KTH Teknikvetenskap

Lectures on Financial Mathematics

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Preface

My main goal with this text is to present the mathematical modelling of financial markets in a mathematically rigorous way, yet avoiding mathematical technicalities that tends to deter people from trying to access it.

Trade takes place in discrete time; the continuous case is considered as the limiting case when the length of the time intervals tend to zero. However, the dynamics of asset values are modelled in continuous time as in the usual Black-Scholes model. This avoids some mathematical technicalities that seem irrelevant to the reality we are modelling.

The text focuses on the price dynamics of *forward* (or futures) prices rather than spot prices, which is more traditional. The rationale for this is that forward and futures prices for *any* good—also consumption goods—exhibit a Martingale property on an arbitrage free market, whereas this is not true in general for spot prices (other than for pure investment assets.) It also simplifies computations when derivatives on investment assets that pay dividends are studied.

Another departure from more traditional texts is that I avoid the notion of “objective” probabilities or probability distributions. I think they are suspect constructs in this context. We can in a meaningful way assign probabilities to outcomes of experiments that can be repeated under similar circumstances, or where there are strong symmetries between possible outcomes. But it is unclear to me what the “objective” probability distribution for the price of crude oil, say, at some future point in time would be. In fact, I don’t think this is a well defined concept.

The text presents the *mathematical modelling* of financial markets. In order to get familiar with the workings of these markets in practice, the reader is encouraged to supplement this text with some text on financial economics. A good such text book is John C. Hull’s: *Options, Futures, & Other Derivatives* (Prentice Hall,) which I will refer to in some places.

2/4–2010

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I: Introduction to Present-, Forward- and Futures Prices

Assume that we want to buy a quantity of coffee beans with delivery in nine months. However, we are concerned about what the (spot) price of coffee beans might be then, so we draw up a contract where we agree on the price today. There are now at least three ways in which we can arrange the payment: 1) we pay now, in advance. We call this price the *present* price of coffee beans with delivery in nine months time, and denote it by P. Note that this is completely different from the spot price of coffee beans, i.e., the price of coffee for immediate delivery. 2) we pay when the coffee is delivered, i.e., in nine months time. This price is the *forward* price, which we denote by G. 3) we might enter a *futures* contract with delivery in nine months time. A futures contract works as follows:

Let us denote the days from today to delivery by the numbers $0, 1, \ldots$ $\ldots, T$, so that day 0 is today, and day T is the day of delivery. Each day n a futures price F_n of coffee beans with delivery day T is noted, and F_T equals the prevailing spot price of coffee beans day T. The futures price F_j is not known until day j; it will depend on how the coffee bean crop is doing, how the weather has been up to that day and the weather prospects up till day T, the expected demand for coffee, and so on. One can at any day enter a futures contract, and there is no charge for doing so. The long holder of the contract will each day j receive the amount $F_j - F_{j-1}$ (which may be negative, in which he has to pay the corresponding amount,) so if I enter a futures contract at day 0, I will day one receive $F_1 - F_0$, day two $F_2 - F_1$ and so on, and day T, the day of delivery, $F_T - F_{T-1}$. The total amount I receive is thus $F_T - F_0$. There is no actual delivery of coffee beans, but if I at day T buy the beans at the spot price F_T, I pay F_T, get my coffee beans and cash the amount $F_T - F_0$ from the futures contract. In total, I receive my beans, and pay F_0, and since F_0 is known already day zero, the futures contract works somewhat like a forward contract. The difference is that the value $F_T - F_0$ is paid out successively during the time up to delivery rather than at the time of delivery.

Here are timelines showing the cash flows for "pay now", forwards and futures contracts:

Pay now

day	0	1	2	3	$\cdots$	$T-1$	T
cash flow	$-P$	0	0	0	$\cdots$	0	X

Forward contract

day	0	1	2	3	$\cdots$	$T-1$	T
cash flow	0	0	0	0	$\cdots$	0	$X-G$

Futures contract

day	0	1	2	3	⋯	$T-1$	T
cash flow	0	F_1-F_0	F_2-F_1	F_3-F_2	⋯	$F_{T-1}-F_{T-2}$	$X-F_{T-1}$

The simplest of these three contracts is the one when we pay in advance, at least if the good that is delivered is non-pecuniary, since in that case the interest does not play a part. For futures contracts, the interest rate clearly plays a part, since the return of the contract is spread out over time.

We will derive some book-keeping relations between the present prices, forward prices and futures prices, but first we need some interest rate securities.

Zero Coupon Bonds

A *zero coupon bond* with maturity T and *face value* V is a contract where the long holder pays $Z_T V$ in some currency day 0 and receives V in the same currency day T. The ratio Z_T of the price today and the amount received at time T is the discount factor converting currency at time T to currency today. One can take both long and short positions on zero coupon bonds.

We need a notation for the currency, and we use a dollar sign \$, even though the currency may be Euro or any other currency.

Money Market Account

A *money market account* (MMA) is like a series of zero coupon bonds, maturing after only one day (or whatever periods we have in our futures contracts—it might be for example a week.) If I deposit the amount \$1 day 0, the balance of my account day 1 is $\$e^{r_1}$, where r_1 is the short interest rate from day 0 to day 1. The rate r_1 is known already day 0. The next day, day 2, the balance has grown to $\$e^{r_1+r_2}$, where r_2 is the short interest rate from day 1 to day 2; it is random as seen from day 0, and its outcome is determined day 1. The next day, day 3, the balance has grown to $\$e^{r_1+r_2+r_3}$, where r_3 is the short interest rate from day 2 to day 3; it is random as seen from day 0 and day 1, and its outcome occurs day 2, and so on. Day T the balance is thus $\$e^{r_1+\cdots+r_T}$ which is a *random variable.* In order to simplify the notation, we introduce the symbol $R(t,T) = r_{t+1} + \cdots + r_T$.

Relations between Present-, Forward- and Futures Prices

Let $P_0^{(T)}[X]$ be the present price today of a contract that delivers the random value X (which may take negative values) at time T. Likewise, let $G_0^{(T)}[X]$ denote the forward price today of the value X delivered at time T and $F_t^{(T)}[X]$ the futures price as of time t of the value X delivered at time T. We then have the following theorem:

Theorem

The following relations hold:

a) $P_0^{(T)}$, $G_0^{(T)}$ *and* $F_0^{(T)}$ *are linear functions, i.e., if* X *and* Y *are random payments made at time* T*, then for any constants* a *and* b
$P_0^{(T)}[aX + bY] = aP_0^{(T)}[X] + bP_0^{(T)}[Y]$,
and similarly for $G_0^{(T)}$ *and* $F_0^{(T)}$.

b) *For any deterministic (i.e., known today) value* V, $G_0^{(T)}[V] = V$, $F_0^{(T)}[V] = V$ *and* $P_0^{(T)}[V] = Z_T V$

c) $P_0^{(T)}[X] = Z_T G_0^{(T)}[X]$

d) $P_0^{(T)}[X e^{R(0,T)}] = F_0^{(T)}[X]$

e) $P_0^{(T)}[X] = F_0^{(T)}[X e^{-R(0,T)}]$

Proof

The proof relies on an assumption of the model: the *law of one price.* It means that there can not be two contracts that both yield the same payoff X at time T, but have different prices today. Indeed, if there were two such contracts, we would buy the cheaper and sell the more expensive, and make a profit today, and have no further cash flows in the future. But so would everyone else, and this is inconsistent with market equilibrium. In Ch. IV we will extend this model assumption somewhat.

To prove c), note that if we take a long position on a forward contract on X and at the same time a long position on a zero coupon bond with face value $G_0^{(T)}[X]$, then we have a portfolio which costs $Z_T G_0^{(T)}[X]$ today, and yields the income X at time T. By the law of one price, it hence must be that c) holds.

To prove d), consider the following strategy: Deposit F_0 on the money market account, and take e^{r_1} long positions on the futures contract on X for delivery at time T.

The next day the total balance is then $F_0 e^{r_1} + e^{r_1}(F_1 - F_0) = F_1 e^{r_1}$. Deposit this on the money market account, and increase the futures position to $e^{r_1+r_2}$ contracts.

The next day, day 2, the total balance is then $F_1 e^{r_1+r_2} + e^{r_1+r_2}(F_2 - F_1) = F_2 e^{r_1+r_2}$. Deposit this on the money market account, and increase the futures position to $e^{r_1+r_2+r_3}$ contracts.

And so on, up to day T when the total balance is $F_T e^{r_1+\cdots+r_T} = X e^{r_1+\cdots+r_T}$. In this way we have a strategy which is equivalent to a contract where we pay F_0 day zero, and receive the value $F_T e^{r_1+\cdots+r_T} = X e^{r_1+\cdots+r_T}$ day T. This proves d).

Since relation d) is true for any random variable X whose outcome is known day T, we may replace X by $Xe^{-R(0,T)}$ in that relation. This proves e).

It is now easy to prove b). The fact that $P_0^{(T)}[V] = Z_T V$ is simply the definition of Z_T. The relation $G_0^{(T)}[V] = V$ now follows from c) with $X = V$. In order to prove that $F_0^{(T)}[V] = V$, note that by the definition of money market account, the price needed to be paid day zero in order to receive $Ve^{R(0,T)}$ day T is V; hence $V = P_0^{(T)}[Ve^{R(0,T)}]$. The relation $F_0^{(T)}[V] = V$ now follows from d) with $X = V$.

Finally, to prove a), note that if we buy a contracts which cost $P_0^{(T)}[X]$ day zero and gives the payoff X day T, and b contracts that gives payoff Y, then we have a portfolio that costs $aP_0^{(T)}[X]+bP_0^{(T)}[Y]$ day zero and yields the payoff $aX + bY$ day T; hence $P_0^{(T)}[aX + bY] = aP_0^{(T)}[X] + bP_0^{(T)}[Y]$. The other two relations now follow immediately employing c) and d). This completes the proof.

Comparison of Forward- and Futures Prices

Assume first that the short interest rates r_i are deterministic, i.e., that their values are known already day zero. This means that $e^{R(0,T)}$ is a constant (non-random,) so

$$\$1 = P_0^{(T)}[\$e^{R(0,T)}] = P_0^{(T)}[\$1]e^{R(0,T)} = \$Z_T e^{R(0,T)},$$

hence

$$Z_T = e^{-R(0,T)}.$$

Therefore

$$\begin{aligned} Z_T G_0^{(T)}[X] &= P_0^{(T)}[X] = F_0^{(T)}[Xe^{-R(0,T)}] \\ &= F_0^{(T)}[X]e^{-R(0,T)} = F_0^{(T)}[X]Z_T, \end{aligned}$$

and hence

$$G_0^{(T)}[X] = F_0^{(T)}[X].$$

We write this down as a corollary:

Corollary

If interest rates are deterministic, the forward price and the futures price coincide: $G_0^{(T)}[X] = F_0^{(T)}[X]$

The equality of forward- and futures prices does not in general hold if interest rates are random, though. To see this, we show as an example that if $e^{R(0,T)}$ is random, then $F_0^{(T)}[\$e^{R(0,T)}] > G_0^{(T)}[\$e^{R(0,T)}]$.

Indeed, note that the function $y = \frac{1}{x}$ is convex for $x > 0$. This implies that its graph lies over its tangent. Let $y = \frac{1}{m} + k(x - m)$ be the tangent line through the point $(m, \frac{1}{m})$. Then $\frac{1}{x} \geq \frac{1}{m} + k(x - m)$ with equality only for $x = m$ (we consider only positive values of x.) Now use this with $x = e^{-R(0,T)}$ and $m = Z_T$. We then have

$$e^{R(0,T)} \geq Z_T^{-1} + k(e^{-R(0,T)} - Z_T)$$

where the equality holds only for one particular value of $R(0,T)$. In the absence of arbitrage (we will come back to this in Ch. IV,) the futures price of the value of the left hand side is greater than the futures price of the value of the right hand side, i.e.,

$$F_0^{(T)}[\$e^{R(0,T)}] > \$Z_T^{-1} + k(F_0^{(T)}[\$e^{-R(0,T)}] - \$Z_T)$$

But $F_0^{(T)}[\$e^{-R(0,T)}] = P^{(T)}[\$1] = \$Z_T$, so the parenthesis following k is equal to zero, hence

$$F_0^{(T)}[\$e^{R(0,T)}] > \$Z_T^{-1}$$

On the other hand,

$$Z_T G_0^{(T)}[\$e^{R(0,T)}] = P_0^{(T)}[\$e^{R(0,T)}] = \$1$$

so $G_0^{(T)}[\$e^{R(0,T)}] = \$Z_T^{-1}$, and it follows that

$$F_0^{(T)}[\$e^{R(0,T)}] > G_0^{(T)}[\$e^{R(0,T)}].$$

In general, if X is positively correlated with the interest rate, then the futures price tends to be higher than the forward price, and vice versa.

Spot Prices, Storage Cost and Dividends

Consider a forward contract on some asset to be delivered at a future time T. We have talked about the *forward* price, i.e., the price paid at the time of delivery for the contract, and the *present* price, by which we mean the price paid for the contract today, but where the underlying asset is still delivered at T. This should not be confused with the spot price today of the underlying asset. The present price should equal the spot price under the condition that the asset is an *investment asset* , and that there are no storage costs or dividends or other benefits of holding the asset. As an example: consider a forward contract on a share of a stock to be delivered at time T. Let r be the interest rate (so that $Z_T = e^{-rT}$) and S_0 the spot price of the share. Since \$1 today is equivalent to $\$e^{rT}$ at time T, the forward price should then be $G_0^{(T)} = S_0 e^{rT}$. But only if there is no dividend of the share between now and T, for if there is, then one could make an arbitrage by buying the share today, borrow for the cost and take a short position on a forward contract. There is then no net payment today, and none at T (deliver the share, collect the delivery price $G_0^{(T)}$ of the forward and pay the $S_0 e^{rT} = G_0^{(T)}$ for the loan.) But it would give the trader the dividend of the share for free, for this dividend goes to the holder of the share, not the holder of the forward. Likewise, the holder of the share might have the possibility of taking part in the annual meeting of the company, so there might be a *convenience yield.*

Comments

You can read about forward and futures contracts John Hull's book *"Options, Futures, & other Derivatives*[1] *".* He describes in detail how futures contracts work, and why they are specified in the somewhat peculiar way they are.

The *mathematical modelling* of a futures contract is a slight simplification of the real contract. We disregard the maintenance account, thus avoiding any problems with interest on the balance. Furthermore, we assume that the delivery date is defined as a certain day, not a whole month. We also disregard the issues on accounting and tax.

We use "continuous compounding" of interest rates (use of the exponential function.) If you feel uncomfortable with this, you may want to read relevant chapters in Hull's book. We will use continuous compounding unless otherwise explicitly stated, since it is the most convenient way to handle interest. It is of course easy to convert between continuous compounded interest and any other compounding.

[1] Prentice Hall

It is important not to confuse the *present price* with the *spot price* of the same type of good. The *spot price* is the price of the good for *immediate* delivery.

It is also extremely important to distinguish between *constants* (values that are currently known) and *random variables.* For instance, assume that $Z_T = e^{-rT}$ (where r hence is a number.) Then it is true that $P_0^{(T)}[X] = e^{-rT} G_0^{(T)}[X]$ (Theorem c,) however, the relation $P_0^{(T)}[X] = e^{-R(0,T)} G_0^{(T)}[X]$ **is invalid and nonsense!** Indeed, $R(0,T)$ is a random variable; its outcome is not known until time T–1, whereas $G_0^{(T)}$ and $P_0^{(T)}$ are known prices today.

II: Forwards, FRA:s and Swaps

Forward Prices

In many cases the theorem of Ch. I can be used to calculate forward prices. As we will see later, in order to calculate option prices, it is essential to first calculate the forward price of the underlying asset.

Example 1.

Consider a share of a stock which costs S_0 today, and which gives a known dividend amount d in t years, and whose (random) spot price at time $T > t$ is S_T. Assume that there are no other dividends or other convenience yield during the time up to T. What is the forward price G on this stock for delivery at time T?

Assume that we buy the stock today, and sell it at time T. The cash flow is

day	0	t	T
cash flow	$-S_0$	d	S_T

The present value of the dividend is $Z_t d$ and the present value of the income S_T at time T is $Z_t G$. Hence we have the relation

$$S_0 = Z_t\, d + Z_T\, G$$

from which we can solve for G

Example 2.

Consider a share of a stock which costs S_0 now, and which gives a known dividend yield $d\, S_t$ i t years, where S_t is the spot price immediately before the dividend is paid out. Let the (random) spot price at time $T > t$ be S_T. Assume that there are no other dividends or other convenience yield during the time up to T. What is the forward price G on this stock for delivery at time T?

Consider the strategy of buying the stock now, and sell it at time t immediately before the dividend is paid out.

day	0	t
cash flow	$-S_0$	S_t

With the notation of Ch. I, we have the relation

$$P_0^{(t)}[S_t] = S_0 \tag{1}$$

Consider now the strategy of buying the stock now, cash the dividend at time t, and eventually sell the stock at time T.

day	0	t	T
cash flow	$-S_0$	dS_t	S_T

The present value of the dividend is $d\,P^{(t)}[S_t]$ and the present value of the income S_T at time T is $Z_T\,G$. Hence we have the relation

$$S_0 = d\,P_0^{(t)}[S_t] + Z_T\,G$$

If we combine with (1) we get

$$(1-d)S_0 = Z_T\,G$$

from which we can solve for G

Example 3.

With the same setting as in example 2, assume that there are dividend yield payments at several points in time $t_1 < \ldots < t_n$, where $t_n < T$, each time with the amount $d\,S_{t_j}$. As in the above example, we can buy the stock today and sell it just before the first dividend is paid out, so the relation

$$P_0^{(t_1)}[S_{t_1}] = S_0 \tag{2}$$

holds. For any $k = 2, 3, \ldots, n$ we can buy the stock at time t_{k-1} immediately before the payment of the dividend, collect the dividend, and sell the stock immediately before the dividend is paid out at time t_k.

day	0	t_{k-1}	t_k
cash flow	0	$-S_{t_{k-1}} + dS_{t_{k-1}}$	S_{t_k}

The price of this strategy today is zero, so we have

$$\begin{aligned} &0 = -P_0^{(t_{k-1})}[S_{t_{k-1}}] + d\,P_0^{(t_{k-1})}[S_{t_{k-1}}] + P_0^{(t_k)}[S_{t_k}], \quad \text{i.e.,} \\ &P_0^{(t_k)}[S_{t_k}] = (1-d)P_0^{(t_{k-1})}[S_{t_{k-1}}] \end{aligned}$$

and repeated use of this relation and $P_0^{(T)}[S_T] = (1-d)P_0^{(t_n)}[S_{t_n}]$ gives

$$P_0^{(T)}[S_T] = (1-d)^n P_0^{(t_1)}[S_{t_1}] = (1-d)^n S_0$$

where we have used (2) to obtain the last equality. Hence, by the theorem of Ch. I, we have the relation

$$Z_T\,G = (1-d)^n S_0 \tag{3}$$

Example 4.

We now consider the setting in example 3, but with a continuous dividend yield ρ, i.e., for any small interval in time $(t, t+\delta t)$ we get the dividend $\rho S_t\,\delta t$. If we divide the time interval $(0,T)$ into a large number n of intervals of length $\delta t = T/n$, we see from (3) that

$$Z_T\,G = (1-\rho\,\delta t)^n S_0$$

and when $n \to \infty$ we get

$$Z_T\,G = e^{-\rho T} S_0$$

Derivation of the limit: $\ln\big((1-\rho\,\delta t)^n\big) = n\,\ln(1-\rho\,\delta t) = n\big(-\rho\,\delta t + \mathcal{O}(\delta t^2)\big) = n\big(-\rho T/n + \mathcal{O}(1/n^2)\big) \to -\rho T$. Taking exponential gives the limit.

Example 5.

Assume we want to buy a foreign currency in t years time at an exchange rate, the *forward rate,* agreed upon today. Assume that the interest on the foreign currency is ρ and the domestic rate is r per year. Let X_0 be the exchange rate now (one unit of foreign currency costs X_0 in domestic currency,) and let X_t be the (random) exchange rate as of time t. Let G be the forward exchange rate.

Consider now the strategy: buy one unit of foreign currency today, buy foreign zero coupon bonds for the amount, so that we have $e^{\rho t}$ worth of bonds in foreign currency at time t when we sell the bonds.

day	0	t
cash flow	$-X_0$	$e^{\rho t} X_t$

Since the exchange rate at that time is X_t, we get $X_t\,e^{\rho t}$ in domestic currency. Since the price we have paid today is X_0 we have the relation

$$X_0 = P_0^{(t)}\big[X_t\,e^{\rho t}\big] = e^{-rt}\,G\,e^{\rho t} = G\,e^{(\rho-r)t}$$

i.e.,

$$G = X_0 e^{(r-\rho)t}$$

Forward Rate Agreements

A *forward rate agreement* is a forward contract where the parties agree that a certain interest rate will be applied to a certain principal during a future time period. Let us say that one party is to borrow an amount L at time t and later pay back the amount $Le^{f(T-t)}$ at time $T > t$. The cash flow for this party is thus L at time t and $-Le^{f(T-t)}$ at time T. Since this contract costs nothing now, we have the relation

$$0 = Z_t L - Z_T L e^{f(T-t)}.$$

From this we can solve for f. The interest rate f is the *forward rate* from t to T.

Plain Vanilla Interest Rate Swap

The simplest form of an interest swap is where one party, say A, pays party B:

the floating interest on a principal L_1 from time t_0 to t_1 at time t_1
the floating interest on a principal L_2 from time t_1 to t_2 at time t_2
the floating interest on a principal L_3 from time t_2 to t_3 at time t_3
...
the floating interest on a principal L_n from time t_{n-1} to t_n at time t_n.

The floating rate between t_j and t_k is the zero coupon rate that prevails between these two points in time. The amount that A pays at time t_k is thus $L_k \cdot (1/Z(t_{k-1}, t_k) - 1)$, where $Z(t_{k-1}, t_k)$ of course is the price of the zero coupon bond at time t_{k-1} that matures at t_k. Note that this price is unknown today but known at time t_{k-1}. The total amount that B will receive, and A will pay is thus random.

On the other hand, party B pays A a fixed amount c at each of the times $t_1, \ldots, t_n$. The question is what c ought to be in order to make this deal "fair".

The notation with continuous compounding is here bit inconvenient. Let us introduce the *one period floating rate* $\hat{r}_j$: the interest from time t_{j-1} to time t_j, i.e., if I deposit an amount a on a bank account at time t_{j-1} the balance at time t_j is $a + a\hat{r}_j$. This is the same as to say that a zero coupon bond issued at time t_{j-1} with maturity at t_j is $1/(1 + \hat{r}_j)$. Note that $\hat{r}_j$ is *random* whose value becomes known at time t_{j-1}. The cash flow that A pays to B is then

day	t_0	t_1	t_2	t_3	$\cdots$	t_n
cash flow	0	$\hat{r}_1 L_1$	$\hat{r}_2 L_2$	$\hat{r}_3 L_3$	$\cdots$	$\hat{r}_n L_n$

In order to calculate the present value of this cash flow, we first determine the present value $P_0^{(t_k)}(\hat{r}_k L_k)$.

Consider the strategy: buy L_k worth of zero coupon bonds at time t_{k-1} with maturity at t_k and face value $(1 + \hat{r}_k)L_k$. This costs nothing today, so the cash flow is

day	t_0	$\cdots$	t_{k-1}	t_k
cash flow	0	$\cdots$	$-L_k$	$(1 + \hat{r}_k)L_k$

hence $0 = -P_0^{(t_{k-1})}[L_k] + P_0^{(t_k)}[L_k] + P_0^{(t_k)}[\hat{r}_k L_k] = -Z_{t_{k-1}}L_k + Z_{t_k}L_k + P_0^{(t_k)}[\hat{r}_k L_k]$, so we get

$$P_0^{(t_k)}[\hat{r}_k L_k] = (Z_{t_{k-1}} - Z_{t_k})L_k.$$

It is now easy to calculate the present value of the cash flow from A to B: it is

$$P_{AB} = \sum_1^n (Z_{t_{k-1}} - Z_{t_k})L_k$$

The present value of B:s payments to A is, on the other hand,

$$P_{BA} = c\sum_1^n Z_{t_k}$$

so we can calculate the fair value of c by solving the equation we get by setting $P_{BA} = P_{AB}$.

Exercises and Examples

Interest rates always refer to continuous compounding. Answers are given in parenthesis; solutions to some problems are given in the next section

1. A share is valued at present at 80 dollars. In nine months it will give a dividend of 3 dollars. Determine the forward price for delivery in one year given that the rate of interest is 5% a year. (81.06 dollars)

2. A share is valued at present at 80 dollars. In nine months it will give a dividend of 4% of its value at that time. Determine the forward price for delivery in one year given that the rate of interest is 5% a year. (80.74 dollars)

3. The current forward price of a share to be delivered in one year is 110 dollars. In four months the share will give a dividend of 2 dollars and in ten months will give a dividend of 2% of its value at that time. Determine the current spot price of the share given that the rate of interest is 6% a year. (107.67 dollars)

4. The exchange rate of US dollars is today 8.50 SEK per dollar. The forward price of a dollar to be delivered in six months is 8.40 SEK. If the Swedish six month interest rate is 4% a year, determine the American six month interest rate. (6.367%)

5. The forward price of a US dollar the first of August with delivery at the end of December is 0.94630 EUR. The forward price of a dollar to be delivered at the end of June next year is 0.95152 EUR. Assuming a flat term structure for both currencies and that the Euro interest rate is 4% a year—what is the American rate of interest? (2.90%)

6. Determine the forward price of a bond to be delivered in two years. The bond pays out 2 EUR every 6-months during $4\frac{1}{2}$ years (starting in six months), and 102 EUR after five years. Thus the bond is, as of today, a 5-year 4%-coupon bond with a coupon dividend every six months with a 100 EUR face value.

   The bond is to be delivered in two years immediately after the dividend has been paid. The present term structure is given by the following rates of interest (on a yearly basis)

6 months	5.0%	18, 24 months	5.6%
12 months	5.4%	30–60 months	5.9%

   (94.05 EUR)

7. A one-year forward contract of a share which pays no dividend before the contract matures is written when the share has a price of 40 dollars and the risk-free interest rate is 10% a year.

   a) What is the forward price? (44.207 dollars)

   b) If the share is worth 45 dollars six months later, what is the value of the original forward contract at this time? If another forward contract is to be written with the same date of maturity, what should the forward price be? (2.949 dollars, 47.307 dollars)

8. Determine the forward price in SEK of a German stock which costs 25 EUR today. The time of maturity is in one year, and the stock pays a dividend in nine months of 5% of the current stock price at that time. The interest rate of the Euro is 4.5% per year, and the crown's rate of interest is 3% per year. One Euro costs 9.40 SEK today. (230.05 SEK)

9. Let r_i be the random daily rate of interest (per day) from day $i-1$ to day i, and $R(0,t) = r_1 + \cdots + r_t$. The random variable X_t is the stock exchange index day t (today is day 0.) The random variables X_t and $R(0,t)$ are *not* independent.

   The forward price of a contract for delivery of the payment $X_t\, e^{R(0,t)}$ EUR day t is 115 EUR, a zero coupon bond which pays 1 EUR day t costs 0.96 EUR.

   Determine the futures price of a contract for delivery of X_t EUR day t. The stock exchange index is today $X_0 = 100$.

10. A share of a stock currently costs 80 EUR. One year from now, it will pay a dividend of 5% of its price at the time of the payment, and the same happens two years from now. Determine the forward price of the asset for delivery in 2.5 years. The interest rate for all maturities is 6% per year. (83.88 EUR)

11. The 6 month zero rate is 5% per year and the one year rate is 5.2% per year. What is the forward rate from 6 to 12 months? (5.4% per year)

12. Show that the present value of a cash flow at time T that equals the floating interest from t to T ($t < T$) on a principal $\$L$ is the same as if the floating rate is replaced by the current forward rate. (Note that this is an easy way to value interest rate swaps: just replace any floating rate by the corresponding forward rate.)

Solutions

3. (The problem is admittedly somewhat artificial, but serves as an exercise.)
   If we buy the stock today and sell it after one year, the cash flow can be represented:

month	0	4	10	12
cash flow	$-S_0$	2	$0.02S_{10}$	S_{12}

i.e.,

$$S_0 = 2e^{-0.06\cdot 4/12} + 0.02\, P_0^{(10)}\big[S_{10}\big] + P_0^{(12)}\big[S_{12}\big]. \qquad (1)$$

On the other hand, if we sell the stock after 10 months, before the dividend, then the cash flow is

month	0	4	10
cash flow	$-S_0$	2	S_{10}

i.e.,

$$S_0 = 2e^{-0.06\cdot 4/12} + P_0^{(10)}\left[S_{10}\right]$$

If we here solve for $P_0^{(10)}(S_{10})$ and substitute into (1) we get

$$0.98S_0 = 1.96e^{-0.06\cdot 4/12} + P_0^{(12)}\left[S_{12}\right]$$

But $P_0^{(12)}\left[S_{12}\right] = e^{-0.06}G_0^{(12)}\left[S_{12}\right] = \$e^{-0.06}110$, hence

$$0.98S_0 = 1.96e^{-0.06\cdot 4/12} + e^{-0.06}110$$

which yields $S_0 \approx \$107.67$.

5. See *example 5.* We have ($X_0 =$ current exchange rate, $r =$ American interest rate)

$$X_0 = e^{(r-0.04)5/12}\,0.94630 \text{ EUR}$$

and

$$X_0 = e^{(r-0.04)11/12}\,0.95152 \text{ EUR}$$

hence

$$e^{(r-0.04)5/12}\,0.94630 = e^{(r-0.04)11/12}\,0.95152$$

from which we can solve for r (take logarithms,) $r \approx 2.90\%$.

6. The present value P_0 of the cash flow of the underlying bond after 2 years (that is, after the bond has been delivered) is

$$\begin{aligned} P_0 &= 2\,e^{-0.059\cdot 2.5} + 2\,e^{-0.059\cdot 3} + 2\,e^{-0.059\cdot 3.5} + 2\,e^{-0.059\cdot 4} \\ &\quad + 2\,e^{-0.059\cdot 4.5} + 102\,e^{-0.059\cdot 5} \approx 84.0836 \text{ EUR} \end{aligned}$$

The forward price $G_0 = P_0/Z_2$, so

$$G_0 = 84.0836 \cdot e^{0.056\cdot 2} \approx 94.05 \text{ EUR}$$

Note that we don't need the interest rates for shorter duration than two years!

7a. Since there are no dividends or other convenience yield, the present price $P_0^{(1)}$ is the same as the spot price: $P_0^{(1)} = \$40$. Hence $G_0^{(1)} = Z_1^{-1}P_0^{(1)} = e^{0.1}\cdot 40 \approx \$44.207$.

b. By the same argument, six months later the present price $P_6^{(12)}\big[S_{12}\big] = \$45$; S_{12} is the (random) spot price of the stock at 12 months. The cash flow of the forward contract is $S_{12} - \$44.207$ at 12 months, so the present value of this cash flow at 6 months is $P_6^{(12)}\big[S_{12}\big] - e^{-0.1\cdot 0.5}\,44.207 \approx \$2.949$.

If a forward contract were drawn up at six months, the forward price would be $G_6^{(12)} = Z_{6,12}^{-1}\,S_6 = e^{0.1\cdot 0.5}\,45 \approx \$47.303$, given that the interest rate is still 10% per year.

9. $F_0^{(t)}[X] = P_0^{(t)}[Xe^R] = 0.96 \cdot G_0^{(t)}[Xe^R] = 0.96 \cdot 115 = 110.40$ EUR.

12. The cash flow at time T is $\$\big(e^{r(T-t)} - 1\big)\,L$ where r is the zero coupon interest rate that pertain at time t for the duration up to time T. The present value today of this cash flow can be written as

$$P_0 = P_0^{(T)}\big[\$e^{r(T-t)} - \$1\big]\,L = \big(P_0^{(T)}\big[\$e^{r(T-t)}\big] - \$Z_T\big)\,L$$

Now consider the strategy: we do nothing today, but at time t we deposit the amount \$1 at the rate r up to time T when we withdraw the amount $\$e^{r(T-t)}$. The present value of this cash flow is zero, so $-\$Z_t + P_0^{(T)}\big[\$e^{r(T-t)}\big] = \$0$ i.e., $P_0^{(T)}\big[\$e^{r(T-t)}\big] = \$Z_t$. Substitute this into the relation above:

$$P_0 = \$(Z_t - Z_T)\,L$$

The deterministic cash flow at T when the forward rate is applied is $\$\big(e^{f(T-t)} - 1\big)\,L$ where f is the forward rate, and its present value is of course

$$\widetilde{P}_0 = \$Z_T\big(e^{f(T-t)} - 1\big)\,L = \$\big(Z_T e^{f(T-t)} - Z_T\big)\,L$$

But $Z_T e^{f(T-t)} = Z_t$ (see "Forward Rate Agreements",) hence

$$\widetilde{P}_0 = \$(Z_t - Z_T)\,L = P_0$$

Q.E.D. ("Quad Erat Demonstrandum".)

III: Optimal Hedge Ratio

A company that knows that it is due to buy an asset in the future can *hedge* by taking a long futures position on the asset. Similarly, a firm that is going to sell may take a short hedge. But it may happen that there is no futures contract on the market for the exact product or delivery date. For instance, the firm might want to buy petrol or diesel, whereas the closest futures contract is on crude oil, or the delivery date of the futures contract is a month later than the date of the hedge. In this case one might want to use several futures on different assets, or delivery times, to hedge. Let S be the price of the asset to be hedged at the date t of delivery, and $F_t^1, \ldots, F_t^n$ the futures prices at date t of the contracts that are being used to hedge. All these prices are of course random as seen from today, but we assume that there are enough price data so that it is possible to estimate their variances and covariances.

Assume that for each unit of volume of S we use futures contracts corresponding to β_i units of volume for contract F^i. The difference between the spot price S and the total futures price at the date of the end of the hedge is $e = S - \sum_{i=1}^n \beta_i F_t^i$, or

$$S = \sum_{i=1}^n \beta_i F_t^i + e$$

The total price paid for the asset including the hedge is $S - \sum_{i=1}^n \beta_i (F_t^i - F_0^i) = e + \sum_{i=1}^n \beta_i F_0^i$, where only e is random, i.e., unknown to us today. The task is to choose β_i such that the variance of the residual e is minimised.

Lemma

If we choose the coefficients β_i *such that* $\mathrm{Cov}(F_t^i, e) = 0$ *for* $i = 1, \ldots, n$, *then the variance* $\mathrm{Var}(e)$ *is minimised.*

Proof

Assume that we have chosen β_i such that $\mathrm{Cov}(F_t^i, e) = 0$ for $i = 1, \ldots, n$, and consider any other choice of coefficients:

$$S = \sum_{i=1}^n \gamma_i F_t^i + f$$

For notational convenience, let $\delta_i = \beta_i - \gamma_i$. We can write the residual f as

$$f = S - \sum_{i=1}^{n} \gamma_i F_t^i = \sum_{i=1}^{n} \beta_i F_t^i + e - \sum_{i=1}^{n} \gamma_i F_t^i$$
$$= \sum_{i=1}^{n} \delta_i F_t^i + e$$

Note that since $\mathrm{Cov}(F_t^i, e) = 0$ it holds that $\mathrm{Cov}(\sum_{i=1}^n \delta_i F_t^i, e) = 0$, hence

$$\mathrm{Var}(f) = \mathrm{Var}\left(\sum_{i=1}^{n} \delta_i F_t^i\right) + \mathrm{Var}(e) \geq \mathrm{Var}(e)$$

Q.E.D.

Theorem

The set of coefficients β_i *that minimises the variance* $\mathrm{Var}(e)$ *of the residual is the solution to the system*

$$\sum_{i=1}^{n} \mathrm{Cov}(F_t^j, F_t^i)\beta_i = \mathrm{Cov}(F_t^j, S) \quad j = 1, \ldots, n.$$

Proof

The condition that $\mathrm{Cov}(F_t^j, e) = 0$ means that

$$0 = \mathrm{Cov}(F_t^j, S - \sum_{i=1}^{n} \beta_i F_t^i)$$
$$= \mathrm{Cov}(F_t^j, S) - \sum_{i=1}^{n} \beta_i \mathrm{Cov}(F_t^j, F_t^i)$$

from which the theorem follows. *Q.E.D.*

If we regard S and F_t^i as any random variables, then the coefficients β_i are called the *regression coefficients* of S onto $F_t^1, \ldots, F_t^n$; in the context here they are the *optimal hedge ratios* when the futures $F_t^1, \ldots, F_t^n$ are used to hedge S: for each unit of volume of S we should use futures F_t^i corresponding to β_i units of volume in the hedge.

Let us consider the case $n = 1$. In this case we have that

$$\beta = \frac{\mathrm{Cov}(S, F_t)}{\mathrm{Var}(F_t)}$$

Using the lemma, we have the following relation of variances:

$$\mathrm{Var}(S) = \beta^2 \mathrm{Var}(F_t) + \mathrm{Var}(e) = \frac{\mathrm{Cov}^2(S, F_t)}{\mathrm{Var}(F_t)} + \mathrm{Var}(e)$$
$$= \rho^2(S, F_t) \mathrm{Var}(S) + \mathrm{Var}(e)$$

where $\rho(S, F_t) = \mathrm{Cov}(S, F_t)/\sqrt{\mathrm{Var}(S)\,\mathrm{Var}(F_t)}$ is the *correlation coefficient* between S and F_t.

Solving for $\mathrm{Var}(e)$ we get the pleasant relation $\mathrm{Var}(e) = \mathrm{Var}(S)\big(1 - \rho^2(S, F_t)\big)$ which means that the *standard deviation* of the hedged position is $\sqrt{1 - \rho^2(S, F_t)}$ times that of the unhedged position.

Exercises and Examples

Interest rates always refer to continuous compounding. Answers are given in parenthesis; solutions to the problems are given in the next section

1. Suppose that the standard deviation of the spot price of a commodity a quarter into the future is estimated to be \$0.65, the standard deviation of quarterly changes in a futures price on the commodity with maturity in four months is \$0.81, and the coefficient of correlation between the two prices is 0.8. What is the optimal hedge ratio for a three month contract, and what does it mean. (0.642)

   In addition: After the hedge, what is the standard deviation per unit of the commodity of the hedged position? (\$0.39)

2. On July 1, an investor holds 50'000 shares of a certain stock. The market price is \$30 per share. The investor is interested in hedging against movements in the market over the next months, and decides to use the September Mini S&P 500 futures contract. The index is currently 1'500 and one contract is for delivery of \$50 times the index. The beta of the stock is 1.3. What strategy should the investor follow? (short 26 futures)

   Comment. The *beta* of a stock is the regression coefficient of the return of the stock on the return of the index. The investor thinks that this particular share will do better than the index, but he wants to hedge against a general decline of the market. The strategy is to capitalise on this better performance, even if there is a general decline.

Solutions

1. The optimal hedge ratio β is the solution to

$$\mathrm{Cov}(F, F)\,\beta = \mathrm{Cov}(F, S)$$

   where S = spot price and F = futures price at the time of the hedge. Since $\mathrm{Cov}(F, F) = \mathrm{Var}(F)$ we get (SD = standard deviation; ρ = correlation coefficient)

$$\beta = \frac{\mathrm{Cov}(F,S)}{\mathrm{Var}(F)} = \frac{\mathrm{SD(F)}\,\mathrm{SD}(S)\,\rho(S,F)}{\mathrm{Var}(F)}$$
$$= \frac{\mathrm{SD}(S)}{\mathrm{SD(F)}}\rho(S,F) = \frac{0.65}{0.81}\,0.8 \approx 0.642$$

This means that for every unit of the commodity to be hedged we should take short futures positions on 0.642 units. The hedged position will have a standard deviation of $\$0.65\sqrt{1-\rho^2} = \$0.39$ per unit.

2. Let X be the value of one share some months from now, $X_0 = \$30$ is the current value. Likewise, let I be the value of the index some months from now, $I_0 = 1'500$ is the current value. The beta value β is by definition such that

$$\frac{X}{X_0} = \beta\,\frac{I}{I_0} + e$$

where the covariance $\mathrm{Cov}(I,e) = 0$. In other words,

$$X = \beta\,\frac{X_0}{I_0}\,I + X_0\,e$$

The idea of the strategy is to hedge the term containing I; this is the impact on the value of the share due to general market movements. We want to capitalise on the idiosyncratic term containing e.

Assume now that we own one share and short k futures. Each future is for delivery of $n = \$50$ times the index. The value of the hedged position after some months is then

$$X - knI = \bigl(\beta\,\frac{X_0}{I_0} - kn\bigr)I + X_0\,e$$

so, obviously, in order to get rid if the I-term we should have $\beta\,\frac{X_0}{I_0} - kn = 0$, i.e.,

$$k = \frac{1}{n}\,\beta\,\frac{X_0}{I_0} = 0.00052$$

If we own 50'000 shares, we should thus short $50'000 \cdot 0.00052 = 26$ futures.

IV: Conditions for No Arbitrage

Let us consider a forward contract drawn up today, at time 0, on some underlying asset with value X at maturity t. The contract is defined by the random variable X and the forward price—which is determined today—$G = G_0^{(t)}[X]$; we describe it by the pair (G, X). The total cash flow for the long holder at time t is thus $X - G$, which is random, whereas the cash flow today is 0. There are no other cash flows in relation to entering the contract.

The problem under study in this chapter is the following: assume that we draw up a another *derivative* contract with the same underlying value X. A (European) derivative on X is a contract where the payoff for the long holder at maturity date t is some function $f(X)$ of X. A typical one is a *call option* on X with a *strike price* K. This means that the long holder has the *option* (but not obligation) at time t to buy the asset for the pre-determined price K. Hence, in this case $f(X) = \max(X - K, 0)$. It is obvious that the value *today* for a call option is positive, and normally the long holder pays for the contract today, but in this chapter we will assume that all cash flows occur at time t, i.e., the derivative is also a forward contract with forward price $G = G_0^{(t)}[f(X)]$. A (European) derivative is hence a forward contract which we describe by the pair (G, Y), where $G = G_0^{(t)}[f(X)]$ and $Y = f(X)$, i.e., the contract is a usual forward contract with underlying value $Y = f(X)$. A special case is of course $Y = f(X) = X$, in which case the derivative is the usual forward contract on X.

Assume that there are already a number of derivative contracts on the market, and we want to draw up yet another one. The problem is to choose a forward price for this new contract such that it is "compatible" in some sensible way with the contracts already existing on the market, e.g., such that the law of one price is maintained.

The market consists of a set of such derivative contracts, and we assume that we can compose arbitrary portfolios of contracts, i.e., if we have contracts $1, \ldots, m$ with payoffs $Y_1, \ldots, Y_m$ with forward prices $G_1, \ldots, G_m$, (where lower index now refer to *contract*) then we can compose a portfolio consisting of λ_i units of contract i, $i = 1, \ldots, m$, where $\lambda_1, \ldots, \lambda_m$ are arbitrary real numbers. I.e., we assume that we can take a short position in any contract, and ignore any divisibility problems. The total forward price of such a portfolio is of course $\sum_1^m \lambda_i G_i$ and the total payoff is $\sum_1^m \lambda_i Y_i$. We call also such portfolios "contracts", so the set of contracts, defined as pairs (G, Y), constitutes a *linear space.*

We simplify the analysis by first assuming that the underlying value X can take on only a finite number n of values, $x_1, \ldots, x_n$ (where of course

n may be huge!). Then the payoff Y of any derivative can also take on only n values $y_1, \ldots, y_n$ (some of which may coincide.)

We assume that the market is *arbitrage free* in the sense that the *law of one price* prevails (Ch. I,) but also in the sense that

$$\text{If } y_k \geq 0 \text{ for all } k, \quad \text{then} \quad G[Y] \geq 0,$$

$$\text{If } y_k \geq 0 \text{ for all } k \text{ and } y_k > 0 \text{ for some } k, \quad \text{then} \quad G[Y] > 0.$$

These conditions avoid obvious arbitrage opportunities.

Theorem (The No Arbitrage Theorem)

Under the assumptions above, there is a probability distribution $p_X(x_k) > 0$ for X such that the forward price $G[f(X)]$ of any derivative contract is the expected value $G[f(X)] = \mathrm{E}[f(X)] = \sum_1^n f(x_k)p_X(x_k)$ with respect to this probability distribution.

Note that the converse is obvious: if forward prices are set such that $G[f(X)] = \sum_1^n f(x_k)p_X(x_k)$, then the market is arbitrage free in the sense above.

Proof

We associate with each contract (G, Y) the vector $(-G, y_1, \ldots, y_n)$ in $\mathbf{R}^{n+1}$. The set of such vectors constitutes a linear subspace V of $\mathbf{R}^{n+1}$. We now prove that there is a vector $\bar{q}$ which is orthogonal to V, all of whose coordinates are positive.

Let K be the subset of $\mathbf{R}^{n+1}$: $K = \{\bar{u} = (u_0, \ldots, u_n) \in \mathbf{R}^{n+1} \mid u_0 + \cdots + u_n = 1 \text{ and } u_i \geq 0 \text{ for all } i\}$. Obviously K and V have no vector in common; indeed, it is easy to see that any such common vector would represent an arbitrage. Now let $\bar{q}$ be the vector of shortest Euclidean length such that $\bar{q} = \bar{u} - \bar{v}$ for some vectors $\bar{u}$ and $\bar{v}$ in K and V respectively. The fact that such a vector exists needs to be proved, however, we will skip that proof. We write $\bar{q} = \bar{u}^* - \bar{v}^*$ where $\bar{u}^* \in K$ and $\bar{v}^* \in V$.

Now note that for any $t \in [0,1]$ and any $\bar{u} \in K$, $\bar{v} \in V$, the vector $t\bar{u} + (1-t)\bar{u}^* \in K$ and $t\bar{v} + (1-t)\bar{v}^* \in V$, hence $|(t\bar{u} + (1-t)\bar{u}^*) - (t\bar{v} + (1-t)\bar{v}^*)|$ as a function of t on $[0,1]$ has a minimum at $t = 0$, by definition of $\bar{u}^*$ and $\bar{v}^*$, i.e., $|t(\bar{u} - \bar{v}) + (1-t)\bar{q}|^2 = t^2|\bar{u} - \bar{v}|^2 + 2t(1-t)(\bar{u} - \bar{v}) \cdot \bar{q} + (1-t)^2|\bar{q}|^2$ has minimum at $t = 0$ which implies that the derivative w.r.t. t at $t = 0$ is ≥ 0. This gives $(\bar{u} - \bar{v}) \cdot \bar{q} - |\bar{q}|^2 \geq 0$ or, equivalently, $\bar{u} \cdot \bar{q} - |\bar{q}|^2 \geq \bar{v} \cdot \bar{q}$ for all $\bar{v} \in V$ and $\bar{u} \in K$. But since V is a linear space, it follows that we must have $\bar{v} \cdot \bar{q} = 0$ for all $\bar{v} \in V$ (the inequality must hold if we replace $\bar{v}$ by $\lambda\bar{v}$ for any scalar λ.) It remains to prove that $\bar{q}$ has strictly positive entries. But we have $\bar{u} \cdot \bar{q} - |\bar{q}|^2 \geq 0$ for all $\bar{u} \in K$, in particular we can take $\bar{u} = (1, 0, \ldots, 0)$ which shows that the first entry of $\bar{q}$ is > 0 and so on.

Hence, we have found positive numbers $q_0, \dots, q_n$ such that

$$-q_0 G + q_1 y_1 + \cdots + q_n y_n = 0$$

for all contracts, and by scaling the q_k:s we can arrange that q_0 is equal to one, hence

$$G = \sum_1^n q_k y_k.$$

Since $G[1] = 1$ we have $\sum_1^n q_k = 1$, so $p_X(x_k) = q_k$ defines a probability distribution for X such that for any payoff function $f(X)$ it holds that

$$G[f(X)] = \mathrm{E}[f(X)].$$

Q.E.D.

A Generalisation

In the discussion above we have considered only one underlying asset X. But the arguments apply equally well to the situation when X represents a vector of underlying values $X = (X_1, \dots, X_j)$. Hence, if we have a finite number of underlying assets $X_1, \dots, X_j$, all of which can take on only a finite number of values, then, on an arbitrage free market, there is a joint probability distribution for $X = (X_1, \dots, X_j)$ such that for any derivative on X with payoff $f(X)$ it holds that

$$G[f(X)] = \mathrm{E}[f(X)].$$

Even though the assumption that the value of the underlying assets can take on only a finite (but possibly huge) number of values is not entirely unrealistic, it is inconvenient for the theoretical modelling. So we do away with this assumption and henceforth assume the following No Arbitrage Condition:

The No Arbitrage Assumption

If $X = (X_1, \dots, X_j)$ are asset values at the future time t, then there exists a joint probability distribution for X such that the forward price $G_0^{(t)}[f(X)]$ of any derivative contract with payoff $f(X)$ is given by the expected value w.r.t. this probability distribution: $G_0^{(t)}[f(X)] = \mathrm{E}[f(X)]$.

It is important to recognise that the expectation above is *not* with respect to some "objective" probability distribution; in fact, the probability distribution has been constructed from *prices* on contracts already existing on the market (this probability distribution need not be uniquely determined, though!) Furthermore, these prices are measured in some "currency" or *numeraire* and the probability distribution may depend on which numeraire we employ. More about this later. For now we denote the expectation $\mathrm{E}^{(t)}$ to emphasise that the expectation equals the forward price (in some currency) with maturity date t:

$$G_0^{(t)}[f(X)] = \mathrm{E}^{(t)}[f(X)] \tag{1}$$

$$P_0^{(t)}[f(X)] = Z_t \mathrm{E}^{(t)}[f(X)] \tag{2}$$

and we call the probability distribution the *forward probability distribution.*

The procedure for pricing a new derivative contract is now as follows: first we find a probability distribution such that prices on existing contracts are given by (1) and (2). Then we price any new derivative employing the same formulas.

V: Pricing European Derivatives

Black's Model

A *European* derivative on some underlying value X is a contract which at a certain date, the *date of maturity,* pays the long holder of the contract a certain function $f(X)$ of the value X. A typical example is an option to buy X for the *strike price* K. The strike price K is a deterministic value written in the contract, and the payoff is $f(X) = \max(X - K, 0)$.

In order to compute the price of a European derivative on some underlying asset value X, we assume that there is already a forward contract for X on the market, or that we can compute the forward price of X, and denote the current forward price for delivery at t by $G_0^{(t)}[X]$. Black's model assumes that the value X has a log-Normal forward probability distribution (see comments below:)

$$X = Ae^{\sigma\sqrt{t}\,z} \quad \text{where} \quad z \in N(0,1). \tag{1}$$

In accordance with (1) of Ch. IV we can express the forward value of the underlying asset as

$$\begin{aligned} G_0^{(t)}[X] &= \mathrm{E}^{(t)}[X] = \mathrm{E}^{(t)}[Ae^{\sigma\sqrt{t}\,z}] = A\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\sigma\sqrt{t}\,x}\, e^{-\frac{1}{2}x^2}\,dx \\ &= A\,e^{\frac{1}{2}\sigma^2 t}. \end{aligned}$$

Hence $A = G_0^{(t)}[X]\,e^{-\frac{1}{2}\sigma^2 t}$ which we substitute into (1):

$$X = G_0^{(t)}[X]\,e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,z}. \tag{2}$$

The current price of any derivative $f(X)$ of X is thus, by (1) of Ch. IV,

$$\begin{aligned} P_0^{(t)}\big[f(X)\big] &= Z_t\,\mathrm{E}^{(t)}[f(X)] = Z_t\,\mathrm{E}^{(t)}\big[f\big(G_0^{(t)}[X]\,e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,z}\big)\big] \\ &= \frac{Z_t}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\big(G_0^{(t)}[X]e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,x}\big)\,e^{-\frac{1}{2}x^2}\,dx. \end{aligned} \tag{3}$$

This is *Black's pricing formula.*

The Black-Scholes Pricing Formula

Let us consider the case when the underlying asset is an investment asset with no dividends or convenience yield, for example, it might be a share of a stock which pays no dividend before maturity of the contract. In this case there is a simple relationship between the forward price $G_0^{(t)}$ and the current spot price S_0 of the underlying asset. Indeed, $G_0^{(t)} = e^{rt}S_0$ where r is the t-year zero coupon interest rate. (To see this, note that

$S_0 = P_0^{(t)} = Z_t G_0^{(t)}$, i.e., $S_0 = e^{-rt} G_0^{(t)}$, see Ch. II, example 1 with $d = 0$.) Hence the present price of the derivative can be written

$$\text{price} = \frac{e^{-rt}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(S_0 e^{rt - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,x}\right) e^{-\frac{1}{2}x^2}\, dx$$

This is the *Black-Scholes pricing formula.*

Put and Call Options

A call option with strike price K is specified by $f(X) = \max(X - K, 0)$, and a put option by $f(X) = \max(K - X, 0)$. In these cases the expectation in Black's formula can be evaluated, and the result is that the call option price c and put option price p are

$$c = Z_t\left(G_0^{(t)}\Phi(d_1) - K\Phi(d_2)\right)$$
$$p = Z_t\left(K\Phi(-d_2) - G_0^{(t)}\Phi(-d_1)\right)$$

where

$$d_1 = \frac{\ln(G_0^{(t)}/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$
$$d_2 = d_1 - \sigma\sqrt{t}$$

and Φ is the distribution function of the standard Normal distribution.

The Interpretation of σ and the Market Price of Risk.

The parameter σ is called the *volatility* of the asset X. It is the standard deviation of the logarithm of X divided by the square-root of the time to maturity. It is defined here as a parameter of the underlying forward probability distribution, but is usually considered to be the same under the "true" (objective; the one given by "nature") probability distribution, if any such probability distribution exists.

The true expected value of X, or the market's subjective assessment of this expected value, $\mathrm{E}[X]$, may differ from the forward value $G_0^{(t)} = \mathrm{E}^{(t)}[X]$. The difference may depend on the balance of supply and demand; if many traders want to take long positions on the underlying value X but few wants to take short positions, then the forward price will be bid up so as to draw speculators into the short side of the market to clear it. The difference between $\mathrm{E}[X]$ and $G_0^{(t)}[X]$ then measures the expected gain a speculator will get as a compensation for the risk he takes. The number λ, defined by

$$\mathrm{E}[X] = G_0^{(t)}[X]\, e^{\lambda\sigma t}$$

is called the *market price of risk.* It measures the expected rate of return on a forward contract on X per unit of σ.

The situation when $\lambda > 0$ is called *backwardation,* and when $\lambda < 0$ the situation is known as *contango.*

Comments

The specification (1) may need some clarification. The idea is this. Let X be the value at some future time t of some asset, for example a share of a stock, or a commodity like a barrel of oil. There are several different random events which can influence this value. If $G_0^{(t)}$ is the forward price today, then tomorrow the forward price may change by a factor $(1 + w_1)$, i.e. tomorrow the forward price is $G_1^{(t)} = G_0^{(t)}\,(1 + w_1)$. Here w_1 is a random variable with rather small standard deviation.

The day after tomorrow, the forward price has changed by a new factor $(1 + w_2)$ such that the forward price then is $G_2^{(t)} = G_1^{(t)}\,(1 + w_2) = G_0^{(t)}\,(1 + w_1)(1 + w_2)$ and so on. Eventually, at time t when the forward expires, the "forward" price is the same as the spot price, and we have

$$
\begin{aligned}
X &= G_0^{(t)}\,(1 + w_1)(1 + w_2)\cdots(1 + w_t), \quad \text{i.e.,} \\
\ln X &= \ln\left(G_0^{(t)}\right) + \ln(1 + w_1) + \ln(1 + w_2) + \cdots + \ln(1 + w_t)
\end{aligned}
$$

If we assume that the random variables w_k are independent and identically distributed, then the Central Limit Theorem says that the sum in the right hand side has a tendency to be Normally distributed, and the variance is proportional to the time interval up to t (the number of terms.) The logarithm $\ln X$ is then Normally distributed with a standard deviation proportional to $\sqrt{t}$. This is the rationale for the specification (1).

Black's model is applicable on a variety of European options. It contains the well known "Black-Scholes" pricing formula as a special case, and it can be used to price options on shares of stocks which pay dividends, stock indices, currencies, futures and bonds. In all cases the procedure is in two steps: first we must find the forward price $G_0^{(t)}$ of the underlying asset. We have seen some examples of such computations in Ch. I–III. The next step is to decide on a volatility σ. There is no obvious way to do this. One way is to estimate it from historical data, another is to use "implied volatilities". The last method could be as follows: Assume we want to price a call option on a particular stock with strike price K. There is already an option on the market with a different strike price L. Since this option already has a price given by the market, we can use Black's model on that option to solve for σ; then use that same σ, the "implied volatility", for the option we want to price.

Note, however, that Black's model (or Black-Scholes) doesn't perfectly conform with empirical data; you may read about "volatility smiles" in the book by Hull[1].

There is one problem with Black's model applied to futures options. The problem is that it is the *forward* price that should go into the formula, not the futures price. However, the forward price equals the futures price if the value of the asset is essentially independent of the interest rate. This is proven in Ch. VII.

Exercises and Examples

Interest rates always refer to continuous compounding. Answers are given in parenthesis; solutions to the problems are given in the next section.

1. Determine the price of a European futures call option on a barrel of crude oil to be delivered in 4 months. This is also the time of maturity of the option. The futures price today is $F_0 = \$25.00$, the strike price is \$23.00 and the volatility of the futures price is estimated to be 25% for one year. The risk-free interest rate is 9% per year. (\$2.527)

2. Determine the price of a European call option on a share which does not pay dividends before the maturity of the option. The present spot price of the share is 45 SEK, the option matures in 4 months, the volatility is 25% in one year, the option's strike price is 43 SEK and the risk-free interest rate is 9% a year. (4.463 SEK)

3. The same question as above, but now we assume the share pays a dividend of 0.50 SEK in 3 months, all other assumptions are the same. (4.115 SEK)

4. Determine the price of a European put option on 1 GBP with a strike price of 14 SEK in 6 months. The exchange rate is 13 SEK for 1 GBP, and the pound's volatility is assumed to be 14% in one year. The pound's rate of interest is 11% and the Swedish crown's rate of interest is 7% a year. (1.331 SEK)

[1] John C. Hull: *Options, Futures, & Other Derivatives;* Prentice Hall.

5. Determine the price of a European call option on an index of shares which is expected to give a dividend of 3% a year, continuously. The current value of the index is 93 SEK, the strike price is 90 SEK and the option matures in two months. The risk-free interest rate is 8% a year and the index has a volatility of 20% in one year. (5.183 SEK)

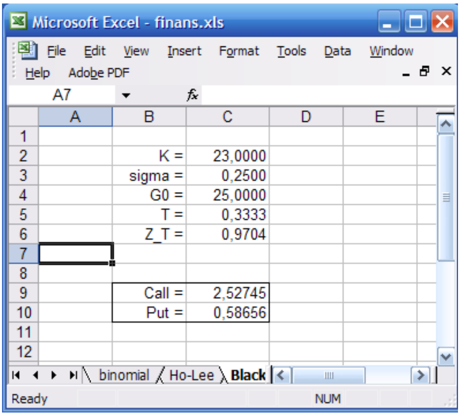
6. Let $S(t)$ be the spot price of a share at time t (year) which does not pay dividends the following year. Determine the price of a contract which after one year gives the owner $\dfrac{S(1)^2}{S(0)}$. The risk-free rate of interest is 6% a year, and the share's volatility is assumed to be 30% for one year. $(S(0)\, e^{0.15})$

7. Let $S(t)$ be the spot price of a share at time t (year) which does not pay dividends the following year. Determine the price of a contract which after one year gives the owner \$100 if $S(1) > \$50$ and nothing if $S(1) \leq \$50$. The current spot price is $S_0 = \$45$, the share's volatility is assumed to be 30% for one year and the risk-free rate of interest is 6% a year. (\$35.94)

8. Show the *put-call parity:* If c and p are the prices of a call and a put option with the same strike price K, then

$$c - p = Z_t\,(G_0 - K)$$

Solutions



I suggest that you make a spreadsheet in Excel (or some such) where you can calculate call- and put options with Black's formula. You input the parameters K, G_0, σ, time to maturity T and discount factor Z_T. You find the relevant formulas in this chapter. You can then use this spreadsheet to solve most of the problems below; the major problem is to find the correct value for G_0 to go into the formula.

1. Since we assume that the price of oil is essentially unaffected by interest rates, we set $G_0 = F_0 = \$25$ in Black's formula. Of course $Z_T = e^{-0.09/3}$. In my spreadsheet the answer appears as in the figure above.

2. Since the share doesn't pay any dividend and there is no convenience yield, we get the forward price $G_0^{(4)}$ from the relation (see example 1 of Ch. II with $d = 0$) 45 SEK $= e^{-0.09/3}\, G_0^{(4)}$, i.e., $G_0^{(4)} \approx 46.37$ SEK.

3. See example 1 in Ch. II. We get $G_0 \approx 45.867$ SEK.

4. See example 5 in Ch. II. The forward price of 1 GBP is ≈ 12.7426 SEK. Note that $Z_T = e^{-0.035}$, i.e., the discounting should use the Swedish interest rate (why?)

5. See example 4 in Ch. II. The index yields a continuous dividend of 3% per year, and $S_0 = 93$ SEK.
   We get $93\, e^{(0.08-0.03)/6}$ SEK ≈ 93.7782 SEK.

6. We determine the forward price $G_0^{(1)}$ as in problem 2: $G_0^{(1)}\big(S(1)\big) = S(0)\, e^{0.06}$. Hence Black's formula (3) yields

$$
\begin{aligned}
p &= e^{-0.06}\, \mathrm{E}\Big[\tfrac{1}{S(0)}\big(G_0^{(1)} e^{-\frac{1}{2}0.3^2 + 0.3\sqrt{1}\, w}\big)^2\Big] \\
&= e^{-0.06}\, \tfrac{1}{S(0)}\, \big(G_0^{(1)}\big)^2\, e^{-2\,\frac{1}{2}0.3^2}\, \mathrm{E}\big[e^{2\cdot 0.3\, w}\big] \\
&= e^{-0.06}\, \tfrac{1}{S(0)}\, \big(G_0^{(1)}\big)^2\, e^{-2\,\frac{1}{2}0.3^2}\, e^{2\cdot 0.3^2} = S(0)\, e^{0.15}.
\end{aligned}
$$

7. First we determine the forward price of the underlying asset as before: $G_0^{(1)} = e^{0.06}\, S(0) = \$45 \cdot e^{0.06}$. Next, let A be the event $S > \$50$ and $I_A(S)$ the *indicator function* of this event, i.e.,

$$I_A(S) = \begin{cases} 1 & \text{if } S > \$50 \\ 0 & \text{if } S \le \$50 \end{cases}$$

The payoff of the derivative is then $\$100 \cdot I_A(S(1))$, hence the value of the contract is, with the usual notation

$$\begin{aligned} p &= Z_1\, \mathrm{E}\big[100 \cdot I_A(G_0^{(1)}\, e^{-\frac{1}{2}\sigma^2 1 + \sigma\sqrt{1}\, w})\big] \\ &= e^{-0.06}\, \mathrm{E}\big[100 \cdot I_A(45 \cdot e^{0.06 - 0.045 + 0.3w})\big] \\ &= e^{-0.06}\, 100 \cdot \Pr\big(45 \cdot e^{0.06-0.045+0.3w} > 50\big) \\ &= e^{-0.06}\, 100 \cdot \Pr\Big(w > \frac{1}{0.3}\, \ln\big(\frac{50}{45}\big) - 0.05\Big) \approx \$35.94 \end{aligned}$$

8. Let x^+ denote $\max(x, 0)$. Then the price of a call option with maturity t and strike price K on the underlying asset value X is $P_0^{(t)}[(X-K)^+]$, and similarly, the price of the corresponding put option is $P_0^{(t)}[(K - X)^+]$. Hence

$$\begin{aligned} c - p &= P_0^{(t)}[(X-K)^+] - P_0^{(t)}[(K-X)^+] \\ &= P_0^{(t)}[(X-K)^+ - (K-X)^+] \\ &= P_0^{(t)}[X - K] = P_0^{(t)}[X] - P_0^{(t)}[K] \\ &= Z_t\, \big(G_0^{(t)}[X] - K\big) \end{aligned}$$

Remark: This relation is called the *put-call parity formula.*

VI: Yield and Duration

The value of a bond, or a portfolio of bonds, depends on the current term structure. The situation is particularly simple for a zero coupon bond; indeed, the value at time $t = 0$ of a zero coupon bond which matures at time $t = D$ with a face value of P_D is of course $P_0(y) = P_D e^{-yD}$ where y is the relevant zero coupon interest rate. The *duration* D measures how sensitive the value is to changes in the *yield* y :

$$D = -\frac{P_0'(y)}{P_0(y)}$$

For coupon bearing bonds, and portfolios of bonds, the situation is more complicated, since the value depends on more than one point on the zero rate curve. The consequence is that the sensitivity of the value to changes in the zero rate curve can not be exactly measured by a single number, and furthermore that it is hard to come up with a model for pricing options on such instruments.

The remedy is approximation. We will show that a coupon bearing bond, or portfolio of bonds, can be approximated by a zero coupon bond with a certain maturity, the *duration* of the bond or portfolio.

Note that for a zero coupon bond with time to maturity D and corresponding zero rate y_0, it holds that

$$D = -\frac{1}{P_0} P_0'(y_0) \tag{1}$$

$$P_0(y) = P_D e^{-yD} \tag{2}$$

$$P_t(y) = P_D e^{-y(D-t)} \tag{3}$$

where $P_s(y)$ is the value at time s, if the zero coupon interest rate at that time equals y, which may differ from y_0.

The idea of the concepts *yield* (y) and *duration* (D) is to define a number D for a coupon paying bond, or a portfolio of bonds, such that the relations (2) and (3) still hold approximately. Hence, let P_0 be the present value of a bond, or a portfolio of bonds:

$$P_0 = \sum_{i=1}^{n} Z_{t_i} c_i$$

where Z_{t_i} is the price today of a zero coupon bond maturing at time t_i and c_i is the payment received at time t_i. Now define the *yield* y_0 by the relation

$$P_0 = \sum_{i=1}^{n} e^{-y_0 t_i} c_i = P_0(y_0) \tag{4}$$

and the *duration* D by

$$D = -\frac{1}{P_0} P_0'(y_0) = \frac{1}{P_0} \sum_{i=1}^{n} t_i \, e^{-y_0 t_i} c_i \tag{5}$$

Now define $\hat{P}_D(y) = P_0(y)e^{yD}$ [see (4).] We then have

$$\hat{P}_D'(y_0) = P_0'(y_0)e^{y_0 D} + DP_0(y_0)e^{y_0 D} = 0,$$

hence we can consider $\hat{P}_D(y)$ to be constant regarded as a function of y, to a first order approximation. Henceforth we will treat $\hat{P}_D = \hat{P}_D(y_0) = P_0 e^{y_0 D}$ as constant. We now have

$$P_0(y) = \hat{P}_D e^{-yD} \tag{2'}$$

$$P_t(y) = \hat{P}_D e^{-y(D-t)} \tag{3'}$$

Here (2′) follows from the definition of $\hat{P}_D$ and (3′) holds if the yield is y at time t, as is seen as follows:

$$P_t(y) = \sum_{i=1}^{n} e^{-y(t_i - t)} c_i = e^{yt} \sum_{i=1}^{n} e^{-yt_i} c_i = e^{yt} P_0(y) = \hat{P}_D e^{-y(D-t)}$$

Summary

A bond, or a portfolio of bonds, has a yield y_0, defined by (4), and a duration D, defined by (5). At any time t before any coupon or other payments have been paid out, the value $P_t(y)$ of the asset is approximately equal to

$$P_t(y) = \hat{P}_D e^{-y(D-t)} \quad \text{where} \quad \hat{P}_D = P_0 e^{y_0 D}$$

and y is the yield prevailing at that time. In particular, the value of the asset at time $t = D$ is $\hat{P}_D = P_0 e^{y_0 D}$, and hence independent of any changes in the yield to a first order approximation.

Duration is a measure of how long on average the holder of the portfolio has to wait before receiving cash payments, but more importantly, it is a measure of the sensitivity of the portfolio's value to changes in the rate of interest. It can also be thought of as a time when the value of the portfolio is nearly unaffected by the rate of interest: If the interest (yield) falls today, then the value of the portfolio immediately increases (if it is a portfolio of bonds,) but it's growth rate goes down, and at the time $t =$ duration the two effects nearly net out. Similarly, of course, if the rate of interest goes up. All this is captured in (3′). Note, however, that in the meantime coupon dividends have been paid out, and when these are re-invested, the duration goes up, so in order to keep a certain date as a target for the value of the portfolio, it has to be re-balanced.

Example

Consider a portfolio of bonds that gives the payment 1'000 after one year, 1'000 after two years and 2'000 after three years. Assume that $Z_1 = 0.945$, $Z_2 = 0.890$ and $Z_3 = 0.830$. The present value of the portfolio is then

$$P_0 = 0.945 \cdot 1'000 + 0.890 \cdot 1'000 + 0.830 \cdot 2'000 = 3'495$$

The yield y_0 is obtained by solving for y_0 in the relation

$$3'495 = P_0 = e^{-y_0} 1'000 + e^{-2y_0} 1'000 + e^{-3y_0} 2'000$$

which gives $y_0 = 0.06055$. Now

$$-P_0'(y_0) = e^{-y_0} 1'000 + 2\, e^{-2y_0} 1'000 + 3\, e^{-3y_0} 2'000 = 7'716.5$$

so the duration is $D = \frac{7'716.5}{3'495}$ years $= 2.208$ years. The value of the portfolio at time t, if the yield at that time is y and t is less than one year, can thus be approximated by

$$P_t(y) = 3'995\, e^{-y(2.208-t)}$$

(where $3'995 = P_0 e^{y_0 D}$.)

Forward Yield and Forward Duration

Consider a bond, or any interest rate security, that after time t gives the holder deterministic payments $c_1, c_2, \ldots, c_n$ at times $t_1 < t_2 < \cdots < t_n$ (where $t_1 > t$.) Today is time $0 < t$ and we want to compute the forward price of the security to be delivered at time t. Note that the security may give the holder payments also before time t, but this is of no concern to us, since we only are interested in the forward value of the security.

Let P_t be the (random) value of the security at time t and consider the strategy of buying the security at time t for this price and then cash the payments $c_1, \ldots, c_n$. The pricing methods of Ch. II shows that the *forward* price $G_0^{(t)} = G_0^{(t)}[P_t]$ of the value P_t is given by

$$Z_t G_0^{(t)} = \sum_{i=1}^{n} Z_{t_i} c_i$$

The *forward yield* y_F is defined by

$$G_0^{(t)} = \sum_{i=1}^{n} c_i e^{-y_F (t_i - t)} \tag{6}$$

When we arrive at time t it may be that the prevailing yield y at that time is not the same as y_F. Let y be the prevailing yield at time t. Then the value of the security at that time is

$$P_t = P_t(y) = \sum_{i=1}^{n} c_i e^{-y(t_i - t)}$$

In particular, we see that $P_t(y_F) = G^{(t)}$. We define the *forward duration* D_F of the portfolio for time t as

$$D_F = -\frac{P_t'(y_F)}{P_t(y_F)} = \frac{1}{G_0^{(t)}} \sum_{i=1}^{n} c_i(t_i - t) e^{-y_F (t_i - t)}. \tag{7}$$

Now define P_F by $P_F(y) = P_t(y) e^{D_F\, y}$. The same calculation as in the previous section shows that P_F is independent of y, to a first order approximation, so we will regard $P_F = P_F(y_F) = G_0^{(t)} e^{D_F\, y_F}$ as a constant. So we have

$$P_t = P_F e^{-D_F\, y}$$

Summary

An interest rate security that after time t gives the deterministic payments $c_1, c_2, \ldots, c_n$ at times $t_1 < t_2 < \cdots < t_n$ has a forward yield y_F, defined by (6), and a forward duration D_F defined by (7). The value of the asset as of time t, $0 < t < t_1$ is then, to a first order approximation

$$P_t = P_F e^{-D_F\, y} \quad \text{where} \quad P_F = G_0^{(t)} e^{D_F\, y_F}$$

and y is the yield prevailing at that time.

Example

In the previous example, assume that $Z_{1.5} = 0.914$. The forward price $G_0(1.5)$ of the portfolio for delivery in one and a half years is obtained from the relation (note that the first payment of 1'000 has already been paid out, and hence does not contribute to the forward value:)

$$0.914\, G_0^{(1.5)} = 0.890 \cdot 1\text{'}000 + 0.830 \cdot 2\text{'}000 \quad \Rightarrow \quad G_0^{(1.5)} = 2\text{'}790$$

The forward yield y_F is obtained from

$$2\text{'}790 = G_0^{(1.5)} = e^{-0.5 y_F}\, 1\text{'}000 + e^{-1.5 y_F}\, 2\text{'}000 \quad \Rightarrow \quad y_F = 0.06258$$

Now, by (7)

$$D_F\, G_0^{(1.5)} = 0.5\, e^{-0.5 y_F}\, 1\text{'}000 + 1.5\, e^{-1.5 y_F}\, 2\text{'}000 = 3\text{'}215.8$$

so the forward duration is $D_F = \frac{3\text{'}215.8}{2\text{'}790}$ years $= 1.153$ years.

Black's Model for Bond Options

Consider again the bond or security in the section *"Forward Yield and Forward Duration"* above. In this section we want to price a European option on the value P_t, a *bond option.*

First we specify a random behaviour of y, the yield that will prevail for the underlying security at time t. We will assume that as seen from today, y is a Normally distributed random variable:

$$y = \alpha - \sigma\sqrt{t}\, z, \quad \text{where } z \in \mathrm{N}(0,1)$$

The minus sign is there for notational convenience later on; note that z and $-z$ have the same distribution. Hence, the present value P_t at time t of the asset is

$$P_t = P_F e^{-D_F(\alpha - \sigma\sqrt{t}\, z)} = A\, e^{D_F\, \sigma\sqrt{t}\, z}$$

where $A = P_F e^{-D_F\, \alpha}$.

Note that this is exactly the same specification as (1) in Ch. V with σ replaced by $D_F\, \sigma$. We can thus use Black's formula for European options of Ch. V, where $\sigma\sqrt{t}$ in that formula stands for the product of the standard deviation of the yield and the forward duration of the underlying asset.

Portfolio Immunisation

Assume that we have portfolio P of bonds whose duration is D_P. The duration reflects its sensitivity to changes in the yield, so we might well want to change that without making any further investments. One way to do this is to add a *bond futures* to the portfolio (or possibly short such a futures.)

Consider again the security discussed under *"Forward Yield and Forward Duration"* above. We will look at a futures contract on the value P_t. Using the notation of Ch. I, the futures price is $F_0 = F_0[P_t] = F_0[P_F e^{-D_F\, y}]$, where the yield y at time t is random as seen from today. Assume now that the present yield y_0 of the portfolio P changes to $y_0 + \delta y$. The question arises what impact this has on y. If $y =$ (current yield) + (random variable independent of current yield), then y should simply be replaced by $y + \delta y$. The new futures price should then be

$$F_0[P_F e^{-D_F\,(y+\delta y)}] = e^{-D_F\,\delta y} F_0[P_F e^{-D_F\,y}] = e^{-D_F\,\delta y} F_0$$

The"marking-to-market" is hence $(e^{-D_F\,\delta y} - 1)F_0$.

Assume now that we add such a futures contract to our initial portfolio P. We let $P = P(y_0)$ denote its original value, and $P(y_0 + \delta y)$ its value after the yield has shifted by δy. The value of the total portfolio P_{tot} before the shift is equal to P, since the *value* of the futures contract is zero. However, after the yield shift, the marking-to-market is added to the portfolio, so after the shift the value of the total portfolio is

$$P_{tot}(y_0 + \delta y) = P(y_0 + \delta y) + (e^{-D_F\,\delta y} - 1)F_0$$

If we differentiate this w.r.t. δy we get for $\delta y = 0$

$$-P'_{tot}(y_0) = -P'(y_0) + D_F F_0 = D_P P + D_F F_0$$

The duration of the total portfolio is thus

$$D_{tot} = -\frac{1}{P} P'_{tot}(y_0) = D_P + D_F \frac{F_0}{P}$$

If we take N futures contracts (where negative N corresponds to going short the futures,) we get

$$D_{tot} = D_P + N D_F \frac{F_0}{P}$$

Exercises and Examples

Interest rates always refer to continuous compounding. Answers are given in parenthesis; solutions are given in the next section.

1. Determine the

   a) forward price (1'993.29 SEK)

   b) forward yield (9.939%)

   c) forward duration (1.861 years)

   two years into the future for a bond that pays out 100 SEK in 2.5, 3 and 3.5 years and 2'100 SEK in 4 years. Zero-coupon interest rates are at present 6%, 6.5%, 7%, 7.5% and 8% for the duration of 2, 2.5, 3, 3.5 and 4 years.

2. Calculate approximately the duration of a portfolio containing a coupon bearing-bond which matures in two years with face value 100'000 SEK and pays a 6%-coupon (this means that the coupon is paid every six month at 3% of the face value,) plus a short position of a futures contract with maturity in two years on a three year (at the time of maturity of the futures) 6% coupon-bearing bond (the first coupon payment is six months after the maturity of the futures) with face value 50'000 SEK. Interest rates are today 5.5% a year with continuous compounding for any length of duration. (Approximate the futures price with the forward price.) (0.514 years)

3. Determine the price of a European put option with maturity in two years on a five year 6% coupon bond issued today (the first coupon payment after the maturity of the option occurs six months after the maturity of the option) with face value 50'000 SEK, which is also the option's strike price. The forward yield of the bond is 6.5% a year and the standard deviation of the yield is 0.012 in one year. The two year zero-coupon rate of interest is 5.5% a year. (1'253 SEK)

4. Calculate the value of a one-year put option on a ten year bond issued today. Assume that the present price of the bond is 1'250 SEK, the strike price of the option is 1'200 SEK, the one-year interest rate is 10% per year, the yield has a standard deviation of 0.013 in one year, the forward duration of the bond as of the time of maturity for the option is 6.00 years and the present value of the coupon payments which will be paid out during the lifetime of the option is 133 SEK. (20.90 SEK)

5. We want to determine the price of a European call bond option. The maturity of the option is in 1.5 years and the underlying bond yields coupon payments \$100 in 2 years, \$100 in 2.5 years and matures with a face value (including coupon) of \$2'000 3 years from today. We assume that the standard deviation of the change in the one-year yield is 0.01 in one year. Current zero coupon rates are (% per year)

1 yr	1.5 yr	2 yr	2.5 yr	3 yr
3.5	4.0	4.5	4.8	5.0

Use Black's model to find the price of the option when the strike price is \$1'950. (≈ \$65.35)

Solutions

1. First we calculate the present price P_0 of the bond:

$$P_0 = e^{-2.5\cdot 0.065}\,100 + e^{-3\cdot 0.07}\,100 + e^{-3.5\cdot 0.075}\,100 + e^{-4\cdot 0.08}\,2'100 \approx 1'767.886 \text{ SEK}$$

The forward price is thus $G_0 = Z_2^{-1} \cdot P_0 = e^{2\cdot 0.06}\,1'767.886$ SEK ≈ 1'993.29 SEK.

Next we determine the forward yield y_F from the relation

$$1'993.29 = G_0 = 100\,e^{-y_F\cdot 0.5} + 100\,e^{-y_F\cdot 1} + 100\,e^{-y_F\cdot 1.5} + 2'100\,e^{-y_F\cdot 2}$$

which we solve numerically (trial and error) to $y_F = 0.09939$. Finally, the forward duration D_F is obtained from

$$1'993.29 \cdot D_F = G_0\,D_F = 0.5\cdot 100\,e^{-y_F\cdot 0.5} + 1\cdot 100\,e^{-y_F\cdot 1} + 1.5\cdot 100\,e^{-y_F\cdot 1.5} + 2\cdot 2'100\,e^{-y_F\cdot 2}$$

and gives $D_F = 1.861$ years.

2. First we determine the present value P_0 of the portfolio. Note that the *value* of the futures contract is zero.

$$P_0 = 3'000\,e^{-0.055\cdot 0.5} + 3'000\,e^{-0.055\cdot 1} + 3'000\,e^{-0.055\cdot 1.5} + 103'000\,e^{-0.055\cdot 2} \approx 100'791.43 \text{ SEK}$$

We get the duration of the bond D_P in the portfolio from the relation

$$P_0 D_P = 0.5\cdot 3'000\,e^{-0.055\cdot 0.5} + 1\cdot 3'000\,e^{-0.055\cdot 1} + 1.5\cdot 3'000\,e^{-0.055\cdot 1.5} + 2\cdot 103'000\,e^{-0.055\cdot 2} \approx 192'984.25$$

which yields $D_P = 1.915$ years.

Next we calculate the present value of the cash flow from the bond underlying the futures:

$$\begin{aligned} p &= 1\text{'}500\, e^{-0.055\cdot 2.5} + 1\text{'}500\, e^{-0.055\cdot 3} + 1\text{'}500\, e^{-0.055\cdot 3.5} \\ &\quad + 1\text{'}500\, e^{-0.055\cdot 4} + 1\text{'}500 e^{-0.055\cdot 4.5} + 51\text{'}500\, e^{-0.055\cdot 5} \\ &\approx 45\text{'}309.35 \text{ SEK} \end{aligned}$$

so the futures price, which we approximate with the forward price, is

$$F_0^{(2)} = Z_2^{-1} p = e^{0.055\cdot 2} \cdot 45\text{'}309.35 \approx 50\text{'}577.84 \text{ SEK}$$

The forward yield y_F is determined by

$$\begin{aligned} F_0^{(2)} &= 1\text{'}500\, e^{-y_F\cdot 0.5} + 1\text{'}500\, e^{-y_F\cdot 1} + 1\text{'}500\, e^{-y_F\cdot 1.5} \\ &\quad + 1\text{'}500\, e^{-y_F\cdot 2} + 1\text{'}500\, e^{-y_F\cdot 2.5} + 50\text{'}500\, e^{-y_F\cdot 3} \end{aligned}$$

which yields $y_F = 0.055$, which we should have anticipated: if the term structure is flat, all yields, also the forward yield, are equal to the prevailing interest rate. The forward duration D_F of the bond underlying the futures is now determined from

$$\begin{aligned} F_0^{(2)} D_F &= 0.5 \cdot 1\text{'}500\, e^{-y_F\cdot 0.5} + 1 \cdot 1\text{'}500\, e^{-y_F\cdot 1} + 1.5 \cdot 1\text{'}500\, e^{-y_F\cdot 1.5} \\ &\quad + 2 \cdot 1\text{'}500\, e^{-y_F\cdot 2} + 2.5 \cdot 1\text{'}500\, e^{-y_F\cdot 2.5} + 3 \cdot 50\text{'}500\, e^{-y_F\cdot 3} \end{aligned}$$

which gives $D_F = 2.791$ years. Finally, we can calculate the duration of the portfolio:

$$D_{tot} = D_P - D_F \frac{F_0}{P_0} = 1.915 - 2.791 \frac{50\text{'}577.84}{100\text{'}791.43} \text{ years} \approx 0.514 \text{ years.}$$

3. We will employ Black's model for put options—see Ch. V—so we need the forward price G of the underlying asset, and the relevant volatility σ.

Since we know the forward yield $y_F = 0.065$ we can easily calculate G:

$$\begin{aligned} G &= 1\text{'}500 e^{-0.5\cdot 0.065} + 1\text{'}500 e^{-1\cdot 0.065} + 1\text{'}500 e^{-1.5\cdot 0.065} \\ &\quad + 1\text{'}500 e^{-2\cdot 0.065} + 1\text{'}500 e^{-2.5\cdot 0.065} + 51\text{'}500 e^{-3\cdot 0.065} \\ &\approx 49\text{'}186.44 \text{ SEK} \end{aligned}$$

The forward duration D_F is obtained from

$$\begin{aligned} G\,D_F &= 0.5\cdot 1\text{'}500e^{-0.5\cdot 0.065} + 1\cdot 1\text{'}500e^{-1\cdot 0.065} + 1.5\cdot 1\text{'}500e^{-1.5\cdot 0.065} \\ &\quad + 2\cdot 1\text{'}500e^{-2\cdot 0.065} + 2.5\cdot 1\text{'}500e^{-2.5\cdot 0.065} + 3\cdot 51\text{'}500e^{-3\cdot 0.065} \\ &\approx 137122.40 \end{aligned}$$

and we get $D_F = 2.788$ years. The relevant value of σ in Black's model is thus $\sigma = 0.012\cdot D_F = 0.0335$. Finally, the price of the put option is 1'252.76 SEK.

4. Again we need the forward price of the underlying asset and the appropriate value of σ. The value of G is easy; the present price of the cash flow from the bond after the maturity of the option is 1'250 – 133 = 1'117 SEK, hence $G = 1117e^{0,1} \approx$ 1'234.48 SEK. The relevant value of σ is $\sigma = 0.013\cdot 6 = 0.0780$. We get the option's price 20.90 SEK from Black's formula.

5. The forward price of the underlying bond is

$$\begin{aligned} G &= e^{1.5\cdot 0.04}\big(100\,e^{-0.045\cdot 2} + 100\,e^{-0.048\cdot 2.5} + 2\text{'}000\,e^{-0.050\cdot 3}\big) \\ &= \$2\text{'}019.08 \\ &= 100\,e^{-y_F\cdot 0,5} + 100\,e^{-y_F\cdot 1} + 2\text{'}000\,e^{-y_F\cdot 1.5} \\ &\text{for } y_F = 0.06 \end{aligned}$$

The forward duration D_F is calculated from

$$\begin{aligned} G\,D_F &= 0.5\cdot 100\,e^{-y_F\cdot 0,5} + 1\cdot 100\,e^{-y_F\cdot 1} + 1.5\cdot 2\text{'}000\,e^{-y_F\cdot 1.5} \\ &= 2\text{'}891.00, \end{aligned}$$

hence $D_F = 1.429$ years. Now $G = \$2\text{'}019.08$ and $\sigma = 0.01\cdot D_F = 0.01429$. The option's price is \$65.35.

VII: Risk Adjusted Probability Distributions

In Ch. IV we introduced the notion that a forward price can be expressed as an expectation: $G^{(t)}[X] = \mathrm{E}^{(t)}[X]$. This can be generalised; the means of payment need not be a currency, but may be any asset with a positive value. For instance, we can use gold or cigarettes as means of payment. Let us say that we use one pack of cigarettes as the unit of payment. If X represents, say, one kilo of coffee, then the value of X expressed in packs of cigarettes is $\frac{X}{N}$, where now X stands for the value of one kilo of coffee in some currency, and N, the *numeraire,* is the value of one pack of cigarettes expressed in the same currency. The ratio $\frac{X}{N}$ is of course independent of which currency we employ—we need not even think of any currency at all: if $\frac{X}{N} = 2$, it simply means that I can exchange, on the market, two packs of cigarettes for one kilo of coffee.

The "forward price" of one kilo of coffee, measured in packs of cigarettes, to be delivered at time t can thus be expressed as an expectation: $\mathrm{E}^N[\frac{X}{N}]$. Here N is the value of one pack of cigarettes at time t, and X the value of one kilo of coffe at that time. The upper index N on the expectations operator E indicates that the probability distribution pertains to the *numeraire* N*:* one pack of cigarettes at time t. The number of packs of cigarettes I have to deliver at time t in order to get one kilo of coffee at that time, if we draw up a forward contract today, is $n = \mathrm{E}^N[\frac{X}{N}]$. If I want to do the payment today, I can pay for the n packs of cigarettes (to be delivered at time t) today at the cost $P_0^{(t)}[N]\,n$. Hence, the present price of one kilo of coffee to be delivered at time t, $P_0^{(t)}[X]$, is

$$P_0^{(t)}[X] = P_0^{(t)}[N]\,\mathrm{E}^N[\tfrac{X}{N}] \tag{1}$$

Here $P_0^{(t)}[\cdot]$ is the present price in any currency. In this way we have defined a new probability distribution Pr^N such that the probability of any event A, whose outcome is known at time t, is defined by

$$Pr^N(A) = \mathrm{E}^N[I_A] = \frac{P_0^{(t)}[I_A \cdot N]}{P_0^{(t)}[N]}$$

where I_A is the *indicator function* of A, i.e., I_A is equal to one if A occurs and zero otherwise (set $X = I_A \cdot N$ in (1).) This relation is easy to interpret:

> *The probability of any event is the ratio of the value of receiving one unit of N conditional upon the happening of the event, and the value of receiving one unit of N unconditionally.*[1]

This seems to be a reasonable definition of the market's assessment of the probability of an event. It is called the *risk adjusted probability with respect to the numeraire* N. However, this probability may depend on the numeraire N employed. For example, if the event A is "the price of one kilo of gold exceeds 10'000 EUR", then the probability of A, defined by employing the numeraire "one kilo of gold" is reasonably higher than if the numeraire "one EUR" is employed.

If the numeraire $N =$ one unit of some currency, i.e., the numeraire is a zero coupon bond, then the probability distribution is the "usual" forward probability distribution in that currency.

An Example

Assume that we want to price a call option on 1'000 barrels of crude oil when the "strike price" is 3 kilos of gold and the date of maturity is t. We model the value of one barrel of crude oil in terms of gold to be log-Normal with respect to the forward-gold distribution:

$$\frac{X}{N} = Ae^{\sigma\sqrt{t}\,w} \quad w \in N(0,1)$$

where X is the value at time t of one barrel of crude oil, and N is the value at time t of one kilo of gold. The forward-gold distribution of w is $N(0,1)$. The parameter σ has to be somehow assessed by the writer of the contract, e.g. by studying historical data (this parameter is assumed to be the same for the "true" probability distribution as for the forward-oil distribution.) The value of A can be computed using (1):

$$\frac{P_0^{(t)}[X]}{P_0^{(t)}[N]} = \mathrm{E}^N[\tfrac{X}{N}] = Ae^{\frac{1}{2}\sigma^2 t}, \text{ i.e.,}$$

$$A = \frac{P_0^{(t)}[X]}{P_0^{(t)}[N]} e^{-\frac{1}{2}\sigma^2 t}.$$

Hence

$$\frac{X}{N} = \frac{P_0^{(t)}[X]}{P_0^{(t)}[N]} e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w} \quad w \in N(0,1)$$

The value-flow at time t for the long holder of the call option is (x^+ denotes $\max(x, 0)$)

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[1] Thomas Bayes' (1702–1761) definition was: *"The probability of any event is the ratio of the value of which an expectation depending on the event ought to be computed, and the value of the thing expected upon its happening."*

$$(1\text{'}000X - 3N)^+ = \left(1\text{'}000\frac{P_0^{(t)}[X]}{P_0^{(t)}[N]}e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\,w} - 3\right)^+ N$$

The present price of this value-flow is, according to (1)

$$\begin{aligned}
c &= P_0^{(t)}[N]\,\mathrm{E}^N\left[1\text{'}000\left(\frac{P_0^{(t)}[X]}{P_0^{(t)}[N]}e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\,w} - 3\right)^+\right] \\
&= \frac{P_0^{(t)}[N]}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1\text{'}000\frac{P_0^{(t)}[X]}{P_0^{(t)}[N]}e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\,x} - 3\right)^+ e^{-\frac{1}{2}x^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1\text{'}000P_0^{(t)}[X]\,e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\,x} - 3\,P_0^{(t)}[N]\right)^+ e^{-\frac{1}{2}x^2}\,dx
\end{aligned}$$

It may be preferable to replace $P_0^{(t)}[\cdot]$ by $Z_t G_0^{(t)}[\cdot]$ and factor out Z_t to get

$$c = \frac{Z_t}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1\text{'}000G_0^{(t)}[X]\,e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\,x} - 3\,G_0^{(t)}[N]\right)^+ e^{-\frac{1}{2}x^2}\,dx$$

The values $G_0^{(t)}[X]$ and $G_0^{(t)}[N]$ can be observed on the futures market (we assume that forward-price = futures price for these commodities.) The expression for the option value is thus the same as Black's "usual" formula, when K is replaced by $KG_0^{(t)}[N]$, but the volatility σ is now the volatility of the *relative* price of X compared to N.

Forward Distributions for Different Maturities

Now we focus on the situation when the numeraire is a zero coupon bond for some currency. The forward expectation $\mathrm{E}^{(t)}$ will in general depend on the time of maturity t; i.e., in general, if $t < T$, then $\mathrm{E}^{(t)}[X] \neq \mathrm{E}^{(T)}[X]$. However, if interest rates are deterministic, then the two coincide; more specifically, we have the following theorem: let $Z(t,T)$ be the price at time t of a zero coupon bond maturing at T;

Theorem 1

Assume that $t < T$ and that the zero coupon bond price $Z(t,T)$ is known at the present time 0. Then, for any random variable X whose outcome is known at time t,

$$\mathrm{E}^{(t)}[X] = \mathrm{E}^{(T)}[X]$$

Proof:

Consider the following strategy: Enter a forward contract on X maturing at t (so $G^{(t)} = \mathrm{E}^{(t)}[X]$,) invest the net payment $X - G^{(t)}$ in zero coupon bonds maturing at T. In this way we have constructed a contract which pays $Z(t,T)^{-1}\,(X - G^{(t)})$ at time T and costs 0 today. Hence

$$\begin{aligned}
&0 = Z_T\,\mathrm{E}^{(T)}[Z(t,T)^{-1}\,(X - G^{(t)})], \quad \text{i.e.,}\\
&0 = \mathrm{E}^{(T)}[(X - G^{(t)})], \quad \text{i.e.,}\\
&\mathrm{E}^{(T)}[X] = G^{(t)} = \mathrm{E}^{(t)}[X]
\end{aligned}$$

which proves the theorem.

There is another important situation when $\mathrm{E}^{(t)}[X] = \mathrm{E}^{(T)}[X]$:

Theorem 2

Assume that $t < T$ and that the random interest rate $R(t,T)$ is independent of the random variable X whose outcome is known at time t. Then

$$\mathrm{E}^{(t)}[X] = \mathrm{E}^{(T)}[X]$$

Proof:

$$\begin{aligned}
Z_t\mathrm{E}^{(t)}[X] &= Z_t G_0^{(t)}[X] = P_0^{(t)}[X] \overset{(1)}{=} P_0^{(T)}[e^{R(t,T)}X]\\
&= Z_T G_0^{(T)}[e^{R(t,T)}X] = Z_T\mathrm{E}^{(T)}[e^{R(t,T)}X]\\
&\overset{(2)}{=} Z_T\mathrm{E}^{(T)}[e^{R(t,T)}]\,\mathrm{E}^{(T)}[X]\\
&= \$^{-1} Z_T G_0^{(T)}[\$ e^{R(t,T)}]\,\mathrm{E}^{(T)}[X]\\
&= \$^{-1} P_0^{(T)}[\$ e^{R(t,T)}]\,\mathrm{E}^{(T)}[X] \overset{(3)}{=} \$^{-1} P_0^{(t)}[\$1]\,\mathrm{E}^{(T)}[X]\\
&= Z_t\mathrm{E}^{(T)}[X]
\end{aligned}$$

Here (1) is seen as follows: At time t deposit X on the money market account (MMA), and withdraw the balance $e^{R(t,T)}X$ at time T. The cost today is zero, hence $-P_0^{(t)}[X] + P_0^{(T)}[e^{R(t,T)}X] = 0$. The relation (3) is obtained by the same token. The relation (2) holds because of the assumption that X and $R(t,T)$ are independent random variables (w.r.t. the T-forward distribution.)

Hence $\mathrm{E}^{(t)}[X] = \mathrm{E}^{(T)}[X]$, *Q.E.D.*

When $\mathrm{E}^{(t)}[X]$ is independent of the date of maturity t (as long as the outcome of X precedes time t,) we denote this expectation by a star: $\mathrm{E}^*[X]$.

Theorem 3

Assume the random interest rate $R(0,t)$ is independent of the random variable X whose outcome is known at time t. Then the forward price and the futures price coincide: $G_0^{(t)}[X] = F_0^{(t)}[X]$.

We leave the proof as exercise 4.

Exercises and Examples

Interest rates always refer to continuous compounding. Answers are given in parenthesis; solutions are given in the next section.

1. Show that

$$G_0^{(t)}[X] = G_0^{(t)}[N]\,\mathrm{E}^N\left[\tfrac{X}{N}\right]$$

2. It is now May 1:st. Let N be the value of 1 kg of gold August 31:st. We now write a forward contract on $X = 100$ barrels of crude oil to be delivered August 31:st where the payment should be paid in gold. Show rigorously that the forward price should be

$$\frac{G_0^{(t)}[X]}{G_0^{(t)}[N]} \text{ kg of gold.}$$

where May 1:st $= 0$ and Aug 31:st $= t$.

3. Derive an expression for Black's formula, similar to (3) in Ch. V, for a European *exchange option,* meaning that asset X can be exchanged for asset Y, or vice versa.

4. Prove theorem 3.

Solutions

1. This follows immediately from (1) if we divide both sides of the equality with Z_t (see Ch. I.)

2. Let $g[X]$ be the forward price of X in terms of gold. For the long holder of the contract, the cash flow at maturity is then $X - g[X]N$. The present value of this cash flow is zero (= value of *contract.*) Hence

$$0 = P_0^{(t)}\big[X - g[X]N\big] = P_0^{(t)}[X] - g[X]P_0^{(t)}[N].$$

Hence

$$g[X] = \frac{P_0^{(t)}[X]}{P_0^{(t)}[N]} = \frac{G_0^{(t)}[X]}{G_0^{(t)}[N]}$$

(the last equality follows as in exercise 1.)

3. We choose one of the asset values at time t, the date of maturity of the option, as numeraire. Take Y. We specify the relative prices as

$$\frac{X}{Y} = Ae^{\sigma\sqrt{t}\,w} \tag{2}$$

where $w \in N(0,1)$ with respect to the forward-Y probability distribution. The standard deviation $\sigma\sqrt{t}$ of $\ln X - \ln Y$ is assumed to be the same as under the "true" probability distribution (if there is any such thing) and has to be somehow assessed or estimated. Taking E^Y of (2) gives, as in the example above,

$$A\,e^{\frac{1}{2}\sigma^2 t} = \frac{P_0^{(t)}[X]}{P_0^{(t)}[Y]} = \frac{G_0^{(t)}[X]}{G_0^{(t)}[Y]}$$

so

$$\frac{X}{Y} = \frac{G_0^{(t)}[X]}{G_0^{(t)}[Y]} e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w}$$

The payoff of the option is $|X - Y|$, so the value of the option is

$$\begin{aligned} p &= P_0^{(t)}[Y]\,\mathrm{E}^Y\left[\tfrac{|X-Y|}{Y}\right] = Z_t G_0^{(t)}[Y]\mathrm{E}^Y\left[\left|\tfrac{X}{Y} - 1\right|\right] \\ &= Z_t G_0^{(t)}[Y]\mathrm{E}^Y\left[\left|\frac{G_0^{(t)}[X]}{G_0^{(t)}[Y]} e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w} - 1\right|\right] \\ &= Z_t \mathrm{E}^Y\left[\left|G_0^{(t)}[X] e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w} - G_0^{(t)}[Y]\right|\right] \\ &= \frac{Z_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|G_0^{(t)}[X] e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,x} - G_0^{(t)}[Y]\right| e^{-\frac{1}{2}x^2}\,dx \end{aligned}$$

This integral is the expression we were looking for. It can be evaluated analytically, indeed, it is the value of the sum of one call and one put option on X with strike price $G_0^{(t)}[Y]$ and σ is as described above.

4. The fact that the interest rate $R(0,t)$ is independent of X (under the t-forward distribution) implies that

$$\mathrm{E}^{(t)}[X e^{R(0,t)}] = \mathrm{E}^{(t)}[X]\,\mathrm{E}^{(t)}[e^{R(0,t)}].$$

Hence

$$\begin{aligned} F_0^{(t)}[X] = P_0^{(t)}[X e^{R(0,t)}] &= Z_t G_0^{(t)}[X e^{R(0,t)}] \\ &= Z_t \mathrm{E}^{(t)}[X e^{R(0,t)}] = Z_t \mathrm{E}^{(t)}[X]\,\mathrm{E}^{(t)}[e^{R(0,t)}] \\ &= G_0^{(t)}[X]\,P_0^{(t)}[\$ e^{R(0,t)}]\$^{-1} = G_0^{(t)}[X] \end{aligned}$$

VIII: Conditional Expectations and Martingales

The outcome of random variables may occur at different points in time. For instance, if I toss a coin, the outcome X, which is *heads* or *tails*, is random before the coin is actually tossed, but at the time it has been tossed, the outcome is observed.

Let $0 < t < T$ be three points in time where $0 =$ now, let X a random variable whose outcome is observed at time T, and U a random variable whose outcome is observed at the earlier time t. Assume that for *any event A whose outcome can be observed at time t,* the relation

$$\mathrm{E}[I_A X] = \mathrm{E}[I_A U] \tag{1}$$

holds. We then say that U is the *conditional expectation at time t* of the variable X. It is easy to see that the variable U is essentially uniquely determined by X, i.e., if U_1 and U_2 both satisfy $\mathrm{E}[I_A X] = \mathrm{E}[I_A U_i]$ for all A, then $U_1 = U_2$ with probability $= 1$.

The conditional expectation is denoted $\mathrm{E}_t[X]$, i.e., $U = \mathrm{E}_t[X]$. The conditional expectation $\mathrm{E}_0[X]$ is the same as the unconditional expectation $\mathrm{E}[X]$. To see this, note that I_A for an event whose outcome is observed now, is either the constant 1 or the constant 0.

The definition of $\mathrm{E}_t[X]$ may seem a bit strange, but the interpretation is as follows: if we at time t form the expectation of X, *using all relevant information available at the time,* then $\mathrm{E}_t[X]$ is the sensible choice. Note that as seen from a point in time earlier than t, this expectation is random, since we do not have all the information to be revealed at time t when the expectation is formed.

We can also see that the conditional expectation behaves in accordance with the intuitive notion from the following properties:

Properties

- If X is stochastically independent of any random variable observed at time t, then the conditional expectation equals the unconditional expectation: $\mathrm{E}_t[X] = \mathrm{E}[X]$.

- More generally, if X is stochastically independent of any random variable observed at time t, and Y is a random variable observed at time t, then $\mathrm{E}_t[YX] = Y\mathrm{E}[X]$.

The first property is the special case of the second when $Y \equiv 1$. In order to see the second property, recall that if two random variables are independent, then the expected value of their product is the product of their expected values. Hence, if A is any event whose outcome is observed at time t, then

$$\mathrm{E}[I_A YX] = \mathrm{E}[I_A Y]\mathrm{E}[X] = \mathrm{E}\big[I_A Y \mathrm{E}[X]\big].$$

Hence $\mathrm{E}_t[YX] = Y\mathrm{E}[X]$.

Note in particular the *law of iterated expectations*

$$\mathrm{E}\big[\mathrm{E}_t[X]\big] = \mathrm{E}[X] \tag{2}$$

which is obtained by letting A be the event "anything" in (1) so that $I_A = 1$. It is even true that

$$\mathrm{E}_s\big[\mathrm{E}_t[X]\big] = \mathrm{E}_s[X] \quad \text{if } s < t. \tag{2'}$$

The proof of this relation follows immediately from the definition (1): Let A be any event with outcome at time s, and denote $Y = \mathrm{E}_t[X]$. Note that the outcome of A is known also at time $t > s$, hence

$$\mathrm{E}\big[I_A \mathrm{E}_s[Y]\big] = \mathrm{E}[I_A Y] = \mathrm{E}\big[I_A \mathrm{E}_t[X]\big] = \mathrm{E}[I_A X].$$

It follows, by definition, that $\mathrm{E}_s[Y] = \mathrm{E}_s[X]$, *Q.E.D.*

Assume now that the time interval from 0 to T is discretised by the points $0 = t_0 < t_1, \ldots < t_n = T$, and that we have a sequence of random variables X_j, $j = 0, \ldots, n$, where the outcome of X_j is observed at time t_j. Such a sequence is a *stochastic process.* A stochastic process is a *martingale* if it is true that

$$X_j = \mathrm{E}_{t_j}[X_{j+1}] \quad \text{for } j = 0, \ldots, n-1. \tag{3}$$

It follows immediately from (2′) that in order for a stochastic process to be a martingale, it suffices that $X_j = \mathrm{E}_{t_j}[X_n]$ for $j < n$.

Martingale Prices

Consider a contract that gives the random payoff X at time T. The forward price $G_t^{(T)}$, $0 \le t < T$, of this contract at time t is a random variable whose outcome is determined at time t.

Theorem

For any t, $0 \le t \le T$, the forward price $G_t^{(T)}$ equals the conditional expectation

$$G_t^{(T)} = \mathrm{E}_t^{(T)}[X]$$

Hence, the forward process has the martingale property (3) w.r.t. the forward distribution, i.e.,

$$G_{t_j}^{(T)} = \mathrm{E}_{t_j}^{(T)}[G_{t_{j+1}}^{(T)}]$$

Proof

Consider the following strategy: Let A be any event whose outcome occurs at time t. Wait until time t and then, if A has occurred, enter a forward contract on X maturing at T with forward price $G_t^{(T)}$, but if A has not occurred, do nothing. We have thus constructed a contract which gives payoff $I_A\,(X - G_t^{(T)})$ at time T which costs 0 today. Hence

$$Z_T\,\mathrm{E}^{(T)}[I_A(X - G_t^{(T)})] = 0 \quad \text{i.e.,} \quad \mathrm{E}^{(T)}[I_A X] = \mathrm{E}^{(T)}[I_A G_t^{(T)}]$$

Since this is true for any event A whose outcome is known at time t, we have by the definition of conditional expectation

$$G_t^{(T)} = \mathrm{E}_t^{(T)}[X]$$

Q.E.D.

Remark 1

From this we deduce (c.f. Ch. I) that the present price p_t at time t of the contract yielding X at time T is

$$p_t = Z(t,T)\mathrm{E}_t^{(T)}[X]$$

where $Z(t,T)$ is the price at time t of a zero coupon bond maturing at time T. Especially, if the interest rate is *deterministic* and equal to r, then the present prices satisfy

$$p_j = e^{-r\,\Delta t}\,\mathrm{E}_j^*[p_{j+1}] \tag{4}$$

where p_j is the present price as of time t_j and $\Delta t = t_{j+1} - t_j$; E^* denotes expectation w.r.t. the forward distribution (which is independent of maturity date, according to Theorem 1.)

Remark 2

The Martingale property holds for distributions referring to any numeraire. By the same token as above, it follows that

$$\frac{P_t^{(T)}[X]}{P_t^{(T)}[N]} = \mathrm{E}_t^N[\tfrac{X}{N}]$$

IX: Asset Price Dynamics and Binomial Trees

In Black's model, there are only two relevant points in time: the time when the contract is written, and the time when it matures. In many cases we have to consider also what happens in between these two points in time.

First some notation. Time is discrete, $t = t_0 < \ldots < t_n = T$, and we let for notational simplicity $t_0 = 0$ and all time spells $t_k - t_{k-1} = \Delta t$ be equally long; hence $t_k = k\Delta t$. Let X be some random value whose outcome becomes known at time T; we are interested in evaluating the prices of derivatives of X.

Black-Scholes Dynamics

The Black-Scholes assumption about the random behaviour of X is that

$$X = Ae^{\sigma \, \Sigma_1^n w_j} \tag{1}$$

where A is some constant, σ a parameter called the *volatility,* and the w_j:s are stochastically independent Normal $\mathrm{N}(0, \Delta t)$ random variables (Δt is the variance; the standard deviation is thus $\sqrt{\Delta t}$) where the outcome of w_j occurs at time t_j.

This is the model for the price dynamics under the forward (T) probability distribution. A problem is the assessment of the volatitity σ. Sometimes it is estimated from real data, i.e., as if the specification (1) holds under a historical natural probability distribution. The constant A, in contrast, must be calculated from real prices and may differ depending on which numeraire probability distribution we employ.

Now we take expectations $\mathrm{E}^{(T)}$ of both sides of (1). Since $G_0^{(T)} = \mathrm{E}^{(T)}[X]$, it follows that $A = G_0^{(T)} e^{-\frac{1}{2}\sigma^2 T}$.

Binomial Approximation

In almost all cases when the whole price path has to be taken into account, it is necessary to use some numerical procedure to calculate the price of the derivative. A useful numerical procedure is to approximate the specification (1) by a *binomial tree.* In (1), we replace w_j by a binary variable b_j which takes the value $-\sqrt{\Delta t}$ with probability (under the forward probability distribution) 0.5 and the value $+\sqrt{\Delta t}$ with probability 0.5. For large values of n, and small Δt, this is a good approximation—in fact, as $n \to \infty$ and $\Delta t \to 0$ the price of a derivative calculated from the binomial tree will converge to the theoretical value it would have from the specification (1).

The binomial specification is thus

$$X = B\, e^{\sigma\, \Sigma_1^n b_j}$$

Note that $\mathrm{E}^{(T)}[e^{\sigma b_j}] = \cosh(\sigma\sqrt{\Delta t})$. In order to ease notation, we write G_k for $G_{t_k}^{(T)}$. We know from the previous chapter that $G_k[X] = \mathrm{E}_k^{(T)}[X]$, so

$$G_k = B\, e^{\sigma\, \Sigma_1^k b_j} \cosh^{n-k}(\sigma\sqrt{\Delta t})$$

Combining this relation with the same for G_{k+1}, we get the relation

$$G_{k+1} = G_k \frac{e^{\sigma\, b_{k+1}}}{\cosh(\sigma\sqrt{\Delta t})} = G_k\big(1 \pm \varepsilon\big) \quad \text{where } \varepsilon = \tanh(\sigma\sqrt{\Delta t})$$

and the plus and minus sign occur with probability (under the forward probability distribution) 0.5 each.

Binomial trees were introduced by J. Cox, S. Ross, and M. Rubenstein in "Option pricing: A simplified approach", *Journal of Financial Economics* 7 (1979), 87–106, and has been a very extensively used numerical procedure. The idea is that the binary variable

$$b = \begin{cases} \sqrt{\Delta t} & \text{with probability } 0.5 \\ -\sqrt{\Delta t} & \text{with probability } 0.5 \end{cases}$$

has the same first and second moments (expectation and variance) in common with a Normal $N(0, \Delta t)$ variable. Hence, a sum of many independent such variables will have almost the same distribution as a corresponding sum of $N(0, \Delta t)$ variables (Bernulli's theorem, or the Central Limit Theorem.)

The Binomial Model

In the previous section we derived the binomial specification of the forward price dynamics of the asset that will be underlying a derivative. The forward price refers to a forward contract that matures at the same time T as the derivative, or later. If we write the dynamics in the usual tree style, we have for the forward price dynamics

$$\begin{array}{ccccccc}
G_0 & G_0\,u & G_0\,u^2 & G_0\,u^3 & G_0\,u^4 & \cdots & G_0\,u^n \\
 & G_0\,d & G_0\,du & G_0\,du^2 & G_0\,du^3 & \cdots & G_0\,du^{n-1} \\
 & & G_0\,d^2 & G_0\,d^2u & G_0\,d^2u^2 & \cdots & G_0\,d^2u^{n-2} \\
 & & & G_0\,d^3 & G_0\,d^3u & \cdots & G_0\,d^3u^{n-3} \\
 & & & & \vdots & \cdots & \vdots \\
 & & & & & & G_0\,d^n
\end{array}$$

In this diagram, each column represents an instant of time, and moving from one instant to the next (one step right) means that the forward price either moves "up" or "down". In this diagram "up" means going one step to the right on the same line, and "down" going one step to the right on the line below (this configuration is convenient in a spreadsheet.) Under the forward distribution the probabilities of moving up and down are both $\frac{1}{2}$. The specification of u and d are as follows:

$$\begin{cases} u = 1 + \varepsilon \\ d = 1 - \varepsilon \end{cases} \qquad \text{where } \varepsilon = \tanh(\sigma\sqrt{\Delta t})$$

where Δt is the time spell between columns, and σ is the volatility of the forward price.

Pricing an American Futures Option

As long as we assume that interest rates are deterministic, the forward prices in the binomial tree equals the corresponding futures prices. Hence we can use this tree for calculating the price of an *American futures option.* In order to do so, we construct a tree of prices of the futures option. In this tree the prices are calculated from right to left. If the time for maturity of the option is at $T = t_4$, then we construct the following tree of prices for the futures option:

$$\begin{array}{ccccc} p & p_{1,0} & p_{2,0} & p_{3,0} & p_{4,0} \\ & p_{1,1} & p_{2,1} & p_{3,1} & p_{4,1} \\ & & p_{2,2} & p_{3,2} & p_{4,2} \\ & & & p_{3,3} & p_{4,3} \\ & & & & p_{4,4} \end{array}$$

Here the prices in the rightmost column are known, since they represent the value of the option at maturity when the underlying futures price is that in the corresponding position of the tree of futures prices. Then option prices are calculated backwards:

$$\hat{p}_{j,k} = e^{-r\Delta t}\big(0.5\, p_{j+1,k} + 0.5\, p_{j+1,k+1}\big) \tag{1}$$

$$p_{j,k} = \max(\hat{p}_{j,k}, \text{value if exercised}) \tag{2}$$

Indeed, $\hat{p}_{j,k} = e^{-r\Delta t}\mathrm{E}^*[p_{j+1,\bullet}]$, which is just another way of writing (1), is the value if the option is not exercised (see (4) of Ch. VIII,) and the highest value of this and the value if exercised is the present value. In order to calculate the value if exercised, we employ the original tree of futures prices, of course. The end result p is the price of the option today.

American Call Option on a Share of a Stock

If we want to price an *American* call option on a share of a stock which pays no dividend before maturity, we need a binomial tree for the current stock price, not the futures price of the stock. The assumption is still that the forward price dynamics is as above, and we derive the stock prices from there. This is easy; according to Ch. II, example 1 with $d = 0$, the relation between the current stock price S_t and the forward price at the same date G_t is $S_t = G_t\, e^{-r(T-t)}$. The relevant binomial tree for the stock price is thus

$$\begin{array}{lllll}
S_0 & S_0\, e^{r\Delta t}\, u & S_0\, e^{2r\Delta t}\, u^2 & S_0\, e^{3r\Delta t}\, u^3 & \cdots \\
 & S_0\, e^{r\Delta t}\, d & S_0\, e^{2r\Delta t}\, du & S_0\, e^{3r\Delta t}\, du^2 & \cdots \\
 & & S_0\, e^{2r\Delta t}\, d^2 & S_0\, e^{3r\Delta t}\, d^2u & \cdots \\
 & & & S_0\, e^{3r\Delta t}\, d^3 & \cdots
\end{array}$$

The price of the option is then calculated in the same way as above.

Options on Assets Paying Dividends

Known Dividend Amount

Assume we want to price an option on a share of a stock when just before time i the stock pays a known dividend d. We need to set up a binomial tree for the current price S_t of the stock. Let $n > i$ be the period at which time T the option matures.

For any time $j \le n$ we have the relation

$$S_j = \begin{cases} e^{-r(i-j)\Delta t}\, d + e^{-r(n-j)\Delta t}\, G_j & \text{if } j < i \\ e^{-r(n-j)\Delta t}\, G_j & \text{if } i \le j \le n \end{cases} \tag{3}$$

This follows as in Ch. II, example 1.

In particular, we have

$$G_0 = S_0\, e^{rn\Delta t} - d\, e^{r(n-i)\Delta t} \tag{4}$$

It is now easy to build the tree of stock prices: First compute G_0 from (4), then construct the tree for G_j, and from these prices, compute S_j from (3).

Known Dividend Yield

Now assume instead that at some time between times $i - 1$ and i the stock will pay a dividend $d\, S_i'$ where S_i' is the stock price the moment before the dividend is paid out.

The relation between the stock price S_j and the futures price G_j for times j later or at time i is as before

$$S_j = e^{-r(n-j)\Delta t}\, G_j \quad j \geq i \tag{5}$$

For times j before i we can compute the stock price in accordance with example 2 of Ch. II:

$$S_j(1-d) = e^{-r(n-j)\Delta t}\, G_j$$

from which it follows that

$$S_j = \frac{1}{1-d} G_j\, e^{-r(n-j)\Delta t} \quad j < i \tag{6}$$

In particular,

$$G_0 = (1-d) S_0\, e^{rn\Delta t} \tag{7}$$

It is now easy to build the tree of stock prices: First compute G_0 from (7), then construct the tree for G_j, and from these prices, compute S_j from (5) and (6).

Options on Consumption Assets

Usually an option on a consumption asset is designed as a futures option, but there are exceptions. If we are about to price an option (or other derivative) where the underlying value is the spot price of some consumption asset (as opposed to investment asset,) then there is no obvious relationship between spot prices and forward prices, so a binomial tree of the latter is of no use. Instead, we should construct a binomial tree of spot prices which is consistent with *current* forward prices with maturity date equal to the corresponding future spot price. A tree of the spot prices S_t is

$$\begin{array}{lllll}
S_0 & G_0^1\, u & G_0^2\, u^2 & G_0^3\, u^3 & G_0^4\, u^4 \\
 & G_0^1\, d & G_0^2\, du & G_0^3\, du^2 & G_0^4\, du^3 \\
 & & G_0^2\, d^2 & G_0^3\, d^2 u & G_0^4\, d^2 u^2 \\
 & & & G_0^3\, d^3 & G_0^4\, d^3 u \\
 & & & & G_0^4\, d^4
\end{array}$$

We leave as an exercise to check that in this tree, $G_0^{(t)} = \mathrm{E}^*[S_t]$.

Comments

There is more than one way to specify the binomial model. The one given here is slightly different from the original model by Cox, Ross and Rubenstein. I find the specification given here more appealing, and also easier to generalise to different situations (see for example Ch. XII, where we discuss an interest rate model.) It is important not to mix up different specifications!

One can construct binomial trees for various underlying assets: indices, currencies, futures, and shares paying various forms of dividends. In order to make these appear less ad-hoc, I do this in two steps: First we construct a tree over the *forward* prices of the underlying asset. These trees are constructed the same way independent of underlying asset. Then we construct a tree over the relevant spot prices (unless the underlying is a futures contract,) where we use the techniques from Ch. II. Finally, we construct the tree over the derivative price.

Many texts construct the binomial tree directly for the spot prices underlying the derivative, without first specifying one for the corresponding forward prices. This is a somewhat risky approach: the tree of spot prices must be consistent with the fact that the corresponding forward prices have the Martingale property. In order to ensure that this happens, I prefer to *first* make a tree over the forward prices such that the Martingale property is satisfied. Now I know that there are no arbitrage possibilities lurking in the model; thereafter I derive the relevant spot prices, and I can feel safe that the model is sound.

One can also use *trinomial* trees, where the $N(0, \Delta t)$ variable is approximated by the "trinary" variable

$$u = \begin{cases} \sqrt{3\Delta t} & \text{with probability } \frac{1}{6} \\ 0 & \text{with probability } \frac{2}{3} \\ -\sqrt{3\Delta t} & \text{with probability } \frac{1}{6} \end{cases}$$

This variable shares the same *five* first moments with the $N(0, \Delta t)$ variable, so the approximation is better in a corresponding trinomial tree compared to a binomial tree.

Trinomial trees are also used in many interest rate models (Hull and White.)

Exercises and Examples

Interest rates always refer to continuous compounding. Answers are given in parenthesis; solutions to some problems are given in the next section

In these examples we use binomial trees with very few time steps. In a "real" situation one would of course use many more, maybe 50–100.

1. The futures price of coffee (100 kg) to be delivered in four weeks is \$70. The volatility is 2% in a week. The risk-free rate of interest is 0.1% per week. In the following cases, use a binomial tree with time interval of one week.

   a) Determine the price of a European futures put option on coffee (100 kg) with strike price \$72. The option matures in four weeks (which is also the maturity of the futures.) (\$2.4255)

   b) The same question, though for an American option, ceteris paribus (latin: "everything else being the same".) (\$2.4287)

2. A share costs today \$13.28. The volatility is 2% in a week, and the share pays no dividends the coming month. The risk-free rate of interest is 0.1% a week. In the following cases, use a binomial tree with time interval one week.

   a) Determine the price of an American put option with maturity in four weeks at a strike price of \$13.70. (\$0.464143)

   b) The same question, but the share pays a dividend of \$0.27 in just under two weeks (that is just before time 2 in the binomial tree.) Also, the current price of the share is \$13.55, ceteris paribus. (\$0.462168)

3. Determine the price of an American option to buy 100 GBP at 13 SEK per pound, which is the value of the pound today. The time to maturity is one year, the pound's volatility relative to the Swedish crown is 10% in one year. The rate of interest of the Swedish crown is 3% a year, and the one year forward exchange rate is 12.60 SEK for one GBP. Use a binomial tree to solve the problem, with time interval tree months. (37.70 SEK)

4. Determine the price of an American call option on a share whose present price is 100 SEK with maturity in one year with strike price 98 SEK. The share's volatility is 20% in one year, the rate of interest is 6% a year, and the share will pay a dividend in 5.5 months of 4% of the value of the share at that time. Use a binomial tree with time interval three months. (9.345 SEK)

5. Calculate the price of an eight month American call option on a maize futures when the present futures price is \$26.4, the strike price of the option is 26.67, the risk-free rate of interest is 8% and the volatility of the futures price is 30% over one year. Use a binomial tree with time interval two months. (\$2.3515)

6. A two month American put option on an index of shares has the strike price 480. The present value of the index is 484, the risk-free rate of interest is 10% per year, the dividend of the index is 3% per year and the volatility is 25% over a year. Determine the value of the option by using a binomial tree with time interval of half a month. (15.2336)

7. The spot price of copper is \$0.60 per pound. Assume that, at present, the futures price of copper is

maturity	futures price
3 months	0.59
6 months	0.57
9 months	0.54
12 months	0.50

The volatility of the price of copper is 40% during one year and the risk-free interest rate is 6% a year. Use a suitable binomial tree with time interval three months to estimate the price of an American call option on copper with strike price \$0.60 and maturity in one year. (\$0.063167.)

Solutions

1.

4	75,7695	72,7985	69,9440	67,2015	64,5665
3	74,2840	71,3713	68,5728	65,8840	
2	72,8276	69,9720	67,2284		
1	71,3998	68,6002			
0	70,0000				
4	0,0000	0,0000	2,0560	4,7985	7,4335
3	0,0000	1,0270	3,4238	6,1099	
2	0,5130	2,2232	4,7621		
1	1,3667	3,4891			
0	2,4255				
4	0,0000	0,0000	2,0560	4,7985	7,4335
3	0,0000	1,0270	**3,4272**	**6,1160**	
2	0,5130	2,2249	**4,7716**		
1	1,3676	3,4948			
0	2,4287				

Here are three binomial trees (the display shows decimal *commas*, which is Swedish standard) cut from an Excel spreadsheet. Note that time goes from bottom to top; I think this is the easiest way to arrange binomial trees in a spreadsheet.

The first tree displays the futures (=forward) prices of coffee. Week 0 (today) it is \$70, week 1 it is either \$71.3998 or \$68.6002, each with probability 0.5 *under the forward probability measure.* Week two, the futures price is \$72.8276 or \$69.9720 if the futures price were \$71.3998 week 1, and \$69.9720 or \$67.2284 if the futures price were \$68.6002 week 1; and so on. This tree is generated in accordance with the first tree (forward prices) appearing in Ch. IX.

The second tree displays the value of the European put option. It is generated from top to bottom: week 4 the option is worth 0 if the futures price is more than or equal to \$72, otherwise it is worth \$72 – futures price (where the futures price equals the spot price of coffee, since the futures contract matures week 4.) Hence we get the first row (week 4) easily from the previous tree's week 4. Next we compute the week 3 option prices, employing (4) of Ch. VIII: the price week 3 is the discounted value of the expected value week 4 under the forward probability measure. Hence, each price week 3 is $e^{-0.001}$ times $(0.5 \cdot p_1 + 0.5 \cdot p_2)$ where p_1 and p_2 are the prices above and right-above respectively, For example, $3.4238 = e^{-0.001}(0.5 \cdot 2.0560 + 0.5 \cdot 4.7985)$.

Going down through the tree, we finally find the price week 0 to be \$2.4255.

The third tree displays the values of the corresponding American put option. It is obtained as in the tree on the European option, except that the price is replaced by the exercise value if that is higher. These numbers are in boldface. For instance, the value \$3.4272 is the value if the option is exercised: $3.4272 = 72 - 68.5728$ which is more than $e^{-0.001}(0.5 \cdot 2.0560 + 0.5 \cdot 4.7985)$.

Going down through this tree, we end up with the price \$2.4287 for the American option.

2a.

4	14,4322	13,8663	13,3226	12,8002	12,2983
3	14,1492	13,5944	13,0614	12,5492	
2	13,8718	13,3279	12,8053		
1	13,5999	13,0666			
0	13,3332				

4	14,4322	13,8663	13,3226	12,8002	12,2983
3	14,1351	13,5808	13,0483	12,5367	
2	13,8441	13,3013	12,7797		
1	13,5591	13,0275			
0	13,2800				

4	0,000000	0,000000	0,377435	0,899820	1,401723
3	0,000000	0,188529	**0,651683**	**1,163315**	
2	0,094170	0,419686	**0,920285**		
1	0,256671	**0,672544**			
0	0,464143				

First we compute the forward price of the share according to Ch. II: $G_0 = S_0\, e^{0.004} = \$13.3332$. Starting out with this value, we construct the first tree of forward prices. Now we know that the forward prices of the underlying share indeed have the Martingale property under the forward probability measure.

Next we construct the tree of spot prices of the share: for example, week 2 the spot price $S_2 = G_2\, e^{-2\cdot 0.001}$, so we get the spot prices by just discounting the values in the forward price tree. Of course, week 0 we get the initial spot price of the share.

Now we construct the tree of option prices. We start with the prices week 4. Next week three: these prices are computed as the mean of the two prices above and right-above, discounted by the one period interest rate. However, we must check that it isn't profitable to exercise the option. This happens in a few places; these are shown in boldface. Finally, the option price is found to be \$0.464143.

2b.

4	14,4327	13,8668	13,3231	12,8007	12,2988
3	14,1498	13,5950	13,0619	12,5497	
2	13,8724	13,3284	12,8058		
1	13,6004	13,0671			
0	13,3338				
4	14,4327	13,8668	13,3231	12,8007	12,2988
3	14,1356	13,5814	13,0488	12,5372	
2	13,8447	13,3018	12,7802		
1	13,8294	13,2977			
0	13,5500				
4	0,000000	0,000000	0,376894	0,899300	1,401223
3	0,000000	0,188259	**0,651153**	**1,162805**	
2	0,094035	0,419286	**0,919766**		
1	0,256404	0,668857			
0	0,462168				

First we compute the forward price of the underlying share according to example 1, Ch. II. Starting from this value, we construct the first tree which contains the forward prices.

Next we compute the tree of spot prices, again using example 1 of Ch. II. For the time periods 2 and 3 this means just discounting the forward price (example 1 with d=0.)

Finally, we compute the tree of option prices in the same way as in part a. Early exercise is shown in boldface. The present price of the option is $0.462168.

3.

4	15,3130	13,8557	12,5372	11,3441	10,2646
3	14,5843	13,1965	11,9406	10,8043	
2	13,8904	12,5686	11,3725		
1	13,2295	11,9705			
0	12,6000				
4	15,3130	13,8557	12,5372	11,3441	10,2646
3	14,6987	13,3000	12,0343	10,8891	
2	14,1092	12,7665	11,5516		
1	13,5432	12,2544			
0	13,0000				
4	231,2950	85,5730	0,0000	0,0000	0,0000
3	**169,8736**	42,4668	0,0000	0,0000	
2	**110,9159**	21,0748	0,0000		
1	65,5022	10,4586			
0	37,6966				

First we construct the three of forward exchange rates.

According to example 5, Ch. II, the relation between forward exchange rate G and the spot exchange rate X is $X = G\,e^{\Delta r/12\,\tau}$ where τ is the number of months to maturity and Δr is the foreign interest rate minus the domestic rate. For $\tau = 12$ we have $13 = 12.60\,e^{\Delta r}$ which gives $\Delta r = 0.03125$. We use this value to create the second tree containing the spot exchange rates.

Finally, we compute the last tree, the tree of option prices in the usual way. Here we discount with the SEK interest rate, of course (3% per year.) Early exercise is labeled in boldface. The price of the option is 37.70 SEK.

4.

4	149,0648	122,0440	99,9212	81,8085	66,9792
3	135,5544	110,9826	90,8648	74,3938	
2	123,2685	100,9237	82,6293		
1	112,0961	91,7765			
0	101,9363				
4	149,0648	122,0440	99,9212	81,8085	66,9792
3	133,5363	109,3303	89,5120	73,2863	
2	119,6254	97,9410	80,1873		
1	111,6287	91,3939			
0	100,0000				
4	51,0648	24,0440	1,9212	0,000000	0,000000
3	36,9953	12,7893	0,9463	0,000000	
2	24,5217	6,7655	0,4661		
1	15,4107	3,5620			
0	9,3451				

The forward price of the underlying share is $100\,(1-0.04)\,e^{0.06} = 101.9363$ SEK (example 2, Ch. II.) We construct the first tree: forward prices of the underlying asset.

From the forward prices we compute the spot prices by just discounting (example 1 with d=0, Ch. II) for times later than 5.5 months. For times previous to 5.5 months we discount and divide by 0.96 (example 2, Lecture Note 2.) Thus we obtain the second tree of spot prices.

Finally, we construct the tree of option prices in the usual way. Early exercise is never profitable. Note that for non-paying dividend stocks, early exercise is never profitable for a call option, but in this case we couldn't know, since there is a dividend payment. The option price is 9.3451 SEK.

6.

4	597,4494	539,4810	487,1370	439,8718	397,1926
3	568,4652	513,3090	463,5044	418,5322	
2	540,8871	488,4067	441,0183		
1	514,6469	464,7125			
0	489,6797				
4	597,4494	539,4810	487,1370	439,8718	397,1926
3	566,8096	511,8141	462,1545	417,3133	
2	537,7411	485,5660	438,4532		
1	510,1634	460,6640			
0	484,0000				
4	0,0000	0,0000	0,0000	40,1282	82,8074
3	0,0000	0,0000	19,9807	**62,6867**	
2	0,0000	9,9488	**41,5468**		
1	4,9537	25,6407			
0	15,2336				

The forward value of the index is $484\,e^{(0.10-0.03)\cdot 2/12)} \approx 489.6797$ (Ch. II, example 4.) We use this value to generate the tree of forward values of the index.

Next we construct the tree of spot index values, again using example 5 of Ch. II.

Finally we construct the tree of option prices; early exercise is as usual shown in boldface. The present option value is 15.2336

7.

4	1,027759	0,688927	0,461802	0,309555	0,207501
3	0,927011	0,621394	0,416533	0,279210	
2	0,817213	0,547795	0,367198		
1	0,706451	0,473549			
0	0,600000				
4	0,427759	0,088927	0,000000	0,000000	0,000000
3	**0,327011**	0,043802	0,000000	0,000000	
2	**0,217213**	0,021575	0,000000		
1	0,117617	0,010627			
0	0,063167				

The first tree is a tree on spot prices on copper. We know from Ch. IV that the forward price is the expected value w.r.t. the forward probability measure of the spot price at the time of maturity: $G_0^{(t)} = \mathrm{E}^*[S_t]$. In order to achieve this, we construct the tree like this (t=quarters:)

$$\begin{array}{lllll} G_0^4\,u^4 & G_0^4\,u^3 d & G_0^4\,u^2 d^2 & G_0^4\,ud^3 & G_0^4\,d^4 \\ G_0^3\,u^3 & G_0^3\,u^2 d^1 & G_0^3\,ud^2 & G_0^3\,d^3 & \\ G_0^2\,u^2 & G_0^2\,ud & G_0^2\,d^2 & & \\ G_0^1\,u & G_0^1\,d & & & \\ S_0 & & & & \end{array}$$

You can check that in this tree, $G_0^{(t)} = \mathrm{E}^*[S_t]$ (see notation in Ch. VII.)

The tree of option prices is now calculated as usual. Early exercise in boldface.

X: Random Interest Rates: The Futures Distribution

In order to study interest rate derivatives, or other situations where interest rates are assumed to be random and the underlying asset depends on the interest rate, we need a different probability distribution than the forward distribution. One reason for this is that when interest rates are random, forward distributions for different maturity dates are not the same. Rather than having zero coupon bonds as numeraires we use the money market account, MMA, (see Ch. I) as a numeraire. Hence, let the numeraire N be $N = \$e^{R(0,t)}$ delivered at time t, where we have used the notation introduced in Ch. I and VII. Let $X = \$W$ be a random payment at time t (W is a dimensionless random variable.) Then (see (1) of Ch. VII)

$$\begin{aligned} F_0^{(t)}[X] &= P_0^{(t)}[\$We^{R(0,t)}] = P_0^{(t)}[\$e^{R(0,t)}]\,\widehat{\mathrm{E}}[W] \\ &= \$1\,\widehat{\mathrm{E}}[W] = \widehat{\mathrm{E}}[X] \end{aligned} \tag{1}$$

where $\widehat{\mathrm{E}}[\cdot]$ denotes expectation w.r.t. the probability distribution associated with the numeraire $N = \$e^{R(0,t)}$. It seems natural to call the probability distribution associated with this numeraire the *futures distribution.* In the literature it is often called the *Equivalent Martingale Measure* (Hull uses the term "traditional risk neutral measure.")

It is important to note that the forward distributions (pertaining to expectation $\mathrm{E}^{(T)}$) differ for different maturities T when interest rates are random. This is not the case for the futures distribution, though. Indeed, we will now prove two theorems on the futures distribution:

Theorem 1

The futures distribution as defined above is independent of the date of maturity T in the following sense: if the outcome of X is known at time t_n then $\widehat{\mathrm{E}}^{(n)}[X] = \widehat{\mathrm{E}}^{(m)}[X]$ if $m > n$, where we denote by $\widehat{\mathrm{E}}^{(k)}$ the futures distribution for contracts maturing at time t_k.

Hence there is no need to index the futures distribution by maturity date.

In order to ease notation, we write F_j for F_{t_j}.

Theorem 2

The futures prices $\{F_j\}$ have the martingale property w.r.t. the futures distribution: $F_j = \widehat{\mathrm{E}}_{t_j}[F_k]$ for $j < k$. In particular, the futures price F_0 is the expected value w.r.t. the futures distribution of X, the spot price at delivery.

Proof of Theorem 1

Let P_0 be the present price of the value $Xe^{R(0,n)}$ to be delivered at time t_n. We can convert this contract to one where instead the value $Xe^{R(0,m)}$ is delivered at time t_m by simply depositing the payoff $Xe^{R(0,n)}$ in the money market account up to time t_m. The present price is of course the same. Using the theorem in Ch. I, we have the following equalities from the two contracts:

$$P_0 = P_0^{(t_n)}[Xe^{R(0,n)}] = F_0^{(t_n)}[X] \quad \text{and}$$
$$P_0 = P_0^{(t_m)}[Xe^{R(0,m)}] = F_0^{(t_m)}[X]$$

By (1) this implies that

$$\widehat{\mathrm{E}}^{(n)}[X] = \widehat{\mathrm{E}}^{(m)}[X]$$

Q.E.D.

Proof of theorem 2

Consider the following strategy: Let A be any event whose outcome is known at time $t_j < t_k$. At time t_{k-1} enter a long position of $e^{R(0,k)}$ futures contracts if A has occurred, otherwise do nothing. At time t_k collect $I_A(F_k - F_{k-1})e^{R(0,k)}$ (which may be negative) and close the contract. This gives the payment $I_A(F_k - F_{k-1})e^{R(0,k)}$ at time t_k and no other cash flow. The present price is zero, hence

$$0 = P_0^{(t_k)}[I_A(F_k - F_{k-1})e^{R(0,k)}] = F_0[I_A(F_k - F_{k-1})] = \widehat{\mathrm{E}}[I_A(F_k - F_{k-1})]$$

Hence,

$$\widehat{\mathrm{E}}[I_A F_k] = \widehat{\mathrm{E}}[I_A F_{k-1}]$$

Employing this equality repeatedly gives, for $j < k$:

$$\widehat{\mathrm{E}}[I_A F_j] = \widehat{\mathrm{E}}[I_A F_k]$$

Since A can be any event whose outcome is known at time t_j, this means that (Ch. VIII)

$$F_j = \widehat{\mathrm{E}}_{t_j}[F_k] \quad \text{for } j < k.$$

This proves theorem 2.

Theorem 3

The following formula for the present price p_j as of time t_j for the asset Y to be delivered at time $t_k > t_j$ holds:

$$p_j = \widehat{\mathrm{E}}_{t_j}[Ye^{-R(j,k)}].$$

Proof

With the notation of Ch. I,

$$p_j = F_j^{(k)}[Ye^{-R(j,k)}] = \widehat{\mathrm{E}}_{t_j}[Ye^{-R(j,k)}]$$

Q.E.D.

XI: A Model of the Short Interest Rate: Ho-Lee

We divide time into short time intervals $t_0, < \ldots < t_n = T$, $t_k - t_{k-1} = \Delta t$. Ideally, the points in time t_k should coincide with the times of settlement of futures contracts.

At t_{k-1} there is an interest rate $r_k \Delta t$ prevailing from t_{k-1} to t_k. This interest rate is random, but its value is known at time t_{k-1}. The *Ho-Lee model* of the interest rate is:

$$r_k = \theta_k + \sigma\sqrt{\Delta t}\,(z_1 + \cdots + z_{k-1})$$

where θ_k are some numbers, the factor σ is the *volatility* of the short interest rate. The z_j:s are independent N(0,1)-variables and the outcome of each z_j occurs at time t_j. This is under the futures probability distribution.

It remains to compute the exact value of θ_k, but first a notational simplification: multiplying by Δt yields

$$r_k \Delta t = \theta_k \Delta t + \sigma \Delta t^{\frac{3}{2}}\,(z_1 + \cdots + z_{k-1})$$

We now normalise $\Delta t = 1$. This means that r_k is the one period interest rate—it is proportional to Δt whereas σ is proportional to $\Delta t^{3/2}$. Now the model reads

$$r_k = \theta_k + \sigma(z_1 + \cdots + z_{k-1}) \tag{1}$$

Let $\$Z_{t_k}$ be the price of a zero coupon bond maturing at t_k with face value \$1. Then, according to Theorem 3, Ch. X,

$$Z_{t_k} = \widehat{\mathrm{E}}[1\, e^{-r_1 - \cdots - r_k}]$$

In order for the model (1) to correctly represent the current term structure, we must thus have

$$\begin{aligned} Z_{t_k} &= \widehat{\mathrm{E}}[e^{-\theta_1 - \cdots - \theta_k - \sigma((k-1)z_1 + (k-2)z_2 + \cdots + 1\, z_{k-1})}] \\ &= e^{-\theta_1 - \cdots - \theta_k}\, \widehat{\mathrm{E}}[e^{-\sigma((k-1)z_1 + (k-2)z_2 + \cdots + 1\, z_{k-1})}] \end{aligned}$$

i.e.,

$$Z_{t_k} = e^{-\theta_1 - \cdots - \theta_k}\, e^{\frac{\sigma^2}{2}((k-1)^2 + \cdots + 1^2)}$$

Since this must hold for any k, this is equivalent to

$$\theta_k = \ln\left(\frac{Z_{t_{k-1}}}{Z_{t_k}}\right) + \frac{\sigma^2}{2}\,(k-1)^2$$

The Ho-Lee model thus reads

$$r_t = \ln\left(\frac{Z_{t-1}}{Z_t}\right) + \frac{\sigma^2}{2}\,(t-1)^2 + \ \sigma(z_1 + \cdots + z_{t-1}) \tag{2}$$

under the futures distribution.

The Price of a Zero Coupon Bond

Consider a zero coupon bond with face value \$1 which matures at time T. We will now compute its price $\$Z(t,T)$ at an earlier time $t < T$. At this time, the bond's value is, by Theorem 3, Ch. X (the sub index t on $\widehat{\mathrm{E}}$ indicates conditional expectation as of time t, i.e., the outcomes of z_j for $j \le t$ are already known and are regarded as constants):

$$\begin{aligned}
Z(t,T) &= \widehat{\mathrm{E}}_t[e^{-r_{t+1}-\cdots-r_T}] \\
&= \frac{Z_T}{Z_t}\widehat{\mathrm{E}}_t[e^{-\frac{\sigma^2}{2}(t^2+\cdots+(T-1)^2)-\sigma(T-t)(z_1+\cdots+z_t)} \\
&\quad \times e^{-\sigma((T-t-1)z_{t+1}+(T-t-2)z_{t+2}+\cdots+1\,z_{T-1})}] \\
&= \frac{Z_T}{Z_t}e^{-\frac{\sigma^2}{2}(t^2+\cdots+(T-1)^2)}\ e^{-\sigma(T-t)(z_1+\cdots+z_t)} \\
&\quad \times \widehat{\mathrm{E}}_t[e^{-\sigma((T-t-1)z_{t+1}+(T-t-2)z_{t+2}+\cdots+1\,z_{T-1})}] \\
&= \frac{Z_T}{Z_t}e^{-\sigma(T-t)(z_1+\cdots+z_t)}\ e^{-\frac{\sigma^2}{2}(t^2+\cdots+(T-1)^2)}\ e^{\frac{\sigma^2}{2}((T-t-1)^2+\cdots+1^2)}
\end{aligned}$$

We compute the sums

$$\begin{aligned}
-\frac{\sigma^2}{2}(t^2+\cdots+(T-1)^2) &+ \frac{\sigma^2}{2}((T-t-1)^2+\cdots+1^2) \\
&= -\frac{\sigma^2}{2}\sum_{j=0}^{T-t-1}(t+j)^2 - j^2 \\
&= -\frac{\sigma^2}{2}t\sum_{j=0}^{T-t-1}(2j+t) \\
&= -\frac{\sigma^2}{2}(T-1)(T-t)t
\end{aligned}$$

Hence,

$$Z(t,T) = \frac{Z_T}{Z_t}e^{-\frac{\sigma^2}{2}(T-1)(T-t)t}e^{-\sigma(T-t)(z_1+\cdots+z_t)} \tag{3}$$

Forward and Futures on a Zero Coupon Bond

We will now compute the futures and forward prices of a contract maturing at t on a zero coupon bond maturing at $T > t$ with face value \$1. The forward price is easy, it is

$$G_0^{(t)} = \$\frac{Z_T}{Z_t}$$

regardless of the interest rate model. We leave the proof to the reader. The futures price F_0 is

$$\begin{aligned}
F_0 = \widehat{\mathrm{E}}[\$Z(t,T)] &= \widehat{\mathrm{E}}[\$\frac{Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-1)(T-t)t} e^{-\sigma(T-t)(z_1+\cdots+z_t)}] \\
&= G_0^{(t)} e^{-\frac{\sigma^2}{2}(T-1)(T-t)t}\, \widehat{\mathrm{E}}[e^{-\sigma(T-t)(z_1+\cdots+z_t)}] \\
&= G_0^{(t)} e^{-\frac{\sigma^2}{2}(T-1)(T-t)t}\, e^{\frac{\sigma^2}{2}(T-t)^2 t} = G_0^{(t)} e^{-\frac{\sigma^2}{2}(T-t)(t-1)t}
\end{aligned}$$

Recall that we have normalised $\Delta t = 1$. With no normalisation, we must replace σ by $\sigma \Delta t^{\frac{3}{2}}$ and t and T by $t\Delta t^{-1}$ and $T\Delta t^{-1}$ respectively, hence

$$F_0^{(t)} = G_0^{(t)} e^{-\frac{\sigma^2}{2}(T-t)(t-\Delta t)t} \longrightarrow G_0^{(t)} e^{-\frac{\sigma^2}{2}(T-t)t^2} \quad \text{when } \Delta t \to 0.$$

We see that in contrast to the situation when interest rates are deterministic, the forward price G_0 and the futures price F_0 differ. Since the price of the underlying asset (the bond) is negatively correlated with the interest rate, the futures price is lower; in the Ho-Lee model by factor $e^{-\frac{\sigma^2}{2}(T-t)t^2}$.

The Forward Distribution

The computation of prices on bond options, for example, is simpler if we use the forward distribution rather than the futures distribution. We will see that in the next section. Let us therefore derive the forward distribution for a certain maturity t. Let $D \subset \mathbf{R}^t$, $Pr^{(t)}$ denote t-forward probability, $\bar{z}_t = (z_1, \ldots, z_t)$ and let $I_D(\bar{x}_t)$ denote the indicator function of D, i.e., $I_D(\bar{x}_t) = 1$ if $\bar{x}_t \in D$, $I_D(\bar{x}_t) = 0$ if $\bar{x}_t \notin D$. The constants C_k are independent of the choice of D, and their actual values are of no concern. We have:

$$\begin{aligned}
\$Pr^{(t)}(\bar{z}_t \in D) = \$\mathrm{E}^{(t)}\left[I_D(\bar{z}_t)\right] = G_0^{(t)}\left[\$I_D(\bar{z}_t)\right] &\stackrel{(1)}{=} Z_t^{-1}\, P_0^{(t)}\left[\$I_D(\bar{z}_t)\right] \\
&\stackrel{(1)}{=} Z_t^{-1}\, F_0^{(t)}\left[\$I_D(\bar{z}_t)\, e^{-\Sigma_1^t r_k}\right] \\
&= C_1 F_0^{(t)}\left[\$I_D(\bar{z}_t)\, e^{-\sigma\, \Sigma_1^t (t-j) z_j}\right] \\
&= \$C_2 \int \ldots \int I_D(\bar{x}_t) e^{-\sigma\, \Sigma_1^t (t-j) x_j} e^{-\frac{1}{2}\Sigma x_j^2}\, dx_1 \ldots dx_t
\end{aligned}$$

$$= \$C_3 \int \dots \int_D e^{-\frac{1}{2}\,\Sigma(x_j+\sigma(t-j))^2}\, dx_1 \dots dx_t$$

Here the equalities (1) follow from the theorem in Ch. I. Hence, under the t-forward probability distribution, the random variables z_j $(j \le t)$ are independent and $z_j \in N\big(-\sigma(t-j),1\big)$. In particular, we have from (3) that under the $\mathrm{E}^{(t)}$-distribution, with w_j independent $\in N(0,1)$:

$$\begin{aligned} Z(t,T) &= \frac{Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-1)(T-t)t} e^{-\sigma(T-t)(w_1-\sigma(t-1)+\cdots+w_{t-1}-\sigma\cdot 1+w_t)} \\ &= \frac{Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-t)^2 t} e^{-\sigma(T-t)(w_1+\cdots+w_t)} \end{aligned} \tag{4}$$

Pricing a European Option on a Zero Coupon Bond

It is now easy to calculate the price of a European option maturing at t on a zero coupon bond maturing at time $T > t$. Let $f(\$Z(t,T))$ be the value in \$ of the option at maturity. Then its price today is, using (4) and the fact that $w_1 + \cdots + w_t$ is a Normal distributed variable with standard deviation $\sqrt{t}$,

$$\begin{aligned} p &= Z_t \mathrm{E}^{(t)}\big[f\big(\$Z(t,T)\big)\big] \\ &= \frac{Z_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\big(\frac{\$Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-t)^2 t} e^{\sigma(T-t)\sqrt{t}\,x}\big) e^{-\frac{1}{2}x^2}\, dx \end{aligned}$$

But $\frac{\$Z_T}{Z_t} = G_0$, the forward price of the underlying bond, hence, (also with no normalisation of time,)

$$\begin{aligned} p &= Z_t \mathrm{E}^{(t)}\big[f\big(\$Z(t,T)\big)\big] \\ &= \frac{Z_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\big(G_0\, e^{-\frac{\sigma^2}{2}(T-t)^2 t} e^{\sigma(T-t)\sqrt{t}\,x}\big) e^{-\frac{1}{2}x^2}\, dx \end{aligned}$$

Note that the same formula may be obtained by Black's model for bond options (Ch. VI.)

XII: Ho-Lee's Binomial Interest Rate Model

When we want to price other than European interest rate derivatives, we need a numerical procedure, and one such is again a binomial tree. This is constructed by replacing z_k in Ch. XI by binomial variables b_j (see Ch. IX, "Binomial Approximation".) Hence, time is discrete: $t_0, t_1, \ldots, t_n = T$, $t_k - t_{k-1} = \Delta t$. The interest rate r_k from t_{k-1} to t_k is assumed to be

$$r_k = \theta_k + \sigma \sum_2^k b_j \quad k = 1, \ldots, n$$

where $\{b_k\}$ are binomial random variables which takes the values ± 1, each with probability 0.5 under the futures distribution; they are thus assumed to be statistically independent. The outcome of the variable b_k occurs at time t_{k-1}. The volatility parameter σ is proportional to $\Delta t^{\frac{3}{2}}$ just as in Ch. XI.

We will choose the terms θ_j:s such that the model is consistent with the current term structure—it will be close to, but not exactly, the same as in the Normal distribution case.

Let us compute the price Z_4 at time t_0 of a zero coupon bond maturing at time t_4 with face value 1:

$$\begin{aligned} Z_4 &= \widehat{\mathrm{E}}[1 \cdot e^{-r_1 - r_2 - r_3 - r_4}] = \widehat{\mathrm{E}}[e^{-\theta_1 - \theta_2 - \theta_3 - \theta_4 - \sigma(3b_2 + 2b_3 + b_4)}] \\ &= e^{-\theta_1 - \theta_2 - \theta_3 - \theta_4} \widehat{\mathrm{E}}[e^{-3\sigma b_2}]\, \widehat{\mathrm{E}}[e^{-2\sigma b_3}]\, \widehat{\mathrm{E}}[e^{-\sigma b_4}] \\ &= e^{-\theta_1 - \theta_2 - \theta_3 - \theta_4} \cosh(3\sigma) \cosh(2\sigma) \cosh(\sigma) \end{aligned}$$

And, by the same token, in general

$$Z_k = e^{-\theta_1 - \cdots - \theta_k} \cosh\big((k-1)\sigma\big) \cdot \ldots \cdot \cosh(\sigma)$$

Since this must hold for all k, we must have

$$\theta_k = \ln\left(\frac{Z_{k-1}}{Z_k}\right) + \ln\big(\cosh\big((k-1)\sigma\big)\big)$$

and the dynamics under the futures distribution is thus described by

$$r_k = \ln\left(\frac{Z_{k-1}}{Z_k}\right) + \ln\big(\cosh\big((k-1)\sigma\big)\big) + \sigma \sum_2^k b_j \quad k = 1, \ldots, n$$

$$\text{where} \begin{cases} b_j = 1 & \text{with probability } \frac{1}{2} \\ b_j = -1 & \text{with probability } \frac{1}{2} \end{cases}$$

Once we have the parameters of the model, we can price interest rate derivatives in the binomial tree. We show the procedure by an example, where we want to price a European call option maturing at t_2 with strike price 86 on a zero coupon bond maturing at t_4 with face value 100 when the following parameters are given: $f_1 = 0.06$, $f_2 = 0.06095$, $f_3 = 0.06180$, $f_4 = 0.06255$, $\sigma = 0.01$, where $f_k = \ln\left(Z_{k-1}/Z_k\right)$ are the forward rates. We represent the interest rates in a binomial tree:

t_0	t_1	t_2	t_3
0.06	0.071	0.082	0.093
	0.051	0.062	0.073
		0.042	0.053
			0.033

The interest rate from one period to the next is obtained by going either one step to the right on the same line, or step to the right to the line below; each with a (risk adjusted) probability of 0.5. We can compute the value at t_2 of the bond: since its value at t_4 is 100, the value at t_3 and t_2 is obtained recursively backwards by use of the formula of theorem 3 in Ch. X, i.e., we take the average of the values in the next period and discount by the interest rate given in the above tree:

t_2	t_3
84.794	91.119
88.254	92.960
91.856	94.838
	96.754

The value of the option can now also be obtained by backward recursion:

t_0	t_1	t_2
2.309	1.050	0
	3.853	2.254
		5.856

The value of the option is thus 2.309. It is also easy to price other more exotic derivatives in this binomial tree model.

Comments

The model presented in this and the previous chapter is Ho-Lee's interest rate model. It was originally presented as a binomial model. It is the simplest interest rate model, and it can be calibrated so as to fit the current term structure perfectly. There are two features with this model that many people feel unhappy with: (1) the short rate will eventually far into the future take negative values with a probability that approaches one half, and (2) there is no "mean reversion" which means that all shocks to the short rate are permanent; they don't wear off with time.

A modification of Ho-Lee's model is Hull-White's model, which is a combination of Ho-Lee's model and Vasicek's model. In a time-discrete version it looks like this:

$$r_k = \theta_k + \sigma\sqrt{\Delta t}\,(\beta^{(k-2)\Delta t} z_1 + \cdots + \beta^{\Delta t} z_{k-2} + z_{k-1}),$$

where $0 < \beta \leq 1$ is the mean-reverting factor. As you can see, the shock z_1, for instance, wear off exponentially with time, since $\beta^{(k-2)\Delta t} \to 0$ as $k \to \infty$ if $\beta < 1$. The Ho-Lee model is the special case $\beta = 1$ (compare with the first expression in Ch. XI.) In Hull-White's model, the standard deviation of the short rate is always bounded by $\dfrac{\sigma}{\sqrt{-2\ln\beta}}$, so if β is small, the probability of negative interest rates can stay small.

This model can not be represented by a (recombining) binomial tree (if $\beta \neq 1$,) but it can be implemented as a *trinomial* tree; see the book by Hull[1]. The difference between Ho-Lee's and the Hull-White model is solely on technicalities concerning the construction of the tree of interest rates; the conceptual ideas are the same.

Exercises and Examples

Interest rates always refer to continuous compounding. Answers are given in parenthesis; solutions to some problems are given in the next section.

1. We have the following Ho-Lee binomial tree of the interest rate in % per time-step:

period	0	1	2	3
	3.0	3.3	3.5	3.8
		2.9	3.1	3.4
			2.7	3.0
				2.6

The interest from period 0 to period 1 is hence 3.0%, and so on. An interest rate security is in period 3 worth $400\,r$ where r is the interest (if $r = 3.8\%$ the value is \$1'520, if $r = 2.6$ the value is \$1'040 etc.)

a) Determine today's futures price of the security to two decimal places. (\$1'280.00)

b) Determine today's forward price of the security to two decimal places. (\$1'279.52)

[1] John C. Hull: *Options, Futures, & Other Derivatives;* Prentice Hall.

2. The following zero-coupon rates of interest holds: 1-year: 8%, 2-year: 8.25%, 3-year: 8.5%, 4-year: 8.75%. The volatility of the one-year rate is assumed to be 1.5% during one year. Determine the price of a European call option with maturity in two years on a (at maturity of the option) zero coupon bond with face value 10'000 SEK (thus the bond matures in four years time from the present.) The strike price of the option is 8'000 SEK.

   a) Use a binomial tree based on Ho-Lee's model with a time interval of one year. (302.05 SEK.)

   b) Calculate the value using Black's model. (293.24 SEK.)

3. Determine the price of a "callable bond" by use of a Ho-Lee tree. A callable bond is a bond where the issuer have the option to buy back the bond at certain points in time at predetermined prices (see relevant chapter in Hull's book.) The topical bond is a coupon paying bond which pays $300 each half year and matures with face value $10'000 (i.e., $10'300 is paid out including the coupon) after 2.5 years.

   The issuer has the option to buy back the bond for $9'950 plus the coupon dividend after 18 months.

   Use the following Ho-Lee tree:

0-6	6-12	12-18	18-24	24-30	months
3.0	3.3	3.5	3.8	4.0	
	2.9	3.1	3.4	3.6	
		2.7	3.0	3.2	
			2.6	2.8	
				2.4	

   The numbers are the interest rate in percent per 6 months, i.e., per time step in the tree. ($9'896.85)

   Remark. This problem can also be solved "analytically" with Black's model. Note that the callable bond is equivalent to the corresponding non-callable bond and a short position of a European call option on the bond with maturity 18 months. See Ch. VI, "Black's Model for Bond Options".

4. Calculate the value of a callable zero-coupon bond with maturity in 10 years and strike price price 100 SEK. The bond can be exercised after 3 years at 70 SEK, after 6 years at 80 SEK and after 8 years at 90 SEK. The volatility of the one-year rate of interest is assumed to be 1.5% during one year. The present zero coupon rates of interest are (% per year with continuous compounding):

maturity	interest	maturity	interest
1 year	4.0	6 year	5.0
2 year	4.2	7 year	5.2
3 year	4.4	8 year	5.4
4 year	4.6	9 year	5.6
5 year	4.8	10 year	5.8

Use a binomial tree with a time interval of one year. (53.1950)

5. Calculate the futures price of a ten-year zero-coupon bond with face value 100 and maturity in six years. The volatility of the one year rate of interest is assumed to be 1.5% during one year, and the zero-coupon rate of interest is the same as in the previous question.

   a) Use a binomial tree with time interval of one year based on Ho-Lee's model. (74.5689)

   b) Do the calculation analytically using the continuous time Ho-Lee model (see the last formula in section "Forward and Futures on a Zero Coupon Bond" in Ch. XI.) (74.3639)

Solutions

1. Here is first the tree of interest rates:

3,8	3,4	3	2,6
3,5	3,1	2,7	
3,3	2,9		
3			

The futures prices of the security is calculated simply as the average 0.5(number above + number above-right), since the futures prices are a Martingale under the futures measure; see Ch. VII:

1 520	1 360	1 200	1 040
1 440	1 280	1 120	
1 360	1 200		
1 280			

The futures price is thus $1'280.

In order to calculate the forward price G_0, we first calculate the present price P_0. This is done with discounting; the values in a row is 0.5(number above + number above-right) $\times$ discount;

1 520,00	1 360,00	1 200,00	1040
1 390,47	1 240,93	1 090,16	
1 272,99	1 132,23		
1 167,07			

The present price is thus \$1167.07. In order to calculate the forward price $G_0 = Z_{30}^{-1}$ we need the zero coupon price Z_{30}. But $Z_{30} = P_0^{30}[1]$:

1	1	1	1
0,965605	0,969476	0,97336	
0,936133	0,943652		
0,912114			

We have thus G_0 = \$1'167.07/0.912114 = \$1'279.52. The difference is small but in accordance with theory; the security is positively correlated with the interest rate, so the futures price is higher than the forward price.

2a. The interest rate tree:

3	0,141012	0,111012	0,081012	0,051012
2	0,120450	0,090450	0,060450	
1	0,100112	0,070112		
0	0,080000			

The value of the bond is calculated backwards in time: (0.5(number above + number above-right) $\times$ discount where we get the discounting rate from the above tree:

4	10 000,00	10 000,00	10 000,00	10 000,00	10 000,00
3	8 684,79	8 949,28	9 221,82	9 502,67	
2	7 816,49	8 299,83	8 813,07		

The value of the option is calculated in the same way:

2	0,00	299,83	813,07
1	135,64	518,77	
0	302,05		

The option price is thus 302.05 SEK.

b. This is an ordinary call option where the forward price of the underlying asset is 8'311.04 SEK and the volatility is $0.015 \cdot 2 = 0.03$. The factor 2 is the forward duration (time to maturity of the underlying bond when the option is exercised.)

3. Here is the interest rate tree, now with time going from bottom to top:

30					
24	4,00	3,60	3,20	2,80	2,40
18	3,80	3,40	3,00	2,60	
12	3,50	3,10	2,70		
6	3,30	2,90			
0	3,00				

The values of the bond can be arranged in various ways in a tree. Here I have written the values with the dividend excluded in each period.

30	10 000,00	10 000,00	10 000,00	10 000,00	10 000,00	10 000,00
24	9 896,13	9 935,80	9 975,62	10 015,60	10 055,74	
18	9 835,04	9 912,87	**9 950,00**	**9 950,00**		
12	9 824,03	9 919,13	9 976,95			
6	9 841,39	9 955,12				
0	9 896,85					

After 30 months, the value is \$10'000 plus dividend. In each subsequent row, the value, net of dividend, is
$\big(0.5(\text{number above} + \text{number above-right}) + 300\big) \times \text{discount}$,
except in two places after 18 months (boldface) where the bond is retracted.