LECTURE NOTES

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A. Laptev

1. LECTURE

Sobolev $\&$ Lieb-Thirring Inequalities, Mass Transportation

Let $x_l \in \mathbb{R}^d$, $l = 1, \dots, N$. Consider many-body Schrödinger operator

$$-\sum_{l=1}^{N} \Delta_l + \sum_{k>l} v(x_k - x_l),$$

defined on normalized fermions, functions $\varphi(x_1, \dots, x_N)$ anti-symmetric with respect to x_l. Special important example: let ψ_j be orthonormal functions in $L^2(\mathbb{R}^d)$, then

$$\varphi(x_1, \dots, x_N) = (N!)^{-1/2} \text{ Det } |\psi_j(x_l)|_{j,l=1}^{N}.$$

The condition of orthogonality is caused by the Pauli exclusion principle.

One of the proofs of the problem of stability of matter is based on a so-called generalized Sobolev inequality:

$$\int_{\mathbb{R}^d} \Big[\sum_{j=1}^{N} |\psi_j(x)|^2 \Big]^{\frac{2+d}{d}} dx \le C_{d,N} \sum_{j=1}^{N} \int_{\mathbb{R}^d} |\nabla \psi_j(x)|^2 \, dx. \tag{1.1}$$

If $j = 1$ then (1.1) becomes

$$\int |\psi|^{\frac{2(2+d)}{d}} dx \le C_{d,1} \int |\nabla \psi|^2 \, dx, \qquad \|\psi\| = 1.$$

If $d \geq 3$, then the usual "critical" Sobolev ineq. $+$ Hölder ineq. imply

$$S_d \int |\nabla\psi|^2\,dx \geq \Big(\int |\psi|^{\frac{2d}{d-2}}\,dx\Big)^{\frac{d-2}{d}} \geq \int |\psi|^{\frac{2(2+d)}{d}}\,dx \Big[\int |\psi|^2\,dx\Big]^{-\frac{2}{d}}.$$

Lieb-Thirring's conjecture (open): Prove that $\sup_N C_{d,N} \leq S_d$, $d \geq 3$.

1D case

Let $\{\psi_j\}_{j=1}^n$ be in orthonormal system of function in $L^2(\mathbb{R})$ and let

$$\rho(x) = \sum_{j=1}^n \psi_j^2(x).$$

Generalised Sobolev inequality in 1D case:

Theorem 1.1 (Eden & Foias)**.**

$$\int_{\mathbb{R}} \rho^3(x)\,dx = \int \Big(\sum_{j=1}^n |\psi_j(x)|^2\Big)^3\,dx \leq \sum_{j=1}^n \int_{\mathbb{R}} |\psi'{}_j(x)|^2\,dx.$$

Proof. We first derive a so-called Agmon inequality

$$\|\psi\|_{L^\infty} \leq \|\psi\|_{L^2}^{1/2}\|\psi'\|_{L^2}^{1/2}.$$

Indeed

$$|\psi(x)|^2 = \frac{1}{2}\Big|\int_{-\infty}^x |\psi^2|'\,dt - \int_x^\infty |\psi^2|'\,dt\Big| \leq \int |\psi||\psi'|\,dt \leq \|\psi\|_{L^2}\|\psi'\|_{L^2}.$$

Let now $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$. Then by Agmon inequality

$$\Big|\sum_{j=1}^n \xi_j\psi_j(x)\Big| \leq \Big(\sum_{j,k=1}^n \xi_j\bar{\xi}_k(\psi_j,\psi_k)\Big)^{1/4}\Big(\sum_{j,k=1}^n \xi_j\bar{\xi}_k(\psi_j',\psi_k')\Big)^{1/4}$$

$$\leq \Big(\sum_{j=1}^n \xi_j^2\Big)^{1/4}\Big(\sum_{j,k=1}^n \xi_j\bar{\xi}_k(\psi_j',\psi_k')\Big)^{1/4}.$$

If we set $\xi_j = \psi_j(x)$ then the latter inequality becomes

$$\rho(x) = \sum_{j=1}^{n} |\psi_j(x)|^2 \leq \rho^{1/4}(x)\Big(\sum_{j,k=1}^{n} \psi_j(x)\overline{\psi_k(x)}(\psi_j', \psi_k')\Big)^{1/4}.$$

Thus

$$\rho^3(x) \leq \sum_{j,k=1}^{n} \psi_j(x)\overline{\psi_k(x)}(\psi_j', \psi_k').$$

Integrating both sides we arrive at

$$\int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx \leq \sum_{j=1}^{n} \int |\psi_j'|^2 \, dx.$$

$\square$

Spectrum of Schrödinger operators

Let $\{\psi_j\}_{j=1}^{\infty}$ be the orthonormal system of eigenfunctions corresponding to the negative eigenvalues of the Schrödinger operator

$$-\frac{d^2}{dx^2}\psi_j - V\psi_j = -\lambda_j \psi_j,$$

where we assume that $V \geq 0$. Then by using the latter result and Hölder's inequality we obtain

$$\int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx - \Big(\int V^{3/2} \, dx\Big)^{2/3} \int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx\Big)^{1/3}$$

$$\leq \sum_j \int \Big(|\psi_j'|^2 - V|\psi_j|^2\Big) \, dx = -\sum_j \lambda_j.$$

Denote

$$X = \Big(\int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx\Big)^{1/3},$$

then the latter inequality can be written as

$$X^3 - \left(\int V^{3/2}\,dx\right)^{2/3} X \leq -\sum_j \lambda_j.$$

Maximizing the left hand side we find $X = \frac{1}{\sqrt{3}}\left(\int V^{3/2}\,dx\right)^{1/3}$.
This implies

$$\frac{1}{3\sqrt{3}}\int V^{3/2}\,dx - \frac{1}{\sqrt{3}}\int V^{3/2}\,dx = -\frac{2}{3\sqrt{3}}\int V^{3/2}\,dx \leq -\sum_j \lambda_j$$

and we finally obtain $\sum_j \lambda_j \leq \frac{2}{3\sqrt{3}}\int V^{3/2}\,dx$.
This is the best known constant in Lieb-Thirring's inequality.

Mass Transportation and Functional Inequalities

We shall consider two examples of functional inequalities with sharp constants:
"Critical" Sobolev inequality with $p > 1$, $p^* = \frac{np}{n-p}$:

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |f|^{p^*}\,dx\Big)^{1/p^*}$$

$$\le S_n(p)\Big(\int_{\mathbb{R}^n} |\nabla f|^p\,dx\Big)^{1/p} = \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

Brézis - Lieb - Sobolev inequality:

$$\|f\|_{L^{p^*}(\Omega)} \le S_n(p)\|\nabla f\|_{L^p(\Omega)} + C_n(p)\|f\|_{L^{\tilde p}(\partial\Omega)},$$

where $\tilde p = \frac{(n-1)p}{n-p}$ and $\partial\Omega$ is locally Lipschitz.

Main idea:

We use the following statement:
Let μ and ν be two probability measures on $\mathbb{R}^n$ ($\int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^n} d\nu = 1$).
Let $d\mu(x) = F(x)\,dx$ and $d\nu(y) = G(y)\,dy$. Then there exists a convex function φ such that

$$F(x) = G(\nabla\varphi(x))\det(\nabla^2\varphi(x)), \tag{1.2}$$

where $\nabla^2\varphi$ is a Hessian of φ.
The latter equation is known as Monge-Ampère equation.
It is highly non-linear.

Optimal transportation

Let (X,μ) and (Y,ν) be two measure space with probability measures

$$\mu(X) = 1 \qquad \nu(Y) = 1.$$

1) For any $A \subset X$ and $B \subset Y$, $\mu(A)$ and $\nu(B)$ measure the "mass" of the subsets A and B respectively.

2) $c(x,y)$ - cost function - tells how much it costs to transport one unit of mass from x to y.

Problem: *Minimize the cost of transporting X to Y.*

Mathematical formulation:

Let $d\pi(x,y)$ be a probability measure which measures the amount of mass transfered from x to y.

We say that π is admissible ($\pi \in \Pi$) if

$$\int_Y d\pi(x,y) = d\mu; \qquad \int_X d\pi(x,y) = d\nu.$$

or equivalently

(1.3)

$$\iint_{X\times Y} (\varphi(x)+\psi(y))\, d\pi(x,y) = \int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y).$$

The problem of minimizing the cost of transportation is equivalent to finding

$$\inf_{\pi\in\Pi} I[\pi] = \iint_{X\times Y} c(x,y)\, d\pi(x,y).$$

- L.V. Kantorovich (Nobel Prize 1975).
Finding π gives optimal transport plan.

Theorem 1.2. (Kantorovich Duality)
Let

$$I[\pi] = \iint_{X\times Y} c(x,y)\, d\pi,$$

$$J(\varphi,\psi) = \int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y)$$

and let Φ_c be a class of functions such that

$$\Phi_c = \{(\varphi,\psi);\ \varphi(x)+\psi(y) \le c(x,y)\}.$$

Then

$$\inf_{\pi\in\Pi} I[\pi] = \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi).$$

More restrictive problem:

Consider a subclass of measures $\pi \in \Pi$ such that to each location x we assign a unique location y (no mass split). This means that there exists a measurable vector function $T: X \to Y$ such that

$$d\pi(x,y) = d\pi_T(x,y) = d\mu(x)\delta(y - T(x)).$$

Then obviously

$$I[\pi] = \int_X c(x, T(x))\, d\mu(x)$$

and (1.3) is equivalent to

$$\int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y)\, d\nu(y) = \int_X \Big(\varphi(x) + \psi(T(x))\Big)\, d\mu(x)$$

or

$$\int_X \psi(T(x))\, d\mu(x) = \int_Y \psi(y)\, d\nu(y), \tag{1.4}$$

which is the same as

$$\nu(B) = \mu(T^{-1}(B)), \qquad B \subset Y.$$

Remark 1. If $d\mu(x) = F(x)\, dx$ and $d\nu(y) = G(y)\, dy$, $y = Tx$, then (1.4) is equivalent to

$$\int_X \psi(T(x))F(x)\, dx = \int_Y \psi(y)G(y)\, dy \tag{1.5}$$

$$= \int_X \psi(T(x))G(T(x))|\det \nabla T(x)|\, dx.$$

This implies

$$F(x) = G(T(x))|\det \nabla T(x)|. \tag{1.6}$$

Remark 2. If $T(x) = \nabla\varphi$ then φ is a solution of the Monge-Ampère equation (1.2).

Quadratic cost

Let $X = Y = \mathbb{R}^n$ and the cost function c be quadratic

$$c(x, y) = \frac{|x-y|^2}{2}.$$

Assume that

$$M_2 = \int_{\mathbb{R}^n} \frac{|x|^2}{2}\, d\mu(x) + \int_{\mathbb{R}^n} \frac{|y|^2}{2}\, d\nu(y) < \infty.$$

Then obviously the total cost $I[\pi]$, $\pi \in \Pi$, is bounded

$$I[\pi] = \iint_{\mathbb{R}^{2n}} \frac{|x-y|^2}{2}\, d\pi(x, y) \le \iint_{\mathbb{R}^{2n}} (|x|^2 + |y|^2)\, d\pi(x, y) = 2M_2.$$

In particular,

$$(\varphi, \psi) \in \Phi_c \quad \Leftrightarrow \quad \varphi(x) + \psi(y) \le \frac{|x-y|^2}{2}$$

$$\Leftrightarrow \quad x \cdot y \le \underbrace{\left[\frac{|x|^2}{2} - \varphi(x)\right]}_{\tilde{\varphi}} + \underbrace{\left[\frac{|y|^2}{2} - \psi(y)\right]}_{\tilde{\psi}}.$$

Let us define

$$\tilde{\Phi} = \{(\varphi, \psi) :\ x \cdot y \le \varphi(x) + \psi(y)\}.$$

Then

$$\inf_{\Pi} I[\pi] = M_2 - \sup_{\Pi} \iint_{\mathbb{R}^{2n}} x \cdot y\, d\pi(x, y),$$

and

$$\sup_{\Phi_c} J(\varphi, \psi) = M_2 - \inf_{(\varphi,\psi)\in\tilde{\Phi}} J(\varphi, \psi).$$

Double convexification trick

Kantorovich's duality is equivalent to

$$\sup_{\Pi} \iint_{\mathbb{R}^{2n}} x \cdot y \, d\pi(x,y) = \inf_{(\varphi,\psi)\in\tilde{\Phi}} J(\varphi,\psi).$$

Clearly

$$(\varphi,\psi) \in \tilde{\Phi} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \psi(y) \geq x \cdot y - \varphi(x) \quad \Rightarrow \quad \psi(y) \geq \sup_{x\in X}[x \cdot y - \varphi(x)] := \varphi^*(y),$$

where φ^* is the Legendre transform of φ and therefore convex. We also obtain that

$$J(\varphi,\psi) \geq J(\varphi,\varphi^*).$$

The pair $(\varphi,\varphi^*) \in \tilde{\Phi}$. If we shall go on one more time we arrive at

$$\varphi(x) \geq \sup_{y\in Y}[x \cdot y - \varphi^*(y)] := \varphi^{**}(x) \quad \Rightarrow$$

$$J(\varphi,\varphi^*) \geq J(\varphi^{**},\varphi^*) \quad \text{and} \quad (\varphi^{**},\varphi^*) \in \tilde{\Phi},$$

where both functions φ^{**} and φ^* are convex.

Remarks

If φ is convex then $\varphi = \varphi^{**}$. Moreover

$$\nabla_y \varphi^*(y) = (\nabla\varphi)^{-1}(y).$$

Indeed

$$x \cdot y = \varphi(x) + \varphi^*(y) \quad \Rightarrow$$

$$y = \nabla_x \varphi(x) \qquad x = \nabla_y \varphi^*(y) \quad \Rightarrow$$

$$y = \nabla_x \varphi(\nabla_y \varphi^*(y)) \quad \Rightarrow \quad \nabla_y \varphi^*(y) = (\nabla\varphi)^{-1}(y).$$

Lemma 1.1. *The following equality holds true*

$$\inf_{(\varphi,\psi)\in\tilde{\Phi}} J(\varphi,\psi) = \inf_{\phi} J(\varphi^{**},\varphi^*) = \inf_{\varphi\in Conv.f.} J(\varphi,\varphi^*) \tag{1.7}$$

Corollary 1.1. *If F and G are measurable function*

$$\int_{\mathbb{R}^n} F(x)\,dx = \int_{\mathbb{R}^n} G(y)\,dy = 1,$$

then there exists a convex function φ such that

$$F = G(\nabla\varphi)\det|\nabla^2\varphi|.$$

Proof. Letting $T = \nabla\varphi$ we apply Kantorovich duality, the last Lemma and (1.6).

Functional inequalities

Theorem 1.3. (Critical Sobolev inequality) *Let $p \in (1, n)$ and $p^* = \frac{np}{n-p}$. Then*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \le S_n(p)\|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

In order to prove Theorem 1.3 we need the following statement:

Theorem 1.4. (Mother Inequality I)
Let $p \in (1, n)$, $q = \frac{p}{p-1}$ and $p^* = \frac{np}{n-p}$. Assume that f and g are two normalized functions, such that $\|f\|_{p^*} = \|g\|_{p^*} = 1$. Then

$$\frac{\int |g|^{\frac{p^*(n-1)}{n}}\,dy}{\left(\int |y|^q|g|^{p^*}\,dy\right)^{1/q}} \le \frac{p(n-1)}{n(n-p)}\,\|\nabla f\|_p. \tag{1.8}$$

Proof. Since $|\nabla|f|| = |\nabla f|$ we can assume that both $f, g \ge 0$. Denote

$$d\mu(x) = F(x)\,dx = f^{p^*}(x)\,dx, \qquad d\nu(y) = G(y)\,dy = g^{p^*}(y)\,dy.$$

Mass transportation theory implies that there exists a convex function φ such that

$$F(x) = G(\nabla\varphi(x))\det\nabla^2\varphi(x).$$

Notice that for a Hermitian $n \times n$ matrix A we have $\det^{1/n} A \leq \frac{1}{n}\mathrm{Tr}A$. Therefore

$$G^{-1/n}(\nabla\varphi) = F^{-1/n}(\det \nabla^2\varphi)^{1/n} \leq \frac{1}{n} F^{-1/n}\Delta\varphi.$$

Multiplying both sides by F and integrating the latter inequality we obtain

$$\int G^{-1/n}(\nabla\varphi)F(x)\, dx \leq \frac{1}{n} \int F^{1-1/n}\Delta\varphi\, dx.$$

Identity (1.5) implies

$$\int G^{1-1/n}(y)\, dy \leq \frac{1}{n} \int F^{1-1/n}\Delta\varphi\, dx \leq -\frac{1}{n} \int \nabla\Big(F^{1-1/n}\Big)\nabla\varphi\, dx.$$

Substituting $F = f^{p^*}$ and $G = g^{p^*}$ and using Hölder's inequality we have

$$\int g^{\frac{p^*(n-1)}{n}}\, dy = \int g^{\frac{p(n-1)}{n-p}}\, dy \leq -\frac{p(n-1)}{n(n-p)} \int f^{\frac{p}{p}\frac{n(p-1)}{n-p}}\, \nabla f \cdot \nabla\varphi\, dx$$

$$(1.9) \qquad = -\frac{p(n-1)}{n(n-p)} \int f^{\frac{p^*}{q}}\nabla f \cdot \nabla\varphi\, dx$$

$$\leq \frac{p(n-1)}{n(n-p)} \|\nabla f\|_p \Big(\int f^{p^*}|\nabla\varphi|^q\, dx\Big)^{1/q}.$$

Finally, by using the mass transportation identity (1.5) again we arrive at

$$\int g^{\frac{p^*(n-1)}{n}}\, dy \leq \frac{p(n-1)}{n(n-p)} \|\nabla f\|_p \Big(\int g^{p^*}(y)|y|^q\, dy\Big)^{1/q},$$

which completes the proof.

Let

$$(1.10) \qquad h_p(x) = \frac{1}{(\sigma_p + |x|^q)^{\frac{n-p}{p}}},$$

where σ_p is chosen such that $\|h_p\|_{p^*} = 1$. It is easy to check that if we substitute $f = g = h_p$ into the Mother Inequality I we obtain identity.

Choosing now $g = h_p$ and letting $f \mapsto f/\|f\|_{p^*}$ we prove Theorem 1.3.

Brézis - Lieb - Sobolev Inequality

Theorem 1.5. *Let $p \in (1,n)$, $p^* = \frac{np}{n-p}$ and $\tilde{p} = \frac{(n-1)p}{n-p}$. Then for any $\Omega \subset \mathbb{R}^n$, with locally Lipschitz boundary $\partial\Omega$*

$$\|f\|_{L^{p^*}(\Omega)} \leq S_n(p)\|\nabla f\|_{L^p(\Omega)} + C_n(p)\|f\|_{L^{\tilde{p}}(\partial\Omega)}.$$

For proving this inequality with sharp constants $S_n(p)$ and $C_n(p)$ one needs

Theorem 1.6. (Mother inequality II)
Assume that f and g are two normalized functions, such that $\|f\|_{Lp^*(\Omega)} = \|g\|_{Lp^*(\mathbb{R}^n)} = 1$. Then for any $y_0 \in \mathbb{R}^n$ there exists a constant R such that

$$n\|g\|^{\tilde{p}}_{L^{\tilde{p}}(\mathbb{R}^n)} \leq \tilde{p}\Big(\int_{\mathbb{R}^n} g^{p^*}|y - y_0|^q \, dy\Big)^{1/q}\|\nabla f\|_{L^p(\Omega)} + R\|f\|_{L^{\tilde{p}}(\partial\Omega)}.$$

The proof of this theorem uses the same idea. The boundary term appears when integrating by parts as in (1.9).

One more inequality

If $\|g\|_{p^*} = 1$ then the Mother Inequality I (Theorem 1.4) implies that there exists a constant C such that

$$\int |g|^{\frac{p(n-1)}{n-p}} \, dy \leq C\Big(\int |y|^{\frac{p}{p-1}}|g|^{\frac{np}{n-p}} \, dy\Big)^{\frac{p-1}{p}}.$$

Letting $g \mapsto g/\|g\|_{p^*}$ we have

Corollary 1.2. *There exists a constant C such that following inequality holds true:*

$$\int |g|^{\frac{p(n-1)}{n-p}}\,dy \le C\Big(\int |y|^{\frac{p}{p-1}}|g|^{\frac{np}{n-p}}\,dy\Big)^{\frac{p-1}{p}}\|g\|_{p^*}. \tag{1.11}$$

Remarks.

- The best constant in (1.11) can be found by substituting in (1.8) $f = h_p$ defined by (1.10).
- A.Nazarov has noticed that (1.11) could be considered as known. After rearrangement it reduces to functions depending only on $|x|$ which is a particular case of Bellman Ineq. (Duke, v.10 (1943), 547-550). The sharp constant was found by Levin (DAN SSSR, v. 59 (1948), 635-639).

An Open Problem

Find the best constant $C_{p,d}$ in the inequality

$$\Big(\int_{x_d=0} |u|^{\frac{p(d-1)}{(d-p)}}\,dx'\Big)^{\frac{(d-p)}{p(d-1)}} \le C_{p,d}\Big(\int_{x_d>0} |\nabla u|^p\,dx\Big)^{\frac{1}{p}},$$

where $x = (x', x_d) \in \mathbb{R}^d$, $d \ge 2$, $p \in (1, d)$.

2. LECTURE

Lieb-Thirring inequalities

2.1. Consider a Schrödinger operator in $L^2(\mathbb{R})$

$$Hu = -\frac{d^2}{dx^2}u + Vu = \lambda u,$$

where V is a real function, $V \to 0$ rapidly enough. Then typically the spectrum of H might have negative eigenvalues $\{-\lambda_j\}_{j=1}^{\infty}$ and is continuous on the interval $[0, \infty)$. Lieb-Thirring inequalities

$$\sum_j \lambda_j^\gamma = \sum_j \lambda_j^\gamma(V) \le L_{\gamma,1} \int V_-^{\gamma+1/2}\, dx \tag{2.1}$$

Semi-classical formula

$$\sum_j \lambda_j^\gamma(\alpha V) \sim_{\alpha\to\infty} L_{\gamma,1}^{cl} \int (\alpha V_-)^{\gamma+1/2}\, dx$$

$$= (2\pi)^{-1} \iint (\xi^2 + \alpha V)_-^\gamma \, d\xi dx.$$

In particular this implies $L_{\gamma,1}^{cl} \le L_{\gamma,1}$.
It is known that

- $\gamma = 1/2 \quad \Leftrightarrow \quad L_{\gamma,1} = 2L_{\gamma,1}^{cl} = 1/2$ (Weidl, Hundertmark-Lieb-Thomas).
- $\gamma \ge 3/2 \quad \Leftrightarrow \quad L_{\gamma,1} = L_{\gamma,1}^{cl}$ (Lieb-Thirring, Lieb-Aizenman).
- if $1/2 < \gamma < 3/2$, then $L_{\gamma,1}$ are unknown.

Remark. If $\gamma < 1/2$, then L-Th inequalities (2.1) do not hold. However, if $0 < \gamma \le 1/2$, then there are finite constants $\tilde{L}_{\gamma,1}$, such that

$$\int V^{\gamma+1/2}\, dx \le \tilde{L}_{\gamma,1} \sum_j \lambda_j^\gamma,$$

(Damanik-Remling '05). It known that $\tilde{L}_{1/2,1} = 4$. Other values of $\tilde{L}$ are unknown.

L-Th inequalities for $\gamma = 3/2$

There are three proofs of Lieb-Thirring inequalities for $\gamma = 3/2$.

Buslaev-Faddeev-Zakharov trace formula

Let ψ solves the equation

$$-\frac{d^2}{dx^2}\psi + V\psi = k^2\psi, \qquad \psi(x,k) = \begin{cases} e^{ikx}, & \text{as } x \to \infty \\ a(k)e^{ikx} + b(k)e^{-ikx}, & \text{as } x \to -\infty. \end{cases}$$

Fundamental property:
if $k \in \mathbb{R}$ then $W[\psi, \bar{\psi}] = \psi\bar{\psi}' - \psi'\bar{\psi} =$ const.
This implies $1 = |a|^2 - |b|^2 \Leftrightarrow |a| \geq 1$.
Let

$$-\lambda_j = (i\kappa_j)^2, \qquad \kappa_j > 0.$$

BFZ trace formula

$$\frac{3}{2\pi}\int k^2 \ln|a|^2\, dk + \sum_j \kappa_j^3 = \frac{3}{16}\int V^2\, dx,$$

which , in particular, implies

$$\sum_j \lambda_j^{3/2} = \sum_j \kappa_j^3 \leq \frac{3}{16}\int V^2\, dx.$$

Proof. Let $H_0 = -\frac{d^2}{dx^2}$ and $B(k) = \Pi_j \frac{k+i\kappa_j}{k-i\kappa_j}$. One can prove that

$$a(k) = \det(H - k^2)(H_0 - k^2)^{-1}.$$

Then

$$\ln(B(k)a(k)) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\ln|a(s)|}{s-k}\, ds = -\frac{1}{i\pi}\sum_{n=0}^{\infty} \frac{1}{k^{n+1}}\int_{-\infty}^{\infty} \ln|a(s)| s^n\, ds$$

$$= \ln B + \mathrm{Tr}\ \ln\Big(I + V(H_0 - k^2)^{-1}\Big)$$

$$\sum_j \ln\Big(1 + \frac{2i\kappa_j}{k - i\kappa_j}\Big) + \sum_n \frac{(-1)^{n+1}}{n}\mathrm{Tr}\left[V\Big(H_0 - k^2\Big)^{-1}\right]^n.$$

Letting $k = i\tau$, $\tau > 0$, $\tau \to \infty$, and comparing the terms with the same powers of k, we obtain an infinite number of trace formulae. □

II. Factorization method (Benguria & Loss)

Let $\operatorname{supp} V \subset (-c, c)$, $c > 0$ and let $-\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ be negative eigenvalues. If $\psi_1 > 0$ is the eigenfunction corresponding to λ_1 then

$$\psi_1(x) = \begin{cases} c_1 e^{-\sqrt{\lambda_1}x}, & x > c, \\ c_2 e^{\sqrt{\lambda_1}x}, & x < -c. \end{cases}$$

Let

$$f_1 = \frac{\psi_1'}{\psi_1} \quad \Longrightarrow \quad f_1' + f_1^2 = V + \lambda_1.$$

Besides

$$f_1(x) = \begin{cases} -c_1\sqrt{\lambda_1}, & x > c, \\ c_2\sqrt{\lambda_1}, & x < -c. \end{cases}$$

From the Riccati equation we obtain that

$$H + \lambda_1 = \Big(-\frac{d}{dx} + f_1\Big)\Big(\frac{d}{dx} + f_1\Big) := A_1^* A_1.$$

Commuting A_1^* and A_1 we find

$$\tilde{H} := A_1 A_1^* - \lambda_1 = -\frac{d^2}{dx^2} + V - 2f_1'.$$

$$\int (V - 2f_1')^2\,dx = \int V^2\,dx + 4\int f_1'(f_1' - V)\,dx$$

$$= \int V^2\,dx + 4\int f_1'(\lambda_1 - f_1^2)\,dx$$

$$= \int V^2\,dx + 4\Big(-2\lambda_1\sqrt{\lambda_1} + \frac{2}{3}(\sqrt{\lambda_1})^3\Big) = \int V^2 dx - \frac{16}{3}\lambda_1^{3/2}.$$

Finally we have

$$\sum_{j=1}^{N}\lambda_j^{3/2} - \frac{3}{16}\int V^2\,dx = \sum_{j=2}\lambda_j^{3/2} - \frac{3}{16}\int (V-2f_1')^2\,dx = \dots$$

$$= -\frac{3}{16}\int (V - 2\sum_{j=1}^{N} f_j')^2\,dx \le 0.$$

III. Soliton's approach (Lieb $\&$ Thirring, Lax, Kruskal)

Let us consider the KdV equation

$$U_t = 6UU_x - U_{xxx}, \qquad U|_{t=0} = V.$$

Then

$$U_t = \Big[-\frac{d^2}{dx^2} + U, M\Big], \quad \text{where} \quad M = 4\frac{d^3}{dx^3} - 3\Big(U\frac{d}{dx} + \frac{d}{dx}U\Big).$$

- Discrete spectrum is independent of t:

$$\lambda_j\Big(-\frac{d^2}{dx^2} + U\Big) = \lambda_j\Big(-\frac{d^2}{dx^2} + V\Big).$$

- $a(k,t) = e^{i8k^3t}a(k,0)$.
- $\int U^2(x,t)\,dx = \int V^2(x)\,dx$.

It is known that $U(x,t) \sim_{t\to\infty} \sum_{j=1}^{N} U_j(x-4\lambda_j t) + U_\infty$, where

- $\|U_\infty\|_\infty \le \varepsilon(t) \to_{t\to\infty} 0$ and U_j are solitons

$$U_j(x) = -2\lambda_j\cosh^{-2}(\sqrt{\lambda_j}x).$$

- $\Big(-\frac{d^2}{dx^2} + U_j\Big)\cosh^{-1}(\sqrt{\lambda_j}x) = -\lambda_j\cosh^{-1}(\sqrt{\lambda_j}x)$.

Finally, since $4\int\cosh^{-4}x\,dx = 16/3$, we obtain

$$\int V^2\,dx \ge \sum_{j=1}^{N}\int U_j^2\,dx = \frac{16}{3}\sum_{j=1}^{N}\lambda_j^{3/2}.$$

Lieb-Thirring inequalities for $\gamma = 1/2$

Theorem 2.1. *Let* $H = -d^2/dx^2 + V$, $V \le 0$ *and* $V \in L^1(\mathbb{R})$. *Then*

$$\sum_j \sqrt{\lambda_j} \le 2L^{cl}_{1/2,1} \int |V|\, dx = \frac{1}{2} \int |V|\, dx.$$

The proof is based on a Monotonicity Lemma due to [HLT].
Let $A \ge 0$ be compact in a Hilbert space $\mathcal{H}$ and let $\|A\|_n = \sum_{j=1}^n \mu_j(A)$. The functionals $\|\cdot\|_n$, $n = 1, 2, \dots$, are norms and thus for any unitary in $\mathcal{H}$ operator U we have $\|U^* AU\|_n = \|A\|_n$.
We say that A majorizes B or $B \prec A$, iff

$$\|B\|_n \le \|A\|_n \qquad \text{for all } n \in \mathbb{N}.$$

Lemma 2.1 (Majorization). *Let* $\{U(\omega)\}_{\omega\in\Omega}$ *be a family of unitary operators and let* g *be a probability measure on* Ω. *Then*

$$B := \int_\Omega U^*(\omega) A U(\omega)\, g(d\omega)$$

is majorized by A.

Proof. This is a simple consequence of the triangle inequality

$$\|B\|_n \le \int_\Omega \|U^*(\omega) A U(\omega)\|_n \, g(d\omega) = g(\Omega)\|A\|_n = \|A\|_n \,.$$

□

Let $W = \sqrt{|V|}$ and denote

$$\mathcal{L}_\varepsilon := W \left[2\varepsilon \Big(-\frac{d^2}{dx^2} + \varepsilon^2 \Big)^{-1} \right] W \,.$$

Obviously, $\mathcal{L}_\varepsilon$ is a trace class operator and its trace equals $\mathrm{Tr}\, \mathcal{L}_\varepsilon = \int |V(x)|\, dx$.

Lemma 2.2 (Monotonicity). *If* $0 \le \varepsilon' \le \varepsilon$, *then*

$$\mathcal{L}_\varepsilon \prec \mathcal{L}_{\varepsilon'}$$

Proof. Let A be the operator given by the kernel $A(x,y) := W(x)W(y)$ (rank one operator). Introduce the following probability measure

$$g_\varepsilon(\xi)\,d\xi = \varepsilon(\pi(\xi^2+\varepsilon^2))^{-1}\,d\xi$$

and let $(U(\xi)\psi)(x) = e^{-i\xi x}\psi(x)$. Then

$$\mathcal{L}_\varepsilon = \int_{-\infty}^{\infty} U^*(\xi)AU(\xi)\,g_\varepsilon(\xi)\,d\xi.$$

We have $\hat{g}_\varepsilon(t) = e^{-\varepsilon|t|}$. Thus $g_\varepsilon = g_{\varepsilon'} * g_{\varepsilon-\varepsilon'}$ and we find that

$$\mathcal{L}_\varepsilon = \int U^*(p)\mathcal{L}_{\varepsilon'}U(p)\,g_{\varepsilon-\varepsilon'}(p)dp \prec \mathcal{L}_{\varepsilon'}.$$

Indeed, let $\mathcal{L}_\varepsilon(x,y)$ the kernel of the operator $\mathcal{L}_\varepsilon$. Then

$$\mathcal{L}_\varepsilon(x,y) = \iint e^{i\xi(x-y)}W(x)W(y)g_{\varepsilon'}(\xi-\eta)g_{\varepsilon-\varepsilon'}(\eta)\,d\xi d\eta$$

$$= \iint e^{i(\eta+\rho)(x-y)}W(x)W(y)g_{\varepsilon'}(\rho)g_{\varepsilon-\varepsilon'}(\eta)\,d\rho d\eta$$

$$\int e^{i\eta(x-y)}\underbrace{\int e^{i\rho(x-y)}W(x)W(y)g_{\varepsilon'}(\rho)\,d\rho}_{\mathcal{L}_{\varepsilon'}(x,y)}\,g_{\varepsilon-\varepsilon'}(\eta)\,d\eta.$$

This completes the proof. □

Proof of Theorem 2.1

Let

$$\mathcal{K}_E := \frac{1}{2\sqrt{E}}\mathcal{L}_{\sqrt{E}} = W\left[\Big(-\frac{d^2}{dx^2}+E\Big)^{-1}\right]W.$$

By Birman-Schwinger principle

$$1 = \mu_j(\mathcal{K}_{\lambda_j})$$

for all negative eigenvalues $\{-\lambda_j\}_j$ of the Schrödinger operator H. Multiplying this equality by $2\sqrt{\lambda_j}$ and summing over j we obtain

$$2\sum\sqrt{\lambda_j} = \sum \mu_j\Big(\mathcal{L}_{\sqrt{\lambda_j}}\Big).$$

By using the monotonicity we obtain

$$\|\mathcal{L}_{\sqrt{\lambda_1}}\|_1 = \mu_1\Big(\mathcal{L}_{\sqrt{\lambda_1}}\Big) \le \mu_1\Big(\mathcal{L}_{\sqrt{\lambda_2}}\Big),$$

$$\mu_1\Big(\mathcal{L}_{\sqrt{\lambda_1}}\Big) + \mu_2\Big(\mathcal{L}_{\sqrt{\lambda_2}}\Big) \le \mu_1\Big(\mathcal{L}_{\sqrt{\lambda_2}}\Big) + \mu_2\Big(\mathcal{L}_{\sqrt{\lambda_2}}\Big)$$
$$= \|\mathcal{L}_{\sqrt{\lambda_2}}\|_2 \le \mu_1\Big(\mathcal{L}_{\sqrt{\lambda_3}}\Big) + \mu_2\Big(\mathcal{L}_{\sqrt{\lambda_3}}\Big),$$

etc. In the end this yields

$$\sum_{j\le n}\mu_j(\mathcal{L}_{\sqrt{\lambda_j}}) \le \sum_{j\le n}\mu_j(\mathcal{L}_{\sqrt{\lambda_n}}) \quad \text{for all } n\in\mathbb{N}.$$

Hence

$$2\sum\sqrt{\lambda_j} \le \sum\mu_j(\mathcal{L}_{\sqrt{\lambda_n}}) \le \operatorname{Tr}\mathcal{L}_{\sqrt{\lambda_n}} = \int_{-\infty}^{\infty} W^2(x)\,dx$$
$$= \int_{-\infty}^{\infty}|V(x)|\,dx.$$

Multidimensional Lieb-Thirring inequalities

The main argument is based on 1D matrix Lieb-Thirring ineq. Let $Q\ge 0$ be a $m\times m$ matrix-function and let $H=-\Delta-Q$. Then

$$\sum_j \lambda_j^{3/2}(H) \le \frac{3}{16}\int \operatorname{Tr} Q^2(x)\,dx \qquad (\text{Lapt}\,\&\,\text{Weidl}),$$

(can be proven by using BZF approach or by using factorization (B&L)).

$$\sum_j \lambda_j^{1/2}(H) \le \frac{1}{2}\int \operatorname{Tr} Q(x)\,dx \qquad (\text{Hundertmark, Lapt}\,\&\,\text{Weidl}).$$

Lifting argument with respect to dimension

Let for simplicity $d = 2$, $V \in C_0^\infty(\mathbb{R}^2)$, $V \geq 0$, $x = (x_1, x_2)$. Then

$$H = -\Delta - V = -\partial^2_{x_1x_1} - \underbrace{(\partial^2_{x_2x_2} + V)}_{\tilde{H}(x_1)}.$$

Spectrum $\sigma(\tilde{H})$ of $\tilde{H}(x_1)$ has a finite number of positive eigenvalues $\mu_l(x_1)$. Thus $\tilde{H}_+(x_1)$ has a finite rank. Let, for instance, $\gamma = 3/2$

$$\sum_j \lambda_j^{3/2}(H) \leq \sum_j \lambda_j^{3/2}(-\partial^2_{x_1x_1} - \tilde{H}_+)$$

$$\leq \frac{3}{16}\int \operatorname{Tr} \tilde{H}_+^2(x_1)\,dx_1 \leq \underbrace{\frac{3}{16}L_{2,1}}_{L^{cl}_{3/2,2}} \iint V^{3/2+1}(x)\,dx.$$

Summary. Best known values of the constants $L_{\gamma,d}$:

- Lieb: $L_{0,d}$, ($L_{0,3} \cong 0.1156$, compare with 0.0780 given by the Sobolev ineq.).
- Hundertmark-Lieb-Thomas: $L_{1/2,1} = 2L^{cl}_{1/2,1}$.
- Eden-Foias: $L_{1,1} \leq \frac{2}{3\sqrt{3}} \cong 1.85\,L^{cl}_{1,1}$, whereas [HLT] estimate gives $L_{1,1} \leq 2\,L^{cl}_{1,1}$.
- Aizenmann-Lieb, Lieb-Thirring: $L_{\gamma,1} = L^{cl}_{\gamma,1}$, $\gamma \geq 3/2$.
- Laptev-Weidl: $L_{\gamma,d} = L^{cl}_{\gamma,d}$, for any $d \in \mathbb{N}$, $\gamma \geq 3/2$.
- Hundertmark-Laptev-Weidl: $L_{\gamma,d} \leq 2\,L^{cl}_{\gamma,d}$, $1 \leq \gamma < 3/2$ and $L_{\gamma,d} \leq 4\,L^{cl}_{\gamma,d}$, $1/2 < \gamma \leq 1$, for any $d \in \mathbb{N}$.

3. LECTURE

Inequalities for the Absolute Continuous Spectrum, Some Hardy Inequalities

Let $H^{\mathcal{D}}$ be a selfajoint operator in $L^2(\mathbb{R}_+)$ with Dirichlet boundary condition at zero

$$H^{\mathcal{D}}u = -\frac{d^2}{dx^2}u + Vu, \qquad u|_{x=0} = 0, \tag{3.1}$$

where V is a real function decaying at infinity. Spectrum $\sigma(H^{\mathcal{D}}) \subset \mathbb{R}$ and we are interested in the properties of its a.c. part belonging to $\mathbb{R}_+$.
If $z \in \mathbb{C} \setminus \mathbb{R}$, then the resolvent $(H^{\mathcal{D}} - zI)^{-1}$ is bounded in $L^2(\mathbb{R}_+)$. Let us fix a function $f \in L^2(\mathbb{R}_+)$, $\|f\| = 1$, and consider

$$\Big((H^{\mathcal{D}} - z)^{-1}f, f\Big) = \int \frac{d\mu_f(t)}{t-z}.$$

Properties:

- $\mu_f(\mathbb{R}) = 1$ and $clos\,\{t :\ \mu_f'(t) > 0\} \subset \sigma_{ac}(H^{\mathcal{D}})$.

Theorem 3.1. (Deift $\&$ Killip)
If $V \in L^2(\mathbb{R}_+)$, then there exists f, such that $\mu_f' > 0$ almost everywhere on $\mathbb{R}_+$.

The proof is based on the BZF trace formula. If $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R})$, then (lecture II)

$$\frac{3}{2\pi}\int k^2 \ln|a|^2\,dk + \sum_j \kappa_j^3 = \frac{3}{16}\int V^2\,dx,$$

which allows us to control the absolute continuous spectrum

$$\frac{3}{2\pi}\int k^2 \ln|a|^2\,dk \le \frac{3}{16}\int V^2\,dx.$$

Let us define the *entropy*:

$$S(\mu_f, [\alpha, \beta]) = \int_\alpha^\beta \log \mu_f'(\lambda)\, d\lambda.$$

By Jensen's inequality

$$S(\mu_f, [\alpha, \beta]) \le \log\Big(\int_\alpha^\beta \mu_f'(\lambda)\, d\lambda\Big) \le \log \mu_f[\alpha, \beta] < 0, \qquad [\alpha, \beta] \subset \mathbb{R}_+.$$

Therefore either $S(\mu_f, [\alpha, \beta]) = -\infty$, or $\mu_f'(\lambda) > 0$ a.e. on $[\alpha, \beta]$.

In order to prove Theorem 3.1 it is enough to prove that

$$S(\mu_f, [\alpha, \beta]) > -C_1 \int V^2(x)\, dx - C_2(f).$$

Let $z = \lambda + i\tau$. Then

$$\mathrm{Im}\,\Big((H^{\mathcal{D}} - z)^{-1} f, f\Big) = \mathrm{Im} \int \frac{d\mu_f(t)}{t - z}\, dt = \int \frac{\tau d\mu_f(t)}{(t-\lambda)^2 + \tau^2}\, dt \to \pi \mu_f'(\lambda),$$

as $\tau \to 0$. Let ψ_1 and ψ_2 be solutions of the equation

$$-\psi'' + V\psi = z\psi, \qquad z = k^2, \quad \mathrm{Im}\ k > 0.$$

Assume that $V(x) = 0$, for $0 < x < \delta$, $\delta > 0$. Then for such x the kernel of the integral operator $(H^{\mathcal{D}} - z)^{-1}$ equals

$$K(x, y) = \begin{cases} \psi_1(x)\psi_2(y), & y > x, \\ \psi_2(x)\psi_1(y), & y < x. \end{cases}$$

where

$$\psi_1(x) = \frac{\sin kx}{k}$$

and

$$\psi_2(x) = \cos kx + M(k)\frac{\sin kx}{k},$$

Here M is the Weyl function chosen such that $\psi_2(x) \to 0$, $\mathrm{Im}\, k > 0$, as $x \to \infty$. Let us choose f satisfying

- supp $f \subset [0, \delta)$ and $f = \bar{f}$.

Then

$$\lim_{\tau\to 0} \operatorname{Im}\Big((H^{\mathcal{D}}-z)^{-1}f,f\Big) = \operatorname{Im} M(\sqrt{\lambda})|F(\sqrt{\lambda})|^2 = \pi\mu_f'(\lambda),$$

where

$$F(k) = \int_0^\infty \frac{\sin kx}{k} f(x)\,dx$$

$F(k)$ is an analytic function for $k \in \mathbb{C}\setminus\{0\}$. Therefore for any $(\alpha,\beta)\subset\mathbb{R}_+$ $\int_\alpha^\beta \log|F(\sqrt{\lambda})|\,d\lambda$ is finite and we now obtain

$$S(\mu_f,(\alpha,\beta)) = \int_\alpha^\beta \log\mu_f'(\lambda)\,d\lambda \ge \int_\alpha^\beta \log\operatorname{Im} M(\sqrt{\lambda})\,d\lambda - C_2(f).$$

It only remains to show that

$$\int_\alpha^\beta \log\operatorname{Im} M(\sqrt{\lambda})\,d\lambda \ge -C_1\int V^2(x)\,dx.$$

Let

$$\psi(x) = \begin{cases} e^{ikx}, & x\to\infty, \\ a(k)e^{ikx}+b(k)e^{-ikx}, & x\in(0,\delta). \end{cases}$$

Then

- The Wronskian $W[\psi,\bar\psi] = \psi'\bar\psi - \psi\bar\psi'$ is independent of x and for $k\in\mathbb{R}$

$$\frac{1}{2ik}W[\psi,\bar\psi] = |a(k)|^2 - |b(k)|^2 = 1,$$

- By using trace formula we have

$$\int_\alpha^\beta k^2\log|a(k)|\,dk \le \int V^2(x)\,dx.$$

Now, since both $\psi_2(x),\psi(x)\to 0$, as $x\to\infty$, $\operatorname{Im} k>0$, then $\psi_2 = C\psi$. From $\psi_2(x) = \cos kx + M(k)\frac{\sin kx}{k}$ we find $M(k) = \frac{\psi_2'(0)}{\psi_2(0)} = \frac{\psi'(0)}{\psi(0)}$. If $k\in\mathbb{R}$ then

$$\operatorname{Im} M(k) = \frac{1}{2i}\left[\frac{\psi'(0)}{\psi(0)} - \frac{\overline{\psi'(0)}}{\overline{\psi(0)}}\right] = \frac{1}{2i}\frac{W[\psi,\bar\psi]_{x=0}}{|\psi(0)|^2} = \frac{k}{|\psi(0)|^2}.$$

Thus by using $|a|^2 = |b|^2 + 1$ we have $|b|^2 < |a|^2$ and therefore

$$\frac{\operatorname{Im} M(k)}{k} = \frac{1}{|a+b|^2} \geq \frac{1}{2(|a|^2+|b|^2)} > \frac{1}{4|a|^2}.$$

We finally obtain

$$\int_\alpha^\beta \log \frac{\operatorname{Im} M(k)}{k}\, dk \geq \int_\alpha^\beta \log \frac{1}{4|a|^2}\, dk \geq -C \int V^2(x)\, dx.$$

Open problem (Simon's conjecture): If $d > 1$, then

$$\int_{\mathbb{R}^d} V^2(x)|x|^{1-d}\, dx < \infty,$$

then there is an infinite number of functions f, such that $\mu_f'(\lambda) > 0$.

Hardy's type Inequalities for many particles

Jointly with Maria and Thomas Hoffmann-Ostenhof and Jesper Tidblom

Let $x \in \mathbb{R}^{nN}$, $x = (x_1, x_2, \ldots, x_N)$ where the $x_i = (x_{i,1}, \ldots, x_{i,n})$ are points in $\mathbb{R}^n$. Denote $r_{ij} = |x_i - x_j|$ and let $\mathcal{N}_N = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_i = x_j$ for some $i \neq j\}$.

Theorem 3.2. (1D Hardy ineq. with N particles). *Let $u \in H_0^1(\mathbb{R}^N \setminus \mathcal{N}_N)$, where $\mathcal{N}_N = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_i = x_j$ for some $i \neq j\}$. Then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \Big(\sum_{1 \leq i < j \leq N} \frac{1}{r_{ij}^2} \Big) dx. \tag{3.2}$$

Remark. The constant $1/2$ appearing in (3.2) is better than $\frac{1}{2(N-1)}$ which can be obtained by adding up inequalities $-\frac{\partial^2}{\partial^2 x_i} - \frac{\partial^2}{\partial^2 x_j} \geq \frac{1}{2} \frac{1}{r_{ij}}$.

Let

$$A(n, N) := \sum_{1 \leq i < j \leq N} \frac{1}{r_{ij}^2}$$

and

$$B(n, N) := \sum_{j=1}^{N} \sum_{i \neq k,\, i,k \neq j} \frac{(x_j - x_i) \cdot (x_j - x_k)}{r_{ij}^2 r_{jk}^2}.$$

Define now

$$K(N) = \max \frac{B(3, N)}{2A(3, N)}.$$

Theorem 3.3. (3D Hardy ineq. with N particles) *Let $u \in H^1(\mathbb{R}^{3N})$, then*

$$\int_{\mathbb{R}^{3N}} |\nabla u|^2 dx \geq \frac{1}{2 + 2K(N)} \int_{\mathbb{R}^{3N}} |u|^2 \sum_{1 \leq i < j \leq N} \frac{1}{r_{ij}^2}\, dx,$$

where $(2+2K(N))^{-1} \geq 1/N$.

- Note that this is already a substantial improvement of the factor $(2N-2)^{-1}$ which we would have gotten by adding up.
- For $N = 3$ and 4 the estimate $(2+2K(N))^{-1} \geq 1/N$ is optimal. Things are different for N larger and the asymptotics of $K(N)$ is an interesting problem.
- $K(N) \sim_{N\to\infty} c\,N$. Indeed, we can obtain that if $Q^m = [0,1]^m$

$$\inf_N N\cdot K(N) \geq \iiint_{Q^9} \frac{(x-y)\cdot(x-z)}{|x-y|^2|x-z|^2}\,dxdydz\Big[2\iint_{Q^6}\frac{1}{|x-y|^2}\,dxdy\Big]^{-1}.$$

  Finding the sharp value of $K(N)$ is an interesting problem from geometrical combinatorics.

Theorem 3.4. (3D Coulomb case with N particles)
Let $u \in W^{1,2}(\mathbb{R}^{3N})$. *Then*

$$\int_{\mathbb{R}^{3N}} |\nabla u|^2 dx - 2\int_{\mathbb{R}^{3N}} \Big(\sum_{i<j}\frac{1}{r_{ij}}\Big)|u|^2\,dx$$

$$\geq -\Big(\frac{N(N-1)}{2} + L(N)\Big)\int_{\mathbb{R}^{3N}} |u|^2\,dx,$$

where

$$L(N) = \max \sum_{j=1}^{N} \sum_{i\neq k,\, i,k\neq j} \frac{(x_j-x_i)\cdot(x_j-x_k)}{r_{ij}r_{jk}}.$$

Remark. The sharp value of $L(N)$ is unknown except of $N = 3, 4, 5$. However, we can show that

$$\frac{1}{6}\,N(N-1)(N-2) \leq L(N) \leq \frac{1}{4}\,N(N-1)(N-2).$$

2D Hardy Inequality

Following Alano Ancona we present here a result from his paper J.London Math. Soc (2) **34** (1986) 274-290, page 278, on Hardy's inequality.

Theorem 3.5. *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected domain,* $\varphi \in H_0^1(\Omega)$*. Then*

$$\int_\Omega |\nabla\varphi|^2\, d\xi \geq \frac{1}{16} \int_\Omega \frac{|\varphi|^2}{\delta^2(\xi)}\, d\xi,$$

where $\delta(\xi)$ *is the distance from* ξ *to the boundary* $\partial\Omega$*.*

In order to prove this theorem we apply a version of Koebe's theorem. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{C}_+ = \{z = x + iy \in \mathbb{C} : x \geq 0\}$.

Lemma 3.1. *Let* f *be a conformal map from* $\mathbb{C}_+$ *to* Ω*. Then*

$$\delta(f(z)) \geq \frac{x}{2}\, |f'(z)|, \qquad z \in \mathbb{C}_+.$$

Proof. Indeed, the standard version of Koebe's one quarter theorem claims that if $g : \mathbb{D} \to \Omega$ is a conformal mapping, then

$$\delta(g(0)) \geq \frac{1}{4}\, |g'(0)|. \tag{3.3}$$

For any conformal mapping $f : \mathbb{C}_+ \to \Omega$ we can now consider

$$g_z(w) = f\Big(z \cdot \frac{1 + e^{-i\theta}w}{1 - e^{i\theta}w}\Big),$$

where $w \in \mathbb{D}$, $z = x + iy \in \mathbb{C}_+$ and $\theta = \arg z$, $-\pi/2 < \theta < \pi/2$.
For a fixed $z = |z|e^{i\theta}$

$$h_z(w) = z \cdot \frac{1 + e^{-i\theta}w}{1 - e^{i\theta}w}$$

maps $\mathbb{D}$ onto $\mathbb{C}_+$.

Clearly $h(0) = z$, $g_z(0) = f(z)$ and

$$g_z'(0) = z\, f'(z)\Big(e^{-i\theta} + e^{i\theta}\Big) = 2\cos\theta\, z\, f'(z).$$

Therefore by using (3.3) we obtain

$$\delta(f(z)) = \delta(g_z(0)) \geq \frac{1}{4}\,|g_z'(0)| = \frac{|z|\cos\theta}{2}\,|f'(z)| \geq \frac{x}{2}\,|f'(z)|.$$

□

Proof of Theorem 3.5.
If $f\,:\,\mathbb{C}_+ \to \Omega$ is a conformal mapping, $f(z) = u(x,y) + iv(x,y)$, $\xi = (u,v)$, then by using Hardy's inequality for half-plane we obtain

$$\int_\Omega |\nabla_\xi \varphi|^2\, d\xi = \int_0^\infty \int_{-\infty}^\infty |\nabla_{(x,y)}\varphi|^2\, dydx \geq \frac{1}{4}\int_0^\infty \int_{-\infty}^\infty \frac{|\varphi|^2}{x^2}\, dydx$$

$$= \frac{1}{16}\int_0^\infty \int_{-\infty}^\infty \frac{|\varphi|^2}{2^{-2}x^2|f'(z)|^2}\,|f'(z)|^2\, dydx \geq \frac{1}{16}\int_\Omega \frac{|\varphi|^2}{\delta^2(\xi)}\, d\xi.$$

The proof is complete.

Open problem: for non-convex case find the best constant K in the inequality

$$\int_\Omega |\nabla\varphi|^2\, d\xi \geq K \int_\Omega \frac{|\varphi|^2}{\delta^2(\xi)}\, d\xi.$$

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