WEEK 4
LINEAR TRANSFORMATIONS
CH. 4.6 - 4.7 FROM AVNER FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Definition. Let T be a one-to-one linear operator $T : X \to Y$. Then the inverse operator T^{-1} is defined as

$$T^{-1}y = x \qquad \text{iff} \qquad Tx = y.$$

Note that domain T^{-1} is a linear subspace of Y.

Theorem 4.1. (AFr 4.6.2)
Let X and Y be a Banach spaces and let T be a one-to-one bounded linear map from X onto Y. Then T^{-1} is a bounded map.

Corollary 4.2. (AFr 4.6.3)
Let X be Banach spaces equipped with either $\|\cdot\|_1$ or $\|\cdot\|_2$. Suppose that there exists a constant K s.t.

$$\|x\|_1 \leq K\,\|x\|_2 \qquad \text{for all} \qquad x \in X.$$

Then there exists a constant K' s.t.

$$\|x\|_2 \leq K'\,\|x\|_1 \qquad \text{for all} \qquad x \in X.$$

Definition. Let X and Y be linear vector spaces and let $T : X \to Y$, where T is defined on D_T. Then

$$G_T = \{(x, Tx) : x \in D_T\} \subset X \times Y$$

is called the graph of T.
If G_T is a closed set in $X \times Y$ then we say that T is a closed operator.

Note that T is closed *iff*

$$x_n \in D_T \quad x_n \to x, \quad Tx_n \to y \Longrightarrow x \in D_T, \quad Tx = y.$$

Theorem 4.3. (AFr 4.6.4) (Closed graph theorem)
Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator with $D_T = X$. If T is closed then T is continuous.

Definition. Let X and Y be linear vector spaces and let $T: X \to Y$ defined on D_T. A linear operator $S: X \to Y$ is called an extension of T if $D_T \subset D_S$ and $Tx = Sx$ for all $x \in D_T$.

Definition. Let X be a linear vector space and Y be a normed space. Let $T: X \to Y$ defined on D_T. Suppose that there exists S such that

(i) S is a closed linear operator

(ii) S is an extension of T

(iii) If S' satisfies (i) and (ii), then S' is an extension of S.

Then we say that S is a s closure of T. The closure of T is denoted by $\overline{T}$.

Theorem 4.4. (AFr 4.6.4) (Closed graph theorem)
Let X and Y be Banach spaces. T has a closure $\overline{T}$ *iff*

$$x_n \in D_T \quad x_n \to 0, \quad Tx_n \to y \Longrightarrow y = 0.$$

Example 1. Let $X = L^2(0,1)$ and let $Tf(x) = f(0)$ with $D_T = C^1[0,1]$.
We show that T cannot be closed.
Indeed, let $f_n(x) = (1-x)^n$. Then $f_n \in D_T$, $\|f_n\|_{L^2(0,1)} \to 0$, $t \to \infty$, but $Tf_n = 1$.
Note that in this case $G_T = \{(f, f(0)) : f \in D_T\}$ and $\|(f, Tf)\| = \|f\|_{L^2} + |f(0)|$.

Example 2. Let $X = L^2(0,1)$, $Tf(x) = f'(x)$ and $D_T = C^1[0,1]$.
We show that T can be closed.
The operator T is not closed as it is defined on $C^1[0,1]$ which is not a closed set in $L^2(0,1)$. In order to show that T can be closed if is enough to show that

$$f_n \in D_T, \quad f_n \to 0, \quad Tf_n \to g \quad \Longrightarrow \quad g = 0.$$

Let $\|f_n\|_{L^2} \to 0$, $f_n \in C^1[0,1]$ and let

$$\int_0^1 |f_n'(x) - g(x)|^2 dx \to 0.$$

For any $\varphi \in C_0^1[0,1]$ $(\varphi(0) = \varphi(1) = 0)$ we have

$$\Big| \int_0^1 (f_n'(x) - g(x))\varphi(x)\, dx \Big| \le \Big(\int_0^1 |f_n'(x) - g(x)|^2\, dx \Big)^{1/2} \Big(\int_0^1 |\varphi(x)|^2\, dx \Big)^{1/2} \to 0,$$

which implies

$$\int_0^1 f_n'(x)\varphi(x)\, dx \to \int_0^1 g(x)\varphi(x)\, dx.$$

On the other hand

$$\Big|\int_0^1 f_n'(x)\varphi(x)\,dx\Big| = \Big|\int_0^1 f_n(x)\varphi'(x)\,dx\Big|$$
$$\le \Big(\int_0^1 |f_n(x))|^2\,dx\Big)^{1/2}\Big(\int_0^1 |\varphi'(x)|^2\,dx\Big)^{1/2} \to 0, \qquad n\to\infty.$$

Thus

$$\Big|\int_0^1 g(x)\varphi(x)\,dx\Big| = 0$$

for any $\varphi \in C_0^1[0,1]$ which means that $g = 0$ as an L^2 function.

Linear Functionals.

Definition. Let X be a linear vector space and let $p : X \to \mathbb{R}(\mathbb{C})$ be a linear operator. Then p is called a real(complex) linear functional.

Examples.

- $p : f(x) \to f(0)$, $f \in C(-1,1)$.
- Let $f \in L^(0, 2\pi)$ and let

  $$f(x) = \sum_{-\infty}^{\infty} a_n e^{-inx}/\sqrt{2\pi}$$

  be its Fourier series.
  Then $p_n(f) = a_n$ is a complex linear functional on $L^2(0, 2\pi)$.
- Let $g \in L^2(0,1)$. We define $p = p_g$ such that

  $$p_g(f) = \int_0^1 f(x)g(x)\,dx.$$

Definition. A partially ordered set S is a non-empty set with a relation " $\le$ " satisfying the properties:

- $x \le x$.
- $x \le y, y \le z \Longrightarrow x \le z$.

If for any $x, y \in S$ either $x \le y$ or $y \le x$ then S is called totally ordered.

Definition. Let $T \subset S$, and S be partially ordered. The element $x \in S$ is called an upper bound if for any $y \in S$, we have $y \le x$.

Definition. Let S be partially ordered. The element $x \in S$ is called maximal if for any $y \in S$ the relation $x \le y$ implies $y \le x$.

Theorem. (Zorn's lemma)
If S is a partially ordered set s.t. every totally ordered subset has an upper bound, then S has a maximal element.

Home exercises.

1. Let X be a Banach space. Show that if $T, S, T^{-1}, S^{-1} \in \mathcal{B}(X, X)$ then $(TS)^{-1} \in \mathcal{B}(X, X)$ and $(TS)^{-1} = S^{-1}T^{-1}$.

2. *(ex. 4.6.2 from AFr)*
Let X be a Banach space and let $A \in \mathcal{B}(X, X)$, $\|A\| < 1$. Show that $(I + A)^{-1}$ exists and

$$(I + A)^{-1} = \sum_{n=0}^{\infty} (-1)^n A^n.$$

3. *(ex. 4.6.3 from AFr)*
Let X be a Banach space and let T and T^{-1} belong to $\mathcal{B}(X, X)$. Show that is $S\mathcal{B}(X, X)$ and $\|S - T\| < 1/\|T^{-1}\|$, then S^{-1} exists and

$$\|S^{-1} - T^{-1}\| < \frac{\|T^{-1}\|}{1 - \|S - T\|\|T^{-1}\|}.$$

4. Let $X = L^2(0, 2\pi)$,

$$\varphi_n = \left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n=-\infty}^{\infty}.$$

Let $\psi \in X$ be an element which is not a finite linear combination of φ_n and let D be a set of all finite linear conbinations of $\{\varphi_n\}$. Denote by T the operator

$$T\Big(b\psi + \sum_{i=-M}^{N} c_j \varphi_j\Big) = b\psi,$$

defined on $D_T = \{\beta\psi + \alpha\varphi,\ \varphi \in D\}$. Show that $\overline{G}_T$ is not a graph. (Namely, show that $(\psi, \psi) \in \overline{G}_T$ and $(\psi, 0) \in \overline{G}_T$.)