# WEEK 1
# CH.4.1 FROM AVNER FRIEDMAN
# FOUNDATIONS OF MORDERN ANALYSIS

**Definition.** $X$ is called an *abelian group* with operation " $+$ " if there is a mapping $X \times X \to X : (x, y) \to x + y$ such that for any $x, y, z \in X$

- $x + (y + z) = (x + y) + z$ (associative)
- $x + y = y + x$ (commutative)
- there exists $0$ element such that $x + 0 = x$
- $\forall\, x \in X$, $\exists\, -x$, such that $x + (-x) = 0$.

**Definition.** $X$ is called a *linear vector space* if

- $X$ is an abelian group

Let $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$. There is a mapping $\mathcal{F} \times X \to X : (\lambda, x) \to \lambda x$ such that for any $\lambda, \mu \in \mathcal{F}$

- $\lambda(\mu x) = (\lambda\mu)x$
- $(\lambda + \mu)x = \lambda x + \mu x$
- $\lambda(x + y) = \lambda x + \mu y$
- $1\, x = x$.

**Example.** $n \times m$ matrices with real or complex entries is a linear vector space.

**Property.**
If $\lambda\, x = 0$, $\lambda \neq 0$, then $x = 0$.
Indeed

$$x = 1\, x = \frac{1}{\lambda}\lambda\, x = \frac{1}{\lambda}\, 0 = 0.$$

**Definition.** Elements $x_1, x_2, \ldots, x_n \in X$ are called *linear independent* if $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$ implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

**Definition.** If for any $n$ there exist $n$ linear independent elements $x_1, x_2, \ldots, x_n \in X$ , then $X$ is called *infinite dimensional*. Otherwise, $X$ is called *finite dimensional* and maximal number of linear independent element is called dimension of $X$.

**Definition.** Let dimension $X$ equals $n$. Any linear independent elements $e_1, e_2, \ldots, e_n \in X$ are said to form a basis of $X$.

For any $x \in X$ there are $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n.$$

The constants $\lambda_j$ are called the *coordinates* of $x$ in the basis $e_1, e_2, \dots, e_n$.

**Notations.** For any subsets $A, B \subset X$ and $\lambda \in \mathcal{F}$ we denote

$$A + B = \{x + y : x \in A, \, y \in B\} \qquad \lambda A = \{\lambda x : x \in A\}.$$

**Definition.** A *normed linear space* is a linear vector space $X$ such that there is a mapping $X \to \mathbb{R}_+ = [0, \infty) : x \to \|x\|$ such that

- $\|x\| \geq 0 \;\; \& \;\; \|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \, \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$.

**Definition.** A *metric linear space* is a linear vector space $X$ that is a metric space i.e. $\exists \, \rho : X \times X \to \mathbb{R}_+$ satisfying

- $\rho(x, y) \geq 0 \;\; \& \;\; \rho(x, y) = 0 \Leftrightarrow x = y$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

and also if $\lambda_n \to \lambda$, $\mu_n \to \mu$, $x_n \to x$ and $y_n \to y$, then

- $\lambda_n x_n + \mu_n y_n \to \lambda x + \mu y$.

**Remark.** $x_n \to x$ if $\rho(x_n, x) \to 0$. We say that $X$ is complete if any Cauchy sequence has a limit belonging to $X$.

**Definition.** A *Fréchet space* is a metric linear space $X$ such that

- $\rho(x, y) = \rho(x - y, 0)$
- $X$ is complete.

**Theorem 1.1.** A normed linear space is a metric space such that $\rho(x, y) = \|x - y\|$.

**Proof.** All standard properties are obvious. Moreover if $\lambda_n \to \lambda$, $\mu_n \to \mu$, $x_n \to x$ and $y_n \to y$, then

$$\begin{aligned} &\|\lambda_n x_n + \mu_n y_n - (\lambda x + \mu y)\| \\ &\quad \leq |\lambda_n - \lambda| \, \|x_n\| + \lambda \, \|x_n - x\| + |\mu_n - \mu| \, \|y_n\| + |\mu| \, \|y_n - y\| \to 0 \end{aligned}$$

as $n \to \infty$.

**Theorem 1.2.** If $X$ is a metric linear space whose metric satisfies

- $\rho(x, y) = \rho(x - y, 0)$
- $\rho(\lambda x, \lambda y) = |\lambda| \, \rho(x, y)$,

then $X$ is a normed space whose norm could be defined as $\|x\| = \rho(x, 0)$.

**Theorem 1.3.**
A normed linear space is a metric linear space, s.t. $\rho(x, y) = \|x - y\|$.

**Example of a Fréchet space which is not a normed space.** Let us consider the space of sequences $X = \{x = (\xi_1, \xi_2, \dots, \xi_n, \dots)\}$, $\xi_j \in \mathbb{R}$ and define

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|},$$

where $x = (\xi_1, \xi_2, \dots, \xi_n, \dots)$, $y = (\eta_1, \eta_2, \dots, \eta_n, \dots)$. Then $\rho$ defines a metric in $X$ but it cannot be identified with a norm.

**Definition.** A normed linear complete space is called a *Banach space*.

**Home exercise**.

We define Lorentz classes $L^{p,s}(\mathbb{R}^n)$, $1 \le p < \infty$, $1 \le s < \infty$ as follows:
Let

$$\mu_f(t) = \text{meas}\{x \in \mathbb{R}^n : |f(x)| > t\}.$$

Then if $1 \le s < \infty$

$$f \in L^{p,s}(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad t\, \mu_f^{1/p}(t) \in L^s(\mathbb{R}_+, t^{-1}dt)$$

which means that

$$\left( \int_0^\infty t^{s-1} \mu_f^{s/p}(t)\, dt \right)^{1/s} < \infty.$$

Show that $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

If $s = \infty$, then we define

$$f \in L^{p,\infty}(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad \sup_t \{t\, \mu_f^{1/p}(t)\} < \infty.$$

Show that the function $f(x) = |x|^{-2} \in L^{n/2,\infty}(\mathbb{R}^n)$.