WEEK 3
LINEAR TRANSFORMATIONS
CH. 4.4 - 4.6 FROM AVNER FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Definition. By $\mathcal{L}(X, Y)$ we denote the space of all linear transformations equipped with

- $T + S$ defined by $(T + S)x = Tx + Sx$.
- λT defined by $\lambda T(x) = T(\lambda x)$.

By $\mathcal{B}(X, Y) \subset \mathcal{L}(X, Y)$ we denote the set of all bounded linear transformations $T : X \to Y$, where X and Y are normed spaces.

Theorem 3.1. (AFr 4.4.3)
Let X and Y be normed linear spaces. Then $\mathcal{B}(X, Y)$ is a normed linear space with the norm

$$\|T\| = \sup_{x \in X} \frac{\|Tx\|}{\|x\|}.$$

Definition. Let X and Y be normed linear spaces. The sequence $\{T_n\}_{n=1}^{\infty}$, $T_n : X \to Y$, of bounded operators is said to be uniformly convergent if there exists a bounded operator $T : X \to Y$ s.t. $\|T_n - T\| \to 0$ as $n \to \infty$.

Theorem 3.2. (AFr 4.4.4)
If X is a normed linear space and Y is a Banach space then $\mathcal{B}(X, Y)$ is a Banach space.

Theorem 3.3. (AFr 4.5.1) (Banach-Steinhaus theorem)
Let X be a Banach space and Y be a normed linear space. Let $\{T_\alpha\}$ be a family of bounded linear operators from X to Y. It for each $x \in X$ the set $\{T_\alpha x\}$ is bounded, then the set $\{\|T_\alpha\|\}$ is bounded.

Definition. Let X and Y be normed linear spaces and let $T_n : X \to Y$. The sequence $\{T_n\}_{n=1}^{\infty}$ is said to be strongly convergent if for any $x \in X$ the limit $\lim_{n\to\infty} T_n x$ exists for any $x \in X$.

If there exists a bounded T s.t. $\lim_{n\to\infty} T_n x = Tx$ for any $x \in X$, then $\{T_n\}_{n=1}^{\infty}$ is called strongly convergent to T ($T_n \to T$).

Theorem 3.4. (AFr 4.5.2)
Let X be a Banach space and Y be a normed linear space and let $\{T_n\}_{n=1}^{\infty}$ be the sequence of bounded operators. If the sequence $\{T_n\}_{n=1}^{\infty}$ strongly convergent, then there exists T s.t. $T_n \to T$ strongly.

Theorem 3.5. (AFr 4.6.1) (Open-mapping theorem)
Let X and Y be Banach spaces and let $T: X \to Y$ be a mapping *onto*. Then T maps open sets of X onto open sets of Y.

Home exercises.

1. Let $K: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$Kf(x) = \int_{-\infty}^{\infty} K(x,y) f(y)\, dy,$$

where

$$\int_{-\infty}^{\infty} |K(x,y)|\, dy \le A \quad \text{and} \quad \int_{-\infty}^{\infty} |K(x,y)|\, dx \le B.$$

Prove that

$$\|K\| \le \sqrt{A}\sqrt{B}.$$

2. Show that the operator $K: L^2(0,1) \to L^2(0,1)$ such that

$$Kf(x) = \int_0^1 \frac{1}{x+y} f(y)\, dy$$

is bounded.