COMBINATORICS OF FULTON'S ESSENTIAL SET

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Abstract. Fulton introduced the *essential set* of a permutation, together with a rank function. In this paper we study some combinatorial aspects of the essential set: We present an algorithm that retrieves a permutation from its ranked essential set. Thereby we can characterize the class of ranked sets that arise as ranked essential sets of permutations, answering a question of Fulton. Several classes of permutations are characterized in terms of their essential set. Results in higher dimensions are discussed.

1. Introduction

An n by n permutation matrix can be represented by an n by n array of squares with one dot in each row and column and all other squares empty. The diagram of a permutation matrix (defined in 1800 by Rothe) is obtained by shading every row from the dot and eastwards and shading every column from the dot and southwards. The *essential set* is the set of southeast corners of the connected components of the diagram. The essential set, together with a rank function, was introduced by Fulton [7] in a pioneering paper from 1992, in order to study the irreducible loci in spaces of matrices. Though in the present paper we are not going to much pursue this geometric issue but rather concentrate on the essential set *per se*, we need a short recollection of the background:

Fulton studies varieties given by ideals of minors, subject to certain rank conditions, in a generic matrix. One concern is, given a prescribed rank function $r(i,j)$, to determine if there exist matrices such that the rank of the upper left i by j submatrices is $r(i,j)$ for every position (i,j). He observes that if such matrices exist, then there is in particular some permutation matrix with this property, so it is enough to consider permutation matrices. More generally, he is interested in prescribing the ranks only for a few of the submatrices, such that all the other ranks follow from these. In other words, find a subset of the upper left submatrices of a permutation

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matrix such that the permutation is uniquely determined by their ranks. Fulton shows that the essential set of a permutation matrix is such a set, that is, the permutation is determined by its essential set and the ranks of the corresponding upper left submatrices. In Section 2 we present more detailed definitions and the relevant results of Fulton.

Our first result, in Section 3, is an elementary algorithm that retrieves a permutation from its ranked essential set. The algorithm is flexible, in that the input set may be bigger and include any other squares, as well as be smaller as long as it contains a certain "core" of the essential set. By this flexibility, the algorithm ought to be useful as a practical tool. In general, the core is much smaller than the essential set, so this is a significant strengthening.

Fulton used the essential sets mainly for vexillary (2143-avoiding) permutations. He gave a characterization of the ranked essential set for vexillary permutations, which may be expressed by three simple conditions, and he pointed out that it might be useful to have an analogous characterization of the ranked sets of squares that arise as ranked essential sets of all permutations. As an application of the algorithm we are able to state such a characterization, which keeps two of Fulton's three conditions for the vexillary case, and replaces the third one with a new and, alas, more complicated condition. We also remark on how the algorithm can be used directly to determine if a given ranked set contains the essential set of a permutation. This is done in Section 4.

In Section 6, we use a combinatorial technique to describe several classes of permutations in terms of their essential set. For example, the Baxter permutations are precisely those whose essential set has at most one square in each row and column. Another example is 321-avoiding vexillary permutations, which can be described as having the essential set entirely contained in one row or in one column. As a corollary we obtain a direct interpretation of the formula for the number of 321-avoiding vexillary permutations of Billey, Jockusch and Stanley [2].

Finally, in Section 7 we discuss how some of Fulton's results on minor ideals and the essential set can be generalized to higher dimensions, that is, if we work not with matrices but higher dimensional arrays of variable entries. It turns out that with natural definitions of permutations, determinants and essential set, it is true also in this general situation that a permutation is determined by its essential set, and that the minor ideal defined by the permutation is generated by the minors stemming from the essential set.

There are several interesting problems left open, and we collect some of them at the end of the paper. In another paper [5], the same authors have studied various enumerative aspects of the essential set.

We thank Dan Laksov for drawing our attention to this problem. We are also grateful to a referee for several helpful suggestions.

2. Preliminaries

The combinatorial object that we are studying is the essential set of a permutation, together with its rank function. They are defined as follows. First, let every permutation $w \in S_n$ be represented by its dotted permutation matrix, regarded as an n by n array of squares in the plane, where square $(i, w(i))$ has a dot for all $i \in [1, n]$, and all other squares are white, so there is exactly one dot in each row and column. We will number the rows from north to south and the columns from west to east.

Shading the squares in each row from the dot and eastwards, and shading the squares in each column from the dot and southwards, we get the diagram of the permutation as the white (nonshaded) squares. We call a white square a *white corner* if it has no white neighbor either to the east or to the south. In other words, the white corners are the southeast corners of the components of the diagram. The *essential set* $\mathcal{E}(w)$ of a permutation w is defined to be the set of white corners of the diagram of w. Equivalently,

$$\mathcal{E}(w) = \{(i, j) \in [1, n-1] \times [1, n-1] : w(i) > j, w^{-1}(j) > i, w(i+1) \leq j, w^{-1}(j+1) \leq i\}.$$

For every square (i, j) in the dotted matrix, its *rank* is defined by

$$r_w(i, j) = |\{q \leq i : w(q) \leq j\}| = |\{ \text{ dots northwest of } (i, j)\}|.$$

The name "rank" stems from the fact that $r_w(i, j)$ is equal to the matrix rank of the i by j upper left submatrix of the ordinary permutation matrix of w, where the dots are replaced by ones and the blank squares by zeros.

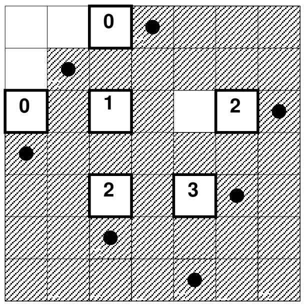

Figure 1. Diagram and ranked essential set of the permutation 4271635.

We will discuss two results of Fulton. First, the property that makes the essential set such a useful subset of the n^2 squares:

Lemma 2.1 (Fulton [7], Lemma 3.10). *(a) For any w in S_n and a generic n by n matrix $A = (x_{ij})$, the ideal in the polynomial ring of the variables $x_{11}, \ldots, x_{nn}$ generated by all minors of size $r_w(i, j) + 1$ taken from the upper left i by j submatrix of A, for all $i, j \in [1, n]$, is generated by these same minors using only those (i, j) that are in $\mathcal{E}(w)$.*

(b) A permutation w in S_n is determined by the restriction of its rank function r_w to $\mathcal{E}(w)$.

Part (a) has a surprisingly simple proof, just by checking in four short steps that nothing outside the essential set adds anything new to the ideal. Part (b) follows from (a), since knowing the ideal implies knowing $r_w(i,j)$ for every position (i,j), and hence knowing the permutation w. Observe the following subtle point: there may be other permutations than w that have coinciding rank functions in $\mathcal{E}(w)$, but no such permutation will have the same essential set as w; for example is $321 \in S_3$ coinciding with the rank of the white corner of $312 \in S_3$. In the next section we will provide an alternative proof of (b) by an algorithm for retrieving the permutation from its ranked essential set. In the final section we will give higher dimensional generalizations of both parts (a), in Proposition 7.1, and (b), in Proposition 7.2.

Second, we present the characterization of the ranked essential sets arising from vexillary permutations. The vexillary permutations are an important class of permutations in the theory of Schubert polynomials and related areas (see Macdonald's book [10]). We should mention that we have reformulated the original conditions of Fulton to suit our purposes.

Proposition 2.2 (Fulton [7], Proposition 9.6). *Let $E \subseteq [1,n] \times [1,n]$ be a set of squares with an integer valued function $r(i,j)$ defined on E. Then E is the essential set with rank function r of some vexillary n by n permutation matrix if and only if:*

(C1) *For each $(i,j) \in E$ we have*
  (a) $\min\{i,j\} > r(i,j) \geq 0$, *and*
  (b) $r(i,j) + n \geq i + j$.

(C2) *For every distinct pair $(i,j),(i',j') \in E$ such that $i \geq i', j \leq j'$ and $E \cap [i',i] \times [j,j'] = \{(i,j),(i',j')\}$ we have*

$$i - i' > r(i,j) - r(i',j') > j - j'$$

(V) *E does not contain any pair $(i_1,j_1),(i_2,j_2)$ where $i_1 < i_2$ and $j_1 < j_2$.*

To be precise, Fulton gave the condition

$$n \geq \max\{i : (i,j) \in \mathcal{E}(w)\} + \max\{j : (i,j) \in \mathcal{E}(w)\} \tag{1}$$

instead of (C1b) for sufficiency. For necessity he just removed (C1b). As stated here, Proposition 2.2 follows from the Chess Theorem presented in Section 4.

Example The permutation $5736241 \in S_7$ is a vexillary permutation, see Figure 2. Note that the essential set of 5736241 does not satisfy (1) above but it does satisfy (C1b).

In Section 4 we will give a characterization of ranked essential sets arising from arbitrary permutations by replacing condition (V) above with a new condition. In Section 6 we will describe several classes of permutations in terms of their essential set (without considering ranks) in the spirit of condition (V).

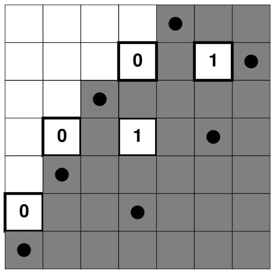

FIGURE 2. The permutation $5736241 \in S_7$ has its essential set in a string from southwest to northeast and is therefore vexillary by Proposition 2.2.

3. The retrieval algorithm

We shall here present and analyze an elementary algorithm for explicitly determining a permutation from its ranked essential set.

It will be convenient to work with concepts from partially ordered sets, so let $\mathbb{P} = \{1, 2, 3, \dots\}$ be the set of positive integers, and let $\mathbb{P}^2$ be partially ordered by $(m_1, m_2) \le (m'_1, m'_2)$ if and only if $m_1 \le m'_1$ and $m_2 \le m'_2$. To stress the analogy with permutation matrices, we shall refer to the elements of $\mathbb{P}^2$ as *squares*.

A finite set B of squares in $\mathbb{P}^2$ is called a *shape*. The *slice* $S_{i,m}$ of B is the set of all squares in B such that the ith coordinate is m. For example, if B is a matrix, then $S_{1,m}$ is the mth row and $S_{2,m}$ is the mth column.

A subset w of a shape B is a *proper dotting* if it contains at most one square of every slice of B. The *rank* of a square c in B with a proper dotting w is

$$r_w(c) \stackrel{\mathrm{def}}{=} |\{c' \in w : c' \le c\}|.$$

Retrieval Algorithm.
INPUT: A shape B with a subset of its squares labeled with non-negative integers.
OUTPUT: A proper dotting w of B.

Let B_0 be a copy of B with labels. We shall obtain a proper dotting w of B while constructing a finite sequence of labeled shapes $B_0, B_1, B_2, \dots$, such that all labels of every B_k will agree with the ranks given by the restriction of the final dotting w to the subshape B_k.

Odd step $2i-1$: The labeled shape B_{2i-1} is obtained from B_{2i-2} by removing every square c such that $c \le c'$ where $c' \in B_{2i-2}$ is a square labeled zero. In these squares, there cannot be any dot in w.

Even step $2i$: After the previous step, B_{2i-1} has no square labeled zero. Put a dot in w in every minimal square (in the partial ordering) of B_{2i-1}. To obtain B_{2i}, we now, for every such minimal square d, first decrease the label by one for every labeled square $c \ge d$ and then remove from B_{2i-1} both slices containing d. This makes sure that the dotting will be proper, and the labels will take into account that some dots have been removed. *If any label becomes negative during the relabeling, then the algorithm halts with no output.* □

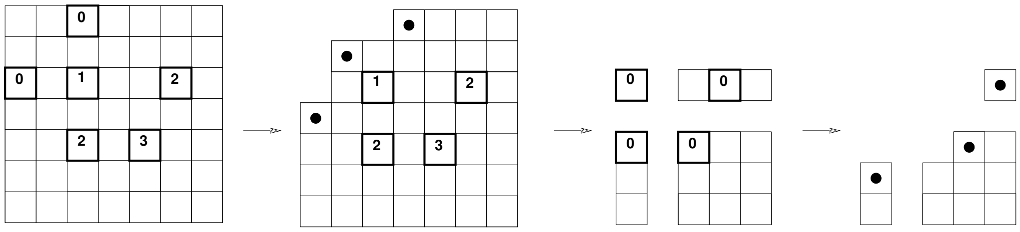

FIGURE 3. A run of the algorithm on the essential set of the permutation in Figure 1. B_0, B_1, B_2 and B_3 depicted, with the revealed dots indicated. B_4 will then be a single square that must have a dot.

We will show that this algorithm is very flexible, and is connected with another construction in algebraic combinatorics, namely Viennot's shadowing procedure [12] (which has an interesting "matrix ball" generalization by Fulton in [8]). Viennot's procedure gives a geometric description of the Robinson–Schensted correspondence and works as follows. Let w be the proper set of dots. Do the shading as we have done before, shading every row and column from the dot and onwards. Then, the first *Viennot layer* V_1 of dots consists of the dots that are not separated from the origin by any shaded slices. If this first layer of dots, and all their slices, are removed, then we see a second Viennot layer V_2, etc. For convenience, extend the poset $\mathbb{P}^2$ with a new minimal element $\hat{0}$ and introduce an artificial Viennot layer $V_0 = \{\hat{0}\}$. A formal, recursive, definition of the Viennot layers is

$$V_i = \{d \in w \setminus \bigcup_{j<i} V_i \mid d > d' \in w \Rightarrow d' \in \bigcup_{j<i} V_j\}, \qquad \text{for } i = 1, 2, 3, \ldots$$

In Viennot's context the shape is a permutation matrix and the connection with the Robinson–Schensted correspondence is that the row number (column number resp.) for the westmost (northmost) dot in Viennot layer number i gives the value in position i in the first row of the first (second) tableaux in the R–S correspondence. However, for our purposes we will allow any proper dotting of any arbitrary shape, and we are only interested in which dots are in each Viennot layer. We shall see that the algorithm reveals the dots one Viennot layer at a time.

Next, we define the core of a given properly dotted shape B. The ith *white filling*, W_i, is the union of connected components of white squares between the two Viennot layers V_{i-1} and V_i:

$$W_i = \{\text{white squares } c \mid \exists d \in V_{i-1} : d < c, \nexists d' \in V_i : d' < c\}, \qquad \text{for } i = 1, 2, 3, \ldots$$

The *core* of the dotting of B is

$$\bigcup_i \{c \in W_i \mid c' > c \Rightarrow c' \notin W_i\},$$

the set of maximal squares of the white fillings.

Lemma 3.1. *For a permutation matrix, the core is contained in the essential set.*

Proof A maximal element of a white filling is necessarily a maximal element of its connected component of white squares, and hence a member of the essential set. □

Example The permutation $4271635 \in S_7$ shown in Figure 1 has core $(3,1),(1,3)$, $(3,6),(5,5)$. We see also in Figure 3 that the core is enough to run the algorithm. For the permutation $5736241 \in S_7$ in Figure 2 the core is equal to the essential set. One can prove that this is true for all vexillary permutations.

Theorem 3.2. *Suppose the algorithm produces a dotting w for a given input shape B with labeled subset E, such that the ranks of squares in E agree with the labels. Then w is the unique such proper dotting with its core contained in E. If the algorithm fails to produce a dotting or produces a dotting whose ranks does not agree with the labels in E, then there is no such dotting.*

Proof Suppose there is some proper dotting w such that the ranks of squares in E agree with the labels, and such that the core of w is contained in E. It is then clear by induction that the algorithm will find this w: The odd step $2i-1$ will remove W_i, the ith white filling of w, its maximal elements being labeled zero at this point. The even step $2i$ will reveal V_i, the ith Viennot layer of w, and remove it and its shaded slices, and adjust the labels so that they agree with the current ranks as determined by the remaining dots. □

Thanks to this theorem and the lemma above, we have immediately the following strengthening of Fulton's result that a permutation is determined by its ranked essential set.

Corollary 3.3. *A permutation matrix is determined by the restriction of its rank function to its core (and hence in particular to its essential set).*

This is a significant strengthening, in the sense that the core in general is much smaller than the essential set. In another article [5], the authors have shown that the average size of the essential set is $n^2/36$. The core, though, is of size less than n.

Lemma 3.4. *A permutation matrix has at most one member of the core for every pair of neighboring dots in a Viennot layer. Therefore, the size of the core is at most n minus the number of (nonempty) Viennot layers.*

Proof Let $(q_1, p_1), (q_2, p_2), \ldots, (q_k, p_k)$ be the dots of ith Viennot layer V_i in order of increasing q:s and decreasing p:s. If you draw a picture, it is obvious that every square in the ith white filling W_i must be, in poset sense, less than or equal to at least one of the (not necessarily white) squares $(q_2-1, p_1-1), (q_3-1, p_2-1), \ldots, (q_k-1, p_{k-1}-1)$. By definition of white filling, there cannot be any square in W_i in a row or column where there is a dot of an earlier Viennot layer V_l, $l < j$, since any white square before the dot must belong to some earlier white filling, and all squares after the dot are shaded. For every $j = 1, \ldots, k-1$, let q'_j and p'_j be largest possible in the intervals $q_j \leq q'_j < q_{j+1}$ and $p_{j+1} \leq p'_j < p_j$ such that no dot of any earlier Viennot layer lies in row q'_j or column p'_j. The square (q'_j, p'_j) must belong to some white filling W_l for $l \leq i$, and they have been chosen maximal. The core members of the W_i are its maximal elements, hence precisely those among the $k-1$ squares $(q'_1, p'_1), \ldots, (q'_{k-1}, p'_{k-1})$ that belong to W_i. Summation over all Viennot layers proves the lemma. □

4. The Chess Theorem

The origin of this article is a question posed twice by Fulton in [7] with a hope that the answer would help to extend his determinantal formula for degeneracy loci from the vexillary case to general permutations: Is there a characterization of which ranked essential sets arise from arbitrary permutations, in analogy with the characterization of ranked essential sets in the case of vexillary permutations?

By using the retrieval algorithm one can prove such a result.

4.1. Fulton's question and the Chess Theorem. Fulton characterized ranked essential sets of vexillary permutations (Proposition 2.2) by the conditions (C1), (C2), and (V). By replacing (V) with a new condition (C3) below, we obtain a characterization of ranked essential sets of arbitrary permutations. We have nicknamed our result the Chess Theorem, for CHaracterization of ranked ESSential sets, inspired by the chessboard-like setting with white squares and dark squares.

Theorem 4.1 (Chess Theorem). *Let $E \subseteq [1, n] \times [1, n]$ be a set of squares with integer labels $r(i, j)$. E is the essential set with rank function r of an n by n permutation matrix if and only if:*

(C1) *For each $(i, j) \in E$ we have*
  (a) $\min\{i, j\} > r(i, j) \geq 0$, *and*
  (b) $r(i, j) + n \geq i + j$.

(C2) *For every distinct pair $(i, j), (i', j') \in E$ such that $i \geq i', j \leq j'$ and $E \cap [i', i] \times [j, j'] = \{(i, j), (i', j')\}$ we have*

$$i - i' > r(i, j) - r(i', j') > j - j'$$

(C3) *For every pair* $(i,j),(i',j') \in E$ *such that* $i < i', j < j'$ *and* $E \cap [i+1,i'] \times [j+1,j'] = \{(i',j')\}$, *and* $r(i'',j'')$, $r(i''',j''')$ *as defined below, we have*

$$r(i',j') \geq i - i'' + j - j''' + r(i'',j'') + r(i''',j''') - r(i,j).$$

In condition (C3), let (i'',j'') be the square of E with the largest i'' satisfying $i'' \leq i$, $j'' \geq j'$ and $E \cap [i'',i] \times [j',j''] = \{(i'',j'')\}$ (if no such square exists, set $r(i'',j'') = 0$); symmetrically, let (i''',j''') be the square of E with the largest j''' satisfying $j''' \leq j$, $i''' \geq i'$ and $E \cap [i',i'''] \times [j''',j] = \{(i''',j''')\}$ (if no such square exists, set $r(i''',j''') = 0$).

The proof is given in Section 5.

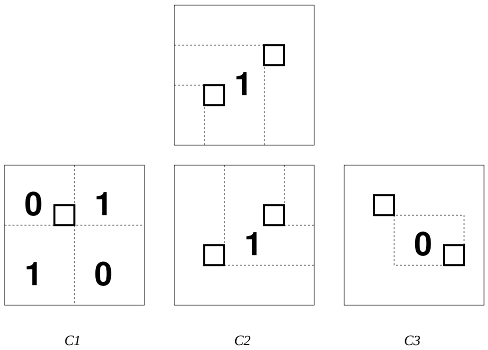

FIGURE 4. The conditions in the "Chess Theorem" are equivalent to lower bounds on how many dots there must be in certain areas. The numbers in the figure are these lower bounds.

Fulton also considers the more general situation when the dotting does not have full rank and the matrix might be a rectangle. The full proof of Theorem 4.1 also shows that this case is covered if we replace (C1b) by condition (R) below.

Corollary 4.2. *Given* $k \leq m,n$, *let* $E \subseteq [1,m] \times [1,n]$ *be a set of squares with integer labels* $r(i,j)$. E *is the essential set with rank function* r *of a properly dotted* $m \times n$ *rectangle with at least* k *dots if and only if conditions* (C1a), (C2) *and* (C3) *of Theorem* 4.1, *as well as* (R) *below, all are fulfilled.*

(R) *For every square* $(i,j) \in E$ *we have* $n + m - k + r(i,j) \geq i + j$.

□

Note that the inequalities in conditions (C1a), (C2) and (C3) are not dependent of k. Hence the number of dots in the $m \times n$ rectangle is the largest number $k \leq m,n$ satisfying (R).

4.2. A related question. The following related question was suggested by a referee: Given a function r on an arbitrary subset E of $[1,m] \times [1,n]$, when does this r determine an irreducible locus in the corresponding space of matrices? Equivalently, when does there exist a subset $E' \subseteq E$ such that restriction of r to E' is the ranked essential set of a permutation, and the other rank conditions on E are implied by those on E'?

It seems to be a difficult task to find a characterization of such sets, but the retrieval algorithm supplies an easy way to answer the question for any given specific example. Take the ranked set E as input to the retrieval algorithm. Thanks to Theorem 3.2 and Lemma 3.1, we know that if E indeed contains the ranked essential set of some permutation w, then the algorithm will produce w as output. So, testing a ranked set E is done by simply running the algorithm and then checking if the essential set of the dotting obtained is a subset of E and if the rank function is correct for all squares in E.

5. Proof of the Chess Theorem

Our goal in this section is to prove the characterization of what ranked essential sets that can arise from arbitrary permutations. To this end, we need to consider also three alternative rank functions *defined for any shape* in $\mathbb{P}^2$.

Definition In an arbitrary shape in $\mathbb{P}^2$ there may be empty rows and columns. Let $h(i)$ be the number of nonempty rows in the shape with row number less than or equal to i; analogously, let $v(j)$ be the number of nonempty columns in the shape with column number less than or equal to j. Let n denote the number of dots in the shape. Define three new rank functions by

$$r^{\mathrm{ne}}(i,j) \stackrel{\mathrm{def}}{=} h(i) - r(i,j),$$

$$r^{\mathrm{sw}}(i,j) \stackrel{\mathrm{def}}{=} v(j) - r(i,j),$$

$$r^{\mathrm{se}}(i,j) \stackrel{\mathrm{def}}{=} n - h(i) - v(j) + r(i,j).$$

It obviously suffices to know $r(i,j)$ but it is convenient to work with all four rank functions. The names of these functions has the following background. Say that (i',j') is *northeast* of (i,j) if $i' \leq i, j' > j$; *southwest* of (i,j) if $i' > i, j' \leq j$; and *southeast* of (i,j) if $i' > i, j' > j$. In the special case when the shape is a permutation matrix we have $h(i) = i$ and $v(j) = j$, so in this case we have the following interpretation of the rank functions: $r^{\mathrm{ne}}(i,j)$, $r^{\mathrm{sw}}(i,j)$ and $r^{\mathrm{ne}}(i,j)$ is the number of dots northeast, southwest, and southeast, respectively, of (i,j). See Figure 5.

For $i \leq i'$, $j \leq j'$ define $[i,i'] \times [j,j']$ to mean the rectangle with corners (i,j), (i,j'), (i',j) and (i',j'). □

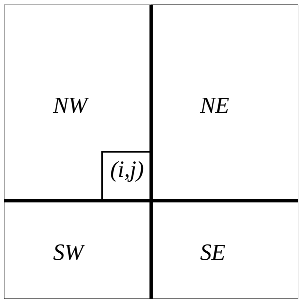

FIGURE 5. How the directions northwest, northeast, southwest and southeast are defined.

In the essential set of a permutation the squares (i'', j'') and (i''', j''') in condition $C3$ of the Chess Theorem have the following important property. See Figure 6.

Lemma 5.1. *Suppose we are given a permutation $w \in S_n$ and $(i, j), (i', j') \in \mathcal{E}(w)$ such that $i < i', j < j'$.*

(a) *Let $(i'', j'') \in \mathcal{E}(w)$ be the square of $\mathcal{E}(w)$ with the largest i'' satisfying $i'' \le i$, $j'' \ge j'$ and $\mathcal{E}(w) \cap [i'', i] \times [j', j''] = \{(i'', j'')\}$. Then all dots in $[1, i] \times [j', n]$ lie northeast of (i'', j''). If no such square exists, then there are no dots in $[1, i] \times [j', n]$.*

(b) *Symmetrically, let (i''', j''') be the square of $\mathcal{E}(w)$ with the largest j''' satisfying $j''' \le j$, $i''' \ge i'$ and $\mathcal{E}(w) \cap [i', i'''] \times [j''', j] = \{(i''', j''')\}$. Then all dots in $[i', n] \times [1, j]$ lie southwest of (i''', j'''). If no such square exist, then there are no dots in $[i', n] \times [1, j]$.*

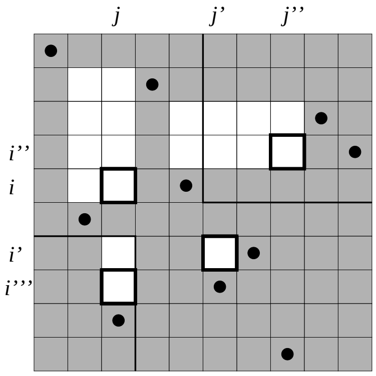

FIGURE 6. Example of squares (i'', j'') and (i''', j''') in Lemma 5.1. Note that $j = j'''$ in this example.

PROOF To prove (a) let (i_1, j_1) be the dot furthest south in $[1, i] \times [j' + 1, n]$, that is with maximal row number. This implies that (i_1, j') is a white square. Hence there is a white corner in that white component with the described properties. If there are no white corners in $[1,i]\times[j',n]$ there cannot be any dots there either. (b) follows by symmetry. $\square$

PROOF OF THE "ONLY IF" PART OF THE CHESS THEOREM We will begin by proving necessity, which is the easy direction. Remember that $C1$ and $C2$ are trivially equivalent to lower bounds on the number of dots in certain areas of the permutation matrix. When the lower bound is zero there is nothing to prove. If (i,j) is a white corner, the square $(i,j+1)$ must be shaded and hence there must be a dot in column $j+1$ on a row with row number less than or equal to i. A symmetric argument shows that there must be a dot in row $i+1$ in a column with column number less than or equal to j. This proves the necessity of the strict inequalities in $C1$ and $C2$. The left hand side of $C3$ counts the number of dots in $[1,i']\times[1,j']$. With a clear understanding of Lemma 5.1 one can deduce that

$r(i'',j'')=\#\{$ dots in $[1,i'']\times[1,j']\}$, since there are no dots in $[1,i'']\times[j'+1,j'']$;

$r(i''',j''')=\#\{$ dots in $[1,i']\times[1,j''']\}$, since there are no dots in $[i'+1,i''']\times[1,j''']$;

$i-i''=\#\{$ dots in $[i''+1,i]\times[1,j']\}$, since there are no dots in $[i''+1,i]\times[j'+1,n]$;

$j-j'''=\#\{$ dots in $[1,i']\times[j'''+1,j]\}$ since there are no dots in $[i'+1,n]\times[j'''+1,j]$;

and

$r(i,j)=\#\{$ dots in $[1,i]\times[1,j]\}$.

See Figure 6. Hence, the right hand side of $C3$ counts the number of dots in $[1,i']\times[1,j']\backslash[i+1,i']\times[j+1,j']$ Thus $C3$ simply says that there is a nonnegative number of dots in $[i+1,i']\times[j+1,j']$. $\square$

To prove that the conditions in the theorem are sufficient we will use the retrieval algorithm. To this end we need to reformulate the conditions so they have meaning during every step of the algorithm. Assume $E\subseteq[1,n]\times[1,n]$ is a set of squares with rank function $r(i,j)$, that we want to test whether it is the essential set for a permutation or not. We add the squares $(0,n)$ and $(n,0)$ to E, both with rank zero. Their meaning will become clear later on. We can now reformulate the Chess Theorem in the following slightly weaker form, using the alternative rank functions r^{ne}, r^{sw} and r^{se}. For an intuitive picture, see Figure 4.

$E\backslash\{(0,n),(n,0)\}$ is the essential set of an $n\times n$ permutation matrix if and only if:

C1'. *For each $(i,j)\in E$ we have*

(a) $r(i,j)\geq 0$ *and*

(b) $r^{\mathrm{se}}(i,j)\geq 0$.

C2'. *For every distinct pair $(i,j),(i',j')\in E$ such that $i\geq i', j\leq j'$ and $E\cap[i',i]\times[j,j']=\{(i,j),(i',j')\}$ we have*

(a) $r^{\mathrm{ne}}(i,j)-r^{\mathrm{ne}}(i',j')\geq\begin{cases}1 & \text{if } i>i'\\ 0 & \text{if } i=i'\end{cases}$ *and*

(b) $r^{sw}(i',j') - r^{sw}(i,j) \geq \begin{cases} 1 & \text{if } j' > j \\ 0 & \text{if } j' = j \end{cases}$.

C3. *For every pair* $(i,j),(i',j') \in E$ *such that* $i < i', j < j'$ *and* $E \cap [i+1,i'] \times [j+1,j'] = \{(i',j')\}$ *let* (i'',j'') *and* (i''',j''') *be as in Lemma* 5.1 *(a) and (b). Then*

$$r(i',j') \geq r(i,j) + r^{\mathrm{ne}}(i,j) + r^{\mathrm{sw}}(i,j) - r^{\mathrm{ne}}(i'',j'') - r^{\mathrm{sw}}(i''',j''').$$

Note that the first inequality $\min\{i,j\} > r(i,j)$ in $C1(a)$ has been encoded into $C2'$ by adding the extra squares $(0,n),(n,0)$ to E. Condition $C2'$ is a slightly weaker statement than $C2$ when $i = i'$ or $j = j'$. However, $C1', C2'$ is implied by $C1, C2$, so it suffices to show sufficiency with $C1, C2$ replaced by $C1', C2'$.

The proof of sufficiency will be done by showing that the conditions is enough to make it possible to run the algorithm and obtain a number of dots in the matrix, which will be shown to constitute a permutation matrix with E as essential set. When we run the algorithm starting with the set E, we only know that E fulfills conditions $C1', C2', C3$ above. To show that we can run the algorithm we need to keep a little more information than we did in section 3. We will not only keep the restriction of E to the shape in the current step and the adjusted rank function for those squares. We will also keep some of the squares in E lying outside the shape and with rank zero. The rules for these so called *false squares* are described below, they play the role of the core here. As described previously we add two false squares $\{(0,n),(n,0)\}$ to E before the algorithm starts. The set E will change during the algorithm just like a variable in a computer program.

The algorithm

So, at every step of the algorithm we have: a shape, which consists of the squares in $[1,n] \times [1,n]$ which have not been deleted in the previous steps; a set E of squares, some of which are so called false squares, i.e. they do not belong to the shape; and a rank function r on the squares in E, where the rank function is zero if the square is false.

In an *odd step* we find squares in $(i,j) \in E$ with label zero (if such squares exist) and delete every square in the shape northwest of that square from the shape. For every such square (i,j) we also remove the squares in $E \cap [1,i] \times [1,j]$ from E including false squares. But we add back into E as false squares, the squares maximal (in the poset sense) among those removed.

In the *even step* of the algorithm we reveal a dot in every minimal square (in the poset sense of section 3) of the shape. For each one of these revealed dots, starting with the northeast most one and proceeding in order, we do the following: Remove from the shape the column and row of the dot. Decrease by one the rank of each square in E southeast of the dot. Also decrease r^{ne} and r^{sw} for squares southwest and northeast respectively of the revealed dot. (Observe that the relationships between the different rank functions are invariant: $r^{\mathrm{ne}}(i,j) + r(i,j) = h(i)$ and $r^{\mathrm{sw}}(i,j) + r(i,j) = v(j)$, since $h(i)$ and $v(j)$ also change.) If there is a square of E to the west of the dot in the row or to the north of the dot in the column we will remove that square from E. But if there are squares of E both in the same row and in the same column we let the square in the same row still be in E; note that here it is important that we proceed in the order specified above.

Properties of false squares

First we note that by the definition of the odd step there are never two false squares in the same row or column. Remember that the square of a revealed dot is minimal in the shape. The location of minimal squares relative false squares is important, as discussed in the following lemma. Say that a square (i,j) is *between* two other squares (i_1,j_1) and (i_2,j_2) if $i_1 \le i \le i_2$ and $j_1 \ge j \ge j_2$. Say that two false squares are *adjacent* if there is no other false square between them, and similarly two minimal squares are adjacent if there is no other minimal square between them.

Lemma 5.2. *At every step of the algorithm, there is exactly one minimal square in the shape between two adjacent false squares (and hence also exactly one false square between two adjacent minimal squares). Moreover, between a minimal square and the false square closest to its left, all columns have been removed from the shape; dually, between a minimal square and the false square closest to its right, all rows have been removed from the shape. See Figure 7.*

Define the *left false square* of a minimal square to be the false square closest to its left, as mentioned in the lemma. Analogously for *right false square*.

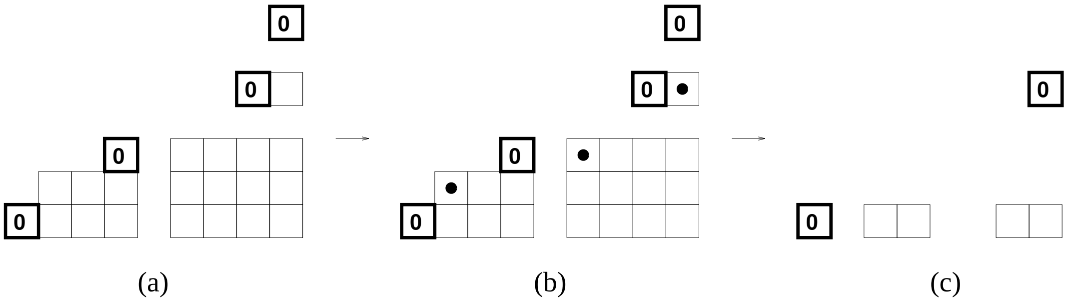

FIGURE 7. Revealed dots and false squares: (a) before, (b) during and (c) after an even step.

REMARK Another way to define the algorithm would be to keep the shape as one connected component by after each even step move all components as far east as possible (but not beyond the n:the column) and then move all components as far south as possible (but not beyond the n:th row). The shape would then always have a straight east edge and a straight south edge and a staircase form from southwest to northeast. This way to view the algorithm interprets the false squares and minimal squares as inner and outer corners. This is intuitively appealing and makes it easier to understand Lemma 5.2. Unfortunately it would cause a book keeping problem of keeping track of the original positions of the squares. □

PROOF OF LEMMA 5.2 The statement is obviously true at the start, when we have two false squares, $(0, n)$ and $(n, 0)$, and one minimal square, $(1, 1)$. The property is clearly invariant under odd steps. In verifying the even step, there are two things to check: that there exists a minimal square between any two false squares, and that there exists a false square between any two minimal squares. See Figure 7.

Consider two false squares, (i, j) and (i', j') not removed during the even step and adjacent after the step. Say that $i' > i$ and $j' < j$. We want to show that there is a minimal square between them after the step. It suffices to show that there exists a square in the shape in $[i + 1, i'] \times [j' + 1, j]$. The number of dots found in minimal squares and removed in $[i, i'] \times [j', j]$ is one more than the number of false squares removed. Every dot cannot have been in the same row as its left square since (i', j') was not removed. Let (i_1, j_1) be the false square not in the same row as any of the dots found in the step. It is easy to see that the square (i_1, j) must still be in the shape.

Consider two minimal squares, (i, j) and (i', j'), after the even step. Say that $i' > i$ and $j' < j$. Since (i, j) was not a minimal square before the even step, there must have used to be a minimal square northwest of (i, j) in the previous step, and this square cannot have been in the same column or same row. Similarly, there must have been a minimal square in $[1, i' - 1] \times [1, j' - 1]$ in the previous step. First we want to rule out the possibility that these minimal squares are positioned in $[1, i - 1] \times [1, j' - 1]$. If this was the case then (i, j') would have been in the shape and could not have been removed by a revealed dot during the even step. This contradicts the fact that (i, j) and (i', j') are minimal. Now consider the southmost of the in the previous step minimal squares northwest of (i, j) and the eastmost of the in the previous step minimal squares northeast of (i', j'); now we claim that in the previous step all the false squares between these minimal squares was totally in $[i, i' - 1] \times [j', j - 1]$. This would guarantee that there is a false square between (i, j) and (i', j') after the step since the even step never removes more false squares than minimal squares. To prove the claim we denote with (i_1, j_1) the left false square of the southmost minimal square in $[1, i-1] \times [j', j-1]$ before this step. Let (i_2, j_2) be the next minimal square southwest of (i_1, j_1). By assumption $i_2 \geq i$, and since (i, j) is still in the shape we must have $i_2 > i$. From the induction hypothesis we know that all rows between i_1 and i_2 were removed from the shape before the even step. We get $i_1 \geq i$. Dually the right false square of the westmost minimal square in $[i, i'] \times [1, j' - 1]$ was in a column east of $j' - 1$.

Finally, if between a minimal square (i_1, j_1) and its left false square (i_2, j_2) there were a column j that had not been removed, $j_2 < j < j_1$, then the square (i_2, j) would be in the shape and hence another minimal square between (i_1, j_1) and (i_2, j_2), a contradiction. □

We also need the following fact about the "staircase" of false squares and minimal squares:

Lemma 5.3. *At every step of the algorithm, there are no columns in the shape east of the false square in the northeast end of the staircase; similarly, there are no rows in the shape south of the false square in the southwest end of the staircase.*

Proof It is easy to see that this property is true at the start and is not ruined in an odd step. Assume that in an even step the false square (i_1, j_1) in the southwest end is removed by a dot found in the minimal square (i_1, j_1'). If the right false square (i_2, j_2) of (i_1, j_1') is not removed in this even step, then we are safe since by Lemma 5.2 all the rows from i_2 to i_1 are not in the shape. If it is removed by a dot (i_2, j_2') found in a minimal square then we just repeat this argument a finite number of times until we find a false square that is not removed (and such a square exists always). The situation is not entirely symmetric, but the same argument works for the northeast end as well. □

Note also that if $i < i'$ and $j < j'$ are such that $(i, j) \in E$ and (i', j') is a square in the shape, then (i', j) is also in the shape, so (i', j') is not minimal. Hence our conventions prevent a dot from being revealed southeast of an existing square in E.

An important fact that we will use frequently is the following lemma.

Lemma 5.4. *(a) If, at an even step of the algorithm, we have $r^{\mathrm{ne}}(i, j) = r^{\mathrm{ne}}(i', j')$ for $(i, j), (i', j') \in E$, $i = i'$ and $j' > j$, then the conditions prevent the algorithm from finding a dot in $[1, i] \times [j, j']$.*

(b) Similarly, if $r^{\mathrm{sw}}(i', j') = r^{\mathrm{sw}}(i, j)$ for $(i, j), (i', j') \in E$, $i > i'$ and $j' = j$, then the conditions prevent the algorithm from finding a dot in $[i', i] \times [1, j]$.

Proof To prove (a), assume that $i = i'$, $r^{\mathrm{ne}}(i, j) = r^{\mathrm{ne}}(i', j')$ and a dot is revealed in $[1, i] \times [j, j']$. Let (i_1, j_1) be the left false square to the dot found. If (i_1, j_1) is (i, j), then $r(i, j) = 0$ which implies $r(i', j') = 0$, a contradiction, since then the square where the dot was found is not in the shape. Hence $i_1 < i$, so we can let $(i'', j''), (i''', j''') \in E$ be as in $C3$ corresponding to (i', j') and (i_1, j_1). Note that $r^{sw}(i''', j''') \geq r^{sw}(i, j) = v(j) - r(i, j)$. We get

$$r(i', j') \geq r(i_1, j_1) + r^{sw}(i_1, j_1) + r^{\mathrm{ne}}(i_1, j_1) - r^{\mathrm{ne}}(i'', j'') - r^{sw}(i''', j''') \geq$$

$$\geq v(j_1) + 1 - (v(j) - r(i, j)) \geq 1 + r(i, j),$$

contradicting the assumptions.

A symmetric argument gives (b). $\square$

PROOF OF THE "IF" PART OF THE CHESS THEOREM First we need to show that the conditions let us perform a step and that the conditions are still valid after the step. The even and odd steps in the retrieval algorithm have to be treated separately; note that it is trivially possible to perform an odd step of the algorithm.

In every even step at least one dot is revealed and the corresponding row and column deleted from the shape, so after a finite number of steps the algorithm will have reached an empty shape and end. The dots that have been revealed make up a proper dotting of the matrix. We have to prove that the essential set of this dotting is the same as the set of squares E that we started with and that the rank function coincide. Finally we have to show that the dotting obtained has full rank so that indeed we have obtained a permutation matrix.

Hence, there are five things to be verified, each in one subsection:

- First, that conditions $C1', C2', C3$ are still valid after an odd step of the algorithm.
- Second, that conditions $C1', C2', C3$ allow us to run an even step of the algorithm.
- Third, that conditions $C1', C2', C3$ are still valid after an even step of the algorithm.
- Fourth, that the dotting obtained by the algorithm really has E as essential set.
- Fifth, that the dotting obtained by the algorithm has full rank.

5.1 Conditions $C1', C2', C3$ are still valid after an odd step of the algorithm.

First we consider a step of the odd type. The only thing that could possibly go wrong in this step is when $(i,j) \in E$ has gotten label zero and removes a square $(i_1, j_1) < (i_1, j_1)$, that was necessary for condition $C3$ as (i'', j'') or (i''', j'''). This could only be the case if $i = i_1$ or $j = j_1$. If $i = i_1$, then $r^{\mathrm{ne}}(i,j) = r^{\mathrm{ne}}(i_1, j_1)$ so when (i,j) replaces (i_1, j_1) in $C3$ it causes no problems. Similarly, if $j = j_1$, then $r^{\mathrm{s}w}(i,j) = r^{\mathrm{s}w}(i_1, j_1)$.

5.2 Conditions $C1', C2', C3$ allow us to run an even step of the algorithm.

It must not happen that a square in E is located east of a dot revealed by the algorithm on the same row or south of the dot in the same column. Assume that the square $(i', j') \in E$, with $r(i', j') > 0$ is located south of a dot found in column j'. Let $(i,j) \in E$ be the left false square of the dot found. If $i \geq i'$ we get a contradiction, since all the columns $j+1, \ldots, j'-1$ are not in the shape, i.e., $v(j') = v(j) + 1$, which together with condition $C2'(b)$ forces $r(i',j')$ to be zero. If $i < i'$ we let (i'',j'') and (i''',j''') be as in $C3$. Since a dot has been found in the rectangle $[1,i] \times [j,j']$, we have from Lemma 5.4 that $r^{\mathrm{ne}}(i,j) - r^{\mathrm{ne}}(i'',j'') \geq 1$ and $r^{\mathrm{sw}}(i''',j''') \leq r^{\mathrm{sw}}(i',j') - 1 = v(j') - r(i',j') - 1$. Also in this case we have $v(j') = v(j) + 1$, so from $C3$ we get

$$r(i',j') \geq r(i,j) + r^{\mathrm{ne}}(i,j) - r^{\mathrm{ne}}(i'',j'') + r^{\mathrm{sw}}(i,j) - r^{\mathrm{sw}}(i''',j''') \geq$$

$$\geq r(i,j) + 1 + v(j) - r(i,j) - (v(j') - r(i',j') - 1) =$$

$$= r(i',j') + 1.$$

Hence the assumption was false and there cannot be any square in column j'. It follows by symmetry that a revealed dot is not west of a square in E on the same row.

5.3 Conditions $C1', C2', C3$ are still valid after an even step of the algorithm.

Now we have proved that the conditions of the theorem suffice to let us perform an even step of the algorithm. It remains to show that the conditions are still valid after the step.

$C1'$: We need to prove that for a given square $(i,j) \in E$ the number of dots revealed northwest of the square in one step is not more than $r(i,j)$.

We prove this by induction on the number of dots revealed. If one dot is revealed then the statement is trivially true. Assume that s dots are found in $[1,i] \times [1,j]$. The current false squares in $[1,i] \times [1,j]$ form a "staircase" with s steps (see Figure 7). Denote these squares $(i_0,j_0), \ldots, (i_s,j_s)$, where $i_0 < i_1 < \cdots < i_{s-1} < i \leq i_s$ and $j_0 \geq j > j_1 > \cdots > j_{s-1} > j_s$. Let (i'',j'') be the square in E that, in the sense of condition C3, corresponds to the pair of squares (i,j) and (i_{s-1},j_{s-1}). Let $k < s$ be such that $i_k \geq i'' > i_{k-1}$. Then the same square (i'',j'') will correspond also to the pair (i,j) and (i_k,j_k). Let (i''',j''') be the other square of E described by $C3$ corresponding to (i,j) and (i_k,j_k). Our aim is to use condition $C3$ and the induction hypothesis to prove that $r(i,j) \geq s$:

- By induction we have $r(i'',j'') \geq k$ which implies $r^{\mathrm{ne}}(i_k,j_k) - r^{\mathrm{ne}}(i'',j'') \geq k$.
- If, say, t of the revealed dots are northwest of (i''',j'''), then we have by induction that $r(i''',j''') \geq t$. The other $s-k-t$ dots are found in columns between (i''',j''') and (i_k,j_k), so these columns are still in the shape and hence we get $r^{\mathrm{sw}}(i_k,j_k) - r^{\mathrm{sw}}(i''',j''') \geq v(j_k) - v(j''') + r(i''',j''') \geq t + s - k - t = s - k$.
- $C3$ now gives

$$r(i,j) \geq r(i_k,j_k) + r^{\mathrm{ne}}(i_k,j_k) - r^{\mathrm{ne}}(i'',j'') + r^{\mathrm{sw}}(i_k,j_k) - r^{\mathrm{sw}}(i''',j''') \geq$$

$$\geq 0 + k + s - k = s.$$

$C2'$: We need to prove that the inequalities are still true after the step. We prove that $C2'(a)$ remains valid. $C2'(b)$ follows by a dual argument. We suppose the inequality becomes false, and we shall obtain a contradiction. Note that by perceiving the revealed dots to be removed one at a time we may assume that only one dot is found causing the inequality to be false.

First we assume that $i > i'$, $r^{\mathrm{ne}}(i,j) = r^{\mathrm{ne}}(i',j') + 1$ and a dot is revealed in $[1,i'] \times [j,j']$. Let (i_1,j_1) be the left false square to the dot found. If (i_1,j_1) is (i,j), then $r(i,j) = 0$ which implies $r(i',j') = h(i') - h(i) + 1 \leq 0$, a contradiction. Since $r^{\mathrm{ne}}(i,j) = r^{\mathrm{ne}}(i',j') + 1$, it cannot be that $(i_1,j_1) \in [i'+1,i] \times [j,j']$. By Lemma 5.4 we get $i_1 \neq i'$, so $i_1 < i'$. Let $(i'',j''),(i''',j''') \in E$ be as in $C3$ corresponding to (i',j') and (i_1,j_1). Note that $r^{\mathrm{sw}}(i''',j''') \geq r^{\mathrm{sw}}(i,j) = v(j) - r(i,j)$. We get

$$r(i',j') \geq r(i_1,j_1) + r^{\mathrm{sw}}(i_1,j_1) + r^{\mathrm{ne}}(i_1,j_1) - r^{\mathrm{ne}}(i'',j'') - r^{\mathrm{sw}}(i''',j''') \geq$$

$$\geq v(j_1) + 1 - (v(j) - r(i,j)) = 1 + h(i) - r^{\mathrm{ne}}(i,j).$$

Since $r(i',j') = h(i') - r^{\mathrm{ne}}(i',j')$, we get

$$h(i') - h(i) \geq 1 + r^{\mathrm{ne}}(i',j') - r^{\mathrm{ne}}(i,j) = 0,$$

contradicting the assumptions.

The second possibility is that $i > i'$, $r^{\mathrm{ne}}(i,j) = r^{\mathrm{ne}}(i',j')+1$ and a dot is revealed in $[i'+1,i] \times [j,j']$. In this case (i,j) and (i',j') are the left and right false squares to the dot. This gives that $r(i,j) = r(i',j') = 0$, so $h(i) = r^{\mathrm{ne}}(i,j) = r^{\mathrm{ne}}(i',j')+1 = h(i')+1$. Hence all the rows $i+1,\ldots,i'-1$ are removed from the shape and the dot must be on row i and (i,j) is removed from E.

The third possibility is that $i = i'$ and $r^{\mathrm{ne}}(i,j) = r^{\mathrm{ne}}(i',j')$. From Lemma 5.4 we know that in this case the algorithm cannot find a dot in $[1,i] \times [j,j']$, so the inequality remains valid.

$C3$: If dots are found in $[1,i] \times [1,j']$, $[i,i'] \times [1,j]$, $[1,i''] \times [j'',n]$ or in $[i''',n] \times [1,j''']$, and none of the squares in the inequality is removed from E, then the inequality is effected as much on both sides, so it is still true. If, say, (i''',j''') is removed by a dot, then this dot was found in column j''' because of the definition of (i''',j'''). Let (i_1,j_1) be the left false square. Then $r^{\mathrm{sw}}(i''',j''') = r^{\mathrm{sw}}(i_1,j_1)$ and (i_1,j_1) will take the place of (i''',j''') in the inequality which will still be true. The only thing left to check is that the conditions prevent the algorithm from finding a dot in the rectangles $[1,i] \times [j+1,j''-1]$ and $[1,j] \times [i+1,i'''-1]$, which could cause problems. Assume that a dot is found in $[1,i'] \times [j+1,j''-1]$. Let $(i_1,j_1) \in E$ be the left false square of the dot found. We have $j_1 \geq j$ and from the definition of (i'',j'') we have $i_1 < i''$. $C3$ applied to $(i'',j''),(i_1,j_1)$ gives squares $(i_2,j_2),(j_3,j_3)$ such that

$$r(i'',j'') \geq r(i_1,j_1) + r^{\mathrm{ne}}(i_1,j_1) + r^{\mathrm{sw}}(i_1,j_1) - r^{\mathrm{ne}}(i_2,j_2) - r^{\mathrm{sw}}(i_3,j_3).$$

We now combine this inequality with the $C3$ inequality for (i,j) and (i',j'), and the following trivial relations: $r^{sw}(i',j') - r^{sw}(i''',j''') \geq 0$, $r^{ne}(i_1,j_1) - r^{ne}(i_2,j_2) \geq 1$, $r(i',j') + r^{ne}(i',j') = h(i')$, $r(i_1,j_1) + r^{sw}(i_1,j_1) = v(j_1)$ and $r^{sw}(i_3,j_3) \geq r^{sw}(i,j) = v(j) - r(i,j)$. We get

$$r(i'',j'') \geq v(j_1)+1-(v(j)-r(i,j)) \geq v(j_1)+1-v(j)+h(i')+0-(h(i'')-r(i'',j'')) \geq$$
$$\geq 1 + r(i'',j''),$$

a contradiction. Similarly, there can be no dots found in $[1,j'] \times [i+1, i'''-1]$.

5.4 E is the essential set of the obtained dotting.

In the rest of the proof E and the rank functions will be as before the algorithm starts. To avoid too much notation, let us drop the initial false squares $\{(0,n),(n,0)\}$ from E, since they are not needed in the following.

In the previous sections 5.1, 5.2, 5.3 of the proof, we have shown that the conditions in the theorem are sufficient for the algorithm to run without problems, giving a properly dotted $n \times n$ matrix. Let $\mathcal{E}$ denote the essential set of this dotted matrix. We must now show that $\mathcal{E} = E$.

We need to prove that any $(i,j) \in E$ is a white corner of the proper dotting obtained. That is, we must verify that there is a dot in column $j+1$ on a row $\leq i$ and a dot on row $i+1$ in a column $\leq j$. If (i,j) is turned into a false square during an odd step, the next step will produce such dots if they were not there previously. Assume that (i,j) is the northmost square in E not having the required dot in the first column to the east and thus not being in the essential set of the dotting. Since the algorithm does not produce the dot in column $j+1$, there must be a square $(i',j') \in E$, $i \leq i'$, $j < j'$, whose rank becomes zero in the same step and prevents (i,j) from becoming a false square. Assume first that $i = i'$. From $C2(a)$ we know that $r(i',j') - r(i,j) = a \geq 1$ dots have been found in $[1,i] \times [j+1, j'-1]$ prior to this step. For this to occur there must be a square in E south of all these dots; let $(i_1,j_1) \in E$, $j < j_1 < j'$, $i_1 < i$, be the southmost such dot ($j < j_1$ due to the choice of (i,j)). Now let (i'',j'') and (i''',j''') be as in $C3$ for (i_1,j_1) and (i',j'). If there are a' dots in $[1,i_1] \times [j,j_1]$ and a'' dots in $[1,i_1] \times [j_1,j']$, then $a' + a'' = a$, $a'' = r^{ne}(i_1,j_1) - r^{ne}(i'',j'')$ and $j_1 > j + a'$. If (i,j) is (i''',j'''), we get an immediate contradiction from

$$r(i',j') \geq r(i_1,j_1) + r^{sw}(i_1,j_1) + r^{ne}(i_1,j_1) - r^{ne}(i'',j'') - r^{sw}(i,j) =$$
$$= j_1 + a'' - (j - r(i,j)) > a + r(i,j).$$

If (i,j) is not (i''',j'''), then let a''' be the number of dots found in $[1,i_1] \times [j''',j_1]$. Then we have $r(i''',j''') \geq r(i,j) + a' - a'''$ and $j_1 > a''' + j'''$, so $j_1 - r^{sw}(i''',j''') = j_1 - j''' + r(i''',j''') > a''' + r(i,j) + a' - a'''$. We get the contradiction $r(i',j') > a + r(i,j)$ again. If $i < i'$, then the "(i,j) is not (i''',j''')" case with $r(i''',j''') \geq r(i,j) + a'' -$

$a''' + \#\{$ dots in $[i+1,i'] \times [1,j]\}$, gives a contradiction just as above. By symmetry there is a dot in the row below every square in E in a required position. We have now shown $E \subseteq \mathcal{E}$.

It remains to show that the essential set of the dotting obtained does not contain any extra squares, that is, $E \supseteq \mathcal{E}$. Assuming this is not the case, let (i,j) be the southmost square in $\mathcal{E}\backslash E$, if not unique take the one furthest east. We have by assumption that E and $\mathcal{E}$ give the same result when the algorithm is used. Since (i,j) is not needed, there must be another square $(i_1,j_1) \in E$, $i_1 \geq i, j_1 \geq j$, that will get rank zero simultaneously.

First assume that $j_1 > j$. We know that $(i,j+1)$ is a shaded square, so the dot in column $j+1$ must be in a row with row number less than or equal to i. Hence, there must be a square $(i',j) \in E$, $i' < i$, otherwise the necessary dot in column $j+1$ would not be found. If there are many possibilities choose i' as large as possible. There are two cases:

Case 1: $i = i_1$. If there are no other white corners in $[i'+1,i] \times [j+1,j_1]$, then there are no dots either since (i,j) and (i_1,j_1) are deleted from the shape in the same step. Hence in the $\mathcal{E}$ case the inequality in $C3$ for (i',j) and (i_1,j_1) will be an equality with (i,j) in the role of (i''',j'''). Since (i,j) is the southmost white corner not in E, any square in $E \subseteq \mathcal{E}$ southwest of (i,j) will have strictly smaller southwest rank than (i,j) and hence will $C3$ be false for E. If there are other white corners in $[i'+1,i] \times [j+1,j_1]$, then let (i_2,j_2) be the westmost such corner that lies in E. The algorithm (using E) cannot find any dots in $[i'+1,i_2] \times [j+1,j_2-1]$. Since there are no dots in that area, we get that $r(i_2,j_2) = r(i',j) + r^{\mathrm{ne}}(i',j) + r^{sw}(i',j) - r^{\mathrm{ne}}(i'',j'') - r^{sw}(i,j)$ for $\mathcal{E}$. Just as above we get that $C3$ fails for E since (i,j) is replaced by a square with strictly smaller southwest rank.

Case 2: $i_1 > i$, $j_1 > j$. There are no dots in $[i,i_1] \times [j,j_1]$ since (i,j) and (i_1,j_1) are deleted simultaneously. Hence (i,j_1) is a white square in the diagram of the dotting, which implies that there exists $(i,j_2) \in \mathcal{E}$ on row i with $j_2 \geq j_1$ and such that there are no dots in $[1,i] \times [j_1,j_2]$. Because of the choice of (i,j) we know that (i,j_2) is also in E. Hence, $r(i,j_2)$ becomes zero in the same step and we are back in Case 1.

If $j_1 = j$, then $i_1 > i$ and we get a contradiction from a symmetric argument.

5.5 The dotting obtained has full rank.

The reader may have noticed that $C1'(b)$ has not been used so far in proving sufficiency. However, we need it now to show that the algorithm does produce a full rank permutation matrix. We know that there is at most one dot on every row, so it remains to show that there cannot be any rows without a dot. Assume there are, and let i be the southmost such row. Let (i,j) be the rightmost white square on row i. Then there are dots north of row i in the columns $j+1,\dots,n$. The dot in row $i+1$ is then west of or on column j. Hence (i,j) is also a white corner. The number of dots in $[1,i]\times[j,n]$ is at most $i-1-r(i,j)$ since row i has no dot. But the number of dots is equal to $n-j$ since every column in $[1,i]\times[j,n]$ has its dot there. This gives $r^{\mathrm{se}}(i,j)=n-i-j+r(i,j)<0$ contradicting $C1'(b)$. □

PROOF OF COROLLARY 4.2 As noted at the end of the proof of Theorem 4.1, $C1(a)$, $C2$ and $C3$ are sufficient for the algorithm to run smoothly. Given a properly dotted $m\times n$ rectangle with k dots we can always extend it in a unique way to an $(n+m-k)\times(n+m-k)$ permutation matrix in the following way. Let $j_1,j_2,\dots,j_{n-k}$ and $i_1,i_2,\dots,i_{m-k}$ be the dot-less columns and rows respectively. We then place dots in squares $(m+1,j_1),\dots,(n+m-k,j_{n-k})$ and in $(i_1,n+1),\dots,(i_{m-k},n+m-k)$. The permutation matrix that arises in this way has exactly the same essential set as the original rectangle. Hence $C1(a),C2$ and $C3$ are necessary and $R1$ replaces $C1(b)$.

The opposite direction is just as easy. Given a set E as in the corollary we get a unique $(n+m-k)\times(n+m-k)$ permutation matrix with no white squares at all in columns $n+1,n+2,\dots,n+m-k$ or in rows $m+1,m+2,\dots,n+m-k$. It is easy to see that there are at least k dots in the $m\times n$ rectangle. The corollary follows. □

6. Essential sets of certain classes of permutations

Certain classes of permutations can be characterized by the shape of the essential set. An easy example is provided by the *alternating permutations*, that is, w such that $w(1)>w(2)<w(3)>w(4)$ etc. A permutation is alternating if and only if there are descents exactly at positions 1, 3, 5, etc. As observed by Fulton, there is a descent at position i if and only if there is a white corner in row i; hence, a permutation $w\in S_n$ is alternating if and only if there are white corners in all odd rows (except for the last one if n is odd), and nowhere else. See Figure 1.

6.1. Vexillary permutations. An important example of characterization by essential set is Fulton's description of the vexillary permutations as having no white corners (i,j) and (i',j') such that $i<i'$ and $j<j'$. His proof is algebraic in nature, but we would like to point out here that the result can be obtained elementarily from the alternative characterization of vexillary permutations as 2143-avoiding, see Macdonald [10]. The argument goes as follows:

Suppose w contains a 2143-pattern: $i_1<i_2<i_3<i_4$ with $w(i_2)<w(i_1)<w(i_4)<w(i_3)$. Then $(i_1,w(i_2))$, the unique square that is at the same time to the left of first dot and above the second one, must be a white square. Thanks to the shading of these dots there must be a white corner (i,j) with $i<i_2, j<w(i_1)$. Similarly, we must have a white square at $(i_3,w(i_4))$, and hence a white corner (i',j') with

$i_3 \leq i', w(i_4) \leq j'$. In particular we have $i < i'$ and $j < j'$. For the other direction, just observe that a white corner (i,j) implies the existence of dots

$$(i_1 \leq i, w(i_1) = j+1) \text{ and } (i_2 = i+1, w(i_2) \leq j),$$

while a white corner $(i',j') > (i,j)$ gives dots

$$(i_3 = i', w(i_3) > j') \text{ and } (i_4 > i', w(i_4) = j'),$$

forming a 2143-pattern.

6.2. Baxter permutations. In a similar way we shall prove some other connections between certain shapes of essential sets and permutations avoiding certain patterns. A permutation $w \in S_n$ is said to be a *Baxter permutation* if for every $1 \leq a < b < c < d \leq n$ we have

(B1) $w(d) + 1 = w(a)$ and $w(c) > w(a) \quad \Longrightarrow w(b) > w(a)$,
(B2) $w(a) + 1 = w(d)$ and $w(b) > w(d) \quad \Longrightarrow w(c) > w(d)$.

This class of permutations was first defined by Baxter [1] in his work on fix point free mappings. Baxter permutations have also been thoroughly studied from an enumerative point of view [3]. Here we will give a new very simple characterization of Baxter permutations in terms of their essential set.

First we characterize the permutations that fulfill (B1).

Lemma 6.1. *A permutation w satisfies* (B1) *if and only if it has at most one white corner in each column.*

Proof First, suppose that w has a column with two white corners: (i,j) and (k,j), with $i < k$. In order for (i,j) to be a white corner, there must be dots early in row $i+1$ and column $j+1$, say in $(i+1, j_1)$ and $(i_1, j+1)$ for some $i_1 \leq i$ and $j_1 \leq j$. In order for (k,j) to be white, the dots in row k and column j must come late, say in (k, j_2) and (k_2, j) for some $k_2 > k$ and $j_2 > j$. Hence, $i_1 < i+1 < k < k_2$, where $w(k_2) + 1 = j+1 = w(i_1)$ and $w(k) > w(i_1)$. But $w(i+1) < w(a)$, which violates (B1).

In the other direction, suppose that we are given a permutation w such that the dots in rows $a < b < c < d$ violates (B1), i.e., $w(d) + 1 = w(a), w(c) > w(a)$ but $w(b) < w(a)$. Then in column $w(d)$ we have white squares at rows a and c while the square at row b is shaded. Since $a < b < c$ and column $w(a)$ is shaded in all relevant rows, there must be two white corners in column $w(d)$. $\square$

Condition (B2) is dual to (B1) in the following sense.

Lemma 6.2. *A permutation w satisfies* (B2) *if and only if w^{-1} satisfies* (B1).

Proof Assume that w does not satisfy (B2), i.e., there exist $a < b < c < d$ such that $w(a) + 1 = w(d), w(b) > w(d)$ but $w(c) < w(d)$. Then there exist two adjacent rows i and $i+1$ such that $b \leq i < i+1 \leq c$ and $a < i < i+1 < d$ also violates (B2). In w^{-1} we now have that the rows $w(i+1) < w(a) < w(d) < w(i)$ violates (B1).

A similar argument gives the other direction. □

Theorem 6.3. *A permutation is a Baxter permutation if and only if its essential set has at most one square in each row and column.*

Proof Since inverting a permutation just interchanges the roles of rows and columns, the theorem follows from Lemma 6.1 and Lemma 6.2. □

Alternating Baxter permutations are also frequently considered in the literature, see eg. [4]. Combining the conditions for being alternating and being Baxter gives:

Corollary 6.4. *A permutation is an alternating Baxter permutation if and only if its essential set has at most one square in each column, exactly one square in each odd row and no square in the even rows.*

6.3. 321-avoiding permutations. Billey, Jockush and Stanley [2] showed that a permutation is 321-avoiding if and only if it has white squares at (i, j') and (i', j), for $i < i'$ and $j > j'$, whenever there are white squares at (i, j) and (i', j'). In this paper, we are more concerned with white corners than with white squares, so what happens when the word "square" is changed to "corner" in the statement? Well, one direction is still valid, the other is not (counterexample 5361724):

Lemma 6.5. *If a permutation w is 321-avoiding, then for every pair (i, j) and (i', j') of white corners such that $i < i'$ and $j > j'$, there must also be white corners at (i, j') and (i', j).*

Proof See Figure 8. There must be a dot in row $i' + 1$ in a column $\leq j'$ and a dot in column $j + 1$ in a row $\leq i$. To avoid a 321-pattern, there must be no dot in the rectangle $[i, i'] \times [j', j]$. Thus, the dot in row $i + 1$ must lie west of column j' and the dot in column $j' + 1$ must lie north of row i. Also, the dot in row i' must be in a column east of column j and the dot in column j must lie in a row south of row i'. This implies white corners in squares (i, j') and (i', j). □

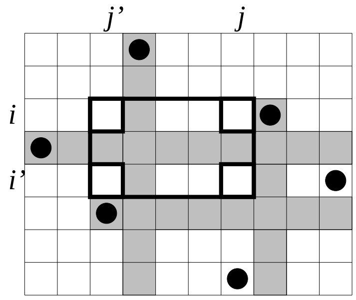

FIGURE 8. Sketch of a 321-avoiding permutation. There must be no dot inside the bold rectangle, whose northeast and southwest corners are (i,j) and (i',j') resp.

6.4. Vexillary 321-avoiding permutations. Billey, Jockusch and Stanley [2] obtained a "curious" enumerative consequence of their analysis of the Schubert polynomials of 321-avoiding permutations: the number of $w \in S_n$ that are both vexillary and 321-avoiding is $1+2(2^n-(n+1))-\binom{n+1}{3}$. By characterizing these permutations in terms of the essential set, we can give an immediate interpretation of each of these terms.

Proposition 6.6. *A permutation w is 321-avoiding and 2143-avoiding (vexillary) if and only if it has all its white corners either in one single row or in one single column.*

PROOF Suppose that all white corners lie in one single row. Then the permutation is vexillary by Fulton's condition, and 321-avoiding, since it has only one descent. By transposition, the same can be said for the case of white corners in one single column

For the converse, suppose that the permutation w is both 321-avoiding and vexillary. Suppose there is a pair of white corners (i,j) and (i',j') where $i < i'$. Then w vexillary implies that $j \geq j'$, but $j > j'$ is impossible since then w 321-avoiding by the lemma above implies the existence of a pair of the type forbidden for vexillary permutations. Hence, all white corners must lie either in one single row or in one single column. $\square$

The identity permutation is both 321-avoiding and vexillary, so it takes care of the first term, 1, in the expression $1+2(2^n-(n+1))-\binom{n+1}{3}$. All other permutations have at least one white corner. Having all white corners in one single row is equivalent to having exactly one descent. The number of permutations with one descent is easily seen to be $2^n-(n+1)$: choose any subset of $[1,n]$ except for intervals $[1,k], k = 0,1,\ldots,n$, and order it in increasing order, then continue with the complement in increasing order. By transposition, there are equally many permutations with all white corners in one single column, so this takes care of the second term, $2(2^n-(n+1))$. We must now subtract the number of permutations that have been added twice;

they are those with only one white square all together. As is most easily seen from the picture (Figure 9), these are the permutations of the word-form

$$1\dots i(j+1)\dots k(i+1)\dots j(k+1)\dots n.$$

We can choose $i<j<k$ arbitrarily in the interval $[0,n]$, so this takes care of the last term, $\binom{n+1}{3}$, of the expression.

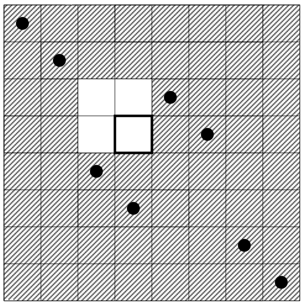

FIGURE 9. Permutation with only one white corner: the dots lie in four diagonal slopes, northwest, east, south, and southeast of the white area.

6.5. Antivexillary permutations. Let us conclude this section with a discussion of *antivexillary* permutations. By antianalogy with the vexillary case, we define a permutation to be antivexillary if it has no white corners (i,j) and (i',j') such that $i<i'$ and $j>j'$. Thus, an antivexillary permutation has its white corners spread in the northwest–southeast direction, while the vexillary permutations have their white corners spread in the southwest–northeast direction. The antivexillary permutations admit a surprising characterization in terms of forbidden patterns:

Proposition 6.7. *A permutation w is antivexillary $\Leftrightarrow$ w is 321-avoiding and 351624-avoiding.*

PROOF ($\Rightarrow$) Assuming the reader is by now familiar with the reasoning (just draw a picture!), we go fast. Suppose w has a 321-pattern, that is, dots $(i_1,j_1),(i_2,j_2),(i_3,j_3)$ with $i_1<i_2<i_3$ and $j_1>j_2>j_3$. Then there must be a white corner somewhere in the rectangle $[i_1,i_2-1]\times[j_2,j_1-1]$ and another one in the rectangle $[i_2,i_3-1]\times[j_3,j_2-1]$, yielding a pair such that w cannot be antivexillary.

Suppose instead w has a 351624-pattern, with dots at $(i_1,j_1),(i_2,j_2),\dots,(i_6,j_6)$, where $i_1<i_2<i_3<i_4<i_5<i_6$ and $j_3<j_5<j_1<j_6<j_2<j_4$. Then there must be a white corner somewhere in the rectangle $[i_2,i_3-1]\times[j_6,j_2-1]$ and another one in the rectangle $[i_4,i_5-1]\times[j_5,j_1-1]$, yielding a pair such that w cannot be antivexillary.

($\Leftarrow$) Suppose w is 321-avoiding but not antivexillary. We must show that w then contains a 351624-pattern. Since w is not antivexillary there exists a pair of white corners $(i_1, j_1), (i_2, j_2)$ with $i_1 < i_2$ and $j_1 > j_2$. Then w being 321-avoiding forces the existence of white corners also at (i_1, j_2) and (i_2, j_1), thanks to Lemma 6.5. Now consider the six dots at: column $j_2 + 1$, column $j_1 + 1$, row $i_1 + 1$, row i_2, column j_2, and column j_1. Draw a picture, or look at Figure 8. For the four white corners to exist, these must lie in one of the patterns 351624, 531624, 351642 and 531642. Since they are not allowed to contain a 321-pattern, the only possibility is a 351624-pattern. □

7. The essential set in higher dimensions

Recall Fulton's Lemma from Section 2, which stated that (a) given a permutation w, the ideal generated by certain minors defined by the rank r_w is generated by the subset of these minors stemming from $\mathcal{E}(w)$; and (b) the permutation is determined by its ranked essential set. In Section 3, we strengthened the (b) part to read that the permutation is determined by its ranked core. We shall now see that both parts (a) and (b) holds also in higher dimensions in a natural way.

7.1. Minors and essential set in higher dimensions. The basic objects will be the d-dimensional permutation matrices, by which we mean a dotting of $[1, n]^d$ with n dots such that when any coordinate is fixed, there is exactly one way of giving values to the other coordinates to find a dot. Equivalently, the d-dimensional permutation matrices can be seen as the geometric description of a sequence of $d-1$ permutations in S_n. Each one of the $d-1$ projections of the d-dimensional permutation matrix onto a two-dimensional coordinate plane that contains the first coordinate axis will be an ordinary permutation matrix, and conversely, every such ordered set of $d-1$ ordinary permutation matrices from S_n determine a unique d-dimensional permutation matrix. Hence we can present a d-dimensional permutation w as $w = (w_1, w_2, \dots, w_{d-1}) \in S_n^{d-1}$.

Accordingly, we define the d-determinant of a hypercubic d-dimensional matrix $M = (m_{i_1,\dots,i_d})$ of size n to be

$$\sum_{w \in S_n^{d-1}} \prod_{j=1}^{n} m_{j,w_1(j),\dots,w_{d-1}(j)} \cdot \operatorname{sgn}(w),$$

where

$$\operatorname{sgn}(w) = \prod_{i=1}^{d-1} \operatorname{sgn}(w_i).$$

So, now we have determinants, and hence minors, in higher dimensions. Observe that the above notion of a higher dimensional determinant is **not** the same as the one proposed in the recent book by Gelfand, Kapranov and Zelevinsky [9]. However, in their book they mention our definition as having been studied during the second half of the 19th century for which they refer to Ernst Pascal's *"Die Determinanten"* [11] from 1900, and further references therein.

Let the *rank* of a d-dimensional matrix be the size of its biggest nonvanishing minor. For a d-dimensional permutation w, let V_w be the algebraic variety of hypercubic d-dimensional matrices A such that the rank of each submatrix $A([1,i_1]\times[1,i_2]\times\ldots\times[1,i_d])$ is at most equal to the rank of the corresponding submatrix of w. Thus, V_w can be described as the zeros of a polynomial ideal generated by minors in a generic matrix of sizes given by the rank function of w. Our goal, in the spirit of Fulton's lemma, is a result stating that this minor ideal is indeed generated by the minors coming from the essential set of w, so we now need a definition of essential set in higher dimensions.

Let $\mathbb{P}^d$, for $d\geq 2$, be partially ordered by $(m_1,\ldots,m_d)\leq(m'_1,\ldots,m'_d)$ if and only if $m_i\leq m'_i$ for all $i\in[1,d]$. For simplicity, we still refer to the elements of $\mathbb{P}^d$ as "squares". As before, a shape is a finite set B of squares in $\mathbb{P}^d$, and the *slice* $S_{i,m}$ of B is the set of all squares in B such that the ith coordinate is m. A dotted subset w of a shape B is a *proper dotting* if it contains at most one square of every slice of B. The *rank* of a square c in B with a proper dotting w is

$$r_w(c)\stackrel{\mathrm{def}}{=}|\{c'\in w: c'\leq c\}|.$$

Clearly, this agrees with the rank function for d-dimensional matrices, if dots are taken as ones and empty squares as zeros.

It should be obvious that the situation of d-dimensional permutation matrices is the special case where the shape is $[1,n]^d$ and the number of dots in w is n, which is maximal for a proper dotting. To complete the analogy, we define the *essential set* of a d-dimensional permutation matrix w by shading, for every dotted square c, every square $c'\geq c$ such that c and c' belong to a common slice; now the essential set $\mathcal{E}(w)$ is the set of "white corners", that is, the maximal elements of the connected components of white squares. Clearly, the rank function is constant on connected components of white squares.

Proposition 7.1. *For any w in S_n^{d-1} and a generic d-dimensional hypercubic matrix A of size n, the ideal generated by all minors of size $r_w(i_1,\ldots,i_d)+1$ taken from the lowest (in poset sense) $i_1\times\ldots\times i_d$ submatrix of A, for all $(i_1,\ldots,i_d)\in[1,n]^d$, is generated by these same minors using only those $(i_1,\ldots,i_d)$ that are in $\mathcal{E}(w)$.*

Proof We will show that the minors that come from any square that is not a white corner lie in the ideal generated by the minors that come from white corners. This is easily seen to be true for a white square c: if c is not a white corner, then there is some white corner $c'\geq c$ in the same connected component of white squares, hence with the same rank r, and of course all minors of size $r+1$ in the submatrix bounded by c are included among the minors of size $r+1$ in the submatrix bounded by c'.

For a shaded square c, there is somewhere a dot shading it, so if we regard as neighbors also neighboring positions outside the matrix (and let them have rank zero), there must always be at least one direction in which the neighbor $c' \leq c$ has rank one less, $r = r_w(c') = r_w(c) - 1$. Since c' is covered by c, any minor of size $r+2$ in the submatrix bounded by c is spanned by minors of size $r+1$ in the submatrix bounded by c'. If c' is a white square, we are done. If c' is outside the matrix, its contribution of minors is none, so we are done. And if c' is shaded we proceed in the same manner to find a neighbor c'', etc., and eventually we must reach either a white square or a position outside the matrix. □

In the two-dimensional case, Fulton is able to verify nice properties of this variety by reducing to Schubert varieties via a natural smooth surjection from GL_n to the flag manifold $Fl(n)$. We do not know if anything analogous can be done in higher dimensions.

In higher dimensions the permutation matrices do not realize all rank distributions (with the above definition of determinant) possible for general matrices. There is some intriguing mathematics going on here which will be studied in a coming paper by the same authors [6].

7.2. The retrieval algorithm in higher dimensions. With the definitions of higher dimensional concepts above, we can take the description of the algorithm *verbatim* from Section 3. Indeed, also the definitions of Viennot layer, white filling and core generalize to higher dimensions in an obvious way. Hence all arguments carry through for the higher dimensional case.

Proposition 7.2. *(a) For a d-dimensional permutation matrix, the core is contained in the essential set.*

(b) Suppose the algorithm produces a dotting w for a given input d-dimensional shape B with labeled subset E, such that the ranks of squares in E agree with the labels. Then w is the unique such proper dotting with its core contained in E. If the algorithm fails to produce such a dotting, then there is no such dotting.

(c) A d-dimensional permutation matrix is determined by the restriction of its rank function to its core (and hence in particular to its essential set).

8. Remarks and open problems

(1) Billey, Jockusch and Stanley [2] defined a class of "heroic" permutations, whose Schubert polynomials have a nice combinatorial property. The heroic permutations are known to include all the vexillary ones. Is there possibly a characterization of the heroic permutations in terms of the essential set?

(2) For which permutations is the core equal to the essential set (instead of being a proper subset)? We noted after Lemma 3.1 that all vexillary permutations

have this property, but there are also some permutations of this kind, e.g. 2143, that are not vexillary. On the other hand, we know that, for large n, the essential set must be larger than the core for most permutations in S_n, since the average size of the essential set is $n^2/36$ while the maximal size of the core is $n-1$.

(3) In [7], Fulton used the characterization of the essential set of vexillary permutations to give an expression for the Schubert polynomial of a vexillary permutation. This was done via a formula for the type of the permutation. Is it possible to describe the Schubert polynomial or at least the type of a permutation in terms of its ranked essential set in general, using the characterization given here?

(4) As we saw in Section 7.2, the fact that a permutation is determined by the restriction of its rank function to its core generalizes to higher dimensions. However, the argument for the linear upper bound on the size of the core in dimension two (Lemma 3.4) does not work in higher dimensions. How big can the core be in general?

(5) A remark on the definition of the higher dimensional determinant: We have chosen the most natural definition (in our opinion) of $\mathrm{sgn}(w)$. There are other possible definitions and the results in Section 7.1 are, as a matter of fact, true for all definitions of $\mathrm{sgn}(w)$ that do not take the value zero.

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