MATH 754
Some theorems on Univalent Functions
Feb 16, 2004

In this class we are going to see some of the properties of holomorphic functions belonging to the following fammilies (as usual $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$).

$$\mathbb{S} = \{f \in H(\mathbb{D}) : f(0) = 0, f^{'}(0) = 1 \text{and } f \text{ is injective}\}$$

$$\sum = \{f \in H(\{|z| > 1\}) : f \text{ is injective and has a simple pole at } \infty \text{ with residue 1 }\}$$

A typical member of $f \in \mathbb{S}$ and $g \in \sum$ will have the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$$

We are interested in the coefficients a_n and b_n.

Area Theorem If $g \in \sum$, $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$, then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

Proof: Let $r > 1, \Gamma_r = f(\{|z| = r\})$. Let E_r be the compact region bounded by Γ_r, then by Green's theorem

$$Area(E_r) = \frac{1}{2i}\int_{\Gamma_r} \bar{w}dw = \frac{1}{2i}\int_{|z|=r} g(\bar{z})dg(z)$$

$$= \pi\{r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\}$$

Letting $r \to 1+$ and using the fact that Area(E_r) ≥ 0, we get the desired result.

Corollary If $g \in \sum$, $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$, then $|b_n| \leq \frac{1}{\sqrt{n}}$ for all n.

Bieberbach's Theorem If $f \in \mathbb{S}, f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ then $|a_2| \leq 2$ with equality if and only if f is a rotation of the function $k(z) = z(1-z)^{-2}$.(The function k is called the Koebe function).

Proof The function

$$g(z) = \{f(\frac{1}{z^2})\}^{-\frac{1}{2}} = z - \frac{a_2}{2}z^{-1} + \cdots$$

is a member of the family $\sum$. Thus by the corollay to the Area theorem, $|\frac{a_2}{2}| \leq 1$

Koebe One-Quarter Theorem The range of every function of class $\mathbb{S}$ contains the disc $\{w : |w| < \frac{1}{4}\}$.

Proof Suppose $w \notin f(\mathbb{D})$, then the function

$$h(z) = \frac{wf(z)}{w - f(z)} = z + (a_2 + \frac{1}{w})z^2 + \cdots$$

is in the family $\mathbb{S}$. Thus by Bieberbach's theorem,

$$|a_2 + \frac{1}{w}| \leq 2.$$

This together with $|a_2| \leq 2$ gives $|w| \geq \frac{1}{4}$.

The following theorem is a particular case of the DeBrange's Theorem(Previously known as Bieberbach's Conjecture).

Theorem If all the taylor coeficients of $f \in \mathbb{S}$ are real then $|a_n| \leq n$.

Proof Let $V(z)$ be the imaginary part of $f(z)$. Then by cauchy's Formula,

$$a_n r^n = \frac{i}{\pi} \int_{-\pi}^{\pi} V(re^{i\theta})e^{-in\theta}d\theta, n = 1, 2, 3, ...$$

Taking real part on both sides and observing the fact that the sign of $V(z)$ is the same as the sign of Imaginary part of z, we get

$$a_n r^n = \frac{2}{\pi} \int_0^{\pi} V(re^{i\theta}) \sin(n\theta)d\theta.$$

Now since $V(re^{i\theta}) \geq 0$ on the upper semi disc and $|\sin(n\theta)| \leq n\sin(\theta)$, we get

$$|a_n|r^n \leq n\frac{2}{\pi} \int_0^{\pi} V(re^{i\theta}) \sin(\theta) = na_1 = n.$$

The following theorem tells us what are the univalent functions on the unit disc with integer Taylor coefficients.

Theorem If $f \in \mathbb{S}$ has all integer Taylor coefficients then f has to be one of the following functions

$$z, \frac{z}{1 \pm z}, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2}.$$

Proof Let

$$g(z) = \frac{1}{f(\frac{1}{z})} = z + b_0 + b_1 z^{-1} + \cdots$$

Then $a_i \in \mathbb{Z}$ implies $b_i \in \mathbb{Z}$. Thus by the corollary to the area theorem $b_n = 0, n \geq 2$ Hence

$$g(z) = z + b_0 + b_1 z^{-1}, b_1 \in \{0, \pm 1\}$$

$$f(z) = \frac{z}{1 + b_0 z + b_1 z^2}.$$

Case(1): $b_1 = 0$

$$f(z) = \frac{z}{1 + b_0 z}.$$

Since f is analytic on the unit disc, the integer b_0 must satisfy $|b_0| \leq 1$. Thus $f(z) = z, \frac{z}{1 \pm z}$ and each of these is in the family $\mathbb{S}$.

Case(2): $b_1 = 1$ In this case

$$f(z) = \frac{z}{1 + b_0 z + z^2}.$$

Since $z^2 + b_0 z + 1 = 0$ has to have both the zeros on the unit circle the only possible values of b_0 are $0, \pm 1, \pm 2$ and correspondingly

$$f(z) = \frac{z}{1 + z^2}, \frac{z}{1 \pm z + z^2}, \frac{z}{(1 \pm z)^2}.$$

Case(3): $b_1 = -1$ In this case the only possible value of b_0 is 0. Thus $f(z) = \frac{z}{1 - z^2}$.

Reference

[1] Univalent Functions, Peter L. Duren, Springer-Verlag

[2] Analytic Function Theory , Vol 2, Einar Hille, Ginn and company

---