WEEK 6
CH. 6.2-6.3 FROM A. FRIEDMAN

Hilbert Spaces

Definition.
H is called a Hilbert space if H is a complex linear space supplied with $(\cdot,\cdot): H \times H \to \mathbb{C}$ s.t.
- $(x, x) \geq 0$, & $(x, x) = 0 \Leftrightarrow x = 0$.
- $(x + y, z) = (x, z) + (y, z)$, $\forall x, y, z \in H$.
- $(\lambda x, y) = \lambda(x, y)$, $\forall x, y \in H$, $\lambda \in \mathbb{C}$.
- $(x, y) = \overline{(y, x)}$, $\forall x, y \in H$.
- If $\{x_n\}$ is a Cauchy sequence and $\lim_{n,m\to\infty}(x_n - x_m, x_n - x_m) = 0$, then there exists $x \in H$, s.t. $\lim_{n\to\infty}(x_n - x, x_n - x) = 0$.

$(\cdot,\cdot)$ is called scalar product.

$\|x\| = \sqrt{(x, x)}$ is called the norm of x.

Theorem. (AFr 6.1.1) (Schwartz inequality)
Let H be a Hilbert space, $x, y \in H$. Then

$$|(x, y)| \leq \|x\|\|y\|.$$

Theorem. (AFr 6.1.2)
Any Hilbert space is a Banach space whose norm is equal to $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

Corollary. (AFr 6.1.3)
The norm in a Hilbert space is strictly convex, namely

$$\|x\| = \|y\| = 1, \quad \|x + y\| = 2 \quad \Rightarrow \quad x = y.$$

Theorem. (AFr 6.1.4)
Let H be a Hilbert space. Then its norm satisfies the identity

$$2\Big(\|x\|^2 + \|y\|^2\Big) = \|x + y\|^2 + \|x - y\|^2.$$

Theorem. (AFr 6.1.5)
If H is a Banach space with norm satisfying Th 6.1.4 then H is a Hilbert space.

Definition.
Let H be a Hilbert space, $x, y \in H$. We say that x is orthogonal to y if $(x, y) = 0$.
Let $M \subset H$. We say that x is orthogonal to M if $(x, y) = 0$, $\forall y \in M$.
Let $N, M \subset H$. We say that N is orthogonal to M if $(x, y) = 0$, $x \in N$, $y \in M$.

Lemma. (AFr 6.2.1)
Let H be a Hilbert space and let $M \subset H$ be a closed convex subset. Then for any x_0 there exists unique element $y_0 \in M$ s.t.

$$\|x_0 - y_0\| = \inf_{y \in M} \|x_0 - y\|.$$

Theorem. (AFr 6.2.2)
Let M be a closed subspace of a Hilbert space H. Then for any $x_0 \in H$ there are elements $y_0 \in M$ and z_0 orthogonal to M s.t.

$$x_0 = y_0 + z_0$$

and this decomposition is unique.

Theorem. (AFr 6.2.4.) (Riesz theorem)
Let H be a Hilbert space. For any bounded linear fuctional $x^* : H \to \mathbb{C}$, there exists $z \in H$ s.t.

- $x^*(x) = (x, z), \qquad x \in H.$
- $\|x^*\| = \|z\|.$

Projections.

Definition.
Let H be a Hilbert space and let $M \subset H$ be a closed linear subspace. Then according to Theorem 6.2.2 for any $x \in H$ there exist unique vectors $y, z \in H$ such that

$$x = y + z, \qquad y \in M, \quad z \text{ orthogonal to } M.$$

We say then that y is the projection of x on M and define the linear operator $P : H \to M$ such that $Px = y$. P is called the projection operator on M.

Definition.
Let T be a bounded linear operator $T : H \to H$. The adjoint T^* of T is defined by the equality

$$(Tx, y) = (x, T^*y), \qquad \forall x, y \in H.$$

If $T = T^*$ then T is called self-adjoint.

Remark. If T is self-adjoint then the scalar product (Tx, y) is real.

Theorem. (AFr 6.3.1.)
Let P be a projection. Then

- P is a self-adjoint linear operator.
- $P^2 = P$.
- $\|P\| = 1$ if $P \neq 0$.

Home exercises.

1. Let a linear operator P satisfies the properties $P^* = P$ and P^2 is a projection. Is P a projection?

2. Confider the operator $Qf(t) = a(t)f(t)$in $L^2(0, 1)$, where $a(t)$ is a scalar function. Find necessary and sufficient conditions on $a(t)$ for Q to be a projection.

3. Let $H = L^2(-\infty, \infty)$ and let

$$\chi(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}.$$

Show that

- the operator $Pf(x) = \chi(x)f(x)$ is a projection.
- Let $\mathcal{F}$ be the Fourier transform
$$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx.$$
  Show that the operator Q defined by
$$Q\,f = \mathcal{F}^{-1}\chi\mathcal{F}\,f$$
  is a projection.
- Is PQP a projection?