WEEK 9

Theorem 1 (AFr Th. 5.1.2.)**.** *Let* $T_n : H \to H$ *be sequence of compact operators uniformly convergent to* T*. Then* T *is also compact.*

Theorem 2. $T \in S_\infty$ *iff* $T^*T \in S_\infty$.

Proof. If $T \in S_\infty$ then T^* is bounded and therefore $T^*T \in S_\infty$. If $T^*T \in S_\infty$ and w-$\lim x_n = 0$, then s-$\lim T^*Tx_n = 0$. Then we obtain $(x_n, T^*Tx_n) = \|Tx_n\|^2 \to 0$. □

Theorem 3. $T \in S_\infty$ *iff* $T^* \in S_\infty$.

Proof. $T^* \in S_\infty$ implies $T^*T \in S_\infty$ and thus $T \in S_\infty$. □

Finite rank operators

Definition. T is said to be of rank r ($r < \infty$) if $\dim T(H) = r$. The class of operators of rank r is denoted by K_r and $K := \cup_r K_r$.

Theorem 4. $T \in K_r$ *iff* $T^* \in K_r$.

Proof. Let $T \in K_r$ and let $u_1, u_2, \dots, u_r$ be an orthonormal basis in $T(H)$. Then for any $x \in H$ we have

$$Tx = \sum_{k=1}^{r} (Tx, u_k)u_k = \sum_{k=1}^{r} (x, T^*u_k)u_k.$$

Denote $v_k = T^*u_k$, then $T = \sum_{k=1}^{r} (\cdot, v_k)u_k$. Moreover

$$(Tx, y) = \sum_{k=1}^{r} ((x, v_k)u_k, y) = \sum_{k=1}^{r} (x, (y, u_k)v_k) = (x, T^*y).$$

Therefore $T = \sum_{k=1}^{r} (\cdot, u_k)v_k$ and thus $T^* \in K_r$. □

Theorem 5. *The uniform closure of the class of finite rank operators* K *coincides with* S_∞*.*

Proof. Let $T \in S_\infty$. Then T maps the set $B = \{x : \|x\| \le 1\}$ onto a relatively compact set. For any $\varepsilon > 0$ there exists a finite set of elements $\{y_k\}_{k=1}^{r}$ such that for any $y \in T(B)$ we have $\min \|y - y_k\| \le \varepsilon$. Let P be the projection on the subspace spanned by y_k. Clearly rank $P \le r$. Thus for any x s.t. $\|x\| \le 1$ we obtain

$$\|Tx - PTx\| \le \min_k \|Tx - y_k\| \le \varepsilon.$$

□

Remark 1. Uniform closure cannot be replaced by the strong closure.

Theorem 6. *The strong closure of* $K(H)$ *coincides with the class of all bounded operators.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be an orthonormal basis in H and let P_n be the projectors on the subspaces spanned by $\{u_k\}_{k=1}^{n}$. Then for any $x \in H$, $\|P_n x - x\| \to 0$ which means that $\text{s-lim}\, P_n = I$. Thus $\text{s-lim}\, P_n T = T$ for any bounded operator T. □

Integral Operators

Theorem 7. *Let* $K : L^2(\Omega) \to L^2(\Omega)$, $\Omega \in \mathbb{R}$, *be an integral operator*

$$Kf(x) = \int_\Omega K(x,y) f(y)\, dy,$$

such that

$$\int_\Omega \int_\Omega |K(x,y)|^2\, dx dy < \infty.$$

Then K *is compact.*

Proof. Let $\{u_j\}_{j=1}^{\infty}$ be an orthonormal basis in $L^2(\Omega)$. Then

$$K(x,y) = \sum_{j=1}^{\infty} K_j(y) u_j(x), \quad \text{where} \quad K_j(y) = \int_\Omega K(x,y)\overline{u_j(x)}\, dx$$

for almost all y. Due to the Parseval identity we have for almost all y

$$\int_\Omega |K(x,y)|^2\, dx = \sum_{j=1}^{\infty} |K_j(y)|^2$$

and

$$\int_\Omega \int_\Omega |K(x,y)|^2\, dx dy = \sum_{j=1}^{\infty} \int_\Omega |K_j(y)|^2\, dy. \tag{1}$$

We now define the following operator of rank N

$$K_N f(x) = \int_\Omega K_N(x,y) f(y)\, dy,$$

where $K_N(x,y) = \sum_{j=1}^{N} K_j(y)u_j(x)$. By Cauchy-Schwartz inequality we obtain

$$\|(K-K_N)f\|^2 = \int_\Omega \Big| \int_\Omega \Big(K(x,y) - K_N(x,y)\Big) f(y)\, dy \Big|^2 dx$$
$$\leq \int_\Omega \int_\Omega |K(x,y) - K_N(x,y)|^2 \, dxdy \, \|f\|^2$$

Thus by using that the right hand side in (**??**) is absolutely convergent, we find

$$\|(K-K_N)\|^2 \leq \int_\Omega \int_\Omega |K(x,y) - K_N(x,y)|^2 \, dxdy$$
$$= \int_\Omega \int_\Omega |K(x,y)|^2 \, dxdy - \int_\Omega \int_\Omega K(x,y) \sum_{j=1}^{N} \overline{K_j(y)u_j(x)} \, dxdy$$
$$- \int_\Omega \int_\Omega \overline{K(x,y)} \sum_{j=1}^{N} K_j(y)u_j(x) \, dxdy + \sum_{j=1}^{N} \int_\Omega |K_j(y)|^2 \, dy$$
$$= \int_\Omega \int_\Omega |K(x,y)|^2 \, dxdy - \sum_{j=1}^{N} \int_\Omega |K_j(y)|^2 \, dy \to 0, \quad \text{as} \quad N \to \infty.$$

□

Bounded Self-adjoint Operators

Definition. A bounded operator $T : H \to H$ is said to be self-adjoint if $\forall x, y \in H$

$$(Tx, y) = (x, Ty), \qquad (A = A^*).$$

Theorem 8 (Av.Fr. 6.5.1)**.** *Let* T : *be a bounded self-adjoint operator in a Hilbert space* H. *Then*

$$\|T\| = \sup_{\|x\|=1} |(Tx, x)|.$$

Proof. Clearly if $\|x\| = 1$, then

$$|(Tx, x)| \leq \|Tx\|\|x\| = \|Tx\| \leq \|T\|$$

and therefore $\sup_{\|x\|=1} |(Tx, x)| \leq \|T\|$.

In order to proof the inverse inequality we consider $z \in H$, $\|z\| = 1$, $Tz \neq 0$ and $u = Tz/\lambda$, where $\lambda = \|Tz\|^{1/2}$. If we denote by $\alpha :=$

$\sup_{\|x\|=1} |(Tx, x)|$, then

$$\begin{aligned}\|Tz\|^2 = \Big(T(\lambda z), u\Big) &= \frac{1}{4}\Big[\Big(T(\lambda z+u), \lambda z+u\Big) - \Big(T(\lambda z-u), \lambda z-u\Big)\Big] \\ &\leq \frac{\alpha}{4}\Big[\|\lambda z+u\|^2 + \|\lambda z-u\|^2\Big] = \frac{\alpha}{2}\Big[\|\lambda z\|^2 + \|u\|^2\Big] \\ &\qquad \frac{\alpha}{2}\Big[\|\lambda\|^2 + \|Tz\|\Big] = \alpha\|Tz\|.\end{aligned}$$

This implies that for any $z \in H$, $\|z\| = 1$ we have $\|Tz\| \leq \alpha$ and hence $\|T\| \leq \alpha = \sup_{\|x\|=1} |(Tx, x)|$. $\square$