= \sum_{\tau \in \mathfrak{S}_{m+n,p}} \tau((x \star y) \otimes_A z) \\ &= \sum_{\tau \in \mathfrak{S}_{m+n,p}} \sum_{\sigma \in \mathfrak{S}_{m,n}} \tau\sigma(x \otimes_A y \otimes_A z) = \sum_{\upsilon \in \mathfrak{S}_{m,n,p}} \upsilon(x \otimes_A y \otimes_A z).\end{aligned}$$

Analogously we obtain that $x \star (y \star z) = \sum_{\upsilon \in \mathfrak{S}_{m,n,p}} \upsilon(x \otimes_A y \otimes_A z)$. Consequently we have that $(x \star y) \star z = x \star (y \star z)$.

In order to show that the product is commutative we define the premutations
n !$\iota$! in $\mathfrak{S}_{m,n}$ by

$$\begin{aligned}&\iota(1) = 1 + n, \iota(2) = 2 + n, \dots, \iota(m) = m + n \\ &\iota(m+1) = 1, \iota(m+2) = 2, \dots, \iota(m+n) = n.\end{aligned}$$

We have that $\iota^{-1}(y \otimes_A x) = x \otimes_A y$ and the correspondence $\mathfrak{S}_{n,m} \to \mathfrak{S}_{m,n}$ that maps $\sigma$ to $\sigma\iota$ is a bijections. Consequently we obtain that

$$y \star x = \sum_{\sigma \in \mathfrak{S}_{n,m}} \sigma(y \otimes_A x) = \sum_{\sigma \in \mathfrak{S}_{n,m}} \sigma\iota(x \otimes_A y) = \sum_{\sigma \in \mathfrak{S}_{m,n}} \sigma(x \otimes_A y) = x \star y.$$

**(4.3) Functoriality in the module.** Let $u : M \to N$ be a homomorphism of
n $A$-modules. It is clear that the map !$T(u) : T(M) \to T(N)$! of tensor algebras
n induces a map !$TS(u) : TS(M) \to TS(N)$! of graded $A$-algebras. Thus $TS$ is a functor from $A$-modules to graded $A$-algebras.

**(4.4) Theorem.** ([R1], Thm. IV p. 272) *Let $M$ be a free $A$-module with basis !$(e_\alpha)_{\alpha\in I}$!. Then $TS(M)$ is a free $A$-module with basis !$(e^{\otimes_A \nu})_{\nu\in \mathbf{N}^{(I)}}$! for $\nu = (\nu_\alpha)_{\alpha\in I}$ in $\mathbf{N}^{(I)}$, where !!$e^{\otimes_A \nu} = \star_{\alpha\in I} e_\alpha^{\otimes_A \nu_\alpha}$.*

*Proof.* Let !$J = I^{[1,n]}$! be the set of maps from the interval ![1, n]! to $I$. Then !!$E = \{e_{\rho(1)} \otimes_A e_{\rho(2)} \otimes_A \cdots \otimes_A e_{\rho(n)}\}_{\rho\in J}$ is an $A$-basis for $T^n(M)$. We have that $\mathfrak{S}_n$ operates on the elements of $E$ by permutation of the factors. Let !$O$! be the set whose elements are the *orbits* under this action. For every orbit $\omega$ in $O$ we let !$\varepsilon_\omega = \sum_{\varepsilon\in\omega} \varepsilon$!. It is clear that the elements $\varepsilon_\omega$ are invariant under $\mathfrak{S}_n$. Consequently they are in $TS^n(M)$. They are also linearly independent since they are sums of different elements of $E$. We shall show that they generate $TS^n(M)$. Let $x$ be in $TS^n(M)$. Then there is a family $(\lambda_\varepsilon)_{\varepsilon\in E}$ in $A^{(E)}$ such that $x = \sum_{\varepsilon\in E} \lambda_\varepsilon \varepsilon$. Since $x$ is in $TS^n(M)$ we must have that $\lambda_{\sigma(\varepsilon)} = \lambda_\varepsilon$ for all $\varepsilon$ in $E$ and $\sigma$ in $\mathfrak{S}_n$. It follows that $x$ is in the submodule of $TS^n(M)$ generated by $\varepsilon_\omega$ for all $\omega \in O$.

It remains to show that the elements $\varepsilon_\omega$ with $\omega\in O$ are the same as the elements $e^\nu$ for $\nu$ in $\mathbf{N}^{(I)}$. To prove this we let $u : J \to \mathbf{N}^{(I)}$ be the maps such that the image of $\rho : [1, n] \to I$ is defined by $u_\rho(\alpha) = \operatorname{card} \rho^{-1}(\alpha)$ for all $\alpha$ in $I$. It is clear that $u_{\rho_1} = u_{\rho_2}$ if and only if $\rho_2 = \rho_1\sigma$ for some $\sigma$ in $\mathfrak{S}_n$. Consequently the elements of $\varepsilon_\omega$ are of the form

$$\varepsilon_\omega = \sum_{\rho\in J, u_\rho=\nu} e_{\rho(1)} \otimes_A e_{\rho(2)} \otimes_A \cdots \otimes_A e_{\rho(n)}$$

for some $\nu$ in $\mathbf{N}^{(I)}$. For $u_\rho = \nu$ we have $\nu_\alpha = u_\rho(\alpha) = \operatorname{card}\rho^{-1}(\alpha)$ for all $\alpha \in I$. Let $\{\rho(1), \rho(2), \ldots, \rho(n)\} = \{\pi(1), \pi(2), \ldots, \pi(m)\}$, where $\pi(1), \pi(2), \ldots, \pi(m)$ are different elements of $I$. Then $\nu_{\pi(i)} = u_{\rho(\pi(i))} = \operatorname{card}\rho^{-1}(\pi(i))$ for $i = 1, 2, \ldots, m$, and we obtain that

$$\begin{aligned}
&\sum_{\rho\in J, u_\rho=\nu} e_{\rho(1)} \otimes_A e_{\rho(2)} \otimes_A \cdots \otimes_A e_{\rho(n)}\\
&= \sum_{\sigma\in\mathfrak{S}_{\nu_{\pi(1)},\nu_{\pi(2)},\ldots,\nu_{\pi(m)}}} e_{\pi(1)}^{\otimes_A \nu_{\pi(1)}} \otimes_A e_{\pi(2)}^{\otimes_A \nu_{\pi(2)}} \otimes_A \cdots \otimes_A e_{\pi(m)}^{\otimes_A \nu_{\pi(m)}}\\
&= e_{\pi(1)}^{\otimes_A \nu_{\pi(1)}} \star e_{\pi(2)}^{\otimes_A \nu_{\pi(2)}} \star \cdots \star e_{\pi(m)}^{\otimes_A \nu_{\pi(m)}}.
\end{aligned}$$

**(4.5) Divided powers and symmetric tensors.** For all natural numbers $n$ we have that $x^{\otimes_A n}$ is in $TS^n(M)$. In the algebra $TS(M)$ we have for all $x$ and $y$ in $M$ and $f$ in $A$ that

(i) $x^0 = 1$.
(ii) $(fx)^{\otimes_A n} = f^n x^{\otimes_A n}$.
(iii) $x^{\otimes_A m} \star x^{\otimes_A n} = \binom{m+n}{m} x^{\otimes_A (m+n)}$.
(iv) $(x+y)^{\otimes_A n} = \sum_{i+j=n} x^{\otimes_A i} \star y^{\otimes_A j}$.

The first two formulas are direct consequences of the definitions. Moreover the third formula follows from the equations

$$x^{\otimes_A m} \star x^{\otimes_A n} = \sum_{\sigma \in \mathfrak{S}_{m,n}} \sigma(x^{\otimes_A m} \otimes_A x^{\otimes_A n})$$

$$= \sum_{\sigma \in \mathfrak{S}_{m,n}} \sigma x^{\otimes_A (m+n)} = \binom{m+n}{m} x^{\otimes_A (m+n)},$$

and the fourth from the equations

$$(x+y)^{\otimes_A n} = \sum_{i+j=n} \sum_{\sigma \in \mathfrak{S}_{i,j}} \sigma(x^{\otimes_A i} \otimes_A y^{\otimes_A j}) = \sum_{i+j=n} x^{\otimes_A i} \star y^{\otimes_A j}.$$

**(4.6) Theorem.** ([R1], Prop. III.1 p. 254) *We have a homomorphism of graded $A$-algebras*

$$\Gamma(M) \to TS(M)$$

*uniquely determined by mapping $\gamma_M^n(x)$ to $x^{\otimes_A n}$ for all $x \in M$ and $n \in \mathbf{N}$.*

([R1], Thm. IV.5 p.272) *When $M$ is free we have that the algebra homomorphism is an isomorphism. In particular, if $(e_\alpha)_{\alpha \in I}$ is a basis for $M$, then the elements $(\gamma_M^\nu(e))_{\nu \in \mathbf{N}^{(I)}} = (\star_{\alpha \in I} \gamma_M^{\nu_\alpha}(e_\alpha))_{\nu \in \mathbf{N}^{(I)}}$ form a basis for the $A$-module $\Gamma(M)$.*

*Proof.* The existence of the map of the Theorem follows from the equalities (i)-
→ (iv) of (4.5). When $M$ is free the map is an $A$-module isomorphism because the elements $e^\nu$ form a basis for the $A$-module $TS(M)$, the elements $\gamma_M^\nu(e)$ generate the $A$-module $\Gamma(M)$, and $\gamma_M^\nu(e)$ maps to $e^\nu$ for all $\nu$ in $\mathbf{N}^{(I)}$.

## 5. The symmetric algebra.

n **(5.1) Notation.** For any $A$-module $M$ we write $!M^*!$ for the dual $\mathrm{Hom}_A(M,A)$
n of $M$ as an $A$-module. Let $!S(M)!$ be the symmetric algebra of the $A$-module
n $M$, and let $!S^n(M)!$ be the symmetric product $n$ times of $M$ with itself over $A$.
n Moreover, let!!

$$!S(M)^*_{\mathrm{gr}}! = \oplus_{n=0}^{\infty} \mathrm{Hom}_A(S^n(M), A) = \oplus_{n=0}^{\infty} S^n(M)^*$$

be the *graded dual* of $S(M)$.

The homomorphism $M \to S(M) \otimes_A S(M)$ of $A$-modules defined by mapping $x$ to $x \otimes_A 1 + 1 \otimes_A x$ for all $x$ in $M$ defines uniquely an $A$-algebra homomorphism

$$!\Delta : S(M) \to S(M) \otimes_A S(M)!.$$

The dual of $\Delta$ gives a multiplication

$$S(M)^*_{\mathrm{gr}} \otimes_A S(M)^*_{\mathrm{gr}} \to (S(M) \otimes_A S(M))^*_{\mathrm{gr}} \xrightarrow{\Delta^*} S(M)^*_{\mathrm{gr}}. \tag{5.1.1}$$

n For $!u!$ and $!v!$ in $S(M)^*_{\mathrm{gr}}$ we denote the image of $u \otimes v$ by the map (5.1.1), that is the product of $u$ and $v$, by $u \star v$.

→ **(5.2) Proposition.** *The multiplication defined in* (5.1.1) *is associative and commutative, and makes $S(M)^*_{\mathrm{gr}}$ to a graded $A$-algebra with identity equal to the identity in* $\mathrm{Hom}_A(S^0(M), A) = \mathrm{Hom}_A(A, A)$.

*Proof.* The only difficult part is to verify commutativity and associativity. To check commutativity and associativity it suffices to check the corresponding properties on elements of $S(M)$, that is, to check that $\Delta$ followed by the map $!\tau : S(M) \otimes_A S(M) \to S(M) \otimes_A S(M)!$ that switches the coordinates is equal to $\Delta$, and that $(1 \otimes_A \Delta)\Delta = (\Delta \otimes_A 1)\Delta$ as maps $S(M) \to S(M) \otimes_A S(M) \otimes_A S(M)$. However, to prove these formulas we only have to check that they hold on elements on $x$ of $M$ because all the maps are $A$-algebra homomorphism and $\Delta$ is determined by its value on $M$. However, for all $x$ in $M$ we have that

$$\tau\Delta(x) = \tau(1 \otimes_A x + x \otimes_A 1) = x \otimes_A 1 + 1 \otimes_A x = \Delta(x)$$

such that $\tau\Delta = \Delta$, and

$$(1 \otimes_A \Delta)\Delta(x) = (1 \otimes_A \Delta)(1 \otimes_A x + x \otimes_A 1)$$
$$= 1 \otimes_A 1 \otimes_A x + 1 \otimes_A x \otimes_A 1 + x \otimes_A 1 \otimes_A 1 = (\Delta \otimes_A 1)(1 \otimes_A x + x \otimes_A 1)$$

such that $(\mathrm{id} \otimes_A \Delta)\Delta = (\Delta \otimes_A \mathrm{id})\Delta$.

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**(5.3) Explicit description.** Let $m$ and $n$ be natural numbers, and let $u$ and $v$ be $A$-module homomorphism in $S^m(M)^* = \mathrm{Hom}_A(S^m(M), A)$ respectively $S^n(M)^* = \mathrm{Hom}_A(S^n(M), A)$. Then, by definition, the images of the elements $(u \otimes_A v)(x_1 x_2 \cdots x_i \otimes_A x_{i+1} x_{i+2} \cdots x_{m+n})$ by the product

$$u \otimes_A v : S^m(M)^* \otimes_A S^n(M)^* \to A$$

where $x_1, x_2, \dots, x_{m+n}$ are elements in $M$ is given by:

$$(u \otimes_A v)(x_1 x_2 \cdots x_i \otimes_A x_{i+1} x_{i+2} \cdots x_{m+n}) = \begin{cases} 0 \quad \text{when} \quad i \neq m \\ u(x_1 x_2 \cdots x_i) \star v(x_{i+1} x_{i+2} \cdots x_{m+n}) \quad \text{when} \quad i = m. \end{cases}$$

Hence the product $u \star v$ is determined by

$$\begin{aligned} u \star v(x_1 x_2 \dots x_{m+n}) &= (u \otimes_A v)(\Delta x_1 \Delta x_2 \cdots \Delta x_{m+n}) \\ &= (u \otimes_A v)(\prod_{i=1}^{m+n} (x_i \otimes_A 1 + 1 \otimes_A x_i) \\ &= (u \otimes_A v) \sum_{i+j=m+n} \sum_{\sigma \in \mathfrak{S}_{i,j}} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(i)} \otimes_A x_{\sigma(i+1)} x_{\sigma(i+2)} \cdots x_{\sigma(i+j)} \\ &= \sum_{\sigma \in \mathfrak{S}_{m+n}} u(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}) \star v(x_{\sigma(m+1)} x_{\sigma(m+2)} \cdots x_{\sigma(m,n)}). \end{aligned} \tag{5.3.1}$$

**(5.4) Divided powers and the symmetric algebra.** For every element $u$ in
n the dual module $M^*$ of $M$ we define the element $!\sigma^n(u)!$ in the $n$'th graded part
n $S^n(M)^*$ of $S(M)^*_{\mathrm{gr}}$ by!!

$$\sigma^n(u)(x_1 x_2 \cdots x_n) = u(x_1) \star u(x_2) \star \cdots \star u(x_n)$$

for all $x_1, x_2, \dots, x_n$ in $M$, and we let $\sigma^0(u) = \mathrm{id}$. For all $u$ and $v$ in $M^*$ and $f$ in $A$ we have that

(i) $\sigma^0(u) = 1$.
(ii) $\sigma^n(fu) = f^n \sigma^n(u)$.
(iii) $\sigma^m(u) \star \sigma^n(u) = \binom{m+n}{m} \sigma^{m+n}(u)$.
(iv) $\sigma^n(u+v) = \sum_{i+j=n} \sigma^i(u) \star \sigma^j(v)$.

The equalities (i) and (ii) follow immediately from the definitions. To show
→ formula (iii) we use formula (5.3.1) and obtain for all $x_1, x_2, \dots, x_{m+n}$ in $M$ the

equalities

$$\begin{aligned}
&(\sigma^m(u) \star \sigma^n(v))(x_1 x_2 \cdots x_{m+n}) \\
&= \sum_{\sigma \in \mathfrak{S}_{m,n}} \sigma^m(u)(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}) \otimes_A \sigma^n(v)(x_{\sigma(m+1)} x_{\sigma(m+2)} \cdots x_{\sigma(m+n)}) \\
&= \sum_{\sigma \in \mathfrak{S}_{m,n}} u(x_{\sigma(1)}) \star u(x_{\sigma(2)}) \star \cdots \star u(x_{\sigma(m)}) \\
&\qquad \star v(x_{\sigma(m+1)}) \star v(x_{\sigma(m+2)}) \cdots \star v(x_{\sigma(m+n)}).
\end{aligned} \tag{5.4.1}$$

→ When $u = v$ we obtain from (5.4.1) that $\sigma^m(u)\sigma^n(u) = \binom{m+n}{m}\sigma^{m+n}(u)$. In
→ order to prove formula (iv) we use (5.4.1) once more and obtain

$$\begin{aligned}
&\sigma^n(u+v)(x_1 x_2 \cdots x_n) = \prod_{i=1}^n (u+v)(x_i) \\
&\sum_{i+j=n} \sum_{\sigma \in \mathfrak{S}_{i,j}} u(x_{\sigma(1)}) u(x_{\sigma(2)}) \cdots u(x_{\sigma(i)}) v(x_{\sigma(i+1)}) v(x_{\sigma(i+2)}) \cdots v(x_{\sigma(i+j)}) \\
&= \sum_{i+j=n} (\sigma^i(u) \star \sigma^j(v))(x_1 x_2 \cdots x_n).
\end{aligned}$$

**(5.5) The symmetric algebra of a free module.** Let $M$ be a free $A$-module
**n** with basis $(e_\alpha)_{\alpha\in I}$, and let $!(e^*_\alpha)_{\alpha \in I}!$ be the dual basis for $M^* = \operatorname{Hom}_A(M, A)$.
**n** Then $S^n(M)^*$ has a basis consisting of the elements $!e^{\nu *} = (\prod_{\alpha \in I} e_\alpha^{\nu_\alpha})^*!$ for all
**n** $\nu \in \mathbf{N}^{(I)}$ with $|\nu| = n$. For $\nu$ in $\mathbf{N}^{(I)}$ we let $!e^{*\nu} = \star_{\alpha\in I} \sigma^{\nu_\alpha}(e^*_\alpha) = \sigma^\nu(e^*)!$.

**(5.6) Theorem.** *We have that $S(M)^*_{\mathrm{gr}}$ is a free $A$-module with basis $(e^{*\nu})_{\nu \in \mathbf{N}^{(I)}}$.*

*Proof.* We have to show that the elements $(e^{*\nu})_{\nu\in\mathbf{N}^{(I)}}$ are linearly independent over $A$.

Let $m_1, m_2, \dots, m_n$ be natural numbers and let $m(i) = m_1 + m_2 + \cdots + m_i$ for
$i = 1, 2, \dots, n$ and $m(0) = 0$. For every set of elements $u_1, u_2, \dots, u_n$ in $M^*$ we
→ obtain from formula (5.4.1), and induction on $n$, that

$$\begin{aligned}
&(\sigma^{m_1}(u_1) \star \sigma^{m_2}(u_2) \star \cdots \star \sigma^{m_n}(u_n))(x_1 x_2 \cdots x_{m_1+m_2+\cdots+m_n}) \\
&\quad = \sum_{\sigma \in \mathfrak{S}_{m_1, m_2, \dots, m_n}} \prod_{i=1}^n u_i(x_{\sigma(m(i-1)+1)}) u_i(x_{\sigma(m(i-1)+2)}) \cdots u_i(x_{\sigma(m(i))})
\end{aligned} \tag{5.6.1}$$

for all elements $x_1, x_2, \dots, x_{m_1+m_2+\cdots+m_n}$ of $M$.

Let $!\rho : [1,p] \to I!$ be a map from the interval $[1,p]$ to $I$. For $\nu \in \mathbf{N}^{(I)}$ with $|\nu| = p$, and $\nu_\alpha = 0$ when $\alpha$ is not in $\{\alpha_1, \dots, \alpha_m\}$ we write $\nu(i) = \nu_{\alpha_1} + \nu_{\alpha_2} + \cdots + \nu_{\alpha_i}$ for $i = 1, 2, \dots, m$ and $\nu(0) = 0$. We obtain from (5.6.1) that

$$e^{*\nu}(e_{\rho(1)}e_{\rho(2)}\cdots e_{\rho(p)}) = (\prod_{\alpha\in I} \sigma^{\nu_\alpha}(e^*_\alpha))(e_{\rho(1)}e_{\rho(2)}\cdots e_{\rho(p)}) =$$
$$\sum_{\sigma\in\mathfrak{S}_{\nu_{\alpha_1},\nu_{\alpha_2},\dots,\nu_{\alpha_m}}} \prod_{i=1}^m e^*_{\alpha_i}(e_{\rho(\sigma(\nu(i-1)+1))})e^*_{\alpha_i}(e_{\rho(\sigma(\nu(i-1)+2))})\cdots e^*_{\alpha_i}(e_{\rho(\sigma((\nu(i)))})).$$

Consequently we obtain that

$$e^{*\nu}(e_{\rho(1)}e_{\rho(2)}\cdots e_{\rho(p)}) = \begin{cases} 1 \quad \text{when} \quad i = \rho(\sigma(\nu(i-1)+1)) \\ = \rho(\sigma(\nu(i-1+2))) = \cdots = \rho(\sigma(\nu(i))) \quad \text{for} \quad i = 1, 2, \dots, n \\ 0 \quad \text{otherwise.} \end{cases}$$

Hence we have that

$$e^{*\nu} = (\prod_{\alpha\in I} e_\alpha^{\nu_\alpha})^* = e^{\nu *}$$

for all $\nu \in \mathbf{N}^{(I)}$, and we have proved the Theorem.

**(5.6) Relation between divided powers and the symmetric algebra.** It follows from the formulas (i)-(iv) of (5.4) that we have a homomorphism of graded $A$-algebras

$$\Gamma(M^*) \to S(M)^*_{\text{gr}}$$

which is uniquely determined by mapping $\gamma^p_M(u)$ to $\sigma^p(u)$ for all natural numbers $p$ and all elements $u$ in $M^*$. When $M$ is a free $A$-module with basis $(e_i)_{i\in I}$ the homomorphism maps the basis $(\gamma^\nu_M(e^*))_{\nu\in\mathbf{N}^{(I)}}$ of $\Gamma(M^*)$ to the basis $(\sigma^\nu(e^*))_{\nu\in\mathbf{N}^{(I)}}$ of $S(M)^*_{\text{gr}}$ and thus is an isomorphism. Clearly the last part of Theorem (4.6) follows.

## 6. Polynomial laws.

n **(6.1) Definition.** Let $M$ and $N$ be $A$-modules. A *polynomial law* $!U!$ on $A$-algebras from $M$ to $N$ is a map

$$!U_B : M \otimes_A B \to N \otimes_A B!$$

for each $A$-algebra $B$ such that, for every homomorphism $\psi : B \to C$ of $A$-algebras, the diagram

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{U_B} & N \otimes_A B \\ {\scriptstyle \mathrm{id} \otimes_A \psi}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes_A \psi} \\ M \otimes_A C & \xrightarrow[U_C]{} & N \otimes_A C. \end{array} \tag{6.1.1}$$

is commutative. We say that $U$ is *homogeneous* of *degree* $n$ if

$$U_B(gz) = g^n U_B(z)$$

for all $z$ in $M \otimes_A B$ and all $g \in B$.

The set of all polynomial laws from the module $M$ to the module $N$ we denote
n by $!\mathcal{P}(M,N) = \mathcal{P}_A(M,N)!$, and the subset of all homogeneous polynomial laws of
n degree $n$ we denote by $!\mathcal{P}^n(M,N) = \mathcal{P}^n_A(M,N)!$.

n **(6.2) Remark.** For every $A$-module $M$ we have a covariant functor $!F_M!$ from $A$-algebras to $A$-modules defined by $F_M(B) = M \otimes_A B$ and $F_M(\psi) = \mathrm{id} \otimes_A \psi$ for all $A$-algebras $B$ and all $A$-algebra homomorphisms $\psi : B \to C$. A polynomial law from $M$ to an $A$-module $N$ is the same as a *natural transformation* of functors $F_M \to F_N$.

**(6.3) Remark.** We have that $\mathcal{P}(M,N)$ and $\mathcal{P}^n(M,N)$ are $A$-modules when we
n define addtion of two polynomial laws $U$ and $!V!$ by

$$(U+V)_B = U_B + V_B,$$

and multiplication by an element $f \in A$ by

$$(fU)_B = fU_B$$

for all $A$-algebras $B$.

n Let $M$, $N$ and $!P!$ be $A$-modules, and let $U$ and $V$ be polynomial laws from $M$
n to $N$, respectively, from $N$ to $P$. We define the composite $!VU!$ of $U$ and $V$ by

$$(VU)_B = V_B U_B$$

for all $A$ algebras $B$. It is clear that $VU$ is a polynomial law from $M$ to $P$, and that the corresponding maps

$$\mathcal{P}(M,N)\otimes_A \mathcal{P}(M,N)\to \mathcal{P}(M,P)$$

and

$$\mathcal{P}^n(N,P)\otimes_A \mathcal{P}^m(M,N)\to \mathcal{P}^{m+n}(M,P)$$

are $A$-modules homomorphisms.

For fixed $M$ the correspondences that maps an $A$-module $N$ to the $A$-module $\mathcal{P}(M,N)$, respectively $\mathcal{P}^n(M,N)$, are covariant functors from $A$-modules to $A$-modules. Similarly, for a fixed $A$-module $N$, the correspondences that maps an $A$-module $M$ to the $A$-module $\mathcal{P}(M,N)$, respectively $\mathcal{P}^n(M,N)$, are contravariant functors from $A$-modules to $A$-modules.

**n** Let $M, N, M', N'$ be $A$-modules, and let $U$ and $!U'!$ be polynomial laws from
**n** $M$ to $N$, respectively from $M'$ to $N'$. Then there is a polynomial law $!U\oplus U'!$ from $M\oplus N$ to $M'\oplus N'$ defined by

$$(U\oplus U')_B = U_B\oplus U'_B$$

for all $A$-algebras $B$.

**(6.4) Example.** Let $u: M\to N$ be a homomorphism of $A$-modules. Then the homomorphisms $u\otimes_A \mathrm{id}: M\otimes_A B\to N\otimes_A B$, for all $A$-algebras $B$, is a polynomial law.

**(6.5) Polynomial laws and functors.** Let $M$ be an $A$-module, and let $G_M$ be a covariant functor from $A$-algebras to $A$-modules. Assume that for every $A$-algebra $B$ we have that $G_M(B)$ is a $B$-module and that there is a $B$-module
**n** homomorphism!!

$$u_B: G_M(B)\to G_M(A)\otimes_A B,$$

**n** and a map!!

$$\beta_B: M\otimes_A B\to G_M(B)$$

such that for every $A$-algebra $C$ and every $A$-algebra homomorphism $\varphi: B\to C$ the digrams

$$\begin{array}{ccc} G_M(B) & \xrightarrow{u_B} & G_M(A)\otimes_A B \\ {\scriptstyle G_M(\varphi)}\downarrow & & \downarrow{\scriptstyle G_M(\mathrm{id})\otimes_A\varphi} \\ G_M(C) & \xrightarrow[u_C]{} & G_M(A)\otimes_A C \end{array}$$

and

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{\beta_B} & G_M(B) \\ {\scriptstyle G_M(\mathrm{id} \otimes_A \varphi)} \downarrow & & \downarrow {\scriptstyle G_M(\varphi)} \\ M \otimes_A C & \xrightarrow[\beta_C]{} & G_M(C). \end{array}$$

Then the composite map

$$u_B \beta_B : M \otimes_A B \to G_M(A) \otimes_A B,$$

for all $A$-algebras $B$ define a polynomial law from $M$ to $G_M(A)$. If we have that

$$\beta_B(gz) = g^n \beta_B(z)$$

for all $z \in M \otimes_A B$ and $g \in B$ we have that the polyomial law is homogeneous of degree $n$. We obtain a unique $A$-module homomorphism

$$\Gamma_A^n(M) \to G_M(A)$$

such that

$$\gamma_A^n(x) = \beta_A(x)$$

for all $x \in M$.

**(6.6) Example.** (The universal polynomial law) Let $M$ be an $A$-module and let $n$ be a non-negative integer. We saw in (3.12) that, for every $A$-algebra $B$, we have a map

$$\gamma_B^n : M \otimes_A B \to \Gamma_A^n(M) \otimes_A B,$$

and it follows from Diagram (3.12.2) that these maps, for all $A$-algebras $B$, define a polynomial law from $M$ to $\Gamma_A^n(M)$ that we denote by $!\gamma^n!$. It follows from (3.12) that $\gamma^n$ is a polynomial law of degree $n$.

**(6.7) Remark.** Let $!(t_\alpha)_{\alpha \in I}!$ be a family of independent variables over the ring $A$, and let $!A[t] = A[t_\alpha]_{\alpha \in I}!$ be the polynomial ring in the variables $t_\alpha$ with coefficients in $A$. It follows from the definition of $\gamma_{A[t]}^n$ in (3.12) that, for every family $(x_\alpha)_{\alpha \in I}$ in $M^{(I)}$, we have that

$$\gamma_{A[t]}^n\Big(\sum_{\alpha \in I} x_\alpha \otimes_A t_\alpha\Big) = \sum_{\nu \in \mathbf{N}^{(I)}, |\nu| = n} \gamma_M^\nu(x) \otimes_A t^\nu. \tag{6.7.1}$$

Let $U$ be a polynomial law from $M$ to $N$. By the definition of a polynomial law it follows that for every family $(x_\alpha)_{\alpha \in I}$ in $M^{(I)}$ there is a unique family with finite support $!(y_\nu(x))_{\nu \in \mathbf{N}^{(I)}}!$ of elements $!y_\nu(x) \in N!$ such that

$$U_{A[t]}\Big(\sum_{\alpha \in I} x_\alpha \otimes_A t_\alpha\Big) = \sum_{\nu \in \mathbf{N}^{(I)}} y_\nu(x) \otimes_A t^\nu. \tag{6.7.2}$$

**(6.8) Lemma.** *Let $B$ be an $A$-algebra and let $(f_\alpha)_{\alpha\in I}$ be in $B^{(I)}$. We denote by*

$$\psi : A[t] \to B$$

*the $A$-algebra homomorphism determined by $\psi(t_\alpha) = f_\alpha$ for all $\alpha \in I$. Then we have that*

$$\gamma_B^n(\sum_{\alpha\in I} x_\alpha \otimes_A f_\alpha) = \sum_{\nu\in\mathbf{N}^{(I)}, |\nu|=n} \gamma_M^\nu(x) \otimes_A f^\nu \tag{6.8.1}$$

*and*

$$U_B(\sum_{\alpha\in I} x_\alpha \otimes_A f_\alpha) = \sum_{\alpha\in\mathbf{N}^{(I)}} y_\nu(x) \otimes_A f^\nu. \tag{6.8.2}$$

*Moreover, when $U$ is homogeneous of degree $n$ we have that*

$$U_B(\sum_{\alpha\in I} x_\alpha \otimes_A f_\alpha) = \sum_{\nu\in\mathbf{N}^{(I)}, |\nu|=n} y_\nu(x) \otimes_A f^\nu. \tag{6.8.3}$$

*Proof.* Since $\gamma^n$ and $U$ are polynomial laws, we obtain from (6.7.1) and (6.7.2) that the equations (6.8.1) and (6.8.2) hold.

Let $s$ be a variable over $A$ that is independent of the variables $t_\alpha$ for all $\alpha$ in $I$. When $U$ is homogeneous of degree $n$ we obtain that

$$s^n U_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A t_\alpha) = U_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A st_\alpha) = \sum_{\nu\in\mathbf{N}^{(I)}} y_\nu \otimes_A s^{|\nu|}t^\nu.$$

It follows from (6.8.5) that the equation (6.8.3) holds.

**(6.9) Theorem.** (Thm IV.1) *Let $M$ and $N$ be $A$-modules, and let $n$ be a non-negative integer. We have an isomorphism of $A$-modules*

$$\mathrm{Hom}_A(\Gamma_A^n(M), N) \to \mathcal{P}_A^n(M, N) \tag{6.9.1}$$

*that maps an $A$-module homomorphism $u : \Gamma^n(M) \to N$ to the polynomial law $u\gamma^n$ defined by*

$$(u\gamma^n)_B = (u \otimes_A \mathrm{id}_B)\gamma_B^n$$

*for all $A$-algebras $B$.*

*For fixed $M$ this is an isomorphism of covariant functors from $A$-modules to $A$-modules.*

*Proof.* We first show that the map (6.9.1) is injective. Let $u : \Gamma^n(M) \to N$ be an $A$-modules homomorphism and let $U = u\gamma^n$. For every family $(x_\alpha)_{\alpha\in I}$ in $M^{(I)}$,

→ and every family $(f_\alpha)_{\alpha\in I}$ in $B^{(I)}$, where $B$ is an $A$-algebra, it follows from (6.8.1) that

$$\begin{aligned} U_B(\sum_{\alpha\in I} x_\alpha \otimes_A f_\alpha) &= (u\otimes_A \mathrm{id})\gamma_B^n(\sum_{\alpha\in I} x_\alpha\otimes_A f_\alpha) \\ &= (u\otimes_A \mathrm{id})(\sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} \gamma_M^\nu(x)\otimes_A f^\nu) = \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} u\gamma_M^\nu(x)\otimes_A f^\nu. \end{aligned}$$

Since the elements of the form $\gamma_M^\nu(x)$ with $|\nu| = n$, and $(x_\alpha)_{\alpha\in I}$ in $M^{(I)}$ generate $\Gamma^n(M)$ it follows that $U_B$ is completely determined by $u$. Consequently the map
→ (6.9.1) is injective.

→ We next show that (6.9.1) is surjective. Assume first that $M$ is a free $A$-module
→ with a basis $(e_\alpha)_{\alpha\in I}$. It follows from (6.7.2) and (6.8.3) that there is a uniquely determined family $(y_\nu)_{\nu\in\mathbf{N}^{(I)}}$ of elements $y_\nu(x)$ in $N$ such that

$$U_B(\sum_{\alpha\in I} e_\alpha\otimes_A f_\alpha) = \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} y_\nu(x)\otimes_A f^\nu$$

→ for all $(f_\alpha)_{\alpha\in I}$ in $B^{(I)}$. Moreover, it follows from Theorem (4.6) that we can define a unique $A$-module homomorphism

$$u : \Gamma^n(M) \to N$$

by

$$u(\gamma_M^\nu(e)) = y_\nu$$

→ for all $\nu$ in $\mathbf{N}^{(I)}$. Then $U$ is the image of $u$ by the map (6.9.1). In fact, for all $(f_\alpha)_{\alpha\in I}$ in $B^{(I)}$, we have that

$$U_B(\sum_{\alpha\in I} e_\alpha\otimes_A f_\alpha) = \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} y_\nu(x)\otimes_A f^\nu,$$

→ and it follows from (6.8.1) that

$$\begin{aligned} (u\otimes_A \mathrm{id}_B)\gamma_B^n(\sum_{\alpha\in I} e_\alpha\otimes_A f_\alpha) &= (u\otimes_A \mathrm{id}_B)(\sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} \gamma_M^\nu(e)\otimes_A f^\nu) \\ &= \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} u\gamma_M^\nu(e)\otimes_A f^\nu = \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} y_\nu(x)\otimes_A f^\nu. \end{aligned}$$

Hence we have proved the theorem when $M$ is a free $A$-module.

When $M$ is not free we choose a surjective homomorphism of $A$-modules $v :$
**n** $M' \to M$ from a free $A$-module $M'$. There is a polynomial law $!U'!$ from $M'$ to $N$ defined by $U'_B = U_B(v \otimes_A \mathrm{id}_B)$ for all $A$-algebras $B$. As we just have shown there
**n** is an $A$-module homomorphism $!u' : \Gamma^n(M') \to N!$ such that $U'_B = (u' \otimes_A \mathrm{id}_B)\gamma^n_B$ for all $A$-algebras $B$. We shall show that $u'$ factors via the homomorphism

$$\Gamma^n(v) : \Gamma^n(M') \to \Gamma^n(M)$$

and an $A$-module homomorphism $u : \Gamma^n(M) \to N$.

→ It follows from (6.7) that for all families $(x'_\alpha)_{\alpha \in I}$ in $(M')^{(I)}$ we have a unique family with finite support $(y_\nu)(x)_{\nu \in \mathbf{N}^{(I)}}$ of elements $y_\nu(x)$ in $N$ such that

$$U'_{A[t]}(\sum_{\alpha \in I} x'_\alpha \otimes_A t_\alpha) = U_{A[t]}(\sum_{\alpha \in I} v(x'_\alpha) \otimes_A t_\alpha) = \sum_{\nu \in \mathbf{N}^{(I)}, |\nu| = n} y_\nu(x) \otimes_A t^\nu. \quad (6.9.2)$$

In particular we see that $y_\nu(x)$ is zero if $\nu = (\nu_\alpha)_{\alpha \in I}$ and $\nu_\alpha \neq 0$ for some $\alpha$ such
→ that $x'_\alpha$ is in the kernel of $v$. It follows from (6.8.1) that

$$U'_{A[t]}(\sum_{\alpha \in I} x'_\alpha \otimes_A t_\alpha) = (u' \otimes_A \mathrm{id}_B)\gamma^n_{A[t]}(\sum_{\alpha \in I} x'_\alpha \otimes_A t_\alpha) = \sum_{\nu \in \mathbf{N}^{(I)}, |\nu| = n} u'\gamma^\nu_M(x') \otimes_A t^\nu. \quad (6.9.3)$$

→ From (6.9.2) and (6.9.3) we obtain that

$$y_\nu(x) = u'\gamma^\nu_M(x').$$

Hence we have that $u'\gamma^\nu_M(x') = 0$ when $\nu = (\nu_\alpha)_{\alpha \in I}$ and $\nu_\alpha \neq 0$ for some $\alpha$ such
→ that $x'_\alpha$ is in the kernel of $v$. It follows from Corollary (3.17) that the elements $\gamma^\nu(x')$ with $x'_\alpha$ in the kernel of $v$ for some $\alpha$ such that $\nu_\alpha \neq 0$ generate the kernel of $\Gamma^n(v)$. Consequently we have that $u'$ factors via $\Gamma^n(v)$ and an $A$-module homomorphism

$$u : \Gamma^n(M) \to N,$$

that is,

$$u' = u\Gamma^n(v),$$

as we wanted to show.

→ In order to show that (6.9.1) is surjective it remains to prove that $U$ is the
→ image of $u$ by (6.9.1). To this end, let $(x_\alpha)_{\alpha \in I}$ be in $M^{(I)}$ and choose $(x'_\alpha)_{\alpha \in I}$ in $(M')^{(I)}$ such that $v(x'_\alpha) = x_\alpha$ for all $\alpha \in I$. For every $A$-algebra $B$, and all

→ $(f_\alpha)_{\alpha\in I}$ in $B^{(I)}$ it follows from (6.8.1) that

$$\begin{aligned}
U_B(\sum_{\alpha\in I} x_\alpha \otimes_A f_\alpha) &= U_B(\sum_{\alpha\in I} v(x'_\alpha)\otimes_A f_\alpha)\\
&= U_B(v\otimes_A \mathrm{id}_B)(\sum_{\alpha\in I} x'_\alpha \otimes_A f_\alpha) = U'_B(\sum_{\alpha\in I} x'_\alpha \otimes_A f_\alpha)\\
&= (u'\otimes_A \mathrm{id}_B)\gamma^n_B(\sum_{\alpha\in I} x'_\alpha\otimes_A f_\alpha) = (u'\otimes_A \mathrm{id}_B)(\sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} \gamma^\nu_{M'}(x')\otimes_A f^\nu)\\
&= \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} u'\gamma^\nu_{M'}(x')\otimes_A f^\nu = \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} u\Gamma^n(v)\gamma^\nu_{M'}(x')\otimes_A f^\nu\\
&= \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} u\gamma^\nu_M(x)\otimes_A f^\nu\\
&= (u\otimes_A \mathrm{id})(\sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n}\gamma^\nu_M(x)\otimes_A f^\nu) = (u\otimes_A \mathrm{id})\gamma^n_B(\sum_{\alpha\in I} x_\alpha\otimes_A f_\alpha).
\end{aligned}$$

Consequently we have that $U_B = (u\otimes_A \mathrm{id})\gamma^n_B$ as we wanted to prove.

The last part of the Theorem is obvious.

**(6.10) Proposition.** *Let $U$ be a polynomial law from $M$ to $N$. Then the corre-*
n *sponding $A$-module homomorphism !$u_U : \Gamma^n(M)\to N$! is uniquely determined by the following condition:*

*For all $\nu\in\mathbf{N}^{(I)}$ and all $x=(x_\alpha)\in M^{(I)}$ we have that*

$$u_U(\gamma^\nu_M(x)) = y_\nu(x)$$

*where the element $y_\nu(x)\in N$ for $\nu\in\mathbf{N}^{(I)}$ are determined by the equation*

$$U_{A[t]}(\sum_{\alpha\in I} x_\alpha\otimes_A t_\alpha) = \sum_{\nu\in\mathbf{N}^{(I)}} y_\nu(x)\otimes_A t^\nu$$

→ *given in* (6.7.2).

*Proof.* The proposition follows from the relation $U_{A[t]} = (u\otimes_A \mathrm{id}_{A[t]})\gamma^n_{A[t]}$ and the
→ equations (6.7.1) and (6.7.2).

**(6.11) Base extension.** A homogeneous polynomial law $U$ on $A$-algebras from $M$ to $N$ induces a polynomial law $U_t$ on $A[t]$-algebras from $M\otimes_A A[t]$ to $N\otimes_A A[t]$, by restriction to $A[t]$-algebras. Consequently we have an $A[t]$-linear homomorphism

$$u_{U_t} : \Gamma^n_{A[t]}(M\otimes_A A[t]) \to N\otimes_A A[t]$$

such that

$$u_{U_t}\gamma^n_{M\otimes_A A[t]} = (U_t)_{A[t]} = U_{A[t]}.$$

→ Consequently it follows from Proposition (6.10) that $u_U : \Gamma^n(M) \to N$ is uniquely determined by

$$u_{U_t}(\gamma^n_{M\otimes_A A[t]}(z))$$

for all $z \in M \otimes_A A[t]$. We paraphrase this by saying that *after base extension to $A[t]$ the homomorphism $u$ is determined by the value of $\gamma^n_M(x)$ for all $x \in M$.*

**(6.12) Homomorphisms to symmetric tensors.** For all $A$-algebras $B$ we
**n** write $!G_M(B) = T^n_B(M \otimes_A B)!$. We have a natural $B$-module homomorphism!!

$$u_B : T^n_B(M \otimes_A B) \to T^n_A(M) \otimes_A B$$

that maps $(x_1 \otimes_A g_1) \otimes_B (x_2 \otimes_A g_2) \otimes_B \cdots \otimes_B (x_n \otimes_A g_n)$ to $x_1 \otimes_A x_2 \otimes_A \cdots \otimes_A x_n \otimes_A g_1 g_2 \cdots g_n$ for all elements $x_1, x_2, \dots, x_n$ in $M$ and all elements $g_1, g_2, \dots, g_n$
**n** in $B$. Moreover, we have a map!!

$$\beta_B : M \otimes_A B \to T^n_B(M \otimes_A B)$$

defined by $\beta_B(z) = z \otimes_B z \otimes_B \cdots \otimes_B z$ for all $z \in M \otimes_A B$. It is clear that

$$\beta_B(gz) = g^m \beta_B(z) \tag{6.12.1}$$

→ for all $g \in B$. It follows from (6.5) that the maps

$$u_B\beta_B : M \otimes_A B \to T^n_A(M) \otimes_A B$$

for all $A$-algebras $B$ gives a homogeneous polynomial law of degree $n$ from $M$ to $T_A(M)$. Hence there is a unique $B$-module homomorphism

$$\varphi : \Gamma^n_A(M) \to T^n_A(M)$$

such that

$$\varphi(\gamma^n_A(x)) = \beta_A(x)$$

for all $x \in M$. It follows from the definition of the homomorphism $\varphi$ that it coincides with the composite map of the inclusion $TS^n_A(M) \to T^n_A(M)$ with the homomorphism $\Gamma^n_A(M) \to TS^n_A(M)$ induced by the homomorphism of Theorem
→ (4.6).

Similarly we obtain a homogeneous polynomial law from $M$ to $S^n_A(M)$ of degree $n$, and a unique $B$-module homomorphism

$$\varphi : \Gamma^n_A(M) \to S^n_A(M)$$

such that

$$\varphi(\gamma^n_A(x)) = \beta^n_A(x) = x^n$$

for all $x \in M$.

**(6.13) Remark.** Theorem (6.9) asserts that for a fixed $A$-module $M$ the $A$-algebra $\Gamma^n_A(M)$ represents the covariant functor that maps an $A$-module $N$ to $\mathcal{P}^n_A(M,N)$ This is a fundamental result on polynomial laws. We therefore give a second proof of Theorem (6.9). Note that in the above proof we used the second part of Theorem (4.6) that describes an explicit basis of $\Gamma^n(M)$ as an $A$-module when we have an explicit basis of the $A$-module $M$. The second proof of Theorem (6.9) does not use this result. We shall show that in fact the last asertion of Theorem (4.6) follows from Theorem (6.9). Hence we obtain another proof of the second part of Theorem (4.6) that does not depend on the theory of the symmetric tensors, or of the symmetric algebra.

**(6.14) Differential operators.** An important ingredient in the second proof of Theorem (6.9) are certain differential operators that we shall introduce next. We shall throughout use the notation of Remark (6.7).

Let $M$ and $N$ be $A$-modules, and let $x$ be an element in $M$. For every $A$-algebra $B$, every element $z \in M \otimes_A B$, and every polynomial law $U \in \mathcal{P}(M,N)$ it follows from (6.7.2) that we have unique elements $!(\frac{\partial^{(n)}}{\partial x^{(n)}}U)_B(z)!$ in $N$ such that

$$U_{B[t_\alpha]}(z + x\otimes_A t_\alpha) = \sum_{n=0}^{\infty}\left(\frac{\partial^{(n)}}{\partial x^{(n)}}U\right)_B(z)\otimes_A t_\alpha^n,$$

where only a finite number of the elements $(\frac{\partial^{(n)}}{\partial x^{(n)}}U)_B(z)$ are different from 0. Consequently we obtain a map

$$!\left(\frac{\partial^{(n)}}{\partial x^{(n)}}U\right)_B : M\otimes_A B \to N\otimes_A B! \quad \text{for} \quad n = 0, 1, \dots.$$

Since $U$ is a polynomial law from $M$ to $N$ it is clear that the maps $(\frac{\partial^{(n)}}{\partial x^{(n)}}U)_B$, for all $A$-algebras $B$, give a polynomial law

$$!\frac{\partial^{(n)}}{\partial x^{(n)}}U!$$

from $M$ to $N$. It follows directly from the definition of $\frac{\partial^{(n)}}{\partial x^{(n)}}U$ that if $!V \in \mathcal{P}_A(M,N)!$ and if $f \in A$ we have that

$$\frac{\partial^{(n)}}{\partial x^{(n)}}(fU) = f\frac{\partial^{(n)}}{\partial x^{(n)}}U \quad \text{and} \quad \frac{\partial^{(n)}}{\partial x^{(n)}}(U+V) = \frac{\partial^{(n)}}{\partial x^{(n)}}U + \frac{\partial^{(v)}}{\partial x^{(v)}}V.$$

We consequently obtain an $A$-linear homomorphism

$$!\frac{\partial^{(n)}}{\partial x^{(n)}} : \mathcal{P}(M,N) \to \mathcal{P}(M,N)!.$$

This homomorphism induces an $A$-linear map

$$!\frac{\partial^{(n)}}{\partial x^{(n)}} : \mathcal{P}^m(M,N) \to \mathcal{P}^{m-n}(M,N)! \quad \text{for} \quad m = 0,1,\dots .$$

In fact, let $U$ be homogeneous of degree $m$. We have for each $z \in M \otimes_A B$ that

$$\sum_{n=0}^{\infty} \left( \frac{\partial^{(n)}}{\partial x^{(n)}} U \right)_{B[t_\beta]} (zt_\beta) \otimes_A (t_\alpha t_\beta)^n = U_{B[t_\alpha,t_\beta]}(zt_\beta + x \otimes_A t_\alpha t_\beta)$$

$$= t_\beta^m U_{B[t_\alpha,t_\beta]}(z + x \otimes_A t_\alpha) = t_\beta^m \sum_{n=0}^{\infty} \left( \frac{\partial^{(n)}}{\partial x^{(n)}} U \right)_B (z) \otimes_A t_\alpha .$$

It follows that

$$\left( \frac{\partial^{(n)}}{\partial x^{(n)}} U \right)_{B[t_\beta]} (zt_\beta) \otimes t_\beta^n = \left( \frac{\partial^{(n)}}{\partial x^{(n)}} U \right)_B (z) \otimes_A t_\beta^m \quad \text{in} \quad B[t_\beta].$$

That is $(\frac{\partial^{(n)}}{\partial x^{(n)}} U)_B(z) = 0$ when $n > m$ and $(\frac{\partial^{(n)}}{\partial x^{(n)}})_{B[t_\beta]}(zt_\beta) = (\frac{\partial^{(n)}}{\partial x^{(n)}})_B(z) t_\beta^{m-n}$ in $B[t_\beta]$ when $n \le m$. The homomorphism $M \otimes_A A[t_\beta] \to M$ obtained by mapping $x \otimes_A g(t_beta)$ to $g(f)x$, for all $g(t_\beta) \in A[t_\beta]$, maps $z \otimes_A t_\beta$ to $fz$. Since $U$ is a polynomial law we consequently obtain that

$$\left( \frac{\partial^{(n)}}{\partial x^{(n)}} U \right)_B (fz) = \begin{cases} 0 & \text{when} \quad n > m \\ f^{m-n} (\frac{\partial^{(n)}}{\partial x^{(n)}} U)_B & \text{when} \quad n \le m. \end{cases}$$

Consequently we have that the polynomial law $\frac{\partial^{(n)}}{\partial x^{(n)}} U$ is homogeneous of degree $m-n$ when $U$ is homogeneous of degree $m$, as asserted.

**(6.15) Theorem.** *Let $M$ and $N$ be $A$-modules. The differential operators $\frac{\partial^{(n)}}{\partial x^{(n)}}$ for $n = 0,1,\dots$, and for all $x \in M$, commute, and for all $x$ and $y$ in $M$, and $f$ in $A$, the following equations hold:*

(1) $\frac{\partial^{(0)}}{\partial x^{(0)}} = \mathrm{id}$.

(2) $\frac{\partial^{(n)}}{\partial (fx)^{(n)}} = f^n \frac{\partial^{(n)}}{\partial x^{(n)}}$.

(3) $\frac{\partial^{(m)}}{\partial x^{(m)}} \frac{\partial^{(n)}}{\partial x^{(n)}} = \binom{m+n}{m} \frac{\partial^{(m+n)}}{\partial x^{(m+n)}}$.

(4) $\frac{\partial^{(n)}}{\partial (x+y)^{(n)}} = \sum_{i+j=n} \frac{\partial^{(i)}}{\partial x^{(i)}} \frac{\partial^{(j)}}{\partial y^{(j)}}$.

→ *Proof.* We shall use the same notation as in Remark (6.7). For all $A$-algebras $B$, all elements $z \in M \otimes_A B$, and all polynomials laws $U$ from $M$ to $N$, we have that

$$U_{B[t_\alpha,t_\beta]}(z + x \otimes_A t_\alpha + y \otimes_A t_\beta) = \sum_{n=0}^{\infty} \left( \frac{\partial^{(n)}}{\partial y^{(n)}} U \right)_{B[t_\alpha]} (z + x \otimes_A t_\alpha) \otimes_A t_\beta^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{\partial^{(m)}}{\partial x^{(m)}} \frac{\partial^{(n)}}{\partial y^{(n)}} U \right)_B (z) \otimes_A t_\alpha^m t_\beta^n . \quad (6.15.1)$$

Similarly we obtain that

$$U_{B[t_\beta,t_\alpha]}(z+y\otimes_A t_\beta+x\otimes_A t_\alpha)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\partial^{(n)}}{\partial y^{(n)}}\frac{\partial^{(m)}}{\partial x^{(m)}}U\right)_B(z)\otimes_A t_\beta^n t_\alpha^m. \quad (6.15.2)$$

→ Comparing the expressions to the right in equation (6.15.1) and (6.15.2) we obtain that $\frac{\partial^{(m)}}{\partial x^{(m)}}\frac{\partial^{(n)}}{\partial y^{(n)}}=\frac{\partial^{(n)}}{\partial y^{(n)}}\frac{\partial^{(m)}}{\partial x^{(m)}}$, and we have proved that the differential operators commute.

(1) Equation (1) clearly holds.

(2) We have that

$$\sum_{n=0}^{\infty}\left(\frac{\partial^{(n)}}{\partial (fx)^{(n)}}U\right)_B(z)\otimes_A t_\alpha^n=U_{B[t_\alpha]}(z+fx\otimes_A t_\alpha)$$

$$=U_{B[t_\alpha]}(z+x\otimes_A ft_\alpha)=\sum_{n=0}^{\infty}\left(\frac{\partial^{(n)}}{\partial x^{(n)}}U\right)_B(z)\otimes_A (ft_\alpha)^n.$$

It follows that $(\frac{\partial^{(n)}}{\partial (fx)^{(n)}}U)_B(z)=f^n(\frac{\partial^{(n)}}{\partial x^{(n)}}U)_B$ for all $A$-algebras $B$, for all $z\in M\otimes_A B$, and for all polynomial laws $U$ from $M$ to $N$. That is, equation (2) holds.

(3) We have that

$$U_{B[t_\alpha,t_\beta]}(z+x\otimes_A(t_\alpha+t_\beta))=\sum_{p=0}\left(\frac{\partial^{(p)}}{\partial x^{(p)}}U\right)_B(z)\otimes_A(t_\alpha+t_\beta)^p$$

$$=\sum_{p=0}^{\infty}\sum_{m+n=p}\left(\frac{\partial^{(p)}}{\partial x^{(p)}}U\right)_B(z)\otimes_A\binom{m+n}{m}t_\alpha^m t_\beta^n. \quad (6.15.3)$$

→ Let $x=y$ in equation (6.15.1) and compare the coefficient of $t_\alpha^m t_\beta^n$ in (613.1) with

→ the coefficient of the same monomial in equation (6.15.3). We obtain that

$$\left(\frac{\partial^{(m)}}{\partial x^{(m)}}\frac{\partial^{(n)}}{\partial x^{(n)}}U\right)_B(z)=\binom{m+n}{m}\left(\frac{\partial^{(m+n)}}{\partial x^{(m+n)}}\right)_B(z).$$

Hence we have proved that equation (3) holds.

(4) We have that

$$U_{B[t_\alpha]}(z+(x+y)\otimes_A t_\alpha)=\sum_{n=0}^{\infty}\left(\frac{\partial^{(n)}}{\partial (x+y)^{(n)}}U\right)_B(z)\otimes_A t_\alpha^n. \quad (6.15.4)$$

→ Let $t_\alpha=t_\beta$ in equation (6.15.1) and compare the coefficient of $t_\alpha^n$ in equation

→ (6.15.1) with the coefficient of $t_\alpha^n$ in (6.15.4). We obtain that $(\frac{\partial^{(n)}}{\partial(x+y)^{(n)}}U)_B(z)=\sum_{i+j=n}(\frac{\partial^{(i)}}{\partial x^{(i)}}\frac{\partial^{(j)}}{\partial y^{(j)}}U)_B(z)$. Consequently equation (4) holds.

**(6.16) Construction of an inverse to** (6.9.1)**.** The reason for introducing and studying differential operators is that they make it possible to construct a natural inverse to the map (6.9.1). This construction is the main ingredient in the second proof of Theorem (6.9.1).

For every $\nu = (\nu_\alpha)_{\alpha\in I}$ in $\mathbf{N}^{(I)}$ and $x = (x_\alpha)_{\alpha\in I}$ in $M^I$ we write $!\frac{\partial^{(\nu)}}{\partial x^{(\nu)}} = \prod_{\alpha\in I}\frac{\partial^{(\nu_\alpha)}}{\partial x^{(\nu_\alpha)}}!$.

Let $!\mathrm{End}_A(\mathcal{P}_A(M,N))!$ be the endomorphism ring of the $A$-module $\mathcal{P}_A(M,N)$, and let $!D!$ be the $A$-subalgebra of $\mathrm{End}_A(\mathcal{P}_A(M,N))$ generated by the elements $\frac{\partial^{(n)}}{\partial x^{(n)}}$ for $n = 0, 1, \ldots$ and for all $x \in M$. It follows from Theorem (6.15) that $D$ is commutative and that we have a canonical $A$-module homomorphism

$$M \to \mathcal{E}_A(D)$$

that maps $x \in M$ to $\sum_{n=0}^{\infty}\frac{\partial^{(n)}}{\partial x^{(n)}}t^n$. This homomorphism corresponds to a canonical $A$-algebra homomorphism

$$\Gamma(M) \to D \tag{6.16.1}$$

that is determined by mapping $\gamma_M^n(x)$ to $\frac{\partial^{(n)}}{\partial x^{(n)}}$ for all $n \in \mathbf{N}$ and all $x \in M$.

For every polynomial law $U$ in $\mathcal{P}(M,N)$ we have a canonical $A$-module homomorphism

$$\mathrm{End}_A(\mathcal{P}(M,N)) \to \mathcal{P}(M,N) \tag{6.16.2}$$

that maps $u$ to $u(U)$. Moreover we have an $A$-linear map

$$\mathcal{P}(M,N) \to N \tag{6.16.3}$$

that maps $U$ to $U_A(0)$. When we compose the maps (6.16.2) and (6.16.3) we obtain a canonical $A$-module homomorphism $\mathrm{End}_A(\mathcal{P}_A(M,N)) \to N$. The latter map restricts, on the subalgebra $D$ of $\mathrm{End}_A(\mathcal{P}_A(M,N)$, to a canonical $A$-linear homomorphism

$$D \to N \tag{6.16.4}$$

that maps $\frac{\partial^{(\nu)}}{\partial x^{(\nu)}}$ to $(\frac{\partial^{(\nu)}}{\partial x^{(\nu)}}U)_A(0)$ for all $\nu \in \mathbf{N}^{(I)}$ and all $x \in M^{(I)}$. The composite of the maps (6.16.1) and (6.16.4) is a canonical $A$-module homomorphism

$$\Gamma(M) \to N$$

that is determined by mapping $\gamma^\nu(x)$ to $(\frac{\partial^{(\nu)}}{\partial x^{(\nu)}}U)_A(0)$ for all $\nu \in \mathbf{N}^{(I)}$ and all $x \in M^{(I)}$. We consequently have defined a canonical $A$-linear homomorphism

$$\mathcal{P}_A(M,N) \to \mathrm{Hom}_A(\Gamma_A(M),N) \tag{6.16.5}$$

that maps $U \in \mathcal{P}_A(M, N)$ to the $A$-module homomorphism

$$u_U : \Gamma_A(M) \to N$$

defined by

$$u_U(\gamma_M^\nu(x)) = \left(\frac{\partial^{(\nu)}}{\partial x^{(\nu)}} U\right)_A (0)$$

for all $\nu \in \mathbf{N}^{(I)}$ and all $x \in M^I$. Since the polynomial law $\frac{\partial^{(\nu)}}{\partial x^{(\nu)}} U$ is homogeneous of degree $n - |\nu|$ when $U$ is homogeneous of degree $n$ we obtain that (6.16.5) induces a canonical $A$-linear homomorphism

$$!\mathcal{P}_A^n(M, N) \to \mathrm{Hom}_A(\Gamma_A^n(M), N)! \tag{6.16.6}$$

that maps $U \in \mathcal{P}_A^n(M.N)$ to the $A$-module homomorphism

$$u_U : \Gamma_A^n(M) \to N$$

defined by

$$u_U(\gamma_M^\nu(x)) = \left(\frac{\partial^{(\nu)}}{\partial x^{(\nu)}} U\right)_A (0) = y_\nu, \tag{6.16.7}$$

where the elements $y_\nu$ in $N$ are defined uniquely by the equation

$$U_{A[t]}(\sum_{\alpha \in I} x_\alpha \otimes_A t_\alpha) = \sum_{\nu \in \mathbf{N}^{(I)}, |\nu| = n} y_\nu \otimes_A t^\nu \tag{6.16.8}$$

for all $x \in M^{(I)}$, with the same notation as in Remark (6.7).

**(6.17)** (Second proof of Theorem (6.9)) We shall show that the map (6.16.6) is the inverse of the map (6.9.1).

Let $v \in \mathrm{Hom}_A(\Gamma^n(M), N)$. Then $v$ is mapped by (6.9.1) to the polynomial law $v\gamma^n \in \mathcal{P}^n(M, N)$ such that $(v\gamma^n)_B = (v \otimes_A \mathrm{id}_B)\gamma_B^n$ for all $A$-algebras $B$. Moreover we have that $v\gamma^n$ is mapped, by the homomorphism (6.16.6), to the homomorphism $u_{v\gamma^n}$ defined by

$$u_{v\gamma^n}(\gamma_M^\nu(x)) = \left(\frac{\partial^{(\nu)}}{\partial x^{(\nu)}} v\gamma^n\right)_A (0).$$

However, it follows from equation (6.7.1) that

$$\begin{aligned}(v_{\gamma^n})_{A[t]}(\sum_{\alpha \in I} x_\alpha \otimes_A t_\alpha) &= (v \otimes_A \mathrm{id}_{A[t]})\gamma_{A[t]}^n(\sum_{\alpha \in I} x_\alpha \otimes_A t_\alpha) \\ = (v \otimes_A \mathrm{id}_{A[t]}) \sum_{\nu \in \mathbf{N}^{(I)}, |\nu| = n} \gamma_M^\nu(x) \otimes_A t^\nu &= \sum_{\nu \in \mathbf{N}^{(I)}, |\nu| = n} v(\gamma_M^\nu(x)) \otimes_A t^\nu. \end{aligned} \tag{6.17.1}$$

It follows from the defining equation (6.16.7) and from the equations (6.16.8) and (6.17.1), that $u_{v\gamma^n_M}(\gamma^\nu_M(x)) = v(\gamma^\nu_M(x))$, thus the composite map of (6.9.1) with (6.16.6) is the identity.

Conversely, let $U \in \mathcal{P}^n_A(M,N)$. Then $U$ is mapped by (6.16.6) to the homomorphism $u_U : \Gamma^n(M) \to N$ defined by $u_U(\gamma^\nu_M(x)) = y_\nu$, where $y_\nu$ is given by (6.16.8). Moreover we have that $u_U$ is mapped by (6.9.1) to the polynomial law $u_U\gamma^n$ defined by $(u_U\gamma^n)_B = (u_U \otimes_A \mathrm{id}_B)\gamma^n_B$ for all $A$-algebras $B$. It follows from the equations (6.7.1) and (6.7.2), and from the defining equations (6.16.7), and (6.16.8) that we have equalities

$$\begin{aligned}
(u_U\gamma^n)_{A[t]}(\sum_{\alpha\in I} x_\alpha \otimes_A t_\alpha) &= (u_U \otimes_A \mathrm{id}_{A[t]})\gamma^n_{A[t]}(\sum_{\alpha\in I} x_\alpha \otimes_A t_\alpha)\\
&= (u_U \otimes_A \mathrm{id}_A) \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} \gamma^\nu_M(x) \otimes_A t^\nu = \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} u_U(\gamma^\nu_M(x)) \otimes_A t^\nu\\
&= \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} \left(\frac{\partial^{(\nu)}}{\partial x^{(\nu)}}U\right)_A (0) \otimes_A t^\nu = \sum_{\nu\in\mathbf{N}^{(I)},|\nu|=n} y_\nu \otimes_A t^\nu\\
&= U_{A[t]}(\sum_{\alpha\in I} x_\alpha \otimes_A t_\alpha). \qquad (6.17.2)
\end{aligned}$$

The homomorphism $M \otimes_A A[t_\alpha] \to M \otimes_A B$ defined by mapping $x \otimes_A g(t_\alpha)$ to $x \otimes_A g(f_\alpha)$ for all $x \in M$ and $g(t_\alpha) \in A[t_\alpha]$ maps $\sum_{\alpha\in I} x_\alpha \otimes_A t_\alpha$ to $z$. Consequently it follows from equation (6.17.2) that $(u_U\gamma^n)_B(z) = U_B(z)$ and we have shown that $u_U\gamma^n = U_B$. That is, the composite of the maps (6.16.6) and (6.9.1) is the identity. Hence we have a second proof of Theorem (6.9).

We can now give the second proof of the second part of Theorem (4.6). For completeness we repeat the statment.

**(6.18) Theorem.** ([R1], Thm. IV.5 p. 272) *When $M$ is a free $A$-module with basis $(e_\alpha)_{\alpha\in I}$ we have that $\Gamma^n(M)$ is a free $A$-module with basis $\gamma^\nu(e)$ for all $\nu \in \mathbf{N}^{(I)}$ with $|\nu| = n$.*

*Proof.* It suffices to show that for every $\nu$ with $|\nu| = n$ there is an $A$-module homomorphism $u_\nu : \Gamma^n(M) \to A$ such that

$$u_\nu(e^\mu) = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu. \end{cases} \qquad (6.18.1)$$

For every $A$-algebra $B$ we have that every element $z \in M \otimes_A B$ can be written uniquely in the form $z = \sum_{\alpha\in I} e_\alpha \otimes_A f_\alpha$ with $(f_\alpha)_{\alpha\in I}$ in $B^{(I)}$. Consequently, by the

$A$-module homomorphism $M \otimes_A A[t_\alpha] \to M \otimes_A B$ defined by mapping $x \otimes_A g(t_\alpha)$ to $x \otimes_A g(f_\alpha)$ for all $x \in M$ and $g(t_\alpha) \in A[t]$, the element $\sum_{\alpha \in J} e_\alpha \otimes_A t_\alpha$ in $M \otimes_A A[t]$, where $J$ is the subset of $I$ where $\nu_\alpha \neq 0$, maps to $z$. It follows that every polynomial law $U \in \mathcal{P}^n(M, A)$ is uniquely determined by the elements $U_{A[t]}(\sum_{\alpha \in J} \sum e_\alpha \otimes_A t_\alpha)$ for all finite subsets $J$ of $I$. Moreover, it is clear that we obtain a polynomial law in $\mathcal{P}^n(M, N)$ by choosing an arbitrary family $(f_\nu)_{\nu \in \mathbf{N}^{(I)}}$ of elements $f_\nu$ in $A$ and defining $U$ by $U_{A[t]}(\sum_{\alpha \in J} e_\alpha \otimes_A t_\alpha) = \sum_{\mu \in \mathbf{N}^{(J)}, |\mu| = n} f_\mu \otimes_A t^\mu$.

For every $\nu \in \mathbf{N}^{(I)}$ with $|\nu| = n$ we let the polynomial law $U_\nu$ in $\mathcal{P}^n(M, A)$ be defined by

$$(U_\nu)_{A[t]}(\sum_{\alpha \in J} e_\alpha \otimes_A t_\alpha) = \sum_{\mu \in \mathbf{N}^{(J)}, |\mu| = n} f_\mu \otimes_A t^\mu = \begin{cases} 1 \otimes_A t^\nu & \text{when} \quad \mu = \nu \\ 0 \quad \text{when} & \mu \neq \nu \end{cases}$$

→ for all finite subsets $J$ of $I$. It follows from Theorem (6.9) that $U_\nu$ corresponds to
→ a homomorphism $u_\nu : \Gamma^n(M) \to A$ with the properties described in (6.18.1).

# 7. Multiplicative polynomial laws.

**(7.1) Bilinear maps.** Let $M$ and $N$ be $A$-modules, and let $B$ be an $A$-algebra. We shall keep the notation of Remark (6.6), so that in particular $(t_\alpha)_{\alpha\in I}$ is a family of independent variables over $A$, and $A[t]$ is the polynomial ring in the variables $t_\alpha$ over $A$. Let $(f_\alpha)_{\alpha\in I}$ and $(g_\alpha)_{\alpha\in I}$ be elements in $B^{(I)}$, and let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be in $M^{(I)}$, respectively in $N^{(I)}$.

We have a canonical isomorphism of $B$-modules

$$(M\oplus N)\otimes_A B \to (M\otimes_A B)\oplus(N\otimes_A B) \tag{7.1.1}$$

that maps the element $\sum_{\alpha\in I}(x_\alpha+y_\alpha)\otimes_A f_\alpha$ to $\sum_{\alpha\in I}(x_\alpha)\otimes_A f_\alpha+(y_\alpha)\otimes_A f_\alpha$. Its inverse maps $(\sum_{\alpha\in I}x_\alpha\otimes_A f_\alpha)+(\sum_{\alpha\in I}y_\alpha\otimes_A g_\alpha)$ to $\sum_{\alpha\in I}(x_\alpha+0)\otimes f_\alpha+\sum_{\alpha\in I}(0+y_\alpha)\otimes g_\alpha$. Moreover we have a canonical $B$-bilinear homomorphism

$$(M\otimes_A B)\times(N\otimes_A B)\to(M\otimes_A B)\otimes_B(N\otimes_A B). \tag{7.1.2}$$

From the canonical isomorphisms of $B$-modules $(M\otimes_A B)\oplus(N\otimes_A B)\xrightarrow{\sim}(M\otimes_A B)\times(N\otimes_A B)$ and $(M\otimes_A B)\otimes_B(N\otimes_A B)\xrightarrow{\sim}M\otimes_A N\otimes_A B$ and the homomorphisms (7.1.1) and (7.1.2) we obtain a canonical map

$$!T_B=(T_{M,N})_B:(M\oplus N)\otimes_A B\to M\otimes_A N\otimes_A B!$$

such that the image of $\sum_{\alpha\in I}(x_\alpha+y_\alpha)\otimes_A f_\alpha$ is $\sum_{\alpha,\beta\in I}x_\alpha\otimes_A y_\beta\otimes_A f_\alpha f_\beta$. It is clear that for all $A$-algebra homomorphisms $\psi:B\to C$ we have a commutative diagram

$$\begin{array}{ccc} (M\oplus N)\otimes_A B & \xrightarrow{T_B} & M\otimes_A N\otimes_A B \\ {\scriptstyle \mathrm{id}_{M\oplus N}\otimes_A\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{M\oplus N}\otimes_A\psi} \\ (M\oplus N)\otimes_A C & \xrightarrow[T_C]{} & M\otimes_A N\otimes_A C. \end{array}$$

That is, the map $T_B$, for all $A$-algebras $B$, is a polynomial law from the $A$-module $M\oplus N$ to the $A$-module $M\otimes_A N$, and it is clear that $T$ is homogeneous of degree 2.

When we compose the polynomial law $T$ with the universal polynomial law $\gamma^n_{M\oplus N}$ from $M\otimes_A N$ to $\Gamma^n(M\otimes_A N)$ we obtain a polynomial law $\gamma^n_{M\oplus N}T$ of degree $2n$ from $M\oplus N$ to $\Gamma^n(M\otimes_A N)$. Correspondingly we have a canonical $A$-algebra homomorphism

$$!\varphi_{M,N}:\Gamma^{2n}(M\oplus N)\to\Gamma^n(M\otimes_A N)! \tag{7.1.3}$$

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such that

$$(\varphi_{M,N} \otimes_A \mathrm{id}_B)(\gamma^{2n}_{M\oplus N})_B = (\gamma^n_{M\otimes_A N})_B T_B \tag{7.1.4}$$

for all $A$-algebras $B$.

→ The inverse of the $A$-algebra isomorphism (3.19.1) is a canonical isomorphism of $A$-algebras

$$\Gamma(M) \otimes_A \Gamma(N) \to \Gamma(M \oplus_A N) \tag{7.1.5}$$

that is determined by mapping $\gamma^n_M(x)\otimes_A 1$ to $\gamma^n_{M\oplus N}(x)$ and $1\otimes_A \gamma^n_N(y)$ to $\gamma^n_{M\oplus N}(y)$ for all $x \in M$ and $y \in N$. Consequently the image of $\gamma^\mu_M(x) \otimes_A \gamma^\nu_N(y)$ is equal to $\gamma^\mu_{M\oplus N}(x)\gamma^\nu_{M\oplus N}(y)$ for all $\nu, \mu$ in $\mathbf{N}^{(I)}$, and for all $x \in M^{(I)}$ and $y \in N^{(I)}$. From
→ (7.1.5) we obtain a canonical isomorphism

$$\oplus_{i+j=n}\Gamma^i(M) \otimes_A \Gamma^j(N) \to \Gamma^n(M \oplus N), \tag{7.1.6}$$

which together with the map $\varphi_{M,N}$ gives a homomorphism of $A$-modules

$$!\varphi^{i,j}_{M,N} : \Gamma^i(M) \otimes_A \Gamma^j(N) \to \Gamma^n(M \otimes_A N)! \tag{7.1.7}$$

for all non-negative integers $i$ and $j$ such that $i + j = 2n$.

**(7.2) Lemma.** *Let $(s_\alpha)_{\alpha\in I}$ and $(t_\alpha)_{\alpha\in I}$ be two families of mutually independent variables over $A$, and let $A[s,t]$ be the ring of polynomials in these variables with coefficients in $A$. Moreover, let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be two families of elements in the $A$-module $M$, respectively $N$. Then we have in $\Gamma^n(M \otimes_A N) \otimes_A A[s,t]$ the equation*

$$\sum_{\mu,\nu,|\mu|+|\nu|=2n} \varphi_{M,N}(\gamma^\mu_{M\oplus N}(x) \star \gamma^\nu_{M\oplus N}(y)) \otimes_A s^\mu t^\nu = \sum_{\xi\in\mathbf{N}^{I\times I},|\xi|=2n} \gamma^\xi_{M\otimes_A N}(x \otimes_A y) \otimes_A (st)^\xi, \tag{7.2.1}$$

*where we have written*

$$\gamma^\xi_{M\otimes_A N}(x \otimes_A y) \otimes_A (st)^\xi = \star_{\alpha,\beta\in I}\gamma^{\xi_{\alpha,\beta}}(x_\alpha \otimes_A y_\beta) \otimes_A (s_\alpha t_\beta)^{\xi_{\alpha,\beta}}.$$

*Let $B$ be an $A$-algebra, and let $z$ and $z'$ be elements in $M \otimes_A B$. Then we have that*

$$(\varphi^{n,n}_{M,N} \otimes_A \mathrm{id}_B)((\gamma^n_M)_B(z) \otimes_B (\gamma^n_N)_B(z')) = (\gamma^n_{M\otimes_A N})_B(z \otimes_A z'). \tag{7.2.2}$$

→ *Proof.* It follows from the equations (3.5)(iv) and (6.6.1) that

$$(\gamma^{2n}_{M\oplus N})_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha + y_\alpha \otimes_A t_\alpha)$$

$$= \sum_{i+j=2n} (\gamma^{i}_{M\oplus N})_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha) \star (\gamma^{j}_{M\oplus N})_{A[s,t]}(\sum_{\alpha\in I} y_\alpha \otimes_A t_\alpha)$$

$$\sum_{i+j=2n} ( \sum_{\mu\in \mathbf{N}^I, |\mu|=i} \gamma^{\mu}_{M\oplus N}(x) \otimes_A s^\mu) \star ( \sum_{\nu\in \mathbf{N}^I, |\nu|=j} \gamma^{\nu}_{M\oplus N}(y) \otimes_A t^\nu)$$

$$= \sum_{\mu,\nu\in \mathbf{N}^{I\times I}, |\mu|+|\nu|=2n} \gamma^{\mu}_{M\oplus N}(x) \star \gamma^{\nu}_{M\oplus N}(y) \otimes_A s^\mu t\nu.$$

Consequently we have that

$$(\varphi_{M,N} \otimes_A \mathrm{id}_{A[s,t]})(\gamma^{2n}_{M\oplus N})_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha + y_\alpha \otimes_A t_\alpha)$$

$$= \sum_{\mu,\nu\in \mathbf{N}^{I\times I}, |\mu|+|\nu|=2n} \varphi_{M,N}(\gamma^{\mu}_{M\oplus N}(x) \star \gamma^{\nu}_{M\oplus N}(y)) \otimes_A s^\mu t^\nu. \quad (7.2.3)$$

→ On the other hand we have, via the isomorphism (7.1.1), that

$$(\gamma^{n}_{M\otimes_A N})_{A[s,t]} T_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha + y_\alpha \otimes_A t_\alpha)$$

$$= (\gamma^{n}_{M\otimes_A N})_{A[s,t]}( \sum_{\alpha,\beta\in I} x_\alpha \otimes_A y_\beta \otimes_A s_\alpha t_\beta) = \sum_{\xi\in \mathbf{N}^{I\times I}} \gamma^{\xi}_{M\otimes_A N}(x \otimes_A y) \otimes_A (st)^\xi. \quad (7.2.4.)$$

→ From (7.1.4) we have that $\varphi_{M,N} \otimes_A \mathrm{id}_{A[s,t]}(\gamma^{2n}_{M\oplus N})_{A[s,t]} = (\gamma^{n}_{M\otimes_A N})_{A[s,t]} T_{A[s,t]}$.
→ Hence the first part of the Lemma follows from (7.2.3) and (7.2.4).

The calculations of the first part of the proof shows that

$$(\varphi_{M,N}\otimes_A \mathrm{id}_{A[s,t]})( \sum_{i+j=2n} (\gamma^{i}_{M\oplus N})_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha) \star (\gamma^{j}_{M\oplus N})_{A[s,t]}(\sum_{\alpha\in I} y_\alpha \otimes_A t_\alpha))$$

$$= (\gamma^{n}_{M\otimes_A N})_{A[s,t]}( \sum_{\alpha,\beta\in I} x_\alpha \otimes_A y_\beta \otimes_A s_\alpha t_\beta),$$

→ and using the isomorphism (7.1.7) with $i = n = j$ we obtain the equation

$$(\varphi^{n,n}_{M,N} \otimes_A \mathrm{id}_{A[s,t]})((\gamma^{n}_{M})_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes s_\alpha) \otimes_{A[s,t]} (\gamma^{n}_{N})_{A[s,t]}(\sum_{\alpha\in I} y_\alpha \otimes_A t_\alpha))$$

$$= (\gamma^{n}_{M\otimes_A N})_{A[s,t]}((\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha) \otimes_{A[s,t]} (\sum_{\alpha\in I} y_\alpha \otimes_A t_\alpha)). \quad (7.2.5)$$

We can clearly find a homomorphism $M \otimes_A A[s,t] \to M \otimes_A B$ that maps the element $\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha$ to $z$ and the elements $\sum_{\alpha\in I} y_\alpha \otimes_A t_\alpha$ to $z'$. Consequently
→ the second part of the Lemma follows from (7.2.5).

**(7.3) Proposition.** *Let $\mu$ and $\nu$ be in $\mathbf{N}^{(I)}$ and let*

$$N_{\mu,\nu} = \{\xi_{\beta,\gamma} \in \mathbf{N}^{I\times I} : \sum_{\gamma\in I} \xi_{\beta,\gamma} = \mu_\beta \quad \text{for all} \quad \beta \in I \quad \text{and} \quad \sum_{\beta\in I} \xi_{\beta,\gamma} = \nu_\gamma \quad \text{for all} \quad \gamma \in I\}. \tag{7.3.1}$$

*Then we have that*

$$\varphi^{|\mu|,|\nu|}_{M,N}(\gamma^\mu_M(x) \otimes_A \gamma^\nu_N(y)) = \begin{cases} 0 & \text{when} \quad |\mu| \neq |\nu| \\ \sum_{\xi\in N_{\mu,\nu}} \gamma^\xi_{M\otimes_A N}(x \otimes_A y) & \text{when} \quad |\mu| = |\nu|. \end{cases} \tag{7.3.2}$$

*In particular we have for all $x \in M^I$ and $y \in N^I$ that*

$$\varphi^{n,n}_{M,N}(\gamma^n_M(x) \otimes_A \gamma^\nu_N(y)) = \star_{\beta\in I}\gamma^{\nu_\beta}_{M\otimes_A N}(x \otimes_A y_{\nu_\beta}). \tag{7.3.3}$$

*Proof.* When we compare the coefficient of the monomial $s^\mu t^\nu$ on each side of the
→ expression (7.2.1) we obtain that

$$\varphi_{M,N}(\gamma^\mu_{M\oplus N}(x) \star \gamma^\nu_{M\oplus N}(y)) = \sum_{\xi\in N_{\mu,\nu}} \gamma^\xi_{M\otimes_A N}(x \otimes_A y). \tag{7.3.4}$$

We note that the set $N_{\mu,\nu}$ is empty when $|\mu| \neq |\nu|$ because we have the equalities
→ $\sum_{\beta\in I} \mu_\beta = \sum_{\beta,\gamma\in I} \xi_{\beta,\gamma} = \sum_{\gamma\in I} \nu_\gamma$. Using the isomorphism (7.1.6) we see that
→ the first part of the Proposition follows from (7.3.4).

The second part of the Proposition follows because when $J = \{\alpha\}$ consists of one element, and $\mu_\alpha = n$, we have that $N_{\mu,\nu}$ consists of the element $(\xi_{\alpha,\beta})_{\beta\in K}$ with $\xi_{\alpha,\beta} = \nu_\beta$ for all $\beta \in I$.

**(7.4) Composition of maps.** Let $M$, $N$ and $P$ be $A$-modules and let

$$u : M \otimes_A N \to P$$

be an $A$-module homomorphism. For every $A$-algebra $B$ we obtain canonical $B$-module homomorphisms

$$(\Gamma^n(M) \otimes_A B) \otimes_B (\Gamma^n(N) \otimes_A B) \xrightarrow{\sim} \Gamma^n(M) \otimes_A \Gamma^n(N) \otimes_A B \xrightarrow{\varphi^{n,n}_{M,N}\otimes\mathrm{id}} \Gamma^n(M \otimes_A N) \otimes_A B \xrightarrow{\Gamma^n(u)\otimes_A\mathrm{id}} \Gamma^n(P) \otimes_A B. \tag{7.4.1}$$

**(7.5) Lemma.** *Let $z \in M\otimes_A B$ and $z' \in N\otimes_A B$. Then the image of $(\gamma^n_M)_B(z)\otimes_B (\gamma^n_N)_B(z')$ by the homomorphism* (7.4.1) *is equal to* $(\Gamma^n(u)\otimes_A \mathrm{id})\gamma^n_{M\otimes_A N}(z\otimes_A z') = (\gamma^n_P)_B(z\otimes_A z')$.

*Proof.* The Lemma follows from and the functoriality of $\Gamma^n(u)$ in $u$.

**(7.6) Divided powers for algebras.** A not necessarily commutative ring !$E$! with a fixed homomorphism of rings $\varphi : A \to E$ such that $\varphi(A)$ is in the center of $E$ is called a *not necessarily commutative* $A$*-algebra* . A homomorphism $\chi : E \to F$ between not necessarily commutative $A$-algebras is a ring homomorphism such that $\psi = \chi\varphi$ where $\psi : A \to F$ defines the algebra structure on $F$.

Let !$E$! be a not necessarily commutative $A$-algebra, and let !$G$! be a left $A$-module. The homomorphism!! $u_G : E\otimes_A G \to G$ that defines the module structure gives, as in (7.4) an $A$-module homomorphism!!

$$v_G : \Gamma^n(E)\otimes_A \Gamma^n(G) \to \Gamma^n(G) \qquad (7.6.1.)$$

We shall show that $v_E$ defines a product on $\Gamma^n_A(E)$ that makes $\Gamma^n_A(E)$ into a not necessarily commutative $A$-algebra, and that the homomorphism $v_G$ makes $\Gamma^n(G)$ into a $\Gamma^n(E)$ module. First we observe that it follows from Proposition (7.3) that

$$v_G(\gamma^\mu_E(x)\otimes_A \gamma^\nu_G(y)) = \sum_{\xi\in N_{\mu,\nu}} \gamma^\xi_G(xy) \qquad (7.6.2)$$

for all $x$ in $E^{(I)}$ and $y$ in $G^{(I)}$, and all $\mu$ and $\nu$ in $\mathbf{N}^{(I)}$ with $|\mu| = |\nu| = n$, where we write !$\star^\xi_G(xy) = \prod_{\alpha,\beta\in I}\gamma_G^{\xi_{\alpha,\beta}}(x_\alpha y_\beta)$!. It follows from (7.6.2) that the *multiplication* of $E$ defined by $v_E$ is associative, and that it is commutative when $E$ is commutative. Moreover it follows from (7.3.3) that

$$v_G(\gamma^n_E(x)\otimes_A \gamma^\nu_E(y)) = \star_{\beta\in I}\gamma_E^{\nu_\beta}(xy_\beta)$$

for all $x$ in $E$ and $y$ in $E^I$ and $n \in \mathbf{N}$. In particular we have that $\gamma^n_E(1)$ is a unit for the multiplication, and it is clear that the homomorphism

$$A \to \Gamma^n_A(E)$$

that maps $f$ to $f\gamma^n_E(1)$ gives $\Gamma^n_A(E)$ a structure of an $A$-algebra. Similarly it follows from (7.6.2) and the $A$-linearity of $v_G$ that $\Gamma^n_A(G)$ becomes a $\Gamma^n_A(E)$ module under the multiplication map $v_G$.

**(7.7) Functoriality and the algebra structure.** Let $E$ be a not necessarily commutative $A$-algebra and let $G$ and $H$ be left $E$-modules. It follows from the definition of the $\Gamma_A^n(E)$-module structure of $\Gamma_A^n(G)$ and $\Gamma_A^n(H)$ given in (7.6) that for every $E$-module homomorphism $u : G \to H$ the resulting map

$$\Gamma_A^n(u) : \Gamma_A^n(G) \to \Gamma_A^n(H)$$

is a $\Gamma_A^n(E)$-module homomorphism.

Let $\varphi : E \to F$ be a homomorphism of not necessarily commutative $A$-algebras. It follows from the definiton of the product on $\Gamma_A^n(E)$ and $\Gamma_A^n(F)$ of (7.6) that the homomorphism

$$\Gamma_A^n(\varphi) : \Gamma_A^n(E) \to \Gamma_A^n(F)$$

is a homomorphism of $A$-algebras. For every $A$-algebra $B$ we obtain that the homomorphism

$$\Gamma_A^n(E) \otimes_A B \to \Gamma_B^n(E \otimes_A B)$$

of (3.11.1) is a $B$-algebra homomorphism.

**(7.8) Definition.** A map $\varphi : E \to F$ between two not necessarily commutative $A$-algebras $E$ and $F$ is called *multiplicative* if $\varphi(1) = 1$ and if $\varphi(xx') = \varphi(x)\varphi(x')$ for all $x, x'$ in $E$. We say that a polynomial law $U$ from $E$ to $F$ is *multiplicative* if the map

$$U_B : E \otimes_A B \to F \otimes_A B$$

is multiplicative for all $A$-algebras $B$.

**(7.9) Example.** Let $E$ be a not necessarily commutative $A$-algebra. Then the polynomial law $\gamma^n$ from $E$ to $\Gamma^n(E)$ is a homogeneous multiplicative law of degree $n$ for $n = 0, 1, \dots$. In fact, this is an immediate consequence of Lemma (7.5).

**(7.10) Proposition.** *Let $E$ and $F$ be not necessarily commutative $A$-algebras, and let $U$ be a polynomial law from $E$ to $F$. With the notation of* (6.6.1) *we write*

$$U_{A[t]}(\sum_{\alpha \in I} x_\alpha \otimes_A t_\alpha) = \sum_{\nu \in \mathbf{N}^{(I)}} z_\nu(x) \otimes_A t^\nu$$

*with $z_\nu(x) \in F$. Then $U$ is multiplicative if and only if*

$$z_\mu(x) z_\nu(y) = \sum_{\xi \in N_{\mu,\nu}} z_\xi(xy) \qquad (7.10.1)$$

*for all $x, y$ in $E^{(I)}$, where $N_{\mu,\nu}$ is defined in Proposition* (7.3.1).

*Proof.* With the same notation as in (7.1) we have the equations

$$\begin{aligned} &U_{A[s,t]}(\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha) U_{A[s,t]}(\sum_{\alpha\in I} y_\alpha \otimes_A t_\alpha) \\ &\quad = (\sum_{\mu\in\mathbf{N}^{(I)}} z_\mu(x)\otimes_A s^\mu)(\sum_{\nu\in\mathbf{N}^{(I)}} z_\nu(x)\otimes_A t^\nu) = \sum_{\mu,\nu\in\mathbf{N}^{(I)}} z_\mu(x)z_\nu(y)\otimes_A s^\mu t^\nu \end{aligned} \tag{7.10.2}$$

and from (6.6.2) we have that

$$\begin{aligned} &U_{A[s,t]}((\sum_{\alpha\in I} x_\alpha \otimes_A s_\alpha)(\sum_{\alpha\in I} y_\alpha \otimes_A t_\alpha)) \\ &\quad = U_{A[s,t]}(\sum_{\alpha,\beta\in I} x_\alpha y_\beta \otimes_A s_\alpha t_\beta) = \sum_{\xi\in\mathbf{F}^{(I)}} z_\xi(x_\alpha y_\beta)\otimes_A (st)^\xi. \end{aligned} \tag{7.10.3}$$

Comparing the coefficients of $s^\mu t^\nu$ on the right hand sides of the equations (7.10.2) and (7.10.3) we obtain the equation (7.10.2).

**(7.11) Theorem.** *Let $E$ and $F$ be not necessarily commutative $A$-algebras. The bijection*

$$\mathrm{Hom}_A(\Gamma^n(E), F) \to \mathcal{P}^n(E,F)$$

*of* (6.8.1) *induces a bijection between $A$-algebra homomorphisms $\Gamma^n(E)\to F$ and homogeneous multiplicative polynomial laws of degree $n$ from $E$ to $F$.*

*Proof.* Let $\varphi : \Gamma^n(E)\to F$ be an $A$-algebra homomorphism. We saw in Example (7.9) that the corresponding polynomial law $\varphi\gamma^n$ from $E$ to $F$, that is given by $(\varphi\gamma^n)_B = (\varphi\otimes_A \mathrm{id})(\gamma^n_M)_B$ for all $A$-algebras $B$, is multiplicative.

Conversely, assume that $U$ is a homogeneous multiplicative polynomial law of degree $n$ from $E$ to $F$, and let $\varphi : \Gamma^n(E)\to F$ be the corresponding $A$-modules homomorphism such that $U=\varphi\gamma^n$. It follows from Example (7.9) and Proposition (7.10) that for all $x,y$ in $E^{(I)}$ and all $\mu$ and $\nu$ in $\mathbf{N}^{(I)}$ with $|\mu|=|\nu|=n$, we have that $\gamma^\mu_E(x)\star\gamma^\nu_E(y) = \sum_{\xi\in N_{\mu,\nu}}\gamma^\xi_E(xy)$. Consequently it follows from Proposition (6.10) that

$$\varphi(\gamma^\mu_E(x)\star\gamma^\nu_E(y)) = \sum_{\xi\in N_{\mu,\nu}} \varphi(\gamma^\xi_E(xy)) = \sum_{\xi\in N_{\mu,\nu}} z_\xi(xy), \tag{7.11.1}$$

where $U_{A[t]}(\sum_{\alpha\in I} x_\alpha\otimes_A t_\alpha) = \sum_{\nu\in\mathbf{N}^{(I)}} z_\nu(x)\otimes_A t^\nu$. Correspondingly it follows from Proposition (6.10) that

$$\varphi(\gamma^\mu_E(x))\varphi(\gamma^\nu_F(y)) = z_\mu(x)z_\nu(y)$$

for all $x$ and $y$ in $E^{(I)}$ and all $\mu$ and $\nu$ in $\mathbf{N}^{(I)}$. Since $U = \varphi\gamma^n$ is multiplicative by assumption it follows from (7.10.2) that

$$\varphi(\gamma_E^\mu(x) \star \gamma_E^\nu(y)) = \varphi(\gamma_E^\mu(x))\varphi(\gamma_F^\nu(y)).$$

Since the elements $\gamma_E^\mu(x)$ for all $x \in E^{(I)}$ and $\mu \in \mathbf{N}^{(I)}$ generate the $A$-module $\Gamma^n(E)$ we have that $\varphi$ is multiplicative, and consequently an $A$-algebra homomorphism.

**(7.12) Example.** let $E$ be a not necessarily commutative $A$-algebra. For every $A$-algebra $B$ we have that $T_B^n(E \otimes_A B)$ is a not necessarily commutative $B$-algebra under the multiplication

$$(y_1 \otimes_B y_2 \otimes_B \cdots \otimes_B y_n)(z_1 \otimes_B z_2 \otimes_B \cdots \otimes_B z_n) = y_1z_1 \otimes_B y_2z_2 \otimes_B \cdots \otimes_B y_nz_n$$

and

$$g(z_1 \otimes_B z_2 \otimes_B \cdots \otimes_B z_n) = gz_1 \otimes_B z_2 \otimes_B \cdots \otimes_B z_n = \cdots = z_1 \otimes_B \cdots \otimes_B z_{n-1} \otimes_B gz_n$$

for all elements $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n$ in $E \otimes_A B$ and all $g$ in $B$.

Similarly we obtain a $B$-algebra structure on $S_B^n(E \otimes_A B)$.

We have that the canonical homomorphism

$$u_B : T_B^n(E \otimes_A B) \to T_B^n(E) \otimes_A B$$

of section (6.12) is a $B$-algebra homomorphism, and the map

$$\beta_B : E \otimes_A B \to T_B^n(E \otimes_A B)$$

from (6.12) is multiplicative and satisfies formula (6.12.1). Consequently the polynomial law from $E$ to $T_B^n(E)$ given in section (6.12) also multiplicative, and the canonical homomorphism

$$\Gamma_A^n(E) \to T_A^n(E)$$

is an $A$-algebra homomorphism. Since each element of the group $\mathfrak{S}_n$ acts on $T_A^n(E)$ as an $A$-algebra homomorphism we have that $TS_A^n(E)$ is a sub $A$-algebra of $T_A^n(E)$, and the induced map

$$\Gamma_A^n(E) \to TS_A^n(E)$$

is a homomorphism of not necessarily commutative $A$-algebras.

Similarly we obtain an $A$-algebra homomorphism

$$\Gamma_A^n(E) \to S_A^n(E).$$

## 8. Norms.

**(8.1) Determinants.** Let $M$ and $N$ be finitely generated free $A$-modules. For every $A$-algebra $B$ we have a natural isomorphism of $B$-modules

$$\mathrm{Hom}_A(M,N)\otimes_A B \to \mathrm{Hom}_B(M\otimes_A B, N\otimes_A B). \tag{8.1.1}$$

We obtain for every positive integer $n$ a natural map

$$\mathrm{Hom}_B(M\otimes_A B, N\otimes_A B) \to \mathrm{Hom}_B(\wedge^n(M\otimes_A B), \wedge^n(N\otimes_A B)) \tag{8.1.2}$$

→ such that the image of $u$ is $\wedge^n u$. The maps (8.1.1) and (8.1.2) together with the isomorphisms $(\wedge^n M)\otimes_A B \to \wedge^n(M\otimes_A B)$ and $(\wedge^n N)\otimes_A B \to \wedge^n(N\otimes_A B)$ give a natural map

$$\mathrm{Hom}_A(M,N)\otimes_A B \to \mathrm{Hom}_B((\wedge^n)M\otimes_A B, (\wedge^n)N\otimes_A B).$$

→ The inverse of the map (8.1.1) for the $A$-modules $\wedge^n M$ and $\wedge^n N$ therefore gives a natural map

$$U_B : \mathrm{Hom}_A(M,N)\otimes_A B \to \mathrm{Hom}_A(\wedge^n M, \wedge^n N)\otimes_A B.$$

It is clear that for all $A$-algebra homomorphisms $\chi : B \to C$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(M,N)\otimes_A B & \xrightarrow{U_B} & \mathrm{Hom}_A(\wedge^n M, \wedge^n N)\otimes_A B \\ {\scriptstyle \mathrm{id}_{\mathrm{Hom}_A(M,N)}\otimes_A \chi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{Hom}_A(M,N)}\otimes_A \chi} \\ \mathrm{Hom}_A(M,N)\otimes_A C & \xrightarrow[U_C]{} & \mathrm{Hom}_A(\wedge^n M, \wedge^n N)\otimes_A C. \end{array}$$

Hence the maps $U_B$ for all $A$-algebras $B$ define a polynomial law $U$ on $A$-algebras from $\mathrm{Hom}_A(M,N)$ to $\mathrm{Hom}_A(\wedge^n M, \wedge^n N)$, and it is clear that $U$ is homogeneous of degree $n$.

**n** The polynomial law $U$ determines a unique $A$-module homomorphism!!

$$\wedge^n_{M,N} : \Gamma^n(\mathrm{Hom}_A(M,N)) \to \mathrm{Hom}_A(\wedge^n M, \wedge^n N))$$

such that

$$U = \wedge^n_{M,N}\gamma^n_{\mathrm{Hom}_A(M,N)}$$

where $\wedge^n_{M,N}\gamma^n_{\mathrm{Hom}_A(M,N)}$ is defined by

$$(\wedge^n_{M,N}\gamma^n_{\mathrm{Hom}_A(M,N)})_B = (\wedge^n_{M,N}\otimes_A \mathrm{id}_B)(\gamma^n_{\mathrm{Hom}_A(M,N)})_B$$

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for all $A$-algebras $B$. In particular we have, for all $A$-module homomorphisms $u : M \to N$, that

$$\wedge^n_{M,N}(\gamma^n_{\mathrm{Hom}_A(M.N)})(u) = \wedge^n u.$$

**(8.2) Composite maps.** Let $L, M, N$ be $A$-modules. We have natural maps

$$\mathrm{Hom}_A(L, M) \otimes_A \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(L, N) \tag{8.2.1}$$

and

$$t : \mathrm{Hom}_A(\wedge^n L, \wedge^n M) \otimes_A \mathrm{Hom}_A(\wedge^n M, \wedge^n L) \to \mathrm{Hom}_A(\wedge^n L, \wedge^n N). \tag{8.2.2}$$

By (8.4) we obtain from (8.2.1) a homomorphism of $A$-modules

$$w : \Gamma^n_A(\mathrm{Hom}_A(L, M)) \otimes_A \Gamma^n_A(\mathrm{Hom}_A(M, N)) \to \Gamma^n_A(\mathrm{Hom}_A(L, N)). \tag{8.2.3}$$

**(8.3) Lemma.** *Let $L, M, N$ be free $A$-modules of finite rank. For every non-negative integer $n$ we have a commutative diagram*

$$\begin{array}{ccc} \Gamma^n_A(\mathrm{Hom}_A(L, M)) \otimes_A \Gamma^n_A(\mathrm{Hom}_A(M, N)) & \xrightarrow{w} & \Gamma^n_A(\mathrm{Hom}_A(L, N)) \\ {\scriptstyle \wedge^n_{L,M} \otimes_A \wedge^n_{M,N}} \downarrow & & \downarrow {\scriptstyle \wedge^n_{L,N}} \\ \mathrm{Hom}_A(\wedge^n L, \wedge^n M) \otimes_A \mathrm{Hom}_A(\wedge^n M, \wedge^n N) & \xrightarrow[t]{} & \mathrm{Hom}_A(\wedge^n L, \wedge^n N) \end{array} \tag{8.3.1}$$

*where $w$ is the homomorphism* (8.2.3) *and $t$ is the natural map* (8.2.2).

*Proof.* By extension of scalars to $A[t]$ it suffices to show that for all $A$-module homomorphisms $u : L \to M$ and $v : M \to N$ we have that the images of $\gamma^n_{\mathrm{Hom}_A(L,M)}(u) \otimes_A \gamma^n_{\mathrm{Hom}_A(M,N)}(v)$ by the clockwise and counter clock wise maps of diagram (8.3.1) are equal. However, we have that

$$\wedge^n_{L,N}(\gamma^n_{\mathrm{Hom}_A(L,M)}(u) \otimes_A \gamma^n_{\mathrm{Hom}_A(M,N)}(v)) = \wedge^n_{L,N}(\gamma^n_{\mathrm{Hom}_A(L,N)}(vu)) = \wedge^n(vu),$$

and

$$t(\wedge^n_{L,M} \otimes_A \wedge^n_{M,N})(\gamma^n_{\mathrm{Hom}_A(L,M)}(u) \otimes_A \gamma^n_{\mathrm{Hom}_A(M,N)}(v)) \\ = \wedge^n_{M,N}\gamma^n_{\mathrm{Hom}_A(M,N)}(v) \wedge^n_{L,M} \gamma^n_{\mathrm{Hom}_A(L,M)}(u) = \wedge^n v \wedge^n u.$$

The lemma consequently follows from the well known formula $\wedge^n(vu) = \wedge^n v \wedge^n u$.

n **(8.4) Notation.** For every $A$-module $M$ we write $!\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)!$
for the ring of $A$-endomorphisms of $M$. Assume that $M$ is a free $A$-module of
rank $n$. Then $\wedge^n M$ is a free $A$-module of rank 1 and $\mathrm{End}_A(\wedge^n M)$ is canonically
→ isomorphic to $A$. It follows from Lemma (8.3) with $L = M = N$ that the map

$$\wedge^n_{M,M} : \Gamma^n_A(\mathrm{End}_A(M)) \to \mathrm{End}_A(\wedge^n M)$$

is an $A$-algebra homomorphism. We consequently have a canonical $A$-algebra
n homomorphism!!

$$\wedge^n_M : \Gamma^n_A(\mathrm{End}_A(M)) \to A$$

n such that for all endomorphisms $u : M \to M$ we have that!!

$$\wedge^n_M \gamma^n_{\mathrm{End}_A(M)}(u) = \det_A(u : M).$$

n where $!!\det_A(u : M)$ is the *determinant* of $u$.

Let $B$ be an $A$-algebra and assume that $M$ is an $B$-module in such a way that
the $A$-module structure on $M$ is given via the $A$-algebra structure on $B$. We have
n a canonical $A$-algebra homomorphism!!

$$\varphi_M : B \to \mathrm{End}_A(M) \tag{8.3.1}$$

n that maps $g \in B$ to the endomorphism $!u_g : M \to M!$ given by $u_g(x) = gx$ for all
$x \in M$. By functoriality we have an $A$-algebra homomorphism $\Gamma^n_A(\varphi_M)\Gamma^n_A(B) \to$
n $\Gamma^n_A(\mathrm{End}_A(M))$. Hence we obtain a canonical $A$-algebra homomorphism!!

$$\mathrm{norm}_M = \wedge^n_M(\Gamma^n_A(\varphi_M)) : \Gamma^n_A(B) \to A$$

such that for all $g \in B$ we have that

$$\mathrm{norm}_M(\gamma^n_B(g)) = \wedge^n_M(\gamma^n_{\mathrm{End}_A(M)}(\varphi_M(g))) = \det_A(u_g : M).$$

**(8.5) Notation.** Let

$$0 \to M' \to M \to M'' \to 0 \tag{8.5.1}$$

be an exact sequence of free $A$-modules $M', M, M''$ of ranks $n', n, n''$ respectively.
n Moreover, let $!E!$ be the $A$-algebra of elements $u \in \mathrm{End}_A(M)$ such that $u(M') \subseteq$
$M'$, and let $!\iota! : E \to \mathrm{End}_A(M)$ be the inclusion map. Then we have an $A$-module
homomorphism

$$v : E \to \mathrm{End}_A(M') \oplus \mathrm{End}_A(M'')$$

that maps $u \in E$ to $!(u', u'')!$, where $u' : M' \to M'$ and $u'' : M'' \to M''$ are the natural maps induced by $u$. We saw in (3.19.1) that we have a canonical $A$-module homomorphism

$$w : \Gamma^n(\mathrm{End}_A(M') \oplus \mathrm{End}_A(M'')) \to \Gamma^{n'}(\mathrm{End}_A(M')) \otimes_A \Gamma^{n''}(\mathrm{End}_A(M''))$$

such that

$$w(\gamma^n(u', u'')) = \gamma^{n'}(u') \otimes_A \gamma^{n''}(u'').$$

From (8.5.2) and (8.5.3) we obtain an $A$-module homomorphism

$$\Gamma^n(E) \to \Gamma^{n'}(\mathrm{End}_A(M')) \otimes_A \Gamma^{n''}(\mathrm{End}_A(M'')).$$

**(8.6) Theorem.** *With the notation of* (8.5) *we have a commutative diagram of $A$-algebras*

$$\begin{array}{ccc} \Gamma^n(E) & \xrightarrow{w\Gamma^n(v)} & \Gamma^{n'}(\mathrm{End}_A(M')) \otimes_A \Gamma^{n''}(\mathrm{End}_A(M'')) \\ {\scriptstyle \Gamma^n(\iota)}\big\downarrow & & \big\downarrow {\scriptstyle \wedge^{n'}_{M'} \otimes_A \wedge^{n''}_{M''}} \\ \Gamma^n(\mathrm{End}_A(M)) & \xrightarrow[\wedge^n_M]{} & A. \end{array} \tag{8.6.1}$$

*Proof.* Extending the scalars to $A[t]$ it suffices to prove that for all $u \in E$ the images of $\gamma^n_E(u)$ in $A$ by the clockwise and anti clockwise maps of diagram (8.6.1) are equal. However we have that

$$\begin{aligned} &(\wedge^{n'}_{M'} \otimes_A \wedge^{n''}_{M''}) w\Gamma^n(v)(u) \\ &\qquad = \wedge^{n'}_{M'} \gamma^{n'}_{M'}(u') \otimes_A \wedge^{n''}_{M''} \gamma^{n''}_{M''}(u'') = \det_A(u' : M') \det_A(u'', M''), \end{aligned}$$

and that $\det_A \Gamma^n(\iota)(u) = \wedge^n_M \gamma^n_{\mathrm{End}_A(M)}(u) = \det_A(u : M)$. The theorem hence follows from the well known equality $\det_A(u', M') \det_A(u'', M'') = \det_A(u : M)$ that is easily proven by choosing a splitting of the exact sequence (8.5.1).

**(8.7) Remark.** Let $B$ be an $A$-algebra, and let $M$ and $N$ be $B$-modules. Then there is a canonical surjective $A$-module homomorphism

$$\otimes^m_A(M \otimes_B \otimes^n_B N) \to \wedge^{mn}_A(M \otimes_B N) \tag{8.7.1}$$

that maps $(\otimes_A)^m_{i=1}(x_i \otimes_B (\otimes_B)^n_{j=1} y_{ij})$ to $(\wedge_A)^m_{i=1}(\wedge_B)^n_{j=1}(x_i \otimes_B y_{ij})$ for all $x_i \in M$ and $y_{ij} \in N$ where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The homomorphism (8.7.1) factors via a canonical homomorphism of $A$-modules

$$! \otimes^m_A (M \otimes_B \wedge^n_B N) \to \wedge^{mn}_A(M \otimes_B N)! \tag{8.7.2}$$

→ that maps $(\otimes_A)_{i=1}^m(x_i \otimes_B (\wedge_B)_{j=1}^n y_{ij})$ to $(\wedge_A)_{i=1}^m(\wedge_B)_{j=1}^n(x_i \otimes_B y_{ij})$, and (8.7.2) factors via a canonical homomorphism of $A$-modules

$$v_N : \wedge_A^m(M \otimes_B \wedge^n N) \to \wedge_A^{mn}(M \otimes_B N)$$

that maps $(\wedge_A)_{i=1}^m(x_i \otimes_B (\wedge_B)_{j=1}^n y_{ij})$ to $(\wedge_A)_{i=1}^m(\wedge_B)_{j=1}^n(x_i \otimes_B y_{ij})$. It is clear that for all $B$-module homomorphisms $u : N \to P$ we have that

$$v_P(\wedge_A^m(\mathrm{id}_M \otimes_B \wedge_B^n u)) = \wedge_A^{mn}(\mathrm{id}_M \otimes_B u)v_N \tag{8.7.3}$$

Assume that $M$ is a free $A$-module of rank $m$ via the $A$-algebra structure on $B$, and that $N$ is a free $B$-module of rank $n$. We have that $M \otimes_B N$ is a free $A$-module of rank $mn$ because a $B$-module isomorphism $N \xrightarrow{\sim} !B^{\oplus n}!$ to the sum of $B$ with itself $n$ times gives $A$-module isomorphisms

$$M \otimes_B N \xrightarrow{\sim} M \otimes_B B^{\oplus n} \xrightarrow{\sim} (M \otimes_B B)^{\oplus n} = M^{\oplus n}.$$

Let $u \in \mathrm{End}_B(N)$.

n **(8.8) Notation.** We denote the determinant of $u$ by !!$\det_B(u : N)$, and the determinant of the $A$-module endomorphism $\mathrm{id}_M \otimes_B u$ on $M \otimes_B N$ we denote by
n $\det_A(\mathrm{id}_M \otimes_B u : M \otimes_B N)$. Finally we let !$\det_A(\det_B(u : N) : M)$! be the determinant of the $A$-module endomorphism of $M$ given by multiplication by $\det_B(u : N)$.

**(8.9) Lemma.** *Let $B$ be an $A$-algebra and let $M$ and $N$ be $B$-modules. Moreover, let $u : N \to N$ be a $B$-module homomorphism. Assume that $M$ is a free $A$-module of rank $m$ via the $A$-algebra structure on $B$, and that $N$ is a free $B$-module of rank $n$. Then*

$$\det_A(\mathrm{id}_M \otimes_B u : M \otimes_B N) = \det_A(\det_B(u : N) : M). \tag{8.9.1}$$

*Proof.* Under the assumptions of the lemma we have that $v_N$ is an isomorphism.
→ Via this isomorphism the identity (8.7.3) can be written on the form (8.9.1).

**(8.10) Notation.** Let $B$ be an $A$-algebra, and $M$ a $B$-module that is free as an $A$-module of rank $m$. Moreover let $C$ be a $B$-algebra, and $N$ a $C$-module that is free as a $B$-module of rank $n$. We denote by $B'$ the image of $B$ by the
→ natural homomorphism $\varphi_M : B \to \mathrm{End}_A(M)$ of (8.3.1) and we let $\iota : B' \to \mathrm{End}_A(M)$ be the inclusion map. Then $B'$ is an $A$-algebra, the map $\iota$ is an $A$-algebra homomorphism, and $M$ is a $B'$-module via $\iota$. We shall consider all $B'$-algebras and $B'$-modules as $B$-algebras, respectively $B$-modules, via the surjection
n !$\varphi'_M : B \to B'$! induced by $\varphi_M$.

Let $N' = N \otimes_B B'$. Then $N'$ is a free $B'$-module of rank $n$. It follws from Example (8.9) that we have a multiplicative homogeneous polynomial law of degree $n$ on $B'$-algebras, from $\mathrm{End}_{B'}(N')$ to $\Gamma^n_{B'}(\mathrm{End}_{B'}(N'))$, and a multiplicative homogeneous polynomial law of degree $m$ on $A$-algebras from $\Gamma^n_B(\mathrm{End}_{B'}(N'))$ to $\Gamma^m_A(\Gamma^n_{B'}(\mathrm{End}_{B'}(N'))$. By composition of polynomial laws we obtain a multiplicative homogeneous polynomal law of degree $mn$ on $A$-algebras from $\mathrm{End}_{B'}(N')$ to $\Gamma^m_A(\Gamma^n_{B'}(\mathrm{End}_{B'}(N'))$. It follows from Theorem (8.11) that we have a canonical $A$-algebra homomorphism!!

$$\varphi : \Gamma^{mn}_A(\mathrm{End}_{B'}(N')) \to \Gamma^m_A\Gamma^n_{B'}(\mathrm{End}_{B'}(N')),$$

and it is clear that for all $u \in \mathrm{End}_{B'}(N')$ we have that

$$\varphi(\gamma^{mn}_{\mathrm{End}_{B'}(N')}(u)) = \gamma^m_{\Gamma^n_{B'}(\mathrm{End}_{B'}(N')}(\gamma^n_{\mathrm{End}_{B'}(N')}(u)).$$

Note that $M \otimes_{B'} N' \xrightarrow{\sim} M \otimes_B N$ is a free $A$-module of rank $mn$ and that we have a natural map of $B'$-algebras

$$\psi : \mathrm{End}_B(N') \to \mathrm{End}_A(M \otimes_{B'} N')$$

that maps $u'$ to $\mathrm{id}_M \otimes_{B'} u'$.

**(8.11) Proposition.** *With the notation and assumptions of* (8.10) *we have a commutative diagram*

$$\begin{array}{ccc} \Gamma^m_A\Gamma^n_{B'}(\mathrm{End}_{B'}(N')) & \xrightarrow{\Gamma^m_A(\wedge^n_{N'})} & \Gamma^m_A(B') \\ \varphi\big\uparrow & & \big\downarrow \det_M \\ \Gamma^{mn}_A(\mathrm{End}_{B'}(N')) & \xrightarrow[\wedge^{mn}_{M\otimes_{B'}N'}\Gamma^{mn}_A(\psi)]{} & A. \end{array} \tag{8.11.1}$$

*Proof.* By extension of scalars to $A[t]$ it suffices to show that for all $u' \in \mathrm{End}_{B'}(N')$ the images of the element $\gamma^{mn}_{\mathrm{End}_{B'}(N')}(u')$ in $A$ are equal by the clockwise map and the bottom map of diagram (8.11.1). We have that

$$\begin{aligned} &\mathrm{norm}_M\, \Gamma^m_A(\wedge^n_{N'})\varphi(\gamma^{mn}_{\mathrm{End}_{B'}(N')}(u')) \\ &\qquad = \mathrm{norm}_M\, \Gamma^m_A(\wedge^n_{N'})\gamma^m_{\Gamma^n_{B'}(\mathrm{End}_{B'}(N')}\gamma^n_{\mathrm{End}_{B'}(N')}(u')) \\ &\qquad = \mathrm{norm}_M\, \gamma^m_{B'}(\wedge^n_{N'}\gamma^n_{\mathrm{End}_{B'}(N')}(u')) = \mathrm{norm}_M\, \gamma^m_{B'}(\det_{B'}(u' : N')) \\ &\qquad = \det_A(u_{\det(u':N')} : M), \end{aligned}$$

where $u_{\det_{B'}(u':N')}$ is the multiplication of $\det_{B'}(u' : N')$ on $M$. On the other hand we have that

$$\begin{aligned}\wedge^{mn}_{M\otimes_B N}\Gamma^{mn}_A(\psi)\gamma^{mn}_{\mathrm{End}_{B'}(N')}(u') &= \wedge^{mn}_{M\otimes_{B'} N'}\Gamma^{mn}_A(\psi)\gamma^{mn}_{\mathrm{End}_{B'}(N')}(u') \\ &= \wedge^{mn}_{M\otimes_{B'} N'}\gamma^{mn}_{\mathrm{End}_A(M\otimes_{B'} N')}(\mathrm{id}_M \otimes_{B'} u') = \det_A((\mathrm{id}_M \otimes_{B'} u') : M \otimes_{B'} N').\end{aligned}$$

→ The commutativity of diagram (8.11.1) consequently follows from Lemma (8.9).

**(8.12) Corollary.** *Let $B$ be an $A$-algebra, and let $M$ be a $B$-module that is free as an $A$-module of rank $m$. Moreover let $C$ be a $B$-algebra, and let $N$ be a $C$-module that is free as a $B$-module of rank $n$. We consider $M \otimes_B N$ as a $C$-module via the action of $C$ on $N$. Then the diagram*

$$\begin{array}{ccc} \Gamma^m_A\Gamma^n_B(C) & \xrightarrow{\Gamma^m_A(\mathrm{norm}_N)} & \Gamma^m_A(B) \\ \varphi\big\uparrow & & \big\downarrow \mathrm{norm}_M \\ \Gamma^{mn}_A(C) & \xrightarrow[\mathrm{norm}_{M\otimes_B N}]{} & A \end{array} \tag{8.12.1}$$

*is commutative*

*Proof.* Since $N' = N \otimes_B B'$ there is a natural $B$-algebra homomorphism $!\chi! :$
→ $\mathrm{End}_B(N) \to \mathrm{End}_{B'}(N')$ and we have from (8.3.1) a natural $B$-algebra homomor-
→ phism $\varphi_C : C \to \mathrm{End}_B(N)$. It follows from (3.9.1) that we have a composite homomorphism $v$ given by

$$\Gamma^m_B(C) \xrightarrow{\Gamma^m_B(\varphi_C)} \Gamma^m_B(\mathrm{End}_B(N)) \xrightarrow{\Gamma^m_B(\chi)} \Gamma^m_B(\mathrm{End}_{B'}(N')) \xrightarrow{\Gamma^m_{\varphi'_M}(\mathrm{id})} \Gamma^m_{B'}(\mathrm{End}_{B'}(N')).$$

It is clear that the diagrams

$$\begin{array}{ccc} \Gamma^m_A\Gamma^n_B(C) & \xrightarrow{\Gamma^m_A\Gamma^n_{\varphi'_M}(v)} & \Gamma^m_A\Gamma^n_{B'}(\mathrm{End}_{B'}(N')) \\ \varphi\big\uparrow & & \big\uparrow\varphi \\ \Gamma^{mn}_A(C) & \xrightarrow[\Gamma^{mn}_A(v)]{} & \Gamma^{mn}_A(\mathrm{End}_{B'}(N')), \end{array} \tag{8.12.2}$$

and

$$\begin{array}{ccc} \Gamma^n_B(C) & \xrightarrow{\mathrm{norm}_N} & B \\ v\big\downarrow & & \big\downarrow\varphi'_M \\ \Gamma^n_{B'}(\mathrm{End}_{B'}(N')) & \xrightarrow[\wedge^n_{N'}]{} & B', \end{array} \tag{8.12.3}$$

→ are commutative. From diagram (8.12.3) we obtain the commutative diagram

$$\begin{array}{ccc} \Gamma_A^m \Gamma_B^n(C) & \xrightarrow{\Gamma_A^m(\mathrm{norm}_N)} & \Gamma_A^m B \\ {\scriptstyle \Gamma_A^m(\upsilon)}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_A^m(\varphi'_M)} \\ \Gamma_A^m \Gamma_{B'}^n(\mathrm{End}_{B'}(N')) & \xrightarrow[\Gamma_A^m(\wedge_{N'}^n)]{} & \Gamma_A^m(B'). \end{array} \tag{8.12.4}$$

→ The commutativity of diagram (8.12.1) follows from the commutativity of the
→ diagrams (8.11.1), (8.12.2) and (8.12.4).

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