Non linear eigenvalues and analytic hypoellipticity.

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Abstract

Motivated by the problem of analytic hypoellipticity, we show that a special family of compact non selfadjoint operators has a non zero eigenvalue. We recover old results obtained by ordinary differential equations techniques and show how it can be applied to the higher dimensional case. This gives in particular a new class of hypoelliptic, but not analytic hypoelliptic operators.

1 Introduction

There is a long history highlighting the links between spectral analysis and the construction of hypoelliptic but not analytic hypoelliptic operators. Since the basic works of [30, 40, 39, 38, 14] and the necessary conditions obtained by [31], there has been a lot of effort in understanding when Hörmander sums of squares operators formed by real-analytic vector fields fail to satisfy the analytic hypoellipticity property. These results more or less may be summarized by the fact that failure of analytic hypoellipticity occurs whenever the characteristic set of the vector fields satisfies a certain condition conjectured by Trèves [41].
We refer to [1, 3, 5, 6, 7, 8, 9, 12, 17, 18, 19, 20, 21, 22, 28, 32, 33, 34] for various examples. Two types of problems appear. The first type is described by the Baouendi-Goulaouic example [1]. For showing that $D_{x_1}^2 + {x_1}^2 D_{x_2}^2 + D_{x_3}^2$ is not hypoanalytic, it is shown that it is enough to find a complex λ such that $D_{x_1}^2 + {x_1}^2 + \lambda^2$ is not injective. It is enough to take $\lambda = i\sqrt{\lambda_j}$ where λ_j is an eigenvalue of the harmonic oscillator. This idea can be used in a quite general context, see [23] and [3] for more recent variants, without any restrictions on the dimension.
The second type was initially proposed by B. Helffer in [21, 22] and solved by Pham The Lai-Robert [34]. For showing that the operator $D_{x_1}^2 + ({x_1}^2 D_{x_2} - D_{x_3})^2$ is not analytic hypoelliptic, one has to show that it is enough to find a complex λ such that $D_{x_1}^2 + ({x_1}^2 - \lambda)^2$ is not injective. This problem is more involved. The proof in [34] although multi-dimensional in principle seems to break down almost immediately when the spectral problem is in dimension greater than 1. The conditions of Theorem 2.3 in [34] (Section 3, Application 1) are not so easy to verify. On the other hand, these authors prove the existence of a complete system of eigenvectors. This property is much stronger but not useful for the problem of non analytic hypoellipticity, which requires only the existence of one eigenvector. After this work, M. Christ (and then many others as recalled in the references above) extended this example. Typically M. Christ can deal with the family $D_{x_1}^2 + (x_1^m - \lambda)^2$ ($m > 1$), in particular with m odd which seems not accessible by the Pham The Lai-Robert method [34] [35].

The method of Christ relies on the Wronskian function and thus seems limited to models which give rise to one dimensional spectral problems. Our aim is to propose a technique permitting to treat many new examples not necessary in dimension 1.

Our family of operators would be of the type

$$H(x, D_x, \lambda) = -\Delta + (\lambda - P(x))^2 \;, \tag{1.1}$$

where $x \mapsto P(x)$ is an homogeneous elliptic polynomial on $\mathbb{R}^n$ of order $m > 1$. Although it could be a rather natural conjecture that in this case there exists always $\lambda \in \mathbb{C}$ such that $H(x, D_x, \lambda)$ is non injective on $\mathcal{S}(\mathbb{R}^n)$, our results will be only true for $n \leq 3$ and $m \geq m(n) > 1$ (See Theorems 5.2 and 6.2).

The spectral result which is considered can first be reduced to a problem for a compact operator.

We rewrite $H(x, D_x, \lambda)$ in the form

$$H(x, D_x, \lambda) = L - 2\lambda M + \lambda^2 \;, \tag{1.2}$$

with

$$L = -\Delta + P(x)^2 \;,\; M = P(x) \;. \tag{1.3}$$

The operator L is invertible and its inverse is a pseudo-differential operator (See appendix C and Helffer [24]). It is also easy to give sufficient condition for determining whether the operator

$$A := L^{-1} \tag{1.4}$$

belongs to a given Schatten class (see [36] and appendix B). The Hilbert-Schmidt character can be deduced from the fact that the Weyl symbol is in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. The restriction $n \leq 3$ appears for example if $m \geq 2$ and if we want to have $A := L^{-1}$ Hilbert-Schmidt. The condition that A is Trace class leads to $m > 1$ and $n = 1$.

Then the initial problem is reduced to the spectral analysis of

$$(I - 2\lambda B + \lambda^2 A)u = 0 \;. \tag{1.5}$$

with

$$B = A^{\frac{1}{2}} P A^{\frac{1}{2}} \tag{1.6}$$

In the spirit of [34], one is led to the study of the so-called operator pencils for which there is a large literature, for e.g. Markus's book [29]. Additional literature was mentioned to us by Markus. However these results do not apply to our situation. Typically one has results where the operator pencils are of the type

$$I - 2\lambda B - \lambda^2 A \;,$$

where A, B are selfadjoint and compact, see Friedman-Shinbrot [13] and reference therein. Our situation is what is called in the literature an elliptic pencil.

A few months ago, one of the authors (S.C.) proved a result [4], which we later realized was a weak version of Lidskii's Theorem. Motivated by [4], we were led to consider the computation of traces in the spectral problems we will deal with in this article. Lidskii's Theorem will systematically be applied in the sequel.

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2 Lidskii's Theorem and applications

Let us show how to use Lidskii's Theorem. We consider the problem of determining if there exists a non trivial pair (λ, v) such that

$$(I - 2\lambda B + \lambda^2 A)u = 0 \; . \tag{2.1}$$

The initial motivating example is the example where :

$$L = D_t^2 + t^{2m} \; , \; A = L^{-1} \; , \; B = A^{\frac{1}{2}} t^m A^{\frac{1}{2}} \tag{2.2}$$

which was solved by Pham The Lai-Robert [34], when $m > 0$ is even and by Christ [5] when $m > 1$ is odd.

We first use the reduction to the linear spectral problem. It is enough to show that the operator $\mathcal{D}$ defined by

$$\mathcal{D} := \begin{pmatrix} 2B & A^{\frac{1}{2}} \\ -A^{\frac{1}{2}} & 0 \end{pmatrix} \tag{2.3}$$

has a non zero eigenvalue μ. The first component of the eigenvector is an eigenvector of the problem (2.1) with $\mu = \frac{1}{\lambda}$.

If B and A are compact, $\mathcal{D}$ is compact but the main difficulty is that $\mathcal{D}$ is not selfadjoint. Standard results as for example explained in [37] do not apply.

We would like to use Lidskii's Theorem (see [37] or [2]) in the form

Theorem 2.1 .
Let $\mathcal{C}$ be a trace class operator then

$$\sum_j \lambda_j(\mathcal{C}) = \operatorname{Tr} \mathcal{C} \,.$$

In particular, if the spectrum $\sigma(\mathcal{C})$ satisfies

$$\sigma(\mathcal{C}) = \{0\} \,,$$

then

$$\operatorname{Tr} \mathcal{C}^k = 0 \,,\ \forall k \in \mathbb{N}^* \,.$$

As an immediate corollary, we get :

Corollary 2.2 *Rank 2 criterion.*
If $\mathcal{D}$ is Hilbert-Schmidt (that is B Hilbert-Schmidt and A positive and Trace class) and if the condition :

$$\operatorname{Tr} (2B^2 - A) \neq 0 \,,$$

is satisfied, then $\mathcal{D}$ has at least one non zero eigenvalue.

Proof.
The proof is by contradiction. If $\mathcal{D}$ has no non zero eigenvalue, the same is true for $\mathcal{C} = \mathcal{D}^2$. We then apply the theorem to $\mathcal{C}$ with $k = 1$.

One could also try to use the criterion for other values of k. If we first consider the case $k = 1$, one gets that if $A^{\frac{1}{2}}$ and B are Trace class and if $\operatorname{Tr} B \neq 0$ then $\mathcal{D}$ has at least one non zero eigenvalue. In our applications (where $A = (-\Delta + P(x)^2)^{-1}$), this is not very useful, because the condition on $A^{\frac{1}{2}}$ is too strong and never satisfied. The consideration of the cases $k = 3$ and $k = 4$ will leads to interesting and new results. One will exploit the two following corollaries.

Corollary 2.3 *Rank 3 criterion.*
If $A^{\frac{3}{2}}$ and B^3 are trace class, then, if

$$\mathrm{Tr}\ \left(4B^3 - 3BA\right) \neq 0\ . \tag{2.4}$$

is satisfied, then $\mathcal{D}$ has at least one non zero eigenvalue.

Corollary 2.4 *Rank 4 criterion.*
If A and B^2 are Hilbert-Schmidt, then, if

$$\mathrm{Tr}\ \left(8B^4 - 8B^2A + A^2\right) \neq 0\ . \tag{2.5}$$

is satisfied, then $\mathcal{D}$ has at least one non zero eigenvalue.

3 Application of the rank 2 criterion

3.1 The Christ–Hanges-Himonas–Pham The Lai-Robert example

Theorem 3.1 .
If $m > 1$, the problem

$$\left(D_t^2 + (t^m - \lambda)^2\right) f = 0\ ,$$

has a solution (λ, f) with $\lambda \in \mathbb{C}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, $f \not\equiv 0$.

Proof.
Let us show that the condition in Corollary 2.2 is satisfied. Using that $(D_t^2 + \gamma t^{2m})$ is isospectral to $\gamma^{\frac{1}{m+1}}(D_t^2 + t^{2m})$, one gets first the identity

$$\mathrm{Tr}\ (D_t^2 + \gamma t^{2m})^{-1} = \gamma^{-\frac{1}{m+1}}\ \mathrm{Tr}\ (D_s^2 + s^{2m})^{-1}$$

Differentiating with respect to γ and taking $\gamma = 1$, leads to

$$\frac{1}{m+1}\ \mathrm{Tr}\ \left((D_t^2 + t^{2m})^{-1}\right) = \ \mathrm{Tr}\ \left((D_t^2 + t^{2m})^{-1} t^{2m} (D_t^2 + t^{2m})^{-1}\right)\ . \tag{3.1}$$

It is indeed enough to see that, if C is Hilbert-Schmidt, then

$$\mathrm{Tr}\ C^2 = \langle C\ ,\ C^* \rangle_{H.S} \leq ||C||_{H.S}\ \cdot\ ||C^*||_{H.S} = \ \mathrm{Tr}\ CC^*. \tag{3.2}$$

Let us see how it is used in our case. We observe that, by cyclicity of the trace (see Proposition A.1 in appendix A), we have

$$\mathrm{Tr}\ B^2 = \ \mathrm{Tr}\ C^2\ ,$$

with $C = t^m(D_t^2 + t^{2m})^{-1}$, if $B = (D_t^2 + t^{2m})^{-\frac{1}{2}} t^m (D_t^2 + t^{2m})^{-\frac{1}{2}}$. We then get that

$$\begin{aligned}&\mathrm{Tr}\ CC^* = \ \mathrm{Tr}\ t^m(D_t^2 + t^{2m})^{-2} t^m = \\ &\mathrm{Tr}\ t^{2m}(D_t^2 + t^{2m})^{-2} = \\ &\mathrm{Tr}\ (D_t^2 + t^{2m})^{-1} t^{2m} (D_t^2 + t^{2m})^{-1}\end{aligned}$$

which is the quantity which was computed in (3.1). We note that this time, we do not have anymore the restriction that m is even for applying the results.

This gives:

$$\mathrm{Tr}\ \left(2B^2 - A\right) = (\frac{2}{m+1} - 1)\ \mathrm{Tr}\ (A) < 0\ , \tag{3.3}$$

if $m > 1$.

3.2 The Hoshiro-Costin-Costin example

Let us now try to recover results by Hoshiro [28] and O. and R. Costin [12]. The goal will be partially achieved by the

Theorem 3.2 .
If

$$2\ell + 1 < m\ , \tag{3.4}$$

then the problem

$$\left(D_t^2 + (t^m - t^\ell \lambda) 2\right) f = 0\ ,$$

has a solution (λ, f) *with* $\lambda \in \mathbb{C}$ *and* $f \in \mathcal{S}(\mathbb{R}^n)$, $f \not\equiv 0$.

We expand the operator in the usual way:

$$\begin{aligned}&I - 2\lambda (D_t^2 + t^{2m})^{-\frac{1}{2}} t^{\ell+m} (D_t^2 + t^{2m})^{-\frac{1}{2}} + \lambda^2 (D_t^2 + t^{2m})^{-\frac{1}{2}} t^{2\ell} (D_t^2 + t^{2m})^{-\frac{1}{2}} \\ &= I - 2\lambda B + \lambda^2 A\ .\end{aligned} \tag{3.5}$$

Here

$$B = (D_t^2 + t^{2m})^{-\frac{1}{2}} t^{\ell+m} (D_t^2 + t^{2m})^{-\frac{1}{2}}$$

and

$$A = (D_t^2 + t^{2m})^{-\frac{1}{2}} t^{2\ell} (D_t^2 + t^{2m})^{-\frac{1}{2}} \ .$$

We note that ℓ should satisfy

$$0 \leq \ell < m \ .$$

We observe that

$$\begin{aligned} \text{Tr } B^2 = &\ \text{Tr } (D_t^2 + t^{2m})^{-\frac{1}{2}} t^{\ell+m} (D_t^2 + t^{2m})^{-1} t^{\ell+m} (D_t^2 + t^{2m})^{-\frac{1}{2}} \\ = &\ \text{Tr } (t^m (D_t^2 + t^{2m})^{-1} t^\ell t^m (D_t^2 + t^{2m})^{-1} t^\ell). \end{aligned}$$

We take

$$C = t^m (D_t^2 + t^{2m})^{-1} t^\ell \ .$$

We get as before the estimate

$$\begin{aligned} \text{Tr } B^2 \leq &\ \text{Tr } t^{2m} (D_t^2 + t^{2m})^{-1} t^{2\ell} (D_t^2 + t^{2m})^{-1} \\ = &\ \text{Tr } t^{2\ell} (D_t^2 + t^{2m})^{-1} t^{2m} (D_t^2 + t^{2m})^{-1} \ . \end{aligned} \tag{3.6}$$

For computing the right hand side, we introduce as before a parameter γ and observe that

$$\text{Tr } (t^{2\ell} (D_t^2 + \gamma t^{2m})^{-1}) = \gamma^{-\frac{\ell+1}{m+1}} \text{ Tr } (s^{2\ell} (D_s^2 + s^{2m})^{-1}) \ .$$

Differentiating with respect to γ, we get

$$\text{Tr } (t^{2\ell} (D_t^2 + t^{2m})^{-1} t^{2m} (D_t^2 + t^{2m})^{-1}) = \frac{\ell+1}{m+1} \text{ Tr } (s^{2\ell} (D_s^2 + s^{2m})^{-1}). \tag{3.7}$$

This finally gives

$$\text{Tr } (2B^2 - A) \leq \left(2\frac{\ell+1}{m+1} - 1 \right) \text{ Tr } A < 0 \ , \tag{3.8}$$

4 The main tools

Four tools were employed in the arguments in the preceding sections. In this section we elaborate briefly on these tools. The tools apply once the trace for our operators is defined. The necessary lemmas needed to prove the existence of the various traces which come up in our arguments are presented in the Appendix. The four tools we need are :

1. Invariance by cyclicity of the trace,
2. Scaling invariance of P and A_γ,
3. Cauchy-Schwarz inequality in the Hilbert-Schmidt spaces and positivity,
4. Invariance by taking the adjoint.

Cyclicity. The justification of the formula

$$\text{Tr }(CD) = \text{ Tr }(DC)\ ,$$

where C and D are Hilbert-Schmidt can be extended slightly using the results of Appendix A. We will systematically identify various non commutative polynomial of P and A giving the same trace.

Scaling. We introduce

$$A_\gamma = (-\Delta + \gamma P^2)^{-1}\ ,\ A_1 = A\ ,\ B = A^{\frac{1}{2}} P A^{\frac{1}{2}}\ .$$

We also observe that P and A are selfadjoint and that A is positive. We shall also use that P is homogeneous of degree m with respect to a dilation and that $-\Delta$ is homogeneous of degree -2. Under this condition, we have immediately by dilation :

Lemma 4.1 .
A_γ is isospectral to $\gamma^{-\frac{1}{m+1}} A_1$.

As a corollary, we get, under the assumption that the objects in consideration are trace class

$$\text{Tr } A_\gamma^\ell = \gamma^{-\frac{\ell}{m+1}} \text{ Tr } A^\ell\ . \tag{4.1}$$

Cauchy-Schwarz and positivity. For a pair of Hilbert-Schmidt operators C, D we will use the properties (with some variants) :

$$\text{Tr } CC^* \geq 0\ , \tag{4.2}$$

and

$$\text{Tr } CD^* \leq \sqrt{\text{Tr } CC^*} \sqrt{\text{Tr } DD^*}\ . \tag{4.3}$$

We recall that we used this with $D = C^*$ in (3.2).

Invariance by taking the adjoint. It is well known, that $\text{Tr}\, C^* = \overline{\text{Tr}\, C}$. If we observe here that **our operators are real operators**, we also have :

$$\text{Tr}\, C = \text{Tr}\, C^* \,. \tag{4.4}$$

5 Application of the rank 3 criterion

In order to apply Corollary 2.3, we need to verify (2.4)

$$4 \text{ Tr}\, B^3 - 3 \text{ Tr}\, BA \neq 0 \,,$$

and to verify that $A^{\frac{3}{2}}$ and B^3 are trace class. We will assume in this section that the homogeneous polynomial P is elliptic. Thus we also have without loss of generality,

$$P \geq 0 \,. \tag{5.1}$$

Using the ellipticity of P and (C.3), we easily see that $A^{\frac{3}{2}}$ and B^3 are trace class provided $n = 2$, $m \geq 4$. We have

Lemma 5.1 .
Assume $n = 2$, $m \geq 4$ and let P be a homogeneous elliptic polynomial. Then

$$\text{Tr}\,(4B^3 - 3BA) \leq \left(2\,\frac{m+2}{m+1} - 3\right) \text{Tr}\,(BA) < 0 \,. \tag{5.2}$$

Proof :
The strict inequality in the statement of Lemma 5.1 follows from the fact that P is elliptic, non negative and $m \geq 4$. The conditions $n = 2$, $m \geq 4$, ensure as noted above that the traces that occur in Lemma 5.1 and in the ensuing proof are all defined. Now,

$$\begin{aligned} \text{Tr}\,(B^3) &= \text{Tr}\,(PA)^3 \,, \\ \text{Tr}\,(BA) &= \text{Tr}\,(PA^2) \,. \end{aligned} \tag{5.3}$$

We will establish,

$$\text{Tr}\,(PA)^3 \leq \frac{1}{2}\left(\frac{m+2}{m+1}\right) \text{Tr}\,(PA^2) \,. \tag{5.4}$$

Combining (5.4) with (5.3) we get

$$\text{Tr}\,(B^3) \leq \frac{1}{2}\left(\frac{m+2}{m+1}\right) \text{Tr}\,(BA) \,. \tag{5.5}$$

Our lemma follows easily from (5.5). We now prove (5.4). The scaling argument is used in the following way :

$$\text{Tr } (PA_\gamma)^3 = \gamma^{-\frac{3}{2}\frac{m+2}{m+1}} \text{ Tr } (PA)^3 \ . \tag{5.6}$$

By differentiation, we get

$$\text{Tr } \big((PA)^3 P^2 A\big) = \frac{1}{2}\frac{m+2}{m+1} \text{ Tr } (PA)^3 \ . \tag{5.7}$$

Since $P \geq 0$, the Cauchy-Schwarz inequality gives :

$$\begin{aligned} \text{Tr } (PA)^3 \ &= \ \text{Tr } \Big((AP^{\frac{1}{2}})(P^{\frac{1}{2}}APAP)\Big) \\ &\leq \Big(\text{ Tr } (AP^{\frac{1}{2}}\ P^{\frac{1}{2}}A)\Big)^{\frac{1}{2}} \Big(\text{ Tr } (P^{\frac{1}{2}}APAP \cdot PAPAP^{\frac{1}{2}})\Big)^{\frac{1}{2}} \\ &= (\text{ Tr } (PA^2))^{\frac{1}{2}} (\text{ Tr } (PA)^3 P^2 A)^{\frac{1}{2}} \ . \end{aligned}$$

Using (5.7), we get

$$\text{Tr } (PA)^3 \leq (\text{ Tr } (PA^2))^{\frac{1}{2}} \left(\frac{1}{2}\frac{m+2}{m+1} \text{ Tr } (PA)^3\right)^{\frac{1}{2}} \ .$$

So this implies (5.4). To summarize, we have proved

Theorem 5.2 .
If $n = 2$, $m \geq 4$ and if P is an elliptic positive homogeneous polynomial of degree m, then there exists a non trivial solution (λ, f) in $\mathbb{C} \times \mathcal{S}(\mathbb{R}^2)$ of

$$(-\Delta + (P(x) - \lambda)^2) f = 0 \ .$$

6 Application of the rank 4 criterion

In this section we will use Corollary 2.4. For the formal part of the argument it is not necessary to assume that P is an elliptic polynomial or positive, in contrast to the previous section. However by assuming ellipticity on P, we easily verify using (C.3) that A is Hilbert-Schmidt and B^4 is trace class when,

$$-4 + n(1 + \frac{1}{m}) < 0 \ .$$

This imposes a dimensional restriction, $n \leq 3$, and $m > 3$. See also Remark 6.3. There is no dimensional restriction in the formal part of the argument. We have,

Lemma 6.1 .
Let $n \leq 3$, $m \geq 6$ and P a homogeneous elliptic polynomial of degree m. Then

$$\mathrm{Tr}\ \left(8B^4 - 8B^2A + A^2\right) \geq \ \mathrm{Tr}\ (8B^4) + \left(\frac{m-7}{m+1}\right)\ \mathrm{Tr}\ A^2\ ,\ \textit{for } m \geq 7\ , \tag{6.1}$$

and

$$\mathrm{Tr}\ \left(8B^4 - 8B^2A + A^2\right) \geq \frac{7m-41}{8(m+1)}\ \mathrm{Tr}\ A^2\ ,\ \textit{for } m \geq 6\ . \tag{6.2}$$

Proof :
As observed above via (C.3) the traces that occur in the statement of Lemma 6.1 and the arguments to follow are all defined since $n \leq 3$ and $m \geq 5$. Our lemma easily follows from,

$$\mathrm{Tr}\ (B^2A) \leq \frac{1}{m+1}\ \mathrm{Tr}\ (A^2)\ , \tag{6.3}$$

and

$$8\ \mathrm{Tr}\ (B^2A) \leq \left(\frac{6}{m+1} + \frac{1}{8}\right)\ \mathrm{Tr}\ A^2 + 8\ \mathrm{Tr}\ B^4\ . \tag{6.4}$$

We begin with the proof of (6.3). We have,

$$\mathrm{Tr}\ B^2A = \ \mathrm{Tr}\ (A^{\frac{1}{2}}PA^{\frac{1}{2}})2A = \ \mathrm{Tr}\ (PA)^2A\ .$$

We will use the Cauchy-Schwarz inequality in two different ways. The first trivial idea is to write

$$\mathrm{Tr}\ B^2A \leq \frac{\alpha}{2}\ \mathrm{Tr}\ B^4 + \frac{1}{2\alpha}\ \mathrm{Tr}\ A^2\ , \tag{6.5}$$

which is true for any $\alpha \in]0,1[$.
Using the cyclicity of the trace, this can equivalently be written in the form

$$\mathrm{Tr}\ (PA)^2A \leq \frac{\alpha}{2}\ \mathrm{Tr}\ (PA)^4 + \frac{1}{2\alpha}\ \mathrm{Tr}\ A^2\ . \tag{6.6}$$

It is immediate to see that this inequality is not sufficient for getting the expected inequality

$$8\ \mathrm{Tr}\ B^2A < 8\ \mathrm{Tr}\ B^4 + \ \mathrm{Tr}\ A^2\ . \tag{6.7}$$

So we try an alternative Cauchy-Schwarz inequality, by writing

$$\begin{aligned} \text{Tr } (PA)^2 A &= \text{ Tr } A^{\frac{1}{2}} PA\, PA^{\frac{3}{2}} \\ &\leq (\text{ Tr } A^{\frac{1}{2}} PAAPA^{\frac{1}{2}})^{\frac{1}{2}} (\text{ Tr } PA^{\frac{3}{2}} A^{\frac{3}{2}} P)^{\frac{1}{2}} \\ &\leq (\text{ Tr } PA^2 PA)^{\frac{1}{2}} (\text{ Tr } P^2 A^3)^{\frac{1}{2}} \,. \end{aligned}$$

This leads to

$$\text{Tr } (PA)^2 A \leq \text{ Tr } P^2 A^3 \,. \tag{6.8}$$

We now use the scaling invariance. As we have seen in (4.1), we have

$$\text{Tr } A_\gamma^2 = \gamma^{-\frac{2}{m+1}} \text{ Tr } A^2 \,, \tag{6.9}$$

and differentiating with respect to γ and taking $\gamma = 1$, we get

$$\text{Tr } A^3 P^2 = \frac{1}{m+1} \text{ Tr } A^2 \,, \tag{6.10}$$

This leads to (6.3). We now prove (6.4). We now combine the inequalities (6.6) and (6.3). We write

$$\begin{aligned} 8 \text{ Tr } AB^2 &= 6 \text{ Tr } AB^2 + 2 \text{ Tr } AB^2 \\ &\leq \tfrac{6}{m+1} \text{ Tr } A^2 + \alpha \text{ Tr } A^2 + \tfrac{1}{\alpha} \text{ Tr } B^4 \,. \end{aligned}$$

The choice of $\alpha = \frac{1}{8}$ gives (6.4). We leave as an exercise for the reader that this idea cannot give a better condition on m. Collecting our results, we have shown the

Theorem 6.2 .
Let $n \leq 3$. Let $P(x)$ be a homogeneous polynomial of degree m, $m \geq 6$, which is elliptic, i.e. $P(\sigma) \neq 0$ if $\sigma \in S^{n-1}$. Then the problem

$$-\Delta f + (P(x) - \lambda)^2 f = 0 \,,$$

has a solution (λ, f) with $f \in \mathcal{S}(\mathbb{R}^n)$, $f \not\equiv 0$.

Remark 6.3 .
The hypothesis that P is elliptic can perhaps be relaxed in the spirit of [3]. For example in two dimensions, if one imposes the condition that the diameter of the tubes $-1 < P(x,y) < 1$ tapers fast enough, one recaptures compactness properties (see also [25]). However one could be then forced to study higher order traces. This is because the p value of the Schatten class $\mathcal{C}_p$ to which the operator L^{-1} belongs to will in general be large. The example when $n = 2$ and $P(x_1, x_2) = x_1 x_2 ({x_1}^2 + x_2^2)^k$ for k large does satisfy the hypotheses of Corollary 2.4 and thus we obtain the conclusions of Theorem 6.2.

7 Application to failure of analytic hypoellipticity

Let us collect some of the standard consequences of our spectral analysis. By applying Theorem 3.2, we get

Proposition 7.1 .
If $2k+1 < m$, the operator $D_t^2 + (t^m D_y - t^k D_z)^2$ is not analytic hypoelliptic.

This recovers for $k = 1$ all the mentioned known results with a unified elementary proof but gives for $k > 1$ only partially results by Hoshiro [28] and O. and R. Costin [12].
A consequence of Theorem 6.2 is the following

Proposition 7.2 .
Let $p = 2$ or 3 and let P be a positive elliptic polynomial of order $m \geq 2p$ in the variables $x = (x_1, \cdots, x_p)$. Then the operator

$$H_k := \sum_{j=1}^{p} D_{x_j}^2 + \big(P(x) D_{x_{p+1}} - D_{x_{p+2}}\big)^2 \ ,$$

is not (germ)- analytic hypoelliptic at 0.

Proof.
The smooth solution to $H_k u = 0$ that is not real-analytic can be constructed in a neighborhood of the origin by means of the formula,

$$u(x, x_{p+1}, x_{p+2}) = \int_0^\infty \exp(i\rho^{2k+1} x_{p+1} + i\rho\lambda x_{p+2}) f(\rho x) \exp(-M\rho) d\rho,$$

where $x = (x_1, \ldots, x_p)$ and f is the eigenfunction we have constructed in Theorem 6.2 and $M > 0$ picked suitably large so that the integral converges for x_{p+2} in some interval centered at the origin. It is elementary to check that u constructed above is a solution to $H_k u = 0$ and the convergence of the integral defining u and other standard estimates follow in a manner analogous to that in [17], Lemma 2.1. Using the fact that the eigenfunction f we have constructed is real-analytic at the origin, we can easily show as in [17], Lemma 2.1 that the function u is in the Gevrey class $2k + 1$ at the origin. This Gevrey order agrees with the formula in [3] that connects the location of the Trèves strata in our example and the number of commutation brackets one needs to descend to the center.

Remark 7.3 .
We emphasize that our statement is stronger than simply saying that H_k is non hypoelliptic analytic. We have indeed proved (see for example the introduction of [31] for a discussion) the existence of a distribution u such that $H_k u$ is analytic in a neighborhood of 0 *and such that u is non analytic in any neighborhood of* 0.
The proof of non hypoanalyticity in a neighborhood of 0 *can be probably obtained by Métivier's result. We also point out that N. Hanges [16] has found an example of a sums of squares operator which is not analytic hypoelliptic but is germ analytic hypoelliptic.*

Remark 7.4 .
A variant of the proof of Theorem 6.2 permits us to show the same result for the model :

$$H_k := \sum_{j=1}^{p} D_{x_j}^2 + \big(P_1(x) D_{x_{p+1}} - D_{x_{p+2}}\big)^2 + P_2(x)^2 D_{x_{p+1}}^2 \ ,$$

where P_1 and P_2 are homogeneous polynomials of degree m in the variables $x = (x_1, \dots, x_p)$, $P_1 \geq 0$, P_1 not identically zero, $P_1^2 + P_2^2$ is elliptic and where we keep the same conditions on p and m.

A Schatten classes

Here we collect a few well known results concerning Schatten classes. We refer to [37] or [2] for more details. We recall that a compact operator A on an Hilbert space $\mathcal{H}$ is in the Schatten class $\mathcal{C}_p$ for some $p \in [1, +\infty[$ if the sequence μ_j of the eigenvalues of $|A| = \sqrt{A^* A}$ satisfy $\sum_j \mu_j^p < +\infty$.
When $p = 1$, we speak about Trace class operators and, when $p = 2$, we recover the standard notion of Hilbert-Schmidt operators.
When $p = 1$, the trace map is defined by

$$\mathcal{C}_1 \ni A \mapsto \ \mathrm{Tr}\, A = \sum_j \langle A e_j \mid e_j \rangle \ , \tag{A.1}$$

where (e_j) is some orthonormal basis. It can be shown that this definition is independent of the choice of the basis and that the Trace map is continuous :

$$| \text{ Tr } A \mid \leq \| \, |A| \, \|_{\mathcal{C}_1} \; . \tag{A.2}$$

We have the Hölder relation, that is the

Proposition A.1 .
If $A \in \mathcal{C}_p$ and $B \in \mathcal{C}_q$, then $AB \in \mathcal{C}_r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.
Moreover, if $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{C}_q$, then $AB \in \mathcal{C}_q$.

When $r = 1$, we will use constantly the so-called cyclicity property of the trace :

$$\text{Tr } (AB) = \text{ Tr } (BA) \ , \ \forall A \in \mathcal{C}_p, \forall B \in \mathcal{C}_q, \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \; . \tag{A.3}$$

The case $p = 1$ is also true, if we replace $\mathcal{C}_\infty$ by $\mathcal{L}(\mathcal{H})$. Various generalizations can be found in the book by M. Birman and M. Solomyak [2].
Note also the property

$$||A||_{\mathcal{C}_1} = ||A^*||_{\mathcal{C}_1} \; . \tag{A.4}$$

The following lemma will be useful for justifying extensions of the cyclicity rule.

Lemma A.2 .
We assume that $\mathcal{H} = L^2(\mathbb{R}^n)$. Let A be of trace class and χ a function in $C_0^\infty(\mathbb{R}^n)$ with compact support in a ball of radius 2 and equal to 1 on the ball of radius 1. Then if $A_j = \chi(\frac{x}{j})A$ for $j \in \mathbb{N}^*$, we have

$$||A - A_j||_{\mathcal{C}_1} \to 0 \ , \quad \textit{as } j \to +\infty \; ; \tag{A.5}$$

and

$$\text{Tr } A = \lim_{j \to +\infty} A_j \; . \tag{A.6}$$

Proof.
Writing $A = |A|^{\frac{1}{2}} C$ with C Hilbert-Schmidt, we immediatly see that it is enough to treat the Hilbert-Schmidt case. If one recalls that the Hilbert-Schmidt operators can be isometrically identified with the operators with distribution kernel in $L^2(\mathbb{R}^k \times \mathbb{R}^k)$, we are reduced to the application of the

dominated convergence Theorem. If K is the kernel of $|A|^{\frac{1}{2}}$, we observe simply that

$$\lim_{j\to+\infty}\int_{\mathbb{R}^n\times\mathbb{R}^n}(\chi(\frac{x}{j})-1)^2|K(x,y)|^2dxdy=0\ .$$

We then conclude by observing that

$$||A-A_j||_{\mathcal{C}_1}\leq||(1-\chi(\frac{\cdot}{j}))|A|^{\frac{1}{2}}||_{\mathcal{C}_2}\ \cdot\ ||C||_{\mathcal{C}_2}\ .$$

Application .
We use this lemma in the following context. We would like to show that

$$\text{Tr }(PC)=\text{ Tr }(CP)\ , \tag{A.7}$$

where P is a polynomial, C is a trace class operator, such that PC and CP are trace class. We first observe that the usual cyclicity trace rule gives :

$$\text{Tr }(\chi(\frac{\cdot}{j})PC)=\text{ Tr }(CP\chi(\frac{\cdot}{j}))\ .$$

The lemma permits to justify the limiting procedure $j\to+\infty$.

Another trick could be to introduce an invertible operator L such that PL^{-1} is bounded and such that LC is trace class. Then one write :

$$\text{Tr }(PC)=\text{ Tr }(PL^{-1}LC)=\text{ Tr }(LCPL^{-1}).$$

If LCP and L^{-1} are in dual Schatten classes, one can reapply the cyclicity rule, and get

$$\text{Tr }(LCPL^{-1})=\text{ Tr }(L^{-1}LCP)=\text{ Tr }(CP)\ .$$

All these conditions are practically easy to verify in the frame work of the pseudo-differential theory.

B Pseudodifferential operators and Schatten classes

The theory of pseudo-differential operators gives an easy way for recognizing that an operator belongs to a Schatten class. Let us recall a few elements of the theory. When a belongs to a suitable class of symbols (see below), the Weyl quantization of the symbol a consists in the introduction of the operator $\mathcal{S}(\mathbb{R}^n) \ni u \mapsto \mathrm{Op}^{\mathrm{w}}(a)u \in \mathcal{S}(\mathbb{R}^n)$ defined by :

$$(\mathrm{Op}^{\mathrm{w}}(a)u)(x) = (2\pi)^{-n} \int\int \exp i < x-y, \xi > a(\frac{x+y}{2}, \xi) u(y) dy d\xi \ . \tag{B.1}$$

As an extension of the Calderon-Vaillancourt theorem giving sufficient conditions for L^2-continuity, we have the following proposition for the Weyl-quantized pseudo-differential operators (See for example [36]).

Theorem B.1 .
There exists k depending only on the dimension such that, if

$$N_{k,p}(a) := \sum_{|\alpha| \leq k} ||D^\alpha_{x,\xi} a(x,\xi)||_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} < +\infty$$

then $\mathrm{Op}^{\mathrm{w}}(a)$ *belongs to* $\mathcal{C}_p$. *Moreover, we have for a suitable constant* C*:*

$$|| \mathrm{Op}^{\mathrm{w}}(a)||_{\mathcal{C}_p} \leq C \ N_{k,p}(a) \ . \tag{B.2}$$

The Hilbert-Schmidt case (corresponding to $\mathcal{C}_2$) is more standard and we recall that :

$$|| \mathrm{Op}^{\mathrm{w}}(a)||^2_{\mathcal{C}_2} = \int\int |a(x,\xi)|^2 \, dx \, d\xi \ . \tag{B.3}$$

The case $p = +\infty$ corresponds, when replacing $\mathcal{C}_\infty$ by $\mathcal{L}(L^2(\mathbb{R}^n))$, to the well known Calderon-Vaillancourt Theorem.

C On globally elliptic operators

The last thing we would like to recall is the class of pseudodifferential operators adapted to our problem of analyzing the inverse of the operators $(-\Delta + P(x)^2)^s$. The reference [27] presents a pseudo-differential calculus which is exactly adapted to the situation. The symbols are indeed C^∞ functions on $\mathbb{R}^n \times \mathbb{R}^n$ for which there exists a real M such that at ∞

$$a(x,\xi) \sim \sum_{j \in \mathbb{N}} a_{M-j}(x,\xi) \ , \tag{C.1}$$

a_{M-j} having the following homogeneity property for suitable $k > 0$ and $\ell > 0$

$$a_{M-j}(\rho^k x, \rho^\ell \xi) = \rho^{M-j} a_{M-j}(x,\xi)\ ,\ \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\ , \forall \rho > 0. \tag{C.2}$$

We call this class $S^M_{k,\ell}$. We denote by $\text{Op}\ S^M_{k,\ell}$ the class of operators defined as $\text{Op}^{\text{w}}\ (a)$ for some a in $S^M_{k,\ell}$. We note that the composition of two operators $A_1 \in\ \text{Op}\ S^{M_1}_{k,\ell}$ and of an operator $A_2 \in\ \text{Op}\ S^{M_1}_{k,\ell}$ gives $A_1 \circ A_2 \in\ \text{Op}\ S^{M_1+M_2}_{k,\ell}$, the principal symbol of the product being simply the product of the principal symbols of A_1 and A_2.

The basic example is $L = -\Delta + P^2$ with P homogeneous of degree m. With $k = \frac{1}{m}$, $\ell = 1$, we see that the symbol of this operator belongs to $S^2_{\frac{1}{m},1}$, so $L \in\ \text{Op}\ S^2_{\frac{1}{m},1}$. This operator is "elliptic" in the sense that its principal symbol does not vanish on the sphere S^{2n-1} and it is shown in [27] that its inverse has a symbol in $S^{-2}_{\frac{1}{m},1}$. Note also that a polynomial of order k belongs to $S^{\frac{k}{m}}_{\frac{1}{m},1}$. The question of determining if a pseudo-differential operator belongs to a Schatten class is then easy. The condition is simply

$$\text{Op}^{\text{w}}\ (a) \in \mathcal{C}_p \text{ if } a \in S^M_{k,\ell} \text{ with } Mp + (k+\ell)n < 0\ . \tag{C.3}$$

Remark C.1 .
We note also that the pseudo-differential calculus gives an easy way for showing that the eigenvector whose existence is proved via Lidskii's Theorem is actually in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

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