WEEK 8

Theorem.
The set $\{x_n\}$ is an orthonormal basis in a separable Hilbert space H *iff*

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, x_n)|^2$$

for any $x \in H$.

Lemma. (AFr 6.4.8.)
Any two infinite dimensional separable Hilbert spaces are isometrical isomorphic.

Corollary. (AFr 6.4.9.)
Any infinite dimensional separable Hilbert space is isometrical isomorphic to $L^2(0, 1)$.

Examples.

1. $H = L^2(-\pi, \pi)$ with an orthonormal basis $u_n(t) = e^{int}/\sqrt{2\pi}$, $n \in \mathbb{Z}$.

2. $H = H^1(-\pi, \pi)$ - Sobolev space with the scalar product:

$$(u, v) = \int_{-\pi}^{\pi} \left(u'(t)\overline{v'(t)} + u(t)\overline{v(t)} \right) dt.$$

The set of functions $u_n(t) = e^{int}/\sqrt{2\pi}$ is orthogonal set but not normal in $H^1(-\pi, \pi)$.
Question: Is the set $\{u_n\}$ basis in the Hilbert space $H^1(-\pi, \pi)$?

3. Let $\mathbb{D}$ be a unit ball in the complex plane

$$\mathbb{D} = \{z = x + iy : |z| < 1\}.$$

Let $A^2(\mathbb{D})$ be the set of $L^2(\mathbb{D})$ analytic functions in $\mathbb{D}$. The space A^2 is a Hillbert space with an orthonormal basis

$$u_k(z) = \pi^{-1/2}(k+1)^{1/2} z^k, \qquad k = 0, \pm 1, \pm 2, \dots.$$

Indeed, let $z = re^{it}$. Then

$$(u_k, u_l) = \pi^{-1}(k+1)^{1/2}(l+1)^{1/2} \int_0^{2\pi} \int_0^1 e^{i(k-l)t} r^{k+l+1} dr dt = \delta_{kl}.$$

4. $H = F^2$ - the Fock space, the space of entire function s.t.

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dx dy < \infty.$$

The set $\{\psi_k\}$

$$\psi_k(z) = \pi^{-1/2}(k!)^{-1/2} z^k, \qquad k = 0, \pm 1, \pm 2, \dots$$

is an orthonormal basis.

Definition.
Let H be a Hilbert space, $x_n, x \in H$.
We say that $x_n \to x$ weakly $\left[x = w\text{-}\lim_{n\to\infty} x_n\right]$ if for any $h \in H$ we have $(x_n, h) \to (x, h)$.

Remark 1. If $\|x_n - x\| \to 0$ then $x = w\text{-}\lim_{n\to\infty} x_n$.

Example. Let $\{u_n\}$ be an orthonormal system. Then $w - \lim_{n\to\infty} u_n = 0$. Indeed, for any $x \in H$, $x_n = (x, u_n)$ are its Fourier coefficients converging to zero.

Lemma 1. *Let* $\{x_n\}$ *be a weakly convergent sequence. Then* $\{x_n\}$ *is bounded.*

Proof. For any $y \in H$ we have $(x_n, y) \to (x, y)$, $x \in H$. Therefore the sequence $\{(x_n, y)\}$ is bounded. Now the lemma follows from the principle of uniform boundedness Th. 4.5.1 (AvFr). □

Lemma 2. *If* $y_n \to y$ *and* $w\text{-}\lim_{n\to\infty} x_n = x$*, then* $(x_n, y_n) \to (x, y)$.

Proof.

$$|(x_n, y_n) - (x, y)| \le \|x_n\| \|y_n - y\| + |(x_n, y) - (x, y)| \to 0.$$

□

Definition. A functional $\Phi : H \times H \to \mathbb{C}$ is called sesqui-linear form if $\Phi(x, y)$ it is linear w.r.t. x, anti-linear w.r.t. y and

$$\|\Phi\| := \sup_{\|x\|=\|y\|=1} |\Phi(x, y)| < \infty.$$

Remark 2. Clearly $|\Phi(x, y)| \le \|\Phi\| \|x\| \|y\|$.

Examples. Let T and S be bounded operators in H. Then $\Phi(x, y) = (Tx, y)$ and $\Phi(x, y) = (x, Sy)$ are sesqui-linear forms.

Definition. $\Phi(x) = \Phi(x, x)$ is called a quadratic form.

Remark 3. By using the same argument as in Th. 6.1.5.(Av.Fr) we find

$$4\Phi(x, y) = \Phi(x + y) - \Phi(x - y) + i\Phi(x + iy) - i\Phi(x - iy). \tag{1}$$

Definition. A form Φ is called Hermitian if

$$\Phi(y, x) = \overline{\Phi(x, y)}.$$

Theorem 1. *The form $\Phi(x, y)$ is Hermitian iff the quadratic form $\Phi(x)$ is real.*

Proof. $\Phi(y, x) = \overline{\Phi(x, y)}$ implies $\Phi(x, x) = \overline{\Phi(x, x)}$. If $\Phi(x)$ is real then by using (**??**) we obtain that $\Phi(x, y)$ is Hermitian. □

Theorem 2. *If $\Phi(x, y) = (Tx, y) = (x, Sy)$, then $\|\Phi\| = \|T\| = \|S\|$.*

Proof. Since $|\Phi(x, y)| \le \|\Phi\|\|x\|\|y\|$ the functional $\Phi(\cdot, y)$ is continuous. Then by Riesz theorem (Th. 6.2.4) there exists $h \in H$ s.t. $\Phi(x, y) = (x, h)$, $\forall x \in H$. Define now $S : y \to h$. The operator S is linear. Indeed

$$\begin{aligned}(\cdot, S(\alpha_1 y_1 + \alpha_2 y_2)) &= \Phi(\cdot, \alpha_1 y_1 + \alpha_2 y_2) \\ &= \bar{\alpha}_1 \Phi(\cdot, y_1) + \bar{\alpha}_2 \Phi(\cdot, y_2) = \bar{\alpha}_1(\cdot, Sy_1) + \bar{\alpha}_2(\cdot, Sy_2) \\ &= (\cdot, \alpha_1 Sy_1 + \alpha_2 Sy_2).\end{aligned}$$

This implies $\|Sy\| = \|h\| \le \|\Phi\|\|y\|$ and thus $\|S\| \le \|\Phi\|$.

On the other hand $|\Phi(x, y)| \le \|x\|\|Sy\| \le \|S\|\|x\|\|y\|$ which gives us $\|\Phi\| \le \|S\|$. □

Corollary 1. *For any bounded operator T in a Hilbert space H $\|T\| = \|T^*\|$.*

Theorem 3. *A linear bounded operator T in H is defined by its quadratic form (Tx, x).*

Proof. Suppose that $(T_1x, x) = (T_2x, x)$. Then by using (**??**) we find that $(T_1x, y) = (T_2x, y)$, $\forall x, y \in H$. Hence $T_1 = T_2$. □

Definition. Let $\{T_n\}$ be a sequence of bonded operators in H

- $T_n \to T$ uniformly if $\|T_n - T\| \to 0$, as $n \to \infty$.
- $T_n \to T$ strongly if $\|T_n x - Tx\| \to 0$, as $n \to \infty$, $\forall x \in H$.
- $T_n \to T$ weakly if $(T_n x, y) \to (Tx, y)$, as $n \to \infty$, $\forall x, y \in H$.

Theorem 4. *If w-lim $T_n = T$, then w-lim $T_n^* = T^*$.*

Proof.

$$\big((T_n - T)x, y\big) \to 0 \quad \Longrightarrow \quad \big((T_n^* - T^*)x, y\big) = \big(x, (T_n - T)y\big)$$
$$= \overline{\big((T_n - T)y, x\big)} \to 0.$$

□

Remark 4. s-lim $T_n = T$ does not imply s-lim $T_n^* = T^*$.
Indeed, let $h \in H$, $\|h\| = 1$ and let $\{u_n\}$ be an orthonormal system in H. Define $T = (\cdot, u_n)h$. Then

$$(T_n x, y) = ((x, u_n)h, y) = (x, (y, h)u_n) \quad \Longrightarrow \quad T^* = (\cdot, h)u_n.$$

Since $\|T_n x\| = |(x, u_n)| \to 0$ we have s-$\lim_{n\to\infty} T_n = 0$. However, $\|T_n^* h\| = \|u_n\| = 1 \not\to 0$.

Definition. (Compact Operators) A bounded operator $T : H \to H$ is called compact if it maps bounded sets onto relatively compact. We shall denote the class of compact operators by S_∞.

Theorem 5 (AFr Th. 5.1.1.). *If* $T \in S_\infty$ *then it maps weakly convergent sequences into convergent sequences.*