Heavy-tailed insurance portfolios: buffer capital and ruin probabilities

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Abstract

We study the risk of insolvency for an insurance company with multiple business lines facing large claims with heavy-tailed distribution. The company is allowed to transfer capital between business lines but such capital transfers are restricted by regulations. The same principles apply to an insurance group with agreements of mutual financial support in case of large losses. The insurance company is considered insolvent when negative positions in one or several lines of business cannot be canceled by means of capital transfer. Under the assumption that the distribution of the vector of claim sizes is multivariate regularly varying we derive the asymptotic decay of the ruin probability as the initial capital tends to infinity. In particular, we analyze the impact of rules for transfer of capital on the ruin probability and we draw conclusions about possible benefits from diversification. We also analyze the asymptotic behavior of the buffer capital, defined as the smallest amount of capital needed to reduce the ruin probability to a prespecified level, as the level tends to zero. A Poisson shock model serves as a useful example for which explicit computations are possible and diversification effects can be quantified.

Key words: Ruin probabilities; buffer capital; regular variation; renewal model; diversification

1 Introduction

Suppose we have a group of insurance companies issuing insurance against losses from catastrophic events (e.g. earthquakes, storms, floods or terrorist attacks). Due to the eventual occurrence of these events each company is exposed to a significant risk of severe losses that may have a considerable impact on the solvency of the company. To reduce the risk of becoming insolvent the company must, in addition to charging appropriate premiums, hold a sufficient amount of buffer capital to cover unexpected losses with a sufficiently high probability. Since it is unprofitable for an insurance company to hold a large buffer capital it may seek agreements with other companies of mutual financial support in case of severe losses or insolvency and thereby diversifying the risk and reducing the required buffer capital. Such agreements may be constructed as a set of rules for capital transfers between the companies. For instance, one may think of an agreement which says that a certain fraction of the reserve in a profitable company may be used to cover losses in another. Alternatively, the participating companies may set up a guarantee fund whose assets can be used to support a company facing unexpected large claims. In this paper we are interested in quantifying the benefits from such agreements. Similar ideas apply when considering a single company with multiple business lines facing the risk of catastrophic losses. If the different lines of business operate in different geographical regions or different segments of the insurance market, then capital transfers may be restricted by regulations. This is the situation we will consider throughout the paper and we aim at quantifying the effect of such restrictions.

We consider a renewal model for the reserves of an insurance company with multiple business lines. The business lines may be interpreted as the activity of the company in different countries or as different types of policies offered by the company. Alternatively, we may think of the business lines as individual companies participating in some agreement of mutual financial support as discussed above. It is assumed that capital may be transferred between the business lines and that such transfers are subject to regulations or transaction costs. By ruin we mean the situation when negative positions in one or several lines of business cannot be eliminated by means of capital transfer. This means that at least one of the business lines cannot be saved by transferring capital from business lines with positive reserves or from a guarantee fund. Such an event may be called a default event and it creates serious problems for the company or insurance group although it does not necessarily imply that the entire company must file for bankruptcy.

As we are primarily interested in catastrophic risks we will assume that the distribution of the claim sizes in the different lines of business is heavy-tailed. This means that there is a nonnegligible probability that one single large claim or one simultaneous occurrence of large claims in several lines of business has a significant effect on the solvency of the insurance company. A standard approach to evaluating the likelihood of insolvency of an insurance company is to compute the ruin probability $\psi(u)$ expressed as a function of the initial capital u. The ruin probability is the probability that the risk reserve process of the company, with initial capital u allocated to the different business lines, eventually exits the so-called solvency region. There is a vast literature on ruin probabilities in the univariate setting, including a large variety of risk processes taking into account for instance effects of interest rates and advanced investment schemes, see e.g. Asmussen (2000), Bühlmann (1970), Embrechts *et al.* (1997), Grandell (1991), Mikosch (2004), Rolski *et al.* (1999) and the references therein. However, the literature on ruin probabilities for multivariate insurance portfolios is relatively sparse. A significant contribution to general multivariate ruin problems in the light-tailed case is the work by Collamore (1996, 2002). In this paper we use the theory developed by Hult *et al.* (2005) for multivariate regularly varying random walks.

The buffer capital based on the probability of ruin may be defined as the minimum capital required to reduce the ruin probability to a prespecified level q. Clearly, solvency concerns of policy holders and regulators require that q is small. We denote the buffer capital by

$$u(q) = \inf\{u > 0 : \psi(u) \leq q\}. \tag{1}$$

The risk reserve process we consider is a multivariate renewal model (claims arrive according to a renewal process) with claim distribution $\mathrm{P}(\mathbf{Z} \in \cdot)$ which is regularly varying with index $\alpha > 1$. For this model the asymptotic decay of the ruin probability $\psi(u)$ is given by

$$\lim_{u\to\infty} \frac{\psi(u)}{u\,\mathrm{P}(|\mathbf{Z}| > u)} = C, \tag{2}$$

where $|\mathbf{Z}|$ denotes the norm of the random vector $\mathbf{Z}$ and C is given a representation in terms of the model parameters (see Proposition 1). Applying a classical result on the decay of inverses to regularly varying functions leads to a similar expression for the asymptotic behavior of the buffer capital:

$$\lim_{q\downarrow 0} \frac{u(q)}{g(q)} = C^{1/(\alpha-1)},$$

where $g(q) = \inf\{u > 0 : u\,\mathrm{P}(|\mathbf{Z}| > u) \leq q\}$. In the special case when the claim surplus process is modeled by a simple Poisson shock model the limiting constant C has a particularly simple form (see Proposition 3). For this model we can explicitly quantify the effect of the rules for capital transfers between business lines on the ruin probability and buffer capital. In particular, we analyze diversification effects and show that the rules for capital transfers have a significant impact on the benefits from diversifying the portfolio.

The paper is organized as follows. In Section 2 we present the multivariate renewal model for the risk reserve process and the rules for capital transfers. Section 3 contains the main results on the asymptotic behavior of the ruin probability and buffer capital. We also consider the ruin problem for a fixed finite time horizon. We work with a simple Poisson shock model in Section 4 which allows us to illustrate diversification effects and effects of different rules for transfer of capital between business lines. Section 5 contains the proofs.

Vectors are assumed to be $\mathbb{R}^d$-valued column vectors and are denoted by bold letters. For example $\mathbf{x} = (x^{(1)}, \dots, x^{(d)})^{\mathrm{T}} \in \mathbb{R}^d$, where T denotes transpose. Moreover, $\mathbf{0} = (0, \dots, 0)^{\mathrm{T}}$ and $\mathbf{1} = (1, \dots, 1)^{\mathrm{T}}$. For $i \in \{1, \dots, d\}$ we use the notation $\mathbf{e}_i$ for the unit vector whose ith component is equal to one.

2 Modeling the insurance portfolio

To describe the evolution of the reserves of an insurance company we consider a multivariate renewal model. The insurance company has d lines of business and claims arrive at the renewal times of a renewal process $(N_t)_{t\geq 0}$ given by $N_t = \#\{n \geq 1 : T_n \leq t\}$ with renewal sequence

$$T_0 = 0, \quad T_n = W_1 + \cdots + W_n \text{ for } n \geq 1, \tag{3}$$

where $(W_k)_{k\geq 1}$ is a sequence of independent and identically distributed random variables. The claim sizes are given by another sequence $(\mathbf{Z}_k)_{k\geq 1}$ of $\mathbb{R}^d$-valued independent and identically distributed random vectors, which is independent of $(W_k)_{k\geq 1}$. Throughout the paper we write $\mathbf{Z}$ for a generic element of $(\mathbf{Z}_k)_{k\geq 1}$ and W for a generic element of $(W_k)_{k\geq 1}$. The **total claim amount process** $(\mathbf{C}_t)_{t\geq 0}$ is a renewal reward process given by

$$\mathbf{C}_t = \sum_{k=1}^{N_t} \mathbf{Z}_k. \tag{4}$$

The total claim amount up to time t for business line j is denoted $C_t^{(j)}$. The insurance company receives premium income at a constant rate $\mathbf{p} \in (0, \infty)^d$. Initially the company has the total capital u which is allocated to the different lines of business according to $u\mathbf{b}$ with $\mathbf{b} \in (0, 1]^d$ and $b^{(1)} + \cdots + b^{(d)} = 1$. That is, business line j has the initial capital $ub^{(j)}$. The **claim surplus process** $(\mathbf{S}_t)_{t\geq 0}$ and the **risk reserve process** $(\mathbf{R}_t)_{t\geq 0}$ are given by

$$\mathbf{S}_t = \sum_{k=1}^{N_t} \mathbf{Z}_k - t\mathbf{p}, \quad \mathbf{R}_t = u\mathbf{b} - \mathbf{S}_t = u\mathbf{b} + t\mathbf{p} - \sum_{k=1}^{N_t} \mathbf{Z}_k. \tag{5}$$

Ruin occurs if the risk reserve process $\mathbf{R}_t$ hits a certain set $F \subset \mathbb{R}^d$ at some time $t \geq 0$. That is

$$\text{Ruin} = \Big\{ \mathbf{R}_t \in F \text{ for some } t \geq 0 \Big\}.$$

Given the **ruin set** F the ruin probability is the function of u is given by

$$\psi_{d,F}(u) = \mathrm{P}(\mathbf{R}_t \in F \text{ for some } t \geq 0) = \mathrm{P}(\mathbf{S}_t \in u\,\mathbf{b} - F \text{ for some } t \geq 0).$$

In the univariate case the ruin set is usually taken to be $(-\infty, 0)$ but in the multivariate setting one can easily motivate several different choices of the ruin set F. For instance, a restrictive choice is to take F as the complement of $[0, \infty)^d$. That is, ruin occurs as soon as one business line has negative reserves. A less restrictive choice is to take $F = \{\mathbf{x} : x^{(1)} + \cdots + x^{(d)} < 0\}$; ruin occurs when the aggregated reserves is negative. We will present a natural class of ruin sets that can be represented as complements of certain cones in $\mathbb{R}^d$. We think of the components of the insurance portfolio as the reserves resulting from activity in different business lines or from different geographical regions (countries). If a big loss occurs in one business line (country) and the loss cannot be covered with capital in that business line, then the company is allowed to transfer capital from one or several of the other business lines to cover the loss.

To specify the rules for capital transfers we use a $d \times d$ matrix Π with entries $(\pi^{(ij)})_{i,j=1}^d$. The entry $\pi^{(ij)}$ is interpreted as the amount of capital in business line i needed to obtain one unit of capital in business line j. It is assumed that Π satisfies the following conditions.

(i) $\pi^{(ij)} > 0$ for $i, j \in \{1, \dots, d\}$,

(ii) $\pi^{(ii)} = 1$ for $i \in \{1, \dots, d\}$,

(iii) $\pi^{(ij)} \leq \pi^{(ik)}\pi^{(kj)}$ for $i, j, k \in \{1, \dots, d\}$.

Condition (iii) means that we cannot gain anything by transferring from i to k and then from k to j instead of transferring directly from i to j. In the financial literature a $d \times d$ matrix Π satisfying (i)-(iii) is called a **bid-ask matrix**. This way of representing capital transfers is consistent with Kabanov's numeraire free approach to modeling foreign exchange rates under proportional transaction costs (Kabanov, 1999; Kabanov and Stricker, 2001) although there the bid-ask matrix evolves randomly in time. In this paper we limit our attention to a fixed matrix Π and use the notation of Schachermayer (2004).

The matrix Π specifies completely the rules for transferring capital between the business lines. If capital can be transferred in such a way that all negative positions are canceled, then we say that the company is solvent and its position is not in the ruin set. Conversely, if it is not possible to cancel all negative positions, then the company is insolvent and ruin occurs. More precisely, the ruin set F is the complement of the closed convex cone $K(\Pi)$ spanned by the vectors $(\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j)$, $i \neq j$, and the vectors $\mathbf{e}_i$. We call $K(\Pi)$ the solvency cone. That is, we have

$$K(\Pi) = \Big\{\mathbf{x} : \mathbf{x} = \sum_{i \neq j} v^{(ij)}(\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j) + \sum_{i=1}^{d} w^{(i)}\mathbf{e}_i,\, v^{(ij)},\, w^{(i)} \geq 0\Big\}, \quad (6)$$

$$F = K(\Pi)^c. \quad (7)$$

An illustration of $K(\Pi)$ in the two-dimensional case is given in Figure 1. Notice that since $u\,F = F$ for every $u > 0$, the ruin probability can be written

$$\psi_{d,F}(u) = \mathrm{P}(\mathbf{S}_t \in u\,(\mathbf{b} - F) \text{ for some } t \geq 0).$$

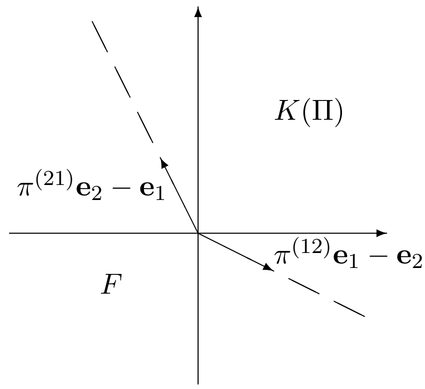

Figure 1: Bivariate illustration of the solvency cone $K(\Pi)$ and ruin set F.

It is convenient for an insurance company to quote all gains and losses occurring in different business lines in a common reference currency (numeraire), e.g. US dollars. If this is the case then we may replace (i) by

(i') $\pi^{(ij)} \geq 1$ for $i, j \in \{1, \ldots, d\}$

because one cannot make money simply by transferring capital from one business line to another. In this case one can also give an alternative interpretation of the matrix Π. Suppose capital transfers between business lines are free of charge but due to restrictions and regulations the company is not allowed to transfer all its capital from one business line to another. Instead only a fraction $\beta^{(ij)} \in [0, 1]$ of the positive capital in business line i may be transferred to business line j. This makes sense because regulators and policy holders in one country would generally not allow the company to transfer the entire reserve to another country. We put $\beta^{ii} = 1$ and impose the condition that $\beta^{ij} \geq \beta^{ik}\beta^{kj}$ for $i, j, k \in \{1, \ldots, d\}$ so one cannot gain anything by transferring from i to j via k instead of directly from i to j. Again ruin occurs if we cannot eliminate negative positions by means of capital transfers. If we put $\pi^{(ij)} = 1/\beta^{(ij)}$ then the corresponding matrix Π satisfies (i'), (ii) and (iii) and the ruin set is given by (7).

Example 1. Consider an insurance company with d business lines. If a business line has a positive reserve, then a fraction $\beta \in [0, 1]$ of this capital may be transferred to cover losses in other business lines. Ruin occurs if a negative position in one line of business cannot be canceled by transferring capital from other business lines. That is, the ruin set is given by

$$F_\beta = \Big\{\mathbf{x} : \beta \sum_{k=1}^{d} (x^{(k)} \vee 0) < -\sum_{k=1}^{d} (x^{(k)} \wedge 0)\Big\},$$

where $\wedge = \min$ and $\vee = \max$. We denote by Π_β the matrix given by $\pi^{(ij)} = \beta^{-1}$ for $i \neq j$ and $\pi^{(ii)} = 1$. It is easy to check that F_β is the complement of the associated solvency cone $K(\Pi_\beta)$. The set F_β is a convenient one-parameter ruin set since it can be viewed as an intermediate ruin set between the most restrictive case $\beta = 0$ (F_0 is the complement of $[0, \infty)^d$) when no capital transfers are allowed and the case $\beta = 1$ when capital can be transferred without restrictions.

3 Ruin probabilities and buffer capital

To evaluate the riskiness of the insurance portfolio we want to compute the ruin probability. For a fixed initial capital u this is typically not possible unless a specific parametric model is assumed and the problem is therefore not tractable from a theoretical point of view. Since a very large initial capital is required by regulators it is reasonable to believe that a good approximation is obtained by studying the limiting behavior of the normalized ruin probability as the initial capital tends to infinity as in (2).

First we turn our attention to the distribution of the claims. As is well known from the univariate case the decay of the ruin probability depends heavily on the rate of decay of the tail probabilities of the claim size distribution. In this paper we will work under the assumption that the distribution of $\mathbf{Z}$ is regularly varying (see e.g. Resnick (1987, 2004)). In the univariate case this simply means that there exists an $\alpha > 0$ such that

$$\frac{\mathrm{P}(Z > zu)}{\mathrm{P}(Z > u)} \to z^{-\alpha} \qquad \text{as } u \to \infty,$$

for every $z > 0$. Formally, multivariate regular variation means that there exist an $\alpha > 0$ and a probability measure σ on the unit sphere $\mathbb{S}^{d-1} = \{\mathbf{x} :$

$|\mathbf{x}| = 1\}$ such that

$$\frac{\mathrm{P}(|\mathbf{Z}| > zu, \mathbf{Z}/|\mathbf{Z}| \in S)}{\mathrm{P}(|\mathbf{Z}| > u)} \to z^{-\alpha}\sigma(S) \qquad \text{as } u \to \infty, \tag{8}$$

for every $z > 0$ and Borel sets $S \subset \mathbb{S}^{d-1}$ with $\sigma(\partial S) = 0$. (By ∂S we denote the boundary of S.) The probability measure σ is called the spectral measure of $\mathbf{Z}$. It describes the most likely direction of extreme observations of $\mathbf{Z}$. Notice that if we put $z = 1$ in (8) then we see that σ is the limiting distribution, as $u \to \infty$, of $\mathbf{Z}/|\mathbf{Z}|$ conditioned on the event $\{|\mathbf{Z}| > u\}$. When spherical coordinates are inconvenient it is useful to consider the following equivalent formulation of regular variation: there exists a measure μ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ so that

$$\frac{\mathrm{P}(\mathbf{Z} \in u\,A)}{\mathrm{P}(|\mathbf{Z}| > u)} \to \mu(A) \qquad \text{as } u \to \infty, \tag{9}$$

for every Borel set $A \subset \mathbb{R}^d$ bounded away from $\mathbf{0}$ with $\mu(\partial A) = 0$. Notice that α, σ and μ are related through $\mu(\mathbf{z} : |\mathbf{z}| > r, \mathbf{z}/|\mathbf{z}| \in S) = r^{-\alpha}\sigma(S)$ for $r > 0$ and $S \subset \mathbb{S}^{d-1}$. As a consequence μ has the homogeneity property $\mu(uA) = u^{-\alpha}\mu(A)$ for every $u > 0$ and Borel set $A \subset \mathbb{R}^d$ bounded away from $\mathbf{0}$. If $\mathbf{Z}$ is regularly varying, then we write $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu)$.

Before formulating our main result we consider a useful reformulation of the ruin probability. If $\mathrm{E}(|\mathbf{Z}|) < \infty$ and $\mathrm{E}(W) < \infty$, then it is well-known that the ruin probability can be expressed as a hitting probability for a random walk (see Lemma 6 for details). More precisely,

$$\begin{aligned}\psi_{d,F}(u) &= \mathrm{P}(\mathbf{S}_t \in u\,(\mathbf{b} - F) \text{ for some } t \geq 0)\\ &= \mathrm{P}(\mathbf{X}_n - n\,\mathbf{c} \in u\,(\mathbf{b} - F) \text{ for some } n \geq 1),\end{aligned}$$

where $\mathbf{c} = \mathrm{E}(W)\,\mathbf{p} - \mathrm{E}(\mathbf{Z})$, F is given by (7) and $\mathbf{X}_n = \sum_{k=1}^{n}(\mathbf{Z}_k - W_k\,\mathbf{p}) + n\,\mathbf{c}$ is a random walk with zero mean. If $\mathbf{c} \in (0,\infty)^d$, then each business line satisfies the net profit condition or equivalently each business line has a positive safety loading (see e.g. Asmussen (2000) and Mikosch (2004)).

We are now ready to state the main result for the asymptotic decay of the ruin probability. It is based on Theorem 3.1 in Hult *et al.* (2005).

Proposition 1. *Let the process* $(\mathbf{S}_t)_{t\geq 0}$ *be given by* (5)*, suppose that* $\mathbf{Z} \in \mathrm{RV}(\alpha, \mu)$ *with* $\alpha > 1$ *and let the set* F *be given by* (7)*. Suppose that* $\mathbf{c} = \mathrm{E}(W)\,\mathbf{p} - \mathrm{E}(\mathbf{Z}) \in (0,\infty)^d$ *and that* $\mathrm{E}(W^\gamma) < \infty$ *for some* $\gamma > \alpha$*. Then*

$$\lim_{u\to\infty} \frac{\mathrm{P}(\mathbf{S}_t \in u\,(\mathbf{b} - F) \textit{ for some } t \geq 0)}{u\,\mathrm{P}(|\mathbf{Z}| > u)} = \int_0^\infty \mu(v\,\mathbf{c} + \mathbf{b} - F)\mathrm{d}v.$$

Remark 1. *Notice that in the univariate case we have $b - F = (1, \infty)$ and hence the asymptotics for the ruin probability is given by*

$$\lim_{u\to\infty} \frac{\mathrm{P}(S_t > u \text{ for some } t \geq 0)}{u\,\mathrm{P}(Z > u)} = \int_0^\infty \mu(vc + 1, \infty)\mathrm{d}v$$
$$= \mu(1, \infty) \int_0^\infty (vc + 1)^{-\alpha}\mathrm{d}v = \frac{1}{c(\alpha - 1)}. \tag{10}$$

That is, we arrive at the classical univariate result (c.f. Embrechts et al. *(1997)).*

Example 2. Suppose that $\pi^{(ij)} = 1$ for $i, j \in \{1, \ldots, d\}$. That is, the capital in all business lines are quoted in the same monetary unit and can be transferred between all business lines without restrictions and costs. Then the ruin set is $F_1 = \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{1} < 0\}$ and

$$v\,\mathbf{c} + \mathbf{b} - F_1 = v\,\mathbf{c} + \Big\{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{1} > \mathbf{b}^{\mathrm{T}}\mathbf{1}\Big\} = \Big\{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{1} > v\,\mathbf{c}^{\mathrm{T}}\mathbf{1} + 1\Big\}.$$

Hence,

$$\int_0^\infty \mu(v\,\mathbf{c} + \mathbf{b} - F_1)\mathrm{d}v = \frac{\mu(\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{1} > 1)}{\mathbf{c}^{\mathrm{T}}\mathbf{1}} \int_1^\infty w^{-\alpha}\mathrm{d}w$$
$$= \frac{\mu(\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{1} > 1)}{\mathbf{c}^{\mathrm{T}}\mathbf{1}(\alpha - 1)}.$$

In particular,

$$\lim_{u\to\infty} \frac{\psi_{d,F_1}(u)}{u\,\mathrm{P}(|\mathbf{Z}| > u)} = \frac{\mu(\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{1} > 1)}{\mathbf{c}^{\mathrm{T}}\mathbf{1}(\alpha - 1)}.$$

Notice that $U_t = \mathbf{R}_t^{\mathrm{T}}\mathbf{1}$ is a univariate risk process with univariate regularly varying claim sizes given by $Z_k = \mathbf{Z}_k^{\mathrm{T}}\mathbf{1}$. Hence, the result can also be derived directly from the univariate result (10) with $c = \mathbf{c}^{\mathrm{T}}\mathbf{1}$.

Using Proposition 1 we arrive at the following result which gives the asymptotic behavior of the buffer capital $u(q)$ as $q \downarrow 0$.

Proposition 2. *Assume the hypotheses of Proposition 1 and let $u_{d,F}(q) = \inf\{u > 0 : \psi_{d,F}(u) \leq q\}$ be the buffer capital. Then*

$$\lim_{q\downarrow 0} \frac{u_{d,F}(q)}{g(q)} = \left(\int_0^\infty \mu(v\,\mathbf{c} + \mathbf{b} - F)\mathrm{d}v\right)^{1/(\alpha-1)},$$

where $g(q) = \inf\{u > 0 : u\,\mathrm{P}(|\mathbf{Z}| > u) \leq q\}$.

3.1 Ruin in finite time

In Proposition 1 we considered the ruin probability over an infinite horizon. Let us now comment upon the case of a finite horizon. Fix $T \in (0,\infty)$ and consider the ruin problem on the interval $[0,T]$. That is, we are interested in the probability

$$\begin{aligned}\psi_{d,F}(u,T) &= \mathrm{P}(\mathbf{R}_t \in F \text{ for some } t \in [0,T]) \\ &= \mathrm{P}(\mathbf{S}_t \in u\,(\mathbf{b}-F) \text{ for some } t \in [0,T])\end{aligned}$$

for large u, where F is given by (7) and $(\mathbf{S}_t)_{t\geq 0}$ and $(\mathbf{R}_t)_{t\geq 0}$ are given by (5). Suppose that (N_t) is a Poisson process, so that $(\mathbf{S}_t)$ is a Lévy process, and that $\mathbf{Z} \in \mathrm{RV}(\alpha,\mu)$ for some $\alpha > 0$. Combining Propositions 4.1 and A.2 in Hult and Lindskog (2005) yields

$$\begin{aligned}\lim_{u\to\infty} \frac{\psi_{d,F}(u,T)}{\mathrm{P}(|\mathbf{Z}|>u)} &= \lim_{u\to\infty} \frac{\mathrm{P}(\mathbf{S}_t \in u\,(\mathbf{b}-F) \text{ for some } t\in[0,T])}{\mathrm{P}(|\mathbf{Z}|>u)} \\ &= \lim_{u\to\infty} \frac{\mathrm{P}(\mathbf{S}_T \in u\,(\mathbf{b}-F))}{\mathrm{P}(|\mathbf{Z}|>u)} = \mathrm{E}(N_T)\mu(\mathbf{b}-F),\end{aligned}$$

where existence of the limit follows, by Lemma 7, from the fact that $\mu(\partial(\mathbf{b}-F))=0$.

4 Diversification effects for an insurance portfolio

From Proposition 1 we obtain an approximation for the ruin probability of an insurance company for large initial capital u:

$$\psi_{d,F}(u) \approx \int_0^\infty \mu(v\,\mathbf{c}+\mathbf{b}-F)\mathrm{d}v\, u\,\mathrm{P}(|\mathbf{Z}|>u). \tag{11}$$

We immediately observe that the rate of decay of the ruin probability is completely determined by $u\,\mathrm{P}(|\mathbf{Z}|>u)$. The effects of the other parameters, such as the dimensionality d, the premium rate $\mathbf{p}$, the initial allocation $\mathbf{b}$, the extremal dependence captured in the measure μ, and the rules for capital transfers expressed in the ruin set F are all captured in the limiting constant $\int_0^\infty \mu(v\,\mathbf{c}+\mathbf{b}-F)\mathrm{d}v$.

The aim of this section is to qualitatively describe the effect of these parameters on the ruin probability and the buffer capital. In particular, we will emphasize the eventual diversification effects. We now consider an example, a simple Poisson shock model, that highlights the interesting effects and at the same time is sufficiently simple so that explicit computations are possible. We assume that the total claim amount process can be represented as a simple Poisson shock model. Consider a nonnegative random variable Z which is regularly varying, i.e. satisfies (8), with $\alpha > 1$. The distribution of the random variable Z will serve as a reference distribution. We assume that there is one type of shock that may affect all lines of business. Shocks of this type arrive at the jump times of a Poisson process $(N_{0,t})_{t\geq 0}$ with intensity λ_0. At the kth arrival the incurred claim in business line $j \in \{1, \ldots, d\}$ has the size $a^{(j)} Z_{0,k}$, where $(Z_{0,k})_{k\geq 1}$ is a sequence of independent and identically distributed random variables with $Z_{0,k} \overset{d}{=} Z$ and $a^{(j)} \geq 0$. Thus, the part of the total claim amount process corresponding to common losses is given by the compound Poisson process

$$\mathbf{C}_{0,t} = \sum_{k=1}^{N_{0,t}} \mathbf{Z}_{0,k},$$

where $\mathbf{Z}_{0,k} = \mathbf{a} Z_{0,k}$ and $\mathbf{a} = (a^{(1)}, \ldots, a^{(d)})^{\mathrm{T}}$. In addition we also take into account business-line-specific claims. Such claims arrive to business line j at the jump times of a Poisson process $(N_{j,t})_{t\geq 0}$ with intensity λ_j. The claim sizes are given by a sequence of independent and identically distributed random variables $(Z_{j,k})_{k\geq 1}$ with $Z_{j,k} \overset{d}{=} \sigma^{(j)} Z$ for some $\boldsymbol{\sigma} \in (0,\infty)^d$. We assume further that all $(N_{i,t})_{t\geq 0}$ and $(Z_{i,k})_{k\geq 1}$, $i \in \{0, \ldots, d\}$, are independent. The total claim amount process can now be written as

$$\mathbf{C}_t = \sum_{k=1}^{N_{0,t}} \mathbf{Z}_{0,k} + \sum_{k=1}^{N_{1,t}} Z_{1,k} \mathbf{e}_1 + \cdots + \sum_{k=1}^{N_{d,t}} Z_{d,k} \mathbf{e}_d \overset{d}{=} \sum_{k=1}^{N_t} \mathbf{Z}_k,$$

where (N_t), with $N_t = N_{0,t} + \cdots + N_{d,t}$, is a Poisson process with intensity $\overline{\lambda} = \lambda_0 + \cdots + \lambda_d$ and $(\mathbf{Z}_k)_{k\geq 1}$ is a sequence of independent and identically distributed random vectors independent of (N_t). Moreover, $\mathbf{Z}_k$ has the stochastic representation

$$\mathbf{Z}_k \overset{d}{=} \mathbf{Z}_{0,1}\delta_0(\xi) + Z_{1,1}\mathbf{e}_1\delta_1(\xi) + \cdots + Z_{d,1}\mathbf{e}_d\delta_d(\xi),$$

where ξ independent of $\mathbf{Z}_{0,1}, Z_{1,1}, \ldots, Z_{d,1}$ and $\mathrm{P}(\xi = k) = \lambda_k/\overline{\lambda}$ for $k \in \{0, \ldots, d\}$. Notice that by independence and regular variation

$$\begin{aligned}
\mathrm{P}(|\mathbf{Z}_k| > u) &\sim \frac{\lambda_0}{\overline{\lambda}} \mathrm{P}(|\mathbf{Z}_{0,1}| > u) + \sum_{j=1}^{d} \frac{\lambda_j}{\overline{\lambda}} \mathrm{P}(|Z_{j,1}\mathbf{e}_j| > u) \\
&= \frac{\lambda_0}{\overline{\lambda}} \mathrm{P}(|\mathbf{a}| Z > u) + \sum_{j=1}^{d} \frac{\lambda_j}{\overline{\lambda}} \mathrm{P}(\sigma^{(j)} Z > u) \\
&\sim \Big(\frac{\lambda_0}{\overline{\lambda}} |\mathbf{a}|^{\alpha} + \sum_{j=1}^{d} \frac{\lambda_j}{\overline{\lambda}} (\sigma^{(j)})^{\alpha} \Big) \mathrm{P}(Z > u) \qquad (12)
\end{aligned}$$

as $u \to \infty$, where $f(u) \sim g(u)$ means that $f(u)/g(u) \to 1$. We also have $\mathbf{Z}_{0,1} \in \mathrm{RV}(\alpha, \mu_0)$ with

$$\mu_0(A) = \int 1_A(r\,\mathbf{a}/|\mathbf{a}|)\alpha r^{-\alpha-1} \mathrm{d}r$$

and $Z_{j,1}\mathbf{e}_j \in \mathrm{RV}(\alpha, \mu_j)$ with

$$\mu_j(A) = \int 1_A(r\,\mathbf{e}_j)\alpha r^{-\alpha-1}\mathrm{d}r,$$

where $1_A(x) = 1$ if $x \in A$ and zero otherwise. Combining the above we find that $\mathbf{Z}_k \in \mathrm{RV}(\alpha, \mu)$ with

$$\mu(A) = \frac{\frac{\lambda_0}{\lambda}|\mathbf{a}|^\alpha \mu_0(A) + \sum_{j=1}^d \frac{\lambda_j}{\lambda}(\sigma^{(j)})^\alpha \mu_j(A)}{\frac{\lambda_0}{\lambda}|\mathbf{a}|^\alpha + \sum_{j=1}^d \frac{\lambda_j}{\lambda}(\sigma^{(j)})^\alpha}. \tag{13}$$

The net-profit condition becomes $\mathbf{c} = (c^{(1)}, \ldots, c^{(d)})^{\mathrm{T}} \in (0,\infty)^d$ with

$$c^{(j)} = \frac{1}{\lambda}\Big(p^{(j)} - \mathrm{E}(Z)(\lambda_0 a^{(j)} + \lambda_j \sigma^{(j)})\Big).$$

To make the expressions for the ruin probability and buffer capital more transparent we assume some symmetry and specialize to the case where $\mathbf{a} = \mathbf{1}$, $\boldsymbol{\sigma} = \mathbf{1}$, $\mathbf{b} = d^{-1}\mathbf{1}$, $\mathbf{p} = p\,\mathbf{1}$ and $\lambda_j = \lambda/d$ for some $\lambda \geq 0$ for $j = 1, \ldots, d$. In particular, $\mathbf{c} = c\mathbf{1}$, where

$$c = \frac{1}{\lambda_0 + \lambda}\Big(p - \mathrm{E}(Z)(\lambda_0 + \lambda/d)\Big),$$

and the components $(S_t^{(j)})$, $j \in \{1, \ldots, d\}$, of the claim surplus process are identically distributed.

Proposition 3. *For the Poisson shock model above the ruin probability $\psi_{d,F_\beta}(u)$ satisfies*

$$\lim_{u\to\infty} \frac{\psi_{d,F_\beta}(u)}{u\,\mathrm{P}(Z > u)} = \frac{\lambda_0}{\lambda}\frac{d^{\alpha-1}}{c(\alpha-1)} + \Big(1 - \frac{\lambda_0}{\lambda}\Big)\Big(\frac{\beta(d-1)+1}{d}\Big)^{-\alpha}\frac{1}{dc(\alpha-1)}.$$

Notice that the ruin probability $\psi_I(u)$ for each individual business line satisfies

$$\frac{\psi_I(u)}{u\,\mathrm{P}(Z > u)} = \frac{\mathrm{P}(S_t^{(1)} > u \text{ for some } t \geq 0)}{u\,\mathrm{P}(Z > u)} \to \frac{1}{c_I(\alpha - 1)}$$

as $u \to \infty$, where $c_I = \big(p - \mathrm{E}(Z)(\lambda_0 + \lambda/d)\big)/(\lambda_0 + \lambda/d)$. We observe the following consequences of Proposition 3.

• If $\lambda_0 = 0$, then the claims in all lines of business are independent and the ruin probability satisfies

$$\lim_{u\to\infty} \frac{\psi_{d,F_\beta}(u)}{u\,\mathrm{P}(Z > u)} = \Big(\frac{\beta(d-1)+1}{d}\Big)^{-\alpha}\frac{1}{dc(\alpha-1)}.$$

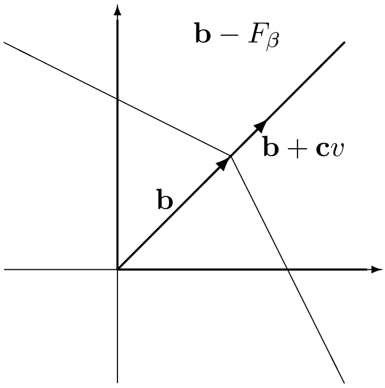

Figure 2: The limiting constant of the normalized ruin probability is given by $\int_0^\infty \mu(\mathbf{c}v + \mathbf{b} - F_\beta)\mathrm{d}v$. The set $\mathbf{c}v + \mathbf{b} - F_\beta$ is illustrated above (for $d = 2$ and $v = 0$). For the Poisson shock model under consideration the measure μ is concentrated on the coordinate axes and on the ray $\{w\,\mathbf{1} : w > 0\}$.

In particular, if capital transfers between business lines are not allowed, then

$$\begin{aligned}\lim_{u\to\infty} \frac{\psi_{d,F_0}(u)}{d\,\psi_I(u/d)} &= \lim_{u\to\infty} \frac{\psi_{d,F_0}(u)}{u\,\mathrm{P}(Z > u)} \frac{(u/d)\,\mathrm{P}(Z > u/d)}{d\,\psi_I(u/d)} \frac{u\,\mathrm{P}(Z > u)}{(u/d)\,\mathrm{P}(Z > u/d)} \\ &= \frac{d^\alpha}{dc(\alpha-1)} \frac{c_I(\alpha-1)}{d} \frac{1}{d^{\alpha-1}} \\ &= 1.\end{aligned}$$

That is, with no capital transfers allowed, the ruin probability for the multi-line insurance company is asymptotically equal to d times the ruin probability of each individual business line.

• If $\lambda = 0$, then the claims in all lines of business are perfectly dependent and the ruin probability satisfies

$$\lim_{u\to\infty} \frac{\psi_{d,F_\beta}(u)}{u\,\mathrm{P}(Z > u)} = \frac{d^{\alpha-1}}{c(\alpha-1)}.$$

In particular, because of the imposed symmetry, the ruin probability does not depend on β. Notice that the normalized limit of the ruin probability for the Poisson shock model considered above is a linear combination of these two terms.

We now compare different rules for capital transfers. Suppose that a multi-line insurance company is required to keep sufficient buffer capital so that the ruin probability is at most q. If no capital may be transferred between business lines, then the rules for capital transfer is governed by Π_0. What is the gain for the company if the rules change and allow for some capital to be transfered, according to Π_β? First note that $\beta' < \beta$ implies

$F_\beta \subset F_{\beta'}$ which implies $u_{d,F_\beta}(q) < u_{d,F_{\beta'}}(q)$. Hence, the buffer capital is decreasing in β. For small q the effect may be quantified by computing

$$\lim_{q \downarrow 0} \frac{u_{d,F_\beta}(q)}{u_{d,F_{\beta'}}(q)}.$$

For the Poisson shock model above with independent claims ($\lambda_0 = 0$) we have

$$\lim_{q \downarrow 0} \frac{u_{d,F_\beta}(q)}{u_{d,F_0}(q)} = [\beta(d-1)+1]^{-\alpha/(\alpha-1)}.$$

For example, if $\beta = 0.1$, $d = 11$ and $\alpha = 2$, then the buffer capital is reduced by 75%. Thus, the company benefits from a considerable diversification effect when at least some capital transfers are allowed.

4.1 Naive buffer capital requirements

Consider for a moment the following naive rule for the buffer capital requirement in an insurance market. Given $q > 0$ and $\beta \in [0, 1]$ each insurance company is obliged to hold the buffer capital $u_{d,F_\beta}(q)$. If β is small this rule may lead a multi-line insurance company to disintegrate, transforming each business line into an individual company running its own business. The reason is that the required buffer capital $u_I(q) = \inf\{u > 0 : \psi_I(u) \leq q\}$ for a business line seen as an individual company may be smaller than $u_{d,F_\beta}(q)/d$ which is the part of the buffer capital for the multi-line insurance company that is allocated to each business line. For the case with independent claims ($\lambda_0 = 0$) this happens (asymptotically as $q \downarrow 0$) if $\beta < \beta^* = (d^{1/\alpha}-1)/(d-1)$, i.e. $\beta < \beta^*$ implies that

$$\lim_{q \downarrow 0} \frac{u_{d,F_\beta}(q)}{du_I(q)} > 1.$$

The disintegration of the company is not advantageous for the policy holders in the sense that the probability of ruin for one business line increases from q to $\psi_{d,F_0}(du_I(q))$. In particular,

$$\lim_{q \downarrow 0} \frac{q}{\psi_{d,F_0}(du_I(q))} = \lim_{q \downarrow 0} \frac{q}{d\psi_I(u_I(q))} = \frac{1}{d} \leq 1$$

so that $\psi_{d,F_0}(du_I(q)) \approx dq$ for q small. Buffer capital requirements where actions that reduces the required buffer capital for an insurance company increases the risk for the policy holders are inappropriate.

4.2 Guarantee fund

Finally, we consider an alternative to the rules for transfer of capital considered so far. A group of insurance companies (or a single multi-line insurance company) may choose to set aside capital to a guarantee fund whose assets will be used to assist one of the participating companies (or business lines) in case of insolvency.

Consider the claim surplus process $(\mathbf{S}_t)$ given by (5). A fraction $\gamma \in [0,1]$ of the initial capital u is allocated to a guarantee fund and can be transferred freely to cancel negative positions in any business line. The remaining initial capital $(1-\gamma)u$ is allocated to the business lines according to $(1-\gamma)u\mathbf{b}$ and cannot be transferred between business lines. Ruin occurs if negative capital in some business line cannot be canceled by transferring capital from the guarantee fund. Hence, the ruin probability is given by

$$\begin{aligned}\psi^g_{d,\gamma}(u) &= \mathrm{P}(\mathbf{S}_t \in u\,((1-\gamma)\mathbf{b} - \gamma\, G) \text{ for some } t \geq 0)\\ &= \mathrm{P}(\mathbf{X}_n - n\,\mathbf{c} \in u\,((1-\gamma)\mathbf{b} - \gamma\, G) \text{ for some } n \geq 1),\end{aligned}$$

where

$$G = \Big\{\mathbf{x} : \sum_{k=1}^{d} x^{(k)} \wedge 0 < -1\Big\}.$$

It is useful to compare the model set up in Section 2 with the current model. We first notice that $\psi^g_{d,0}(u) = \psi_{d,F_0}(u)$ for every $u > 0$, i.e. the models coincide when the capital is allocated to the business lines and no transfer of capital is allowed. However, $\psi^g_{d,1}(u) \neq \psi_{d,F_1}(u)$. For the guarantee fund model with $\gamma = 1$ only the initial capital u can be transferred freely to business lines to cover losses. However, for the model in Section 2 with $\beta = 1$ all capital, including premium income, can be transferred freely between the business lines.

Similar to Propositions 1 and 2 we have

$$\begin{aligned}\lim_{u\to\infty} \frac{\psi^g_{d,\gamma}(u)}{u\ \mathrm{P}(|\mathbf{Z}| > u)} &= \int_0^\infty \mu(v\,\mathbf{c} + (1-\gamma)\mathbf{b} - \gamma\, G)\mathrm{d}v,\\ \lim_{q\downarrow 0} \frac{u^g_{d,\gamma}(q)}{g(q)} &= \Big(\int_0^\infty \mu(v\,\mathbf{c} + (1-\gamma)\mathbf{b} - \gamma\, G)\mathrm{d}v\Big)^{1/(\alpha-1)},\end{aligned}$$

where $u^g_{d,\gamma}(q) = \inf\{u > 0 : \psi^g_{d,\gamma}(u) \leq q\}$ is the buffer capital and $g(q) = \inf\{u > 0 : u\,\mathrm{P}(|\mathbf{Z}| > u) \leq q\}$.

To make the expressions for the ruin probability and buffer capital more transparent we now specialize to the shock model considered previously with $\mathbf{a} = \mathbf{1}$, $\boldsymbol{\sigma} = \mathbf{1}$, $\mathbf{b} = d^{-1}\mathbf{1}$ and $\lambda_j = \lambda/d$ for some $\lambda \geq 0$ for $j = 1,\dots,d$. In particular, $\mathbf{c} = c\mathbf{1}$, and the components $(S_t^{(j)})$ of the claim surplus process are identically distributed.

Proposition 4. *For the Poisson shock model the ruin probability* $\psi^g_{d,\gamma}(u)$ *satisfies*

$$\lim_{u\to\infty} \frac{\psi^g_{d,\gamma}(u)}{u\,\mathrm{P}(Z>u)} = \frac{\lambda_0}{\overline{\lambda}} \frac{d^{\alpha-1}}{c(\alpha-1)} + \Big(1-\frac{\lambda_0}{\overline{\lambda}}\Big)\Big(\frac{\gamma(d-1)+1}{d}\Big)^{-\alpha+1} \frac{1}{c(\alpha-1)}.$$

Notice the similarity with the expression for the ruin probability in Proposition 3. The reason is that the ruin sets are not very different (compare Figure 2 and Figure 3) and that the limiting measure μ only puts mass on the coordinate axes and on the ray $\{w\mathbf{1} : w > 0\}$. The diversification effects for the guarantee fund are can be analyzed as in Section 4.1 and the results are similar.

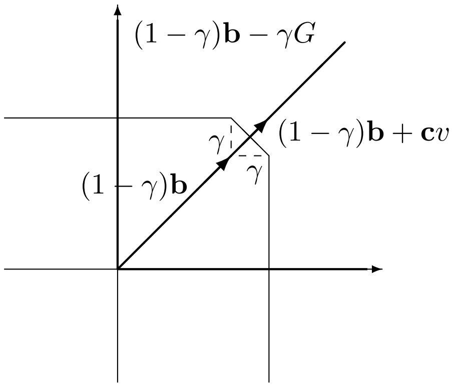

Figure 3: For the guarantee fund the limiting constant is given by $\int_0^\infty \mu(\mathbf{c}v + (1-\gamma)\mathbf{b}-\gamma G)\mathrm{d}v$. The set $\mathbf{c}v+(1-\gamma)\mathbf{b}-\gamma G$ is illustrated above (for $d=2$ and $v=0$). For the Poisson shock model under consideration the measure μ is concentrated on the coordinate axes and on the ray $\{w\,\mathbf{1} : w > 0\}$.

5 Proofs

Let $K(\Pi)$ and F be given by (6) and (7) and denote by $K^*(\Pi)$ the closed convex cone given by

$$K^*(\Pi) = \{\mathbf{y} : \mathbf{y}^{\mathrm{T}}\mathbf{x} \geq 0 \text{ for every } \mathbf{x} \in K(\Pi)\}.$$

This is sometimes referred to as the dual cone of $-K(\Pi)$. We begin this section with a useful representation of the cone $K^*(\Pi)$ and the set $\mathbf{b}-F$.

Lemma 5. *(i)* $K^*(\Pi) = \bigcap_{i\neq j}\{\mathbf{y}\in[0,\infty)^d : \mathbf{y}^{\mathrm{T}}(\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j) \geq 0\}$.

(ii) $\mathbf{b}-F = \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{y} > \mathbf{b}^{\mathrm{T}}\mathbf{y}$ *for some* $\mathbf{y}\in K^*(\Pi)\}$.

(iii) If $\pi^{(ij)} = \beta^{-1} \in [1,\infty)$ *for all* $i \neq j$*, then* $K^*(\Pi) = K^*(\Pi_\beta)$ *with*

$$K^*(\Pi_\beta) = \{\lambda \mathbf{x} : \lambda \geq 0,\, \mathbf{x} \in D_\beta\},$$
$$D_\beta = \{\mathbf{x} \in [0,\infty)^d : \vee_{i=1}^d x^{(i)} = 1, \wedge_{i=1}^d x^{(i)} \geq \beta\}.$$

Proof. (i) Since $\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j \in K(\Pi)$ for all $i \neq j$ it follows that $K^*(\Pi) \subseteq \bigcap_{i\neq j}\{\mathbf{y} \in [0,\infty)^d : \mathbf{y}^{\mathrm{T}}(\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j) \geq 0\}$. To show the reverse inequality take $\mathbf{y} \in [0,\infty)^d$ such that $\mathbf{y}^{\mathrm{T}}(\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j) \geq 0$ for all $i \neq j$. Let $\mathbf{x} \in K(\Pi)$ be arbitrary, with representation $\mathbf{x} = \sum_{i\neq j} v^{(ij)}(\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j) + \sum_i w^{(i)}\mathbf{e}_i$, with $v^{(ij)} \geq 0$ and $w^{(i)} \geq 0$. Then

$$\mathbf{y}^{\mathrm{T}}\mathbf{x} = \sum_{i\neq j} v^{(ij)}\mathbf{y}^{\mathrm{T}}(\pi^{(ij)}\mathbf{e}_i - \mathbf{e}_j) + \sum_i w^{(i)}\mathbf{y}^{\mathrm{T}}\mathbf{e}_i \geq 0,$$

and hence $\mathbf{y} \in K^*(\Pi)$. This proves (i).
(ii) Since $K^*(\Pi) = \{\mathbf{y} : \mathbf{y}^{\mathrm{T}}\mathbf{x} \geq 0 \text{ for every } \mathbf{x} \in K(\Pi)\}$ it follows immediately that

$$K(\Pi) = \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{y} \geq 0 \text{ for every } \mathbf{y} \in K^*(\Pi)\}$$

and hence the ruin set $F = K(\Pi)^c$ can be written as

$$F = \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{y} < 0 \text{ for some } \mathbf{y} \in K^*(\Pi)\}.$$

Hence,

$$\begin{aligned}\mathbf{b} - F &= \mathbf{b} - \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{y} < 0 \text{ for some } \mathbf{y} \in K^*(\Pi)\}\\ &= \{\mathbf{x} : (\mathbf{x} - \mathbf{b})^{\mathrm{T}}\mathbf{y} > 0 \text{ for some } \mathbf{y} \in K^*(\Pi)\}\\ &= \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{y} > \mathbf{b}^{\mathrm{T}}\mathbf{y} \text{ for some } \mathbf{y} \in K^*(\Pi)\}.\end{aligned}$$

(iii) If $\pi^{(ij)} = \beta^{-1}$ for all $i \neq j$ then (i) implies that for $\mathbf{y} \in K^*(\Pi)$ we have $\beta^{-1}y^{(i)} - y^{(j)} \geq 0$ and $\beta^{-1}y^{(j)} - y^{(i)} \geq 0$ for all $i \neq j$. In particular, $\beta \leq (\wedge_{i=1}^d y^{(i)})/(\vee_{i=1}^d y^{(i)})$ Normalizing so that $\vee_{i=1}^d y^{(i)} = \lambda$ we have $\mathbf{y} = \lambda\mathbf{x}$ for some $\mathbf{x} \in D_\beta$. Hence $K^*(\Pi) \subseteq \{\lambda\mathbf{x} : \lambda \geq 0, \mathbf{x} \in D_\beta\}$. The reverse inclusion holds because for each $\mathbf{x} \in D_\beta$ we have $\mathbf{x}^{\mathrm{T}}(\beta^{-1}\mathbf{e}_i - \mathbf{e}_j) = \beta^{-1}x^{(i)} - x^{(j)} \geq 0$ for all $i \neq j$. Hence, $\mathbf{x} \in K^*(\Pi)$ by (i). □

Lemma 6. *Consider the processes* $(\mathbf{S}_t)_{t\geq 0}$ *and* $(\mathbf{R}_t)_{t\geq 0}$ *given by* (5) *and suppose that* $\mathrm{E}(|\mathbf{Z}|) < \infty$*,* $\mathrm{E}(W) < \infty$*, and* $\mathbf{c} = \mathrm{E}(W)\,\mathbf{p} - \mathrm{E}(\mathbf{Z})$*. Let the ruin set* F *be given by* (7)*. Then the ruin probability is given by*

$$\begin{aligned}\psi_{d,F}(u) &= \mathrm{P}(\mathbf{S}_t \in u\,(\mathbf{b} - F) \textit{ for some } t \geq 0)\\ &= \mathrm{P}(\mathbf{X}_n - n\,\mathbf{c} \in u\,(\mathbf{b} - F) \textit{ for some } n \geq 1),\end{aligned}$$

where $\mathbf{X}_n = \sum_{k=1}^n(\mathbf{Z}_k - W_k\,\mathbf{p}) + n\,\mathbf{c}$ *is a random walk with zero mean.*

Proof. First, notice that $u\,F = F$ for $u > 0$. Hence,

$$\begin{aligned}
\text{Ruin} &= \Big\{\mathbf{R}_t \in F \text{ for some } t \geq 0\Big\} \\
&= \Big\{\mathbf{S}_t \in u\,\mathbf{b} - F \text{ for some } t \geq 0\Big\} \\
&= \Big\{\mathbf{S}_t \in u\,(\mathbf{b} - F) \text{ for some } t \geq 0\Big\} \\
&= \Big\{\mathbf{S}_{T_n} \in u\,(\mathbf{b} - F) \text{ for some } n \geq 1\Big\} \\
&= \Big\{\sum_{k=1}^{n} \mathbf{Z}_k - T_n\,\mathbf{p} \in u\,(\mathbf{b} - F) \text{ for some } n \geq 1\Big\} \\
&= \Big\{\underbrace{\sum_{k=1}^{n} (\mathbf{Z}_k - W_k\,\mathbf{p})}_{\widetilde{\mathbf{X}}_n} \in u\,(\mathbf{b} - F) \text{ for some } n \geq 1\Big\} \\
&= \Big\{\underbrace{\widetilde{\mathbf{X}}_n - \mathrm{E}(\widetilde{\mathbf{X}}_n)}_{\mathbf{X}_n} + \underbrace{\mathrm{E}(\widetilde{\mathbf{X}}_n)}_{-n\,\mathbf{c}} \in u\,(\mathbf{b} - F) \text{ for some } n \geq 1\Big\} \\
&= \Big\{\mathbf{X}_n - n\,\mathbf{c} \in u\,(\mathbf{b} - F) \text{ for some } n \geq 1\Big\},
\end{aligned}$$

where $\mathbf{X}_n = \mathbf{Y}_1 + \cdots + \mathbf{Y}_n$ and $(\mathbf{Y}_k)$ is an iid sequence with $\mathrm{E}(\mathbf{Y}_1) = \mathbf{0}$. □

Lemma 7. *Let $F = K(\Pi)^c$ and let μ be given by* (9)*. If $\mathbf{a} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, then $\mu(\partial(\mathbf{a} - F)) = 0$.*

Proof. Notice that $\partial(\mathbf{a} - F) = \mathbf{a} - \partial F = \mathbf{a} - \partial K(\Pi)$ and that

$$\partial K(\Pi) \subset \bigcup_{i \neq j} \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}(\frac{1}{\pi^{(ij)}}\mathbf{e}_i + \mathbf{e}_j) = 0\}$$

we find that

$$\partial(\mathbf{a} - F) \subset \bigcup_{i \neq j} \Big\{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\Big(\frac{1}{\pi^{(ij)}}\mathbf{e}_i + \mathbf{e}_j\Big) = a^{(i)}/\pi^{(ij)} + a^{(j)}\Big\} =: \bigcup_{i \neq j} H_{ij}.$$

With

$$W_{ij} = \Big\{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\Big(\frac{1}{\pi^{(ij)}}\mathbf{e}_i + \mathbf{e}_j\Big) > a^{(i)}/\pi^{(ij)} + a^{(j)}\Big\},$$

we have

$$\mu(W_{ij}) \geq \mu\Big(\bigcup_{q \in \mathbb{Q} \cap (1,\infty)} qH_{ij}\Big) = \sum_{q \in \mathbb{Q} \cap (1,\infty)} \mu(qH_{ij}) = \mu(H_{ij}) \sum_{q \in \mathbb{Q} \cap (1,\infty)} q^{-\alpha}.$$

Since $\mu(W_{ij}) \in (0,\infty)$ and $\sum_{q \in \mathbb{Q} \cap (1,\infty)} q^{-\alpha} = \infty$ we must have $\mu(H_{ij}) = 0$. Hence, $\mu(\partial(\mathbf{a} - F)) \leq \sum_{i \neq j} \mu(H_{ij}) = 0$. □

Proof of Proposition 1

Recall from Lemma 6 that

$$\mathrm{P}(\mathbf{R}_t \in F \text{ for some } t \geq 0) = \mathrm{P}(\mathbf{X}_n - n\,\mathbf{c} \in u\,(\mathbf{b} - F) \text{ for some } n \geq 1),$$

where $\mathbf{X}_n = \mathbf{Y}_1 + \cdots + \mathbf{Y}_n$ and $(\mathbf{Y}_k)$ is an iid sequence with $\mathrm{E}(\mathbf{Y}_1) = \mathbf{0}$. Since $\mathbf{Z}_1 \in \mathrm{RV}(\alpha, \mu)$ and $E(W_1^\gamma) < \infty$ for some $\gamma > \alpha$, we have $\mathbf{X}_1 \in \mathrm{RV}(\alpha, \mu)$. Since $\mathbf{c} \in (0, \infty)^d$ the set $\mathbf{b} - F$ is $\mathbf{c}$-increasing; $\mathbf{x} + t\,\mathbf{c} \in \mathbf{b} - F$ whenever $\mathbf{x} \in \mathbf{b} - F$ and $t \geq 0$. Moreover, by Lemma 7, $\mu(\partial(v\,\mathbf{c} + \mathbf{b} - F)) = 0$ for every $v \geq 0$. The conclusion now follows by combining Theorem 3.1 and Remark 3.2 in Hult *et al.* (2005). □

Proof of Proposition 2

Set $h_1(u) = [\psi(u)]^{-1}$ and $h_2(u) = [u\,\mathrm{P}(|\mathbf{Z}| > u)]^{-1}$. Notice that h_1 and h_2 are nondecreasing, regularly varying with index $\alpha - 1$ and satisfies $\lim_{u\to\infty} h_1(u)/h_2(u) = C \in [0, \infty]$. Hence, Proposition 0.8(vi) in Resnick (1987) yields

$$\lim_{s\to\infty} \frac{h_1^{\leftarrow}(s)}{h_2^{\leftarrow}(s)} = C^{-1/(\alpha-1)},$$

where $h_k^{\leftarrow}(s) = \inf\{u : h_k(u) \geq s\}$ for $k = 1, 2$. Hence,

$$\begin{aligned}
\lim_{q\downarrow 0} \frac{u(q)}{g(q)} &= \lim_{q\downarrow 0} \frac{\inf\{u > 0 : \psi(u) \leq q\}}{\inf\{u > 0 : u\,\mathrm{P}(|\mathbf{Z}| > u) \leq q\}} \\
&= \lim_{q\downarrow 0} \frac{h_1^{\leftarrow}(1/q)}{h_2^{\leftarrow}(1/q)} \\
&= C^{-1/(\alpha-1)}.
\end{aligned}$$

Since $C = \left(\int_0^\infty \mu(v\,\mathbf{c} + \mathbf{b} - F)\mathrm{d}v\right)^{-1}$ we obtain

$$\lim_{q\downarrow 0} \frac{u(q)}{g(q)} = \left(\int_0^\infty \mu(v\,\mathbf{c} + \mathbf{b} - F)\mathrm{d}v\right)^{1/(\alpha-1)}.$$

□

Proof of Proposition 3

Recall from Lemma 5(ii) that

$$\begin{aligned}
&\mathbf{b} - F_\beta = \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{y} > \mathbf{b}^{\mathrm{T}}\mathbf{y} \text{ for some } \mathbf{y} \in K^*(\Pi_\beta)\}, \\
&v\,\mathbf{c} + \mathbf{b} - F_\beta = \{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{y} > (\mathbf{b} + v\,\mathbf{c})^{\mathrm{T}}\mathbf{y} \text{ for some } \mathbf{y} \in K^*(\Pi_\beta)\}.
\end{aligned}$$

Moreover, by Lemma 5,

$$K^*(\Pi_\beta) = \{\lambda \mathbf{x} : \lambda \ge 0, \mathbf{x} \in D_\beta\},$$
$$D_\beta = \{\mathbf{x} \in \mathbb{R}_+^d : \vee_{i=1}^d x^{(i)} = 1, \wedge_{i=1}^d x^{(i)} \ge \beta\}.$$

By Proposition 1, (12) and (13) we have

$$\lim_{u\to\infty} \frac{\psi_{d,F_\beta}(u)}{u\,\mathrm{P}(Z>u)} = \frac{\lambda_0}{\overline{\lambda}} d^{\alpha/2} \int_0^\infty \mu_0(v\,\mathbf{c} + \mathbf{b} - F_\beta)\mathrm{d}v + \Big(1 - \frac{\lambda_0}{\overline{\lambda}}\Big) \int_0^\infty \mu_I(v\,\mathbf{c} + \mathbf{b} - F_\beta)\mathrm{d}v,$$

where $\mu_I = \mu_1 + \cdots + \mu_d$. For the first term we have

$$\begin{aligned}
&\int_0^\infty \mu_0(v\,\mathbf{c} + \mathbf{b} - F_\beta)\mathrm{d}v \\
&= \int_0^\infty \mu_0(\mathbf{z} = q\mathbf{1}/\sqrt{d} : \mathbf{z}^{\mathrm{T}}\mathbf{y} > (\mathbf{b} + v\,\mathbf{c})^{\mathrm{T}}\mathbf{y} \text{ for some } \mathbf{y} \in K^*(\Pi_\beta))\mathrm{d}v \\
&= \int_0^\infty \mu_0(\mathbf{z} = q\mathbf{1}/\sqrt{d} : \mathbf{z}^{\mathrm{T}}\mathbf{y} > (\mathbf{b} + v\,\mathbf{c})^{\mathrm{T}}\mathbf{y} \text{ for some } \mathbf{y} \in D_\beta)\mathrm{d}v \\
&= \int_0^\infty \mu_0(\mathbf{z} = q\mathbf{1}/\sqrt{d} : q > (1 + vcd)/\sqrt{d})\mathrm{d}v \\
&= d^{\alpha/2}\mu_0(\mathbf{z} = q\mathbf{1}/\sqrt{d} : q > 1) \int_0^\infty (1 + vcd)^{-\alpha}\mathrm{d}v \\
&= \frac{d^{\alpha/2-1}}{c(\alpha-1)}.
\end{aligned}$$

For the second term we have

$$\begin{aligned}
&\int_0^\infty \mu_I(v\,\mathbf{c} + \mathbf{b} - F_\beta)\mathrm{d}v \\
&= \sum_{k=1}^d \int_0^\infty \mu_I\left[\mathbf{x} : x^{(k)} > \frac{(v\,\mathbf{c} + \mathbf{b})^{\mathrm{T}}\mathbf{y}}{\mathbf{e}_k^{\mathrm{T}}\mathbf{y}} \text{ for some } \mathbf{y} \in D_\beta\right]\mathrm{d}v \\
&= \sum_{k=1}^d \int_0^\infty \mu_I\left[\mathbf{x} : x^{(k)} > \min_{\mathbf{y}\in D_\beta} \frac{(vc + 1/d)\,\mathbf{1}^{\mathrm{T}}\mathbf{y}}{\mathbf{e}_k^{\mathrm{T}}\mathbf{y}}\right]\mathrm{d}v \\
&= \sum_{k=1}^d \int_0^\infty \mu_I\left[\mathbf{x} : x^{(k)} > (vcd + 1)(\beta(d-1) + 1)/d\right]\mathrm{d}v \\
&= \Big(\frac{\beta(d-1)+1}{d}\Big)^{-\alpha} \int_0^\infty (vcd+1)^{-\alpha}\mathrm{d}v \sum_{k=1}^d \mu_I(\mathbf{x} : x^{(k)} > 1) \\
&= \Big(\frac{\beta(d-1)+1}{d}\Big)^{-\alpha} \frac{1}{dc(\alpha-1)}.
\end{aligned}$$

□

Proof of Proposition 4

The proof is similar to the proof of Proposition 3. By Proposition 1, (12) and (13) we have

$$\lim_{u\to\infty} \frac{\psi^g_{d,\gamma}(u)}{u\,\mathrm{P}(Z>u)} = \frac{\lambda_0}{\overline{\lambda}} d^{\alpha/2} \int_0^\infty \mu_0(v\,\mathbf{c} + (1-\gamma)\mathbf{b} - \gamma\, G)\mathrm{d}v + \Big(1-\frac{\lambda_0}{\overline{\lambda}}\Big) \int_0^\infty \mu_I(v\,\mathbf{c} + (1-\gamma)\mathbf{b} - \gamma\, G)\mathrm{d}v,$$

where $\mu_I = \mu_1 + \cdots + \mu_d$. For the first term we have

$$\begin{aligned}
&\int_0^\infty \mu_0(v\,\mathbf{c} + (1-\gamma)\mathbf{b} - \gamma\, G)\mathrm{d}v \\
&= \int_0^\infty \mu_0((vc + (1-\gamma)/d)\mathbf{1} - \gamma\, G)\mathrm{d}v \\
&= \int_0^\infty \mu_0(\mathbf{z} = q\mathbf{1}/\sqrt{d} : \mathbf{z}^{\mathrm{T}}\mathbf{1} > (vc + (1-\gamma)/d)\mathbf{1}^{\mathrm{T}}\mathbf{1} + \gamma)\mathrm{d}v \\
&= \int_0^\infty \mu_0(\mathbf{z} = q\mathbf{1}/\sqrt{d} : q > (vcd+1)/\sqrt{d})\mathrm{d}v \\
&= d^{\alpha/2} \int_0^\infty (vcd+1)^{-\alpha}\mathrm{d}v \\
&= \frac{d^{\alpha/2-1}}{c(\alpha-1)}.
\end{aligned}$$

For the second term we have

$$\begin{aligned}
&\int_0^\infty \mu_I(v\,\mathbf{c} + (1-\gamma)\mathbf{b} - \gamma\, G)\mathrm{d}v \\
&= \sum_{k=1}^d \int_0^\infty \mu_I(\mathbf{x} : x^{(k)} > vc + (1-\gamma)/d + \gamma)\mathrm{d}v \\
&= \int_0^\infty (vc + (1-\gamma)/d + \gamma)^{-\alpha}\mathrm{d}v \sum_{k=1}^d \mu_I(\mathbf{x} : x^{(k)} > 1) \\
&= \Big(\frac{\gamma(d-1)+1}{d}\Big)^{-\alpha+1} \frac{1}{c(\alpha-1)}.
\end{aligned}$$

□

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