# WEEK 7

**Theorem.** (AFr 6.3.2.)
If $P$ is a self-adjoint linear operator s.t. $P^2 = P$, then $P$ is a projection.

**Definition.**
Let $P_1$ and $P_2$ be projections in $H$. We say that $P_1$ is orthogonal to $P_2$ is $P_1P_2 = 0$.

**Remark.** Since $(P_1P_2)^* = P_2^*P_1^* = P_2P_1$, then $P_1P_2 = 0$ implies $P_2P_1 = 0$.

**Theorem.** (AFr 6.3.4.)
The operator $P_1 + P_2$ is a projection *iff* $P_1P_2 = 0$.

**Theorem.** (AFr 6.3.5.)
The product $P_1P_2 = 0$ is a projection *iff* $P_1P_2 = P_2P_1$.

**Definition.**
Let $H$ be a Hilbert space. A set $K \subset H$ is called orthonormal if for each element of $x \in K$ we have $\|x\| = 1$ and and any two elements from $K$ are orthogonal. An orthonormal set $K$ is called complete if there exists no nonzero elements that are orthogonal to $K$.

**Theorem.** (AFr 6.4.1.) (Bessel's inequality)
Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set in a Hilbert space $H$. Then for any $x \in H$

$$\sum_{n=1}^{\infty} |(x, x_n)|^2 \leq \|x\|^2.$$

**Theorem.** (AFr 6.4.2.)
Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set in a Hilbert space $H$ and let $\{\lambda_n\}$ be any sequence of scalars. Then for any $m$

$$\left\|x - \sum_{n=1}^{\infty} \lambda_n x_n\right\| \geq \left\|x - \sum_{n=1}^{\infty} (x, x_n) x_n\right\|.$$

**Definition.**

A set $K$ is called an orthonormal basis of $H$ if $K$is an orthonormal set and if for any $x \in H$

$$x = \sum_{y \in K} (x, y) y.$$

**Definition.**
A space $X$ is called separable if it contains a countable dense set.

**Lemma.** (AFr 6.4.7.)
Any orthonormal basis in a separable Hilbert space is countable.