PATTERN AVOIDANCE IN ALTERNATING SIGN MATRICES

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Abstract. We generalize the definition of a pattern from permutations to alternating sign matrices. The number of alternating sign matrices avoiding 132 is proved to be counted by the large Schröder numbers, $1, 2, 6, 22, 90, 394\ldots$. We give a bijection between 132-avoiding alternating sign matrices and Schröder-paths, which gives a refined enumeration. We also show that the $132, 123$-avoiding alternating sign matrices are counted by every second Fibonacci number.

1. Introduction

For some time, we have emphasized the use of permutation matrices when studying pattern avoidance. This and the fact that alternating sign matrices are generalizations of permutation matrices, led to the idea of studying pattern avoidance in alternating sign matrices.

The main results, concerning 132-avoidance, are presented in Section 3. In Section 5, we enumerate sets of alternating sign matrices avoiding two patterns of length three at once. In Section 6 we state some open problems. Let us start with the basic definitions.

Given a permutation π of length n we will represent it with the $n \times n$ permutation matrix with 1 in positions $(i, \pi(i)), 1 \leq i \leq n$ and 0 in all other positions. We will also suppress the 0s.

$$\begin{array}{c|c|c|c|}
\cline{2-4}
1 & 1 & & \\
\cline{2-4}
3 & & & 1 \\
\cline{2-4}
2 & & 1 & \\
\cline{2-4}
\end{array}$$

Figure 1. The permutation matrix of 132.

In this paper, we will think of permutations in terms of permutation matrices.

Definition 1.1. ***(Patterns in permutations)*** *A permutation π is said to contain a permutation τ as a* ***pattern*** *if the permutation matrix of τ is a submatrix of π. That is, if it's possible to remove rows and columns from the permutation matrix of π, so that the remaining matrix corresponds to the permutation τ.*

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Definition 1.2. ***(Pattern avoidance)*** *If a permutation π does not contain a pattern τ, it is said to* ***avoid*** *τ.*

Example The permutation $\pi = 45312$ avoids 132.

Definition 1.3. ***(Alternating sign matrix)*** *An alternating sign matrix (ASM) is a matrix of 0s, 1s and -1s for which the sum of entries in each row and in each column is 1 and the non-zero entries of each row and of each column alternate in sign.*

1			
		1	
	1	-1	1
		1	

Figure 2. An example of an alternating sign matrix (zeros suppressed).

An important observation is that an alternating sign matrix must be a square matrix (the sum in each row and column is 1). Another observation is that every permutation matrix is also an alternating sign matrix, and therefore alternating sign matrices can be seen as generalizations of permutation matrices. The set of alternating sign matrices is denoted $\mathcal{A}_n$. The number of $n \times n$ alternating sign matrices can be described by the following formula

$$|\mathcal{A}_n| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

This formula was conjectured in the early 80's by Mills, Robbins and Rumsey, but was not proved until 1996 by Doron Zeilberger [9]. Further information about alternating sign matrices and the proof can be found in [1].

Remark: This article is based on the Master's Thesis of the first author [5], where a more detailed exposition can be found.

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2. Pattern avoidance in alternating sign matrices

We will now generalize the concept of pattern avoidance to alternating sign matrices.

Definition 2.1. ***(Pattern avoidance in ASMs)*** *Let M be an $n \times n$ alternating sign matrix and π a permutation (matrix). We say that M* **contains** *π if there exists a submatrix $D = (d_{ij})$ of M, where $d_{ij} = 1$ if $\pi(i) = j$. In other words, we demand that there exist 1s in M such that their relative positions are as in π.*

If M does not contain π, we say it **avoids** *π. The set of alternating sign matrices of order n, avoiding π, is denoted $\mathcal{A}_n(\pi)$.*

This means for example that

$\begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline 1 & \text{-}1 & 1 & \\ \hline & 1 & \text{-}1 & 1 \\ \hline & & 1 & \\ \hline \end{array}$ contains $\begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline \end{array}$ in two different ways and avoids $\begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & \\ \hline 1 & & \\ \hline \end{array}$.

Note that one may choose a different definition for pattern avoidance. The definition here seems most natural for alternating sign matrices. It is also very similar to the the definition used in studies of patterns in general 0-1 matrices, see e.g. [6] and [4].

3. Avoiding 132

In this section we prove that the 132-avoiding alternating sign matrices are counted by the *large Schröder numbers*, starting with $1, 2, 6, 22, 90, 394, 1806, ...$ We also give a bijection to *Schröder paths.*

3.1. **Schröder numbers.** The large Schröder numbers, r_n, from now on only called the Schröder numbers, are closely related to the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$, as seen in the following formula,

$$r_n = \sum_{d=0}^{n} \binom{2n-d}{d} C_{n-d} \qquad (n \geq 0). \tag{1}$$

The Schröder numbers are known to enumerate many combinatorial objects, see e.g. the eleven examples in [8, Exercise 6.39]. Egge and Mansour [2] proved that permutations avoiding both 1243 and 2143 are counted by the Schröder numbers, and recently Egge [3] enumerated eleven classes of pattern-avoiding *signed permutations* by these numbers. The Schröder numbers can also be defined by

$$r_n = r_{n-1} + \sum_{k=1}^{n} r_{k-1} r_{n-k} \qquad (n \geq 1), \tag{2}$$

where $r_0 = 1$.

3.2. **General structure.** To simplify counting of $\mathcal{A}_n(132)$, we introduce a lemma, which will be used as a base for our reasoning throughout this text.

Since M is an ASM it will have exactly one 1 in the rightmost column. Let k denote the row containing this 1. Let us define to types of ASMs.

(I) M has no -1 in row k. All 1s and -1s in rows $k+1$ to n are in columns 1 to $n-k$ and the submatrix consisting of these rows and columns is called M_1. All 1s and -1s in rows 1 to $k-1$ are in columns $n-k+1$ to $n-1$. This submatrix is called M_2. Here M_1 or M_2 may be empty.

(II) Row k in M has a -1 in column $(n-k+1)$. All 1s and -1s in rows k to n are in columns 1 to $n-k+1$. This submatrix with the -1 in position $(k, n-k+1)$ changed to a 0 is called M_1. All 1s and -1s in rows 1 to $k-1$ are in columns $n-k+1$ to $n-1$. This submatrix is called M_2.

See Figure 3 for the two types.

Lemma 3.1 (**Structure Lemma for** 132**-avoiding ASMs**)**.** *Let M be a* 132*-avoiding ASM. We may decompose M in exactly one of the following two ways.*

1. *M is of type (I) and $1 \leq k \leq n$. Then M_1 is any 132-avoiding ASM of order $n-k$.*
2. *M is of type (II) and $2 \leq k \leq n-1$. Then M_1 is an* 132*-avoiding ASM of order $(n-k+1)$ and must have a zero in the upper right corner.*

In both these cases, M_2 is any 132*-avoiding ASM of order $k-1$.*

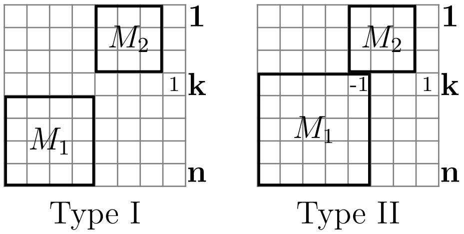


Figure 3. The structure of 132-avoiding alternating sign matrices.

Proof. Recall that M is a 132-avoiding ASM. Hence, all 1s in rows $1, \ldots, k-1$ must be to the right of or in the same column as all 1s in rows $k+1, \ldots, n$. The submatrix called M_2 consists of the nonzero columns of the first $k-1$ rows. The corresponding columns in M will be zero below row $k-1$ except perhaps the leftmost of these columns. M_2 thus satisfies the definition of an alternating sign matrix and must thus be square as noted after Definition 1.3.

If we have no -1 in row k we have case (1).

Assume there is exactly one -1 in row k. Since M_2 is a square matrix this -1 must be in position $(k, n-k+1)$ and we have the second case.

There cannot be two or more -1s in row k since that would imply a 1 in rows $1, \ldots, k-1$ to the left of a 1 in rows $k+1, \ldots, n$ and thus an occurence of a 132-pattern. $\square$

3.3. **Proof of main theorem.**

Theorem 3.2. $|\mathcal{A}_n(132)| = r_{n-1}$ *for all* $n \geq 1$.

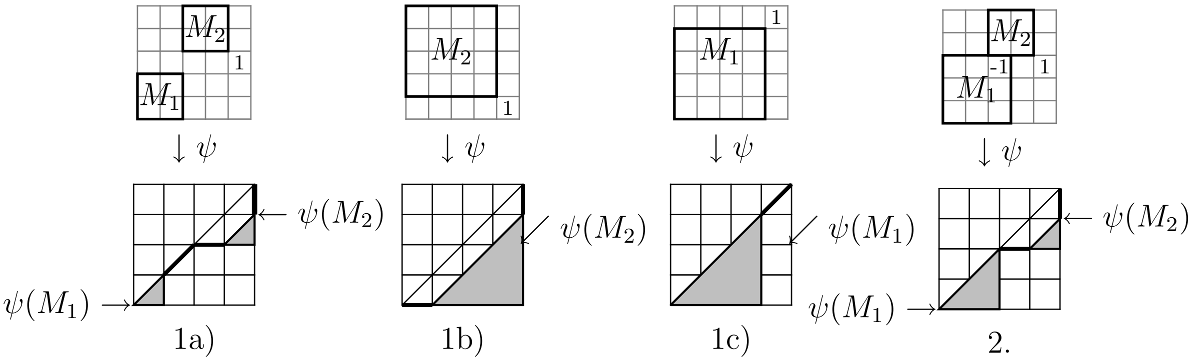
Proof. Let $a_n = |\mathcal{A}_n(132)|$. Recall that we denote the position of the 1 in the rightmost column of an arbitrary 132-avoiding ASM M as k. If $k = 1$ or $k = n$, there are a_{n-1} ways of forming M respectively. By the Structure Lemma we know that M can be decomposed into 132-avoiding ASMs M_1 and M_2 in two ways.

(1) For the first case of the Structure Lemma, there are a_{n-k} ways to form M_1 and a_{k-1} ways for M_2. Thus, there are $a_{k-1} \cdot a_{n-k}$ ASMs for case 1, when $2 \leq k \leq n-1$.

(2) In the other case from the Structure Lemma, there are a_{k-1} ways of forming M_2. Counting M_1 is a little less straightforward. M_1 is of order $n-k+1$, but contains a 0 in its upper right corner. This means that we have to remove the a_{n-k} ASMs of order $(n-k+1)$ which contains a 1 in the top right corner. In this case we thus have $a_{k-1} \cdot (a_{n-k+1} - a_{n-k})$ when $2 \leq k \leq n-1$.

In total we have $a_n = 2 \cdot a_{n-1} + \sum_{k=2}^{n-1} [a_{k-1} \cdot a_{n-k} + a_{k-1} \cdot (a_{n-k+1} - a_{n-k})] = a_{n-1} + \sum_{k=2}^{n} a_{k-1} \cdot a_{n-k+1} = a_{n-1} + \sum_{k=1}^{n-1} a_k \cdot a_{n-k}$.

This implies that $a_n = r_{n-1}$ and this completes the proof. □



1a) 1b) 1c) 2.

FIGURE 4. The construction of a Schröder path from a 132-avoiding alternating sign matrix.

3.4. **A bijection.** In this section we present a bijection between 132-avoiding alternating sign matrices and *Schröder paths*. We use this bijection to give new enumerations according to various statistics. The bijection is illustrated in Figure 4.

Schröder paths are paths in the integer lattice from $(0,0)$ to (n,n) with allowed steps east $(1,0)$, north $(0,1)$ and diagonal $(1,1)$ staying below main diagonal $y = x$. The number of Schröder paths from $(0,0)$ to (n,n) is r_n, see e.g. [8, Exercise 6.39(j)].

We define $\psi : \{M | M \in \mathcal{A}_n(132)\} \rightarrow \{$Schröder paths from $(0,0)$ to $(n-1, n-1)\}$ recursively. Recall that k is the row of the 1 in the rightmost column of M.

(0) *Base case*: If M is the 1×1 matrix with a single 1, then $\psi(M)$ is the empty path.

(1) a) If M is structured as in the first case of the Structure Lemma and $2 \leq k \leq n-1$, we construct $\psi(M)$ by joining the following paths:

  - $*$ A path with the same steps as $\psi(M_1)$ from $(0,0)$ to $(n-k-1, n-k-1)$.
  - $*$ A diagonal step $(1,1)$ followed by an east $(1,0)$ step.
  - $*$ A path using the same steps as $\psi(M_2)$ from $(n-k+1, n-k)$ to $(n-1, n-2)$.
  - $*$ A north $(0,1)$ step from $(n-1, n-2)$ to $(n-1, n-1)$.

   b) If M has a 1 in the bottom right corner, i.e. $k = n$, then $\psi(M)$ is defined to start with an east step $(1,0)$ and end with a north step $(0,1)$. From $(1,0)$ to $(n-1, n-2)$ $\psi(M)$ uses the same steps as $\psi(M_2)$.

   c) If M has a 1 in the top right corner, i.e. $k = 1$, then $\psi(M)$ is defined to begin with the same steps as in $\psi(M_1)$. The path ends with a single diagonal step from $(n-2, n-2)$ to $(n-1, n-1)$.

(2) If M has the structure of the second case of the Structure Lemma, we construct $\psi(M)$ from $(0,0)$ to $(n-k, n-k)$ by using the same steps as in $\psi(M_1)$. The path then continues with an east $(1,0)$ step followed by a path with the same steps as $\psi(M_2)$ from $(n-k+1, n-k)$ to $(n, n-1)$, ending with a north $(0,1)$ step to (n,n).

Example In Figure 5 we show three examples of ψ. For the permutation matrix 21, using step 1c in the definition. For 123, using step 1b twice and the base case. The 5×5 ASM to the right is constructed using step 2 in ψ and the former constructions.

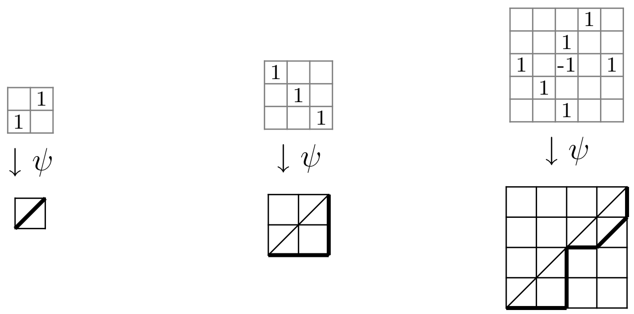


FIGURE 5. Examples of constructions of Schröder paths using ψ.

In the construction above the important parameter is k. This corresponds to the position on the diagonal $(n-k, n-k)$ where the path hits it for the last time before reaching $(n-1, n-1)$. Note that the matrix M_1 in case 2 never has a 1 in the top right position. Thus the step reaching $(n-k, n-k)$ is never a diagonal step and there is thus no overlap with the defintion in case 1a. The inverse of ψ can hence be constructed recursively in a similar way.

As a consequence of the bijective construction above, the "inner corners" formed by a joint north and east step, correspond to the -1s of the ASM. Hence 132-avoiding permutations map to Schröder paths never having a north step directly followed by an east step. Since the former are well-known to be enumerated by the Catalan numbers we get the following.

Corollary 3.3. *The number of Schröder paths from* $(0,0)$ *to* $(n-1, n-1)$ *which have no north step directly followed by an east step is the* n*th Catalan number* C_n.

It is also possible to count Schröder paths by summing over the number of diagonal steps. This yields the formula in Equation 1. Note that there are exactly C_{n-1} paths with no diagonal step. Do also note that diagonal steps are formed in cases 1a) and 1c) of ψ. To classify the 132-avoiding ASMs formed by steps 1b) and 2 we make the following definition. If all 1s on row i in M are to the right of all 1s in row $i+1$ then we call i a **positive descent** of M. It is easy to see that cases 1a) and 1c) occur exactly when k is a positive descent of M.

Corollary 3.4. *The number of 132-avoiding ASMs of size* $n+1$ *with* d *positive descents is* $\binom{2n-d}{d}C_{n-d}$. *In particular, there are* C_n *with no positive descents.*

The alternating sign matrices were originally defined as a generalization of permutation matrices to generalize the notion of the determinant [1]. We find it aesthetically very pleasing that 132-avoiding ASMs offer a bijection with Schröder paths which are a generalization of Dyck paths, well known to be in bijection with 132-avoiding permutations. The shift in index by one seems however to make it difficult to capture both correspondances in one bijection.

4. Avoiding other patterns

If τ' is obtained from τ by a reflection or rotation then clearly $|\mathcal{A}_n(\tau')| = |\mathcal{A}_n(\tau)|$ by applying the same symmetry operation to $\mathcal{A}_n$. There are thus only two cases of permutation patterns of length three. We have made several attempts to count $|\mathcal{A}_n(123)|$, but none succeeded. Computer calculations yielded the following results:

When investigating alternating sign matrices avoiding patterns of length four, the elementary symmetry operations reduce the number

n	1	2	3	4	5	6	7	8	9	10
$\lvert\mathcal{A}_n(123)\rvert$	1	2	6	23	103	514	2785	16132	98897	637192

Table 1. The number of 123-avoiding alternating sign matrices of order $n \leq 10$.

of cases from 24 to seven. We have not been able to find a formula for any of these cases. The computer-generated integer sequences can be found in Table 2.

$\tau\backslash n$	1	2	3	4	5	6	7	8	9
2143	1	2	7	40	320	3152	35551	441280	5885844
2413	1	2	7	41	364	4168	54659	775528	11604671
1243	1	2	7	41	360	4200	59869	990930	18452035
1432	1	2	7	41	361	4234	60723	1009328	18789963
1342	1	2	7	41	367	4455	66403	1138774	21600472
1234	1	2	7	41	370	4638	74093	1423231	31669412
1324	1	2	7	41	376	4985	88985	2024954	56429599

Table 2. The number of alternating sign matrices avoiding τ, a pattern of length four, for $n \leq 9$.

None of the sequences were found in The On-Line Encyclopedia of Integer Sequences [7] in December 2006.

5. Avoiding several patterns at once

The set of alternating sign matrices of order n avoiding a set of patterns $\{\tau_1, \tau_2, \dots, \tau_m\}$ are denoted $\mathcal{A}_n(\tau_1, \tau_2, \dots, \tau_m)$. We will now turn to the number of alternating sign matrices which avoid two different patterns of length three at once. Five equivalence classes modulo rotation and reflection arise, which we can represent by $\mathcal{A}_n(132, 123)$, $\mathcal{A}_n(132, 213)$, $\mathcal{A}_n(132, 231)$, $\mathcal{A}_n(132, 321)$ and $\mathcal{A}_n(123, 321)$. Of these the first two turn out to be equinumerous and it is easy to see that $|\mathcal{A}_n(123, 321)| = 0$ if $n \geq 5$. A summary is given in Table 3.

Theorem 5.1. *$\mathcal{A}_n(132, 123)$ is counted by every second Fibonacci number, $F(2n-1)$, where $F(0) = 0$ and $F(1) = 1$.*

Proof. Let $a_n = |\mathcal{A}_n(132, 123)|$. Let M be an alternating sign matrix avoiding both 132 and 123. We know from the Structure Lemma that it can be decomposed into submatrices M_1 and M_2 in two ways. For M to be 123-avoiding, M_2 must not contain the pattern 12. Thus it must be a permutation matrix of the form $(k-1)(k-2)\cdots 1$, where k is the position of the 1 in the rightmost column of M. This is illustrated in Figure 6.

(1) For the first case from the Structure Lemma, when the 1 in the rightmost column of M is the only element of row k, there are a_{n-k} ASMs, for $2 \leq k \leq n-1$.
(2) In the second case, we know by the same reasoning as in earlier proofs that there are $a_{n-k+1} - a_{n-k}$ ASMs on this form, for $2 \leq k \leq n-1$.

Additionally, there are a_{n-1} 132-123-avoiding ASMs of order n when $k=1$ and a single ASM of order n when $k=n$.

In total we have $a_n = 1 + a_{n-1} + \sum_{k=2}^{n-1}(a_{n-k+1} - a_{n-k} + a_{n-k}) = 1 + a_{n-1} + \sum_{k=2}^{n-1} a_{n-k+1} = a_{n-1} + \sum_{k=1}^{n-1} a_k$, which is a well-known, and easy to deduce, formula for every second Fibonacci number. This implies that $a_n = F(2n-1)$ since the formula holds for $n=1$ and $n=2$. □

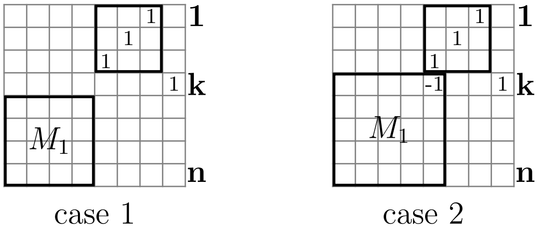


Figure 6. The two cases of $\mathcal{A}_n(132, 123)$.

The proof that $\mathcal{A}_n(132, 213)$ is also counted by $F(2n-1)$ can be constructed analoguosly, with M_2 a permutation matrix of the form $1 \cdot 2 \cdots (k-1)k$. This gives $|\mathcal{A}_n(132, 123)| = |\mathcal{A}_n(132, 213)|$, the only non-trivial identity in Table 3. The sets $\mathcal{A}_n(132, 231)$ and $\mathcal{A}_n(132, 321)$ are counted by $2^{n-3} \cdot 5$ and $(n-1)^2 + 1$ respectively. Detailed proofs for this are provided in [5].

We may now interpret our bijection ψ between 132-avoiding ASMs and Schröder paths to the case of avoiding yet another pattern.

Corollary 5.2. *Both of the following sets of restricted Schröder paths from* $(0,0)$ *to* $(n-1, n-1)$ *are enumerated by every second Fibonacci number* $F(2n-1)$.

(1) *Paths staying between main diagonal* $y = x$ *and off diagonal* $y = x - 1$.
(2) *Paths such that, whenever it leaves the main diagonal* $y = x$ *it has a number of steps east followed by the same number of steps north until it hits the main diagonal again.*

We leave to the reader to verify that these sets of paths are the image of $\mathcal{A}_n(132, 123)$ and $\mathcal{A}_n(132, 321)$ respectively. This enumerative result is also not difficult to obtain directly from the recursion for every second Fibonacci number.

$\tau_1 \backslash \tau_2$	123	132	213	231	312	321
123	$\lvert\mathcal{A}_n(123)\rvert$	$F(2n-1)$	$F(2n-1)$	$(n-1)^2+1$	$(n-1)^2+1$	$0, (n \geq 5)$
132	$F(2n-1)$	$\lvert\mathcal{A}_n(132)\rvert$	$F(2n-1)$	$2^{n-3} \cdot 5$	$2^{n-3} \cdot 5$	$(n-1)^2+1$
213	$F(2n-1)$	$F(2n-1)$	$\lvert\mathcal{A}_n(132)\rvert$	$2^{n-3} \cdot 5$	$2^{n-3} \cdot 5$	$(n-1)^2+1$
231	$(n-1)^2+1$	$2^{n-3} \cdot 5$	$2^{n-3} \cdot 5$	$\lvert\mathcal{A}_n(132)\rvert$	$F(2n-1)$	$F(2n-1)$
312	$(n-1)^2+1$	$2^{n-3} \cdot 5$	$2^{n-3} \cdot 5$	$F(2n-1)$	$\lvert\mathcal{A}_n(132)\rvert$	$F(2n-1)$
321	$0, (n \geq 5)$	$(n-1)^2+1$	$(n-1)^2+1$	$F(2n-1)$	$F(2n-1)$	$\lvert\mathcal{A}_n(123)\rvert$

Table 3. All values on alternating sign matrices of order $n \geq 3$ avoiding two patterns of length three, τ_1 and τ_2.

6. Open problems

There are a number of possible questions for further study. The most obvious is to find a formula for $|\mathcal{A}_n(123)|$.

A second question is to study what happens if we extend the definition of pattern avoidance in ASMs by letting the pattern τ also be an alternating sign matrix. It could be fruitful to study which alternating sign matrices of order n which avoid an alternating sign matrix of order $k \leq n$. For example, the alternating sign matrices which avoid

$$\begin{array}{|c|c|c|}\hline & 1 & \\ \hline 1 & -1 & 1 \\ \hline & 1 & \\ \hline\end{array}$$

are exactly the permutation matrices. It would also be possible to study patterns which are not square matrices.

A third possibility that might be within reach with the above Structure Lemma is to enumerate the ASMs that contain exactly r 132 patterns, for small values of r.

A fourth question is if one can say something about an asymptotic upper bound for the number of pattern-avoiding ASMs. This could possibly be similar to the exponential bound conjectured by Stanley and Wilf and proved by Marcus and Tardos [6].

References

[1] D.M. Bressoud, *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*, Cambridge University Press, 1999.

[2] E.S. Egge and T. Mansour, *Permutations which avoid 1243 and 2143, continued fractions, and Chebyshev polynomials*, Electronic Journal of Combinatorics **9** (2003), no. 2, R7.

[3] E.S. Egge, *Restricted signed permutations counted by the Schröder numbers*, Discrete Mathematics **306** (2006), no. 6, 552–563.

[4] Z. Füredi and P. Hajnal, *Davenport-Schinzel theory of matrices*, Discrete Math. **103** (1992), no. 3, 231–251.

[5] R. Johansson, *Pattern Avoidance in Alternating Sign Matrices*, Master's Thesis, Department of Mathematics, Linköpings Universitet, 2006. Available from `http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-7936`.

[6] A. Marcus and G. Tardos, *Excluded permutation matrices and the Stanley-Wilf conjecture*, J. Combin. Theory Ser. A **107** (2004), no. 1, 153–160.

[7] N. J. A Sloane, *The On-Line Encyclopedia of Integer Sequences* (November 2006). `http://www.research.att.com/~njas/sequences/`.

[8] R.P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge University Press, 1999.

[9] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electronic Journal of Combinatorics **3** (1996), no. 2.

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