WEEK 5
CH. 4.8, 6.1-2 FROM A. FRIEDMAN
FOUNDATIONS OF MORDERN ANALYSIS

Linear Functionals.

Notation.
By X^* we denote the set of all continuous linear functionals on X.

Theorem. (Hahn-Banach lemma, AFr Th 4.8.1.)
Let X be a real linear vector space space and let p be a real functional (not necessary linear) on X s.t.

$$p(x+y) \leq p(x) + p(y) \qquad p(\lambda x) = \lambda p(x), \qquad \lambda > 0, \quad x, y \in X.$$

Let f be a real linear functional on a linear subspace $Y \subset X$ s.t.

$$f(x) \leq p(x), \qquad \forall x \in Y.$$

Then there exists a real linear functional F on X s.t.

$$F(x) = f(x), \qquad x \in X, \qquad \text{and} \qquad F(x) \leq p(x), \quad \forall x \in X.$$

Theorem. (Hahn-Banach) (AFr 4.8.2)
Let X be a normed linear vector space and let $Y \subset X$ be a linear subspace. Then for any $y^* \in Y^*$ there exists $x^* \in X^*$ s.t.

$$\|x^*\| = \|y^*\| \qquad \& \qquad x^*(y) = y^*(y), \quad \forall y \in Y.$$

Theorem. (AFr 4.8.3)
Let X be a normed linear vector space and let $Y \subset X$ be a linear subspace. Let $x_0 \in X$ s.t.

$$\inf_{y \in Y} \|y - x_0\| = d > 0.$$

Then there exists $x^* \in X^*$ s.t.

$$x^*(x_0) = 1, \qquad \|x^*\| = \frac{1}{d} \quad \text{and} \quad x^*(y) = 0, \quad \forall y \in Y.$$

Corollary. (AFr 4.8.4)
If X is a normed linear space, then for any $x \neq 0$ there exists $x^* \in X^*$ s.t. $\|x^*\| = 1$ and $x^*(x) = \|x\|$.

Corollary. (AFr 4.8.5)

If X is a normed linear space and if $y \neq z$, $y, z \in X$, then there exists $x^* \in X^*$ s.t. $x^*(y) \neq x^*(z)$.

Corollary. (AFr 4.8.6)
Let X be a normed linear vector space. Then for any $x \in X$

$$\|x\| = \sup_{x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|} = \sup_{\|x^*\|=1} |x^*(x)|.$$

Corollary. (AFr 4.8.7)
Let X be a normed linear vector space and let $Y \subset X$ be a linear subspace. Assume that $\overline{Y} \neq X$. Then there exists $x^* \neq 0$ s.t. $x^*(y) = 0$, $\forall y \in Y$.

Definition. The null space of $x^* \in X^*$ is the set

$$N_{x^*} = \{x \in X : \, x^*(x) = 0\}.$$

Let $x^* \neq 0$. Then there is $x_0 \neq 0$ s.t. $x^*(x) = 1$ and any $x \in X$ can be written as $x = z + \lambda x_0$, where $\lambda = x^*(x)$ and $z = x - \lambda x_0 \in N_{x^*}$.

Example. Let $f \in L^p(0,1)$ $g \in L^q(0,1)$ $1/p + 1/q = 1$ be real functions and define a linear functional G^* on $L^p(0,1)$ such that

$$G^*(f) = \int_0^1 f(x)g(x)\,dx.$$

Then $N_g = \{f \in L^p(0,1) : \int f(x)g(x)\,dx = 0\}$.

Hilbert Spaces

Definition.
H is called a Hilbert space if H is a complex linear space supplied with $(\cdot,\cdot) : H \times H \to \mathbb{C}$ s.t.

- $(x,x) \geq 0$, & $(x,x) = 0 \Leftrightarrow x = 0$.
- $(x+y,z) = (x,z) + (y,z)$, $\forall x,y,z \in H$.
- $(\lambda x, y) = \lambda(x,y)$, $\forall x,y \in H$, $\lambda \in \mathbb{C}$.
- $(x,y) = \overline{(y,x)}$, $\forall x,y \in H$.
- If $\{x_n\}$ is a Cauchy sequence and $\lim_{n,m\to\infty}(x_n - x_m, x_n - x_m) = 0$, then there exists $x \in H$, s.t. $\lim_{n\to\infty}(x_n - x, x_n - x) = 0$.

$(\cdot,\cdot)$ is called scalar product.

$\|x\| = \sqrt{(x,x)}$ is called the norm of x.

Examples.

- $H = l^2 = \{\mathbf{a} = \{a_n\}_{n=1}^{\infty}\}$, such that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. We define scalar product

$$(\mathbf{a}, \mathbf{b}) = \sum_{n=1}^{\infty} a_n \overline{b_n}, \qquad \mathbf{a}, \mathbf{b} \in H.$$

- $H = L^2(0,1) = \{f : \int_0^1 |f(x)|^2 \, dx < \infty\}$ with scalar product

$$(f, g) = \int_0^1 f(x) \overline{g(x)} \, dx.$$

- Sobolev space

$$H^1(0,1) = \left\{ f : \int_0^1 \left(|f'(x)|^2 + |f(x)|^2 \right) dx < \infty \right\}.$$

The corresponding scalar product is equal to

$$(f, g) = \int_0^1 \left(f'(x) \overline{g(x)} + f(x) \overline{g(x)} \right) dx.$$

Home exercises.

1. Define C_0 as a set of sequences $\{a_k\}_{k=1}^{\infty}$ for which $\lim_{k\to\infty} a_k = 0$. If we introduce the following norm

$$\|\{a_k\}_{k=1}^{\infty}\| = \max_{k \in \mathbb{N}} |a_k|,$$

then C_0 becomes a normed linear space.
Assume that $\{\lambda\}_{k=1}^{\infty} \in l_1 \left(\Leftrightarrow \sum_{k=1}^{\infty} |\lambda_k| \le \infty \right)$.
Show that

$$\Lambda(\{a_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \lambda_k a_k$$

is a linear functional on C_0 and $\|\Lambda\| = \sum_{k=1}^{\infty} |\lambda_k|$.

2. Show that $l_\infty = l_1^*$ but $l_\infty^* \neq l_1$.