LECTURE 2
CH.4.1-4.4 FROM AVNER FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Definition. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in X are equaivalent if there are constants $\beta \geq \alpha > 0$ such that for any $x \in X$

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

Definition. A subset $K \subset X$ of a linear vector space is called convex if for any $x, y \in K$ and $0 \leq t \leq 1$ we have $tx + (1-t)y \in K$.

Definition. Linear subspace.
A subset Y of a liner vector space X is called a linear subspace if for any $\lambda, \mu \in \mathcal{F}$ and any $x, y \in Y$ we have $\lambda x + \mu y \in Y$.

Lemma 2.1. (AFr 4.3.1)
Let $Y \subset X$ be a closed linear subspace of a normed linear space X. Then for any $\varepsilon > 0$ there exists $z \in X$ s.t. $\|z\| = 1$ and $\|z - y\| > 1 - \varepsilon$, for any $y \in Y$.

Theorem 2.2. (AFr 4.3.2)
If Y is a finite-dimensional linear subspace of a normed linear space, then Y is closed.

Theorem 2.3. (AFr 4.3.3)
A normed linear space is of finite dimension *iff* every bounded subset is relatively compact.

Example. Let $X = L^2(0, 2\pi)$ and let $e_k(x) = e^{ikx}/\sqrt{2\pi}$, $k = 0, \pm 1, \ldots$. Then

$$\int_0^{2\pi} |e_k(x)|^2 dx = 1 \qquad \text{and} \qquad \int_0^{2\pi} |e_k - e_j|^2 dx = 2.$$

The set $\{e_k\}_{k\in\mathbb{Z}}$ is not relatively compact (i.e every bounded sequence has a convergent subsequence).

Definition. Linear transformations (operators, mappings).

Let X, Y be two linear spaces, $D_T \subset X$ and let $T: D_T \to Y$ be a mapping s.t. for any $x \in D_T$, $\exists! y \in Y$such that $y = Tx$. Then D_T is called the domain of T and T.
If D_T is a linear subspace of X and if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T x_1 + \lambda_2 T x_2$$

then T is called a linear transformation (operator).

Definition. Let X and Y be metric linear spaces, $D_T = X$ and $T: X \to Y$. $T$is called continuous for any x_n, x such that $x_n \to x$ we have $Tx_n \to Tx$

Theorem 2.4. (AFr 4.4.1)
Let X and Y be normed linear spaces. A linear transformation $T: X \to Y$ is continuous *iff* T is continuous at one point.

Definition. Let X and Y be normed linear spaces, $T: X \to Y$ and let there exists a constant $K > 0$ s.t.

$$\|Tx\| \leq K\|x\| \qquad x \in X.$$

Then T is called a bounded linear map and

$$\|T\| = \sup_{x \in X} \frac{\|Tx\|}{\|x\|}$$

is called the norm of T.

Theorem 2.5. (AFr 4.4.2)
Let X, Y be normed linear spaces. The operator $T: X \to Y$ is continuous *iff* T is bounded.

Home exercises.

1. Let X be a normed liner space. Show that $B = \{x \in X : \|x\| < 1\}$ is convex.

2. Let X be a linear space and $K, F \subset X$ are convex. Show that $K + F$ is convex.

3. A norm $\|\cdot\|$ is called strictly convex if $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$ implies $x = y$.
Show that $L^p(\mathbb{R}^n)$ is strictly convex if $1 < p < \infty$.

4. Prove that the max-norm in $C[a, b]$ is not equivalent to L^p norm if $1 \leq p < \infty$.

5. Find the norm of the operator A given by

$$Af(t) = tf(t), \qquad 0 \leq t \leq 1,$$

where **a.** $X = C[0, 1]$, **b.** $X = L^p[0, 1]$, $1 \leq p \leq \infty$.

6. Let $\mathcal{L}$ be the Laplace transform defined by

$$g(s) = \mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Show that $\mathcal{L} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is bounded and

$$\|\mathcal{L}\| \leq \sqrt{\pi}.$$