A GENERALIZATION OF THE RANDOM ASSIGNMENT PROBLEM

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Abstract. We give a conjecture for the expected value of the optimal k-assignment in an $m \times n$-matrix, where the entries are exp(1)-distributed random variables or zeros. This conjecture generalizes a conjecture of Coppersmith and Sorkin. We prove this conjecture for the case that there is a zero-cost $k-1$-assignment. Assuming our conjecture, we determine some limits, as $k = m = n \to \infty$, of the expected cost of an optimal n-assignment in an n by n matrix with exp(1) variables in a certain region, and zeros outside this region. If we take this region to be a circle inscribed in the square matrix, this limit is thus conjectured to be $\pi^2/24$. We have made a computer-generated verification of a conjecture of Coppersmith and Sorkin for $k \leq 6$, and of a conjecture of Parisi for $k \leq 7$.

1. Introduction

Suppose we are given an m by n array of nonnegative real numbers. A k-*assignment* is a set of k entries, no two of which are in the same row or the same column (we "assign" rows to columns, or vice versa). The *cost* of the assignment is the sum of the entries. A k-assignment is *optimal* if its cost is minimal among all k-assignments.

We let $F_k(m,n)$ denote the expected cost of the optimal k-assignment in an m by n array of independent exponentially distributed random variables with mean 1.

D. Aldous [A92, A01] has recently shown that $\lim_{n\to\infty} F_n(n,n) = \pi^2/6$. Moreover, Parisi [P98] has conjectured that $F_n(n,n) = 1 + 1/4 + 1/9 + \cdots + 1/n^2$. Computer generated verifications of this conjecture have been made for $n \leq 6$ in [AS99], and for $n = 7$ by the authors.

In [CS98], a more precise conjecture is formulated.

Conjecture 1.1.

$$F_k(m,n) = \sum_{\substack{i,j\geq 0 \\ i+j<k}} \frac{1}{(m-i)(n-j)}.$$

It is also shown in [CS98] that this conjecture is in accordance with the conjecture of Parisi if $k = m = n$. We have verified Conjecture 1.1 for $k \leq 6$, (and arbitrary m and n). The cases $k = 3, 4$ are treated in [AS99] and $k = 5$ is given in the appendix. For $k = 6$, about 10,000 cases were analyzed by our computer program.

Date: March 7, 2003.

1.1. **The Main Conjecture.** As is shown in [CS98], [AS99], and in Section 2, the calculation of $F_k(m,n)$ can in some (but not all!) cases be done recursively by reducing the problem to several assignment problems for matrices which have zeros in certain positions.

Let Z be a finite set of matrix-positions, that is,

$$Z = \{(i_1, j_1), \dots, (i_r, j_r)\},$$

for some positive integers $i_1, j_1, \dots, i_r, j_r$. Suppose $m \geq \max(i_1, \dots, i_r)$ and $n \geq \max(j_1, \dots, j_r)$. We define $F_{k,Z}(m,n)$ to be the expected cost of the optimal k-assignment in an m by n matrix with zeros in the positions belonging to Z, and independent exponential random variables with mean 1 in all other positions.

We will consider sets of rows and columns in the $m \times n$-matrix. A set α of rows and columns is said to *cover* Z if every entry in Z is on either a row or on a column in α. A covering with s rows and columns will be called an s-covering. $k-1$-coverings will be of particular importance. By a *partial* $k-1$*-covering* of Z, we mean a set of rows and columns which is a subset of a $k-1$-covering of Z. Let $p_{i,j}$ be the probability that a set of i rows and j columns, chosen from uniform distribution on all such sets, is a partial $k-1$-covering.

We make the following conjecture for the values of $F_{k,Z}(m,n)$.

Conjecture 1.2 (Main Conjecture)**.**

$$F_{k,Z}(m,n) = \sum_{\substack{i,j \geq 0 \\ i+j<k}} \frac{p_{i,j}}{(m-i)(n-j)} \tag{1}$$

Example 1.3. When Z is empty, Conjecture 1.2 clearly specializes to Conjecture 1.1.

Example 1.4. When $k=2$ and Z has just one element, then $p_{1,0} = 1/m$, $p_{0,1} = 1/n$ and $p_{0,0} = 1$, so according to Conjecture 1.2 we get

$$F_{2,Z}(m,n) = \frac{1}{mn} + \frac{1/m}{(m-1)n} + \frac{1/n}{m(n-1)} = \frac{-1}{mn} + \frac{1}{(m-1)n} + \frac{1}{m(n-1)},$$

which is also the correct value, see Example 2.4 below.

We prove in Section 3 that Conjecture 1.2 holds under the assumption that Z contains a $k-1$-assignment. This is our main theorem.

Let P be the poset of all intersections of $k-1$-coverings of Z, ordered by reverse inclusion, and with an artificial minimal element $\hat{0}$ corresponding to the empty intersection. We prove in Section 3 that the following is an equivalent form of Conjecture 1.2:

$$F_{k,Z}(m,n) = \sum_{\alpha \in P \setminus \hat{0}} \frac{-\mu_P(\alpha)}{(m-i_\alpha)(n-j_\alpha)}, \tag{2}$$

where $\mu_P(\alpha)$ is the Möbius function of the interval $(\hat{0}, \alpha)$ in P and i_α, j_α are the number of rows and columns in α respectively.

In [BCR02], an interesting conjecture is formulated, which generalizes the Coppersmith-Sorkin conjecture to matrices of exponential variables whose mean values are given by a matrix of rank 1.

In Section 2 we describe our tools to recursively compute $F_{k,Z}(m,n)$. In Section 4 we discuss some consequences of Conjecture 1.2. One such consequence is that we always can write

$$F_{k,Z}(m,n)=\sum_{i+j<k}\frac{b_{i,j}}{(m-i)(n-j)},$$

for some integers $b_{i,j}$ independent of m and n. We also prove that Conjecture 1.2 implies a conjecture by Olin [O92] on the probability that the smallest element in a row is used in the optimal assignment. Olin's conjecture was proved by Aldous in [A01]. In Section 5 we prove that some asymptotic results follows from Conjecture 1.2.

2. Computation of $F_{k,Z}(m,n)$

We say that a set α of rows and columns is an *optimal covering* of a set Z, if α covers Z, and α has minimal cardinality among all coverings of Z.

If A is a (random) m by n matrix, and u is an assignment, then we let $\Psi_u(A)$ be the cost of u, that is,

$$\Psi_u(A)=\sum_{(i,j)\in u}A_{i,j}.$$

If k is an integer not greater than m or n, we let $\Psi_k(A)$ be the cost of the optimal k-assignment in A.

Our calculations of $F_{k,Z}(m,n)$ are based on the following theorem.

Theorem 2.1. *Let $k\le m,n$ be positive integers. Let A be an m by n matrix, with zeros in a certain set Z of positions, and (possibly random) positive values outside Z. Let α be an optimal covering of Z. Let ϵ be a positive real (possibly random) number, not greater than any of the entries not covered by α. Let A' be the matrix obtained from A by subtracting ϵ from every entry not covered by α, and adding ϵ to every doubly covered entry of A. Then*

$$\Psi_k(A)=\epsilon\cdot(k-|\alpha|)+\Psi_k(A').$$

To prove this theorem, we need the following lemma, which we prove in Section 2.1.

Lemma 2.2. *If α is an optimal covering of a subset of Z, then there is an optimal k-assignment which intersects every row and every column of α.*

Proof of Theorem 2.1. Let u be an optimal k-assignment of A' intersecting every row and column of α. Such an optimal assignment exists by Lemma 2.2. Note that subtracting ϵ from every entry not covered by α, and adding ϵ to every doubly covered entry is the same thing as first subtracting ϵ from all entries not covered by the rows of α, and then adding ϵ to all the entries covered by the columns of α. Let i and j be the number of rows and columns in α, respectively. Then

$$\Psi_u(A)=\Psi_u(A')+\epsilon\cdot(k-i)-\epsilon\cdot j= \\ =\epsilon\cdot(k-|\alpha|)+\Psi_u(A')=\epsilon\cdot(k-|\alpha|)+\Psi_k(A'). \tag{3}$$

For every k-assignment v we have

$$\Psi_v(A)\ge\epsilon\cdot(k-|\alpha|)+\Psi_v(A')\ge\epsilon\cdot(k-|\alpha|)+\Psi_k(A').$$

Hence u is an optimal k-assignment also in A, and $\Psi_k(A)$ is given by (3). ∎

In the typical use of the theorem, ϵ is the minimum of all the non-covered entries, which gives a new zero when we subtract ϵ. The special case of this theorem where all doubly covered entries are known not to be in the optimal k-assignment was treated in [CS98] and [AS99], where it was used in the proof of Conjecture 1.1 for $k \le 4$.

Before proving Lemma 2.2, we give an example of how Theorem 2.1 is used. First we need to state some well-known properties of the exponential distribution. We say that a random variable is $exp(l)$-distributed if it is exponentially distributed with mean l.

Lemma 2.3. *Let $X_1, \dots, X_t$ be independent exponential random variables, with mean $a_1, a_2, \dots, a_t$ respectively. Let $Y = \min\{X_1, \dots, X_t\}$. Then*

1. *Y is exponentially distributed with mean*
$$\frac{1}{1/a_1 + \cdots + 1/a_t}$$
2. *if $Y = X_i$, then $X_j - Y$ is again $exp(a_j)$-distributed, for $i \neq j$, and $\{X_j - Y : j \neq i\}$ are independent, and independent of Y.*
3. *if I is the random variable defined by $Y = X_I$, then I is uniquely defined with probability 1 and*
$$Pr(I = i) = \frac{1/a_i}{1/a_1 + \cdots + 1/a_t}.$$

Example 2.4. Let us show how this is used to compute $F_2(m,n)$. First we take ϵ to be the minimum of all elements of A, which by symmetry we can assume to be $a_{1,1}$. We now subtract $a_{1,1}$ from all entries in A to obtain the matrix A' with a zero in position $(1,1)$ and exp(1)-distributed random variables elsewhere. The covering α is here empty and ϵ is $\exp(1/mn)$-distributed. From Theorem 2.1 we get

$$F_2(m,n) = \frac{2}{mn} + F_{2,Z'}(m,n),$$

where $Z' = \{(1,1)\}$.

To compute $F_{2,Z'}(m,n)$, we have to find an optimal covering of Z' and we can choose $\alpha' = \{r_1\}$, the first row, see Figure 1. We again take the minimum of the entries that are not covered by α', which means that ϵ' is $\exp(1/(m-1)n)$. From Theorem 2.1 we get

$$F_{2,Z'}(m,n) = \frac{1}{(m-1)n} + \frac{1}{n}F_{2,Z''}(m,n) + \frac{n-1}{n}F_{2,Z'''}(m,n),$$

where $Z'' = \{(1,1),(2,1)\}$ and $Z''' = \{(1,1),(2,2)\}$. The minimum is equally likely to occur in any of the entries that is not in the first row. Here Z'' represents the case that the minimum occurs somewhere in the first column, while Z''' represents the case that the minimum occurs in another column. Hence we get only two different non-isomorphic cases.

$F_{2,Z'''}(m,n) = 0$ because Z''' contains a 2-assignment of zeros. Finally we use Theorem 2.1 on Z''. The only optimal covering is the first column, $\alpha'' = \{c_1\}$. The minimum of the other entries is $\exp(1/m(n-1))$, which we use as ϵ''. Note that a new zero in another column than the first will produce a 2-assignment together with one of the zeros in Z''. We get

$$F_{2,Z''}(m,n) = \frac{1}{m(n-1)}.$$

Substituting back we get

$$F_{2,Z'}(m,n) = \frac{-1}{mn} + \frac{1}{(m-1)n} + \frac{1}{m(n-1)},$$

as in Example 1.4, and

$$F_{2,Z}(m,n) = \frac{1}{mn} + \frac{1}{(m-1)n} + \frac{1}{m(n-1)},$$

in accordance with Conjecture 1.2.

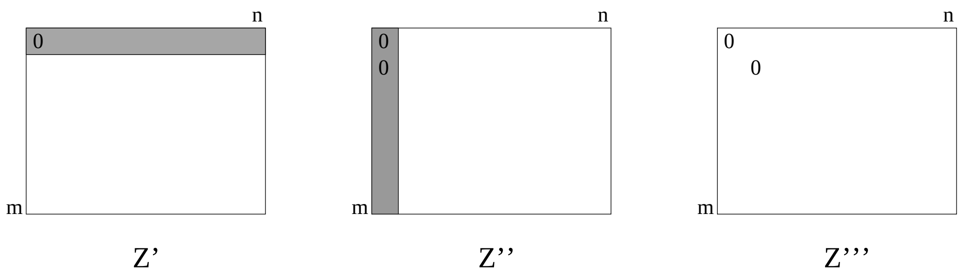

FIGURE 1. The cases needed when computing $F_2(m,n)$. The shaded row and column are the coverings.

Example 2.5. Let us also give an example where there is a doubly covered entry. Let $k = 5$ and $Z = \{(1,2),(1,3),(2,1),(3,1)\}$. This is the simplest case where the recursion will result in a matrix with an entry which is neither $\exp(1)$ nor zero.

The only optimal covering $\alpha = \{r_1, c_1\}$ consists of row one and column one. As usual we let ϵ be the minimum of the non-covered entries, which here is $\exp(1/(m-1)(n-1))$. We get

$$F_{5,Z}(m,n) = \frac{3}{(m-1)(n-1)} + \frac{1}{(m-1)(n-1)} \sum_{(i,j)\in[2,m]\times[2,n]} \Psi_5(A'_{i,j}),$$

where $A'_{i,j}$ has zeros in $Z \cup \{(i,j)\}$ and an entry which is the sum of two r.v.'s, $a_{1,1} = \exp(1) + \exp(1/(m-1)(n-1))$. There are actually four different cases here, and as is shown in the appendix it is possible to continue the recursion in each case to compute $F_{5,Z}(m,n)$ and finally $F_5(m,n)$.

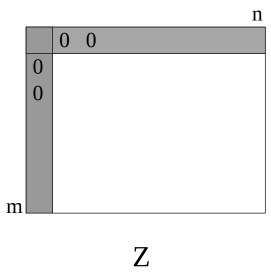

FIGURE 2. The smallest "problem case". The shaded row and column is the covering.

Note that in Section 2.3 we show a way to avoid getting sums of random variables as entries in this particular case. We can thus reduce the number of "problem cases", that is cases when the recursion in Theorem 2.1 gives us entries different from exp(1) and zero. But we have not found any way to avoid them.

2.1. **Proof of Lemma 2.2.** We begin by citing a famous theorem of D. König:

Theorem 2.6. *If a set of matrix positions cannot be covered by $k-1$ rows and columns, then it contains a k-assignment.* □

Lemma 2.7. *Let $k \leq m, n$ be positive integers. Let A be an m by n matrix, with zeros in a certain set Z of positions, and (possibly random) positive values outside Z. Suppose that Z does not contain a $k+1$-assignment. Let α be an optimal covering of Z. Then every row and every column of α contains an element of every optimal k-assignment of A.*

Proof. Suppose that α contains rows $1, \dots, r$ and no other rows. Since α is optimal it follows by König's theorem that there is an r-assignment v in Z containing no element from the columns in α. Suppose (for a contradiction) that there is an optimal k-assignment u which does not use row 1.

We now construct a sequence of matrix positions (all with zeros) as follows: Let v_0 be the element of v which is in the first row. Suppose that we have defined $v_0, \dots, v_h$ and $u_1, \dots, u_h$. Then if there is an element of $u \cap Z$ in the same column as v_h, let this element be u_{h+1}, and let v_{h+1} be the element of v which is in the same row as u_{h+1}. Since u does not contain any element from the first row, the sequences $v_0, v_1, v_2, \dots$ and $u_1, u_2, \dots$ cannot contain any repetitions of the same element. Hence the sequence must end with an element v_h such that there is no element of $u \cap Z$ in the same column.

We now consider two cases. Suppose first that no element in the column of v_h belongs to u. Since Z does not contain a $k+1$-assignment, the cardinality of α is at most k. Each row and column of α covers at most one element of u, and row 1 does not cover any element of u. Consequently there is an element (i, j) of u which is not covered by α, hence does not belong to Z. Then $u \setminus \{u_1, \dots, u_{h-1}, (i, j)\} \cup \{v_0, \dots, v_h\}$ is a k-assignment of smaller cost than u, a contradiction. If on the other hand there is an element (i', j') of u in the first column, then $u \setminus \{u_1, \dots, u_{h-1}, (i', j')\} \cup \{v_0, \dots, v_h\}$ is a k-assignment of smaller cost than u. This contradiction proves the lemma. □

Proof of Lemma 2.2. This follows by a continuity argument. Let the values in the positions of Z not covered by α be ϵ, and let ϵ tend to zero. Since there are only finitely many k-assignments, there has to be an assignment which is optimal for all sufficiently small ϵ, which by Lemma 2.7 has to intersect every row and column of α. This assignment is also optimal for $\epsilon = 0$. □

Note that it is not true that every optimal k-assignment has to intersect the rows and columns of an optimal covering of a subset of Z.

2.2. **Superfluous matrix elements.** It can be of great help to know that a certain entry of the matrix cannot occur in any optimal assignment. Sufficient conditions to draw this conclusion were given in Lemma 26 of [CS98] and Lemma 9 of [AS99]. The following lemma gives a necessary and sufficient condition. The condition here is in fact equivalent to the condition given in [AS99].

Theorem 2.8. *Let $k \leq m, n$ be positive integers. Let $A = (a_{i,j})$ be an m by n matrix. Suppose that a set Z of entries are zero, and the remaining entries are random variables that can take arbitrary positive values. Suppose that $(i,j) \notin Z$. If every set of $k-1$ rows and columns covering Z also covers (i,j), then (i,j) cannot belong to an optimal k-assignment. Conversely, if there is a $k-1$-covering of Z which does not cover (i,j), then it is possible to assign positive values to the matrix entries not in Z in such a way that (i,j) belongs to a unique optimal k-assignment.*

Proof. By the theorem of König, a set contains a k-assignment if and only if it cannot be covered by fewer than k rows and columns. The minimal cost of a k-assignment is of course the same as the minimal cost of a set containing a k-assignment. In trying to minimize the cost of such a set, it suffices to consider sets containing Z, since the entries in Z do not increase the cost.

Suppose that (i,j) is covered whenever Z is covered with fewer than k rows and columns. Then from any set containing Z, which cannot be covered with fewer than k rows and columns, we can delete (i,j) and obtain another such set of smaller cost. This shows that (i,j) can never belong to an optimal k-assignment.

Suppose on the other hand that we can cover Z with a set α of $k-1$ rows and columns, without covering (i,j). Then we assign the value 1 to $a_{i,j}$, and to the entries not in Z which are covered by α, and values larger than k to all other entries not in Z. Since m and n are not smaller than k, the set of entries which are covered by α cannot be covered in any other way by $k-1$ rows and columns. Hence the set of entries which are ≤ 1 cannot be covered with $k-1$ rows and columns. By König's theorem there is a k-assignment of cost at most k. However, since every k-assignment must use at least one entry not covered by α, every k-assignment of cost at most k must use $a_{i,j}$. Hence $a_{i,j}$ belongs to the unique optimal k-assignment. $\square$

In this situation, the fact that $F_{k,Z}(m,n) = F_{k,Z\cup\{(i,j)\}}(m,n)$ can sometimes simplify computation. It is possible that several elements have the property of (i,j) in the above theorem. Indeed, as the following theorem shows, it is sometimes possible to dispose of an entire row or column. If r is a row, we let $Z\backslash r$ denote the set where all positions of Z in row r has been removed and similarly for a column.

Theorem 2.9. *Suppose that a row r is used in every covering of Z by $k-1$ rows and columns (in particular, this is the case if at least k elements in r belong to Z). Let $Z' = Z \setminus r$. Then*

$$F_{k,Z}(m,n) = F_{k-1,Z'}(m-1,n). \tag{4}$$

Similarly, if a column c belongs to every covering of Z with fewer than k rows and columns, and $Z'' = Z \setminus c$, then $F_{k,Z}(m,n) = F_{k,Z''}(m,n-1)$.

Proof. If r is used in every covering of Z by $k-1$ rows and columns, then by Theorem 2.8, no non-zero element in row r, no matter how small, is used in any optimal k-assignment. Hence we do not change the cost of the optimal k-assignment if we replace all non-zero entries in row r by zeros. Let A' be the matrix obtained from A by deleting the row r. Every k-assignment of A must of course contain (at least) a $k-1$-assignment of A'. It is not a problem if Z contains an entry in row m, since we can always permute the rows in Z so row m becomes empty instead of row r.

Conversely, if row r consists of only zeros, then every $k-1$-assignment in A' can be extended to a k-assignment in A of the same cost. Hence $\Psi_k(A) = \Psi_{k-1}(A')$. Passing to expected values, we obtain (4).

The corresponding statement for columns is of course equivalent. □

2.3. **The probability that a certain element belongs to the optimal assignment.** In this section we prove a theorem relating the expected values of certain assignments to the probability that a certain element belongs to such an assignment. We show how this implies a second recursion which can be used to avoid some "problem cases" in the computation of $F_{k,Z}(m,n)$.

Theorem 2.10. *Let $k \le m, n$ be positive integers. Let $A = (a_{i,j})$ be an $m \times n$-matrix, where a set Z of the entries are zero, and the others are independent exponentially distributed random variables with mean 1. Suppose that $(i,j) \notin Z$, and let $Z' = Z \cup \{(i,j)\}$. Then the probability that (i,j) belongs to an optimal k-assignment in A is $F_{k,Z}(m,n) - F_{k,Z'}(m,n)$.*

Proof. We condition on the values of all matrix elements except $a_{i,j}$. Let X be the value of the minimal k-assignment in A which does not use $a_{i,j}$. Let Y be the value of the minimal $k-1$-assignment in A which does not use row i or column j. The optimal k-assignment in A either contains (i,j) and has value $Y + a_{i,j}$, or does not contain (i,j) and has value X. Hence the probability that $a_{i,j}$ belongs to the optimal k-assignment in A is equal to the probability that $a_{i,j} < X - Y$.

We wish to show that this is equal to

$$F_{k,Z}(m,n) - F_{k,Z'}(m,n) = \mathrm{E}(\max(0, \min(X - Y, a_{i,j})).$$

If $X \le Y$, then both are zero. If $X > Y$, then we let $\delta = X - Y$. Then the probability that $a_{i,j}$ is used in the optimal k-assignment in A is $1 - e^{-\delta}$. We compare this to

$$\mathrm{E}(\min(\delta, a_{i,j})) = \delta e^{-\delta} + \int_0^\delta te^{-t}\,dt = \delta e^{-\delta} + 1 - (\delta+1)e^{-\delta} = 1 - e^{-\delta}.$$

This proves the theorem. □

From this we can get another generalization of the basic recursion equation used in [CS98] and [AS99], which is of some help when all the doubly covered entries are nonzero. However, none of the results in this paper depends on this equation.

Corollary 2.11. Let $k \le m, n$ be integers. Let $Z \subset [1,m] \times [1,n]$, be a set of zeros. Let α be an optimal covering of Z and let R be the non-covered entries. Also let the random variable S be the number of positions not covered by α that are in the optimal assignment. Then

$$F_{k,Z}(m,n) = \frac{E[S]}{|R|} + \frac{1}{|R|} \sum_{(i,j)\in R} F_{k,Z\cup(i,j)}(m,n).$$

Proof. By linearity of the expected value we have

$$E[S] = \sum_{(i,j)\in R} (F_{k,Z}(m,n) - F_{k,Z\cup(i,j)}(m,n).$$

□

Example 2.12. Let us give an example of how Corollary 2.11 can be used to simplify the calculations of $F_{k,Z}(m,n)$. As in Example 2.5 we take $k=5$ and $Z=\{(1,2),(1,3),(2,1),(3,1)\}$. The optimal covering α is the first row and the first column. The expected number of non-covered elements used in the optimal k-assignment is 3+ the probability that $a_{1,1}$ is in the optimal assignment which is $F_{k,Z}(m,n)-F_{k,Z\cup\{(1,1)\}}(m,n)$. We get

$$(5)\quad F_{k,Z}(m,n)=\frac{3+F_{k,Z}(m,n)-F_{k,Z\cup\{(1,1)\}}(m,n)}{(m-1)(n-1)}+ \\ +\frac{1}{(m-1)(n-1)}\sum_{(i,j)\in[2,m]\times[2,n]}F_{k,Z\cup\{(i,j)\}}(m,n).$$

This can then be rewritten as

$$F_{k,Z}(m,n)=\frac{3-F_{k,Z\cup\{(1,1)\}}(m,n)+\sum_{(i,j)\in[2,m]\times[2,n]}F_{k,Z\cup\{(i,j)\}}(m,n)}{(m-1)(n-1)-1},$$

which enables us to stay within the set of matrices with all entries exp(1) or zero. However, this recursion does not help in the case

$$Z=\{(1,1),(1,2),(1,3),(2,1),(3,1)\}.$$

For this we would need to know the probability that the zero in position $(1,1)$ is in the optimal k-assignment. We know of no simple way to calculate this, and in general it might not even be well-defined, since in some cases with positive probability there are several optimal k-assignments using the same non-zero entries, but different zeros.

We mention that Theorem 2.10 can be generalized in the following way:

Theorem 2.13. *Let a_i and b_i, $1\le i\le N$, be real numbers. Let X be an exponentially distributed random variable with mean 1. Define the random variable $I\in\{1,\dots,N\}$ by $a_I+b_IX=\min_i a_i+b_iX$. Then $E(b_I)=E(a_I+b_IX)-\min(a_i)$.*

Proof. Let $f(x)=\min_i(a_i+b_ix)$. Then

$$(6)\quad -\min_i(a_i)=-f(0)=\int_0^\infty \frac{d}{dx}\left(e^{-x}f(x)\right)\,dx= \\ =\int_0^\infty e^{-x}f'(x)\,dx-\int_0^\infty e^{-x}f(x)\,dx= \\ =\mathrm{E}(f'(X))-\mathrm{E}(f(X))=\mathrm{E}(b_I)-\mathrm{E}(a_I+b_IX).$$

□

3. Proof of the Main Theorem

In this section we give a proof of Conjecture 1.2 under the assumption that Z contains a $k-1$-assignment.

Let k, m, n and Z be as usual. Let $B(m,n)$ be the Boolean algebra of all sets of rows and columns in A, ordered by reverse inclusion. We define a function $f=f_{k,Z}$ on $B(m,n)$ by

$$f(\alpha)=\begin{cases}1, \text{ if } \alpha \text{ is a partial } k-1\text{-covering of } Z\\ 0, \text{ otherwise}\end{cases}$$

We define $g = g_{k,Z}$ recursively by the equation

$$\sum_{\beta \le \alpha} g(\beta) = f(\alpha).$$

Lemma 3.1. *Let $P = P_{k,Z}(m,n)$ be the sub-poset of $B(m,n)$ consisting of all intersections of $k-1$-coverings, together with an artificial minimal element. Then*

$$g(\alpha) = \begin{cases} -\mu_P(\alpha), \text{ if } \alpha \in P \\ 0, \text{ if } \alpha \notin P \end{cases}$$

Proof. Suppose that α is minimal contradicting the statement. Then $\alpha \notin P$ and $g(\alpha) \neq 0$. Let α' be the intersection of all atoms of P which are smaller than α, that is, which are supersets of α. Then $\alpha' \in P$, and for every $\beta \in P$, if $\beta \le \alpha$, then $\beta \le \alpha'$. Hence $\sum_{\beta<\alpha} g(\beta) = 0$, and consequently, $g(\alpha) = 0$. This is a contradiction. ☐

For $\alpha \in B(m,n)$, we let i_α and j_α be the number of rows and columns in α, respectively. As we will prove, a reformulation of our main conjecture is the following:

Conjecture 3.2.

$$F_{k,Z}(m,n) = \sum_{\alpha \in B(m,n)} \frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}. \tag{7}$$

Example 3.3. Let $k = 3$ and $Z = \{(1,1)\}$. We will compute $F_{3,Z}(m,n)$ using Conjecture 3.2 and Lemma 3.1. The $k-1$-coverings of Z are all sets of two rows and columns including row one or column one. The poset $P = P_{3,Z}(m,n)$ and the Möbius function is shown in Figure 3. The sets with an index i, e.g. $r_1 r_i$, represents all $m-1$ sets with $2 \le i \le m$ and similarly the sets with an index j represents all $n-1$ sets with $2 \le j \le n$. We get

(8)
$$\begin{aligned} F_{3,Z}(m,n) &= (m-1)\frac{1}{(m-2)n} + (n-1)\frac{1}{(m-1)(n-1)} + \frac{1}{(m-1)(n-1)} + \\ &+ (m-1)\frac{1}{(m-1)(n-1)} + (n-1)\frac{1}{m(n-2)} + \frac{-m-n+2}{(m-1)n} + \\ &+ (m-1)\frac{-1}{(m-1)n} + (n-1)\frac{-1}{m(n-1)} + \frac{-m-n+2}{m(n-1)} + \frac{m+n-2}{mn} = \\ &= \frac{1}{(m-2)n} + \frac{1}{m(n-2)} + \frac{1}{(m-1)(n-1)} + \frac{1}{m(n-1)} + \frac{1}{(m-1)n} + \frac{-2}{mn} \end{aligned}$$

We now state a lemma, which will also be useful later. It should be known, but for completeness we include a proof.

Lemma 3.4. *If m and r are positive integers, then*

$$\sum_{i=0}^{r} \frac{(-1)^i \binom{r}{i}}{m-i} = \frac{(-1)^r}{m\binom{m-1}{r}}.$$

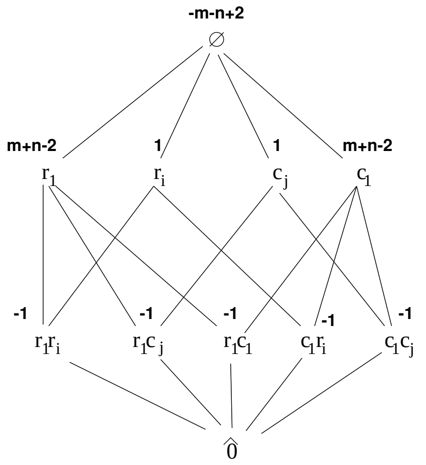

FIGURE 3. The poset $P_{3,Z}(m,n)$ in Example 3.3. The Möbius function is written in boldface.

Proof. We have

$$\sum_{i=0}^{r}\frac{(-1)^i\binom{r}{i}}{m-i}=\sum_{i=0}^{r}\int_0^1(-1)^i\binom{r}{i}x^{m-1-i}\,dx= \tag{9}$$

$$=\int_0^1\left(\sum_{i=0}^{r}(-1)^i\binom{r}{i}x^{m-1-i}\right)\,dx=$$

$$=(-1)^r\int_0^1 x^{m-1-r}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{r-i}x^{r-i}\,dx=$$

$$=(-1)^r\int_0^1 x^{m-1-r}(1-x)^r\,dx.$$

This integral can be interpreted as a probability. Let $Y_1,\dots,Y_m$ be independent random variables uniformly distributed in $[0,1]$. Then the integral in (9) represents the probability that $Y_i<Y_{m-r}$ for $i=1,\dots,m-r-1$, and $Y_j>Y_{m-r}$ for $j=m-r+1,\dots,m$. If we sort the m numbers $Y_1,\dots,Y_m$ according to size, every permutation occurs with equal probability $1/m!$. The number of permutations such that $Y_i, i=1,\dots,m-r-1$ occur before Y_{m-r}, and Y_j, $j=m-r+1,\dots,m$ all occur after Y_{m-r}, is $r!(m-r-1)!$. Therefore, the probability that $Y_i<Y_{m-r}<Yj$ for all $i=1,\dots,m-r-1$ and $j=m-r+1,\dots,m$, is

$$\frac{r!(m-r-1)!}{m!}=\frac{1}{m\binom{m-1}{r}}.$$

This proves the lemma. □

Proposition 3.5. *Conjecture 1.2 ⇔ Conjecture 3.2.*

Proof. Let C be the set of partial $k-1$-coverings. For $\beta\in C$, we define

$$f_\beta(\alpha)=\begin{cases}1, \text{ if } \alpha=\beta\\ 0, \text{ otherwise}\end{cases}$$

Obviously, $f = \sum_{\beta \in C} f_\beta$. Moreover,

$$g = \sum_{\beta \in C} g_\beta,$$

where g_β is defined by

$$\sum_{\gamma \leq \alpha} g_\beta(\gamma) = f_\beta(\alpha).$$

We have $g_\beta(\alpha) = \mu(\beta, \alpha)$, the Möbius function of the interval (β, α) in the Boolean algebra over the set of rows and columns (ordered by reverse inclusion). This interval is itself a Boolean algebra. Hence

$$g_\beta(\alpha) = \begin{cases} (-1)^{|\beta|-|\alpha|}, \text{ if } \alpha \subseteq \beta \\ 0, \text{ otherwise} \end{cases}$$

For the right hand side of (7), we have

$$\sum_{\alpha \in B(m,n)} \frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)} = \sum_{\beta \in C} \sum_{\alpha \geq \beta} \frac{(-1)^{|\beta|-|\alpha|}}{(m-i_\alpha)(n-j_\alpha)} = \tag{10}$$

$$= \sum_{\beta \in C} \left(\sum_{i=0}^{i_\beta} (-1)^i \frac{\binom{i_\beta}{i}}{m-i} \right) \left(\sum_{j=0}^{j_\beta} (-1)^j \frac{\binom{j_\beta}{j}}{n-j} \right) =$$

$$= [\text{by Lemma 3.4}] = \sum_{\beta \in C} \frac{1}{m\binom{m-1}{i_\beta} n \binom{n-1}{j_\beta}} =$$

$$= \sum_{\beta \in C} \frac{1}{\binom{m}{i_\beta}\binom{n}{j_\beta}(m-i_\beta)(n-j_\beta)}.$$

The factor

$$\frac{1}{\binom{m}{i_\beta}\binom{n}{j_\beta}}$$

can be interpreted as the probability of β under the uniform probability measure on all sets of i_β rows and j_β columns. Hence (10) equals (1). $\square$

We need the following lemma.

Lemma 3.6. *Suppose Z contains a $k-1$-assignment u. Let $Z' = Z \cup \{(i,j)\}$, where (i,j) lies in a column not used by u. Then*

$$g_{k,Z'}(\alpha) = \begin{cases} g_{k,Z}(\alpha), \text{ if } \alpha \text{ contains row } i \\ 0, \text{ otherwise} \end{cases}$$

Proof. It follows from the assumptions that in order to cover Z' with $k-1$ rows and columns, one must use row i. This implies that $P_{k,Z'}(m,n)$ consists of precisely those elements of $P_{k,Z}(m,n)$ which contain row i. Since this means that $P_{k,Z'}(m,n)$ is an order-ideal in $P_{k,Z}(m,n)$, the Möbius function on $P_{k,Z'}(m,n)$ is the restriction of the Möbius function on $P_{k,Z}(m,n)$ to $P_{k,Z'}(m,n)$. Now the statement follows from Lemma 3.1. $\square$

We now prove the formula of Conjecture 3.2 under the condition that Z contains a $k-1$-assignment.

Theorem 3.7. *If Z contains a $k-1$-assignment, then*

$$F_{k,Z}(m,n) = \sum_{\alpha \in B(m,n)} \frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}. \tag{11}$$

Proof. If Z contains a k-assignment, the statement is true. We will therefore assume that Z does not contain a k-assignment. We will use induction on k, and for a specific value of k, we will also use induction on the number of elements *not* belonging to Z. Hence we will assume that the statement has been proved for $F_{k,Z'}$, for every superset Z' of Z.

Notice that if there is a row r which belongs to every $k-1$-covering of Z, then by Theorem 2.9,

$$F_{k,Z}(m,n) = F_{k-1,Z\setminus r}(m-1,n).$$

By induction on k, this is

$$\sum_{\alpha \in P_{k-1,Z\setminus r}(m-1,n)\setminus \hat{0}} \frac{g(\alpha)}{(m-1-i_\alpha)(n-j_\alpha)}.$$

But $P_{k-1,Z\setminus r}(m-1,n)$ is isomorphic to $P_{k,Z}(m,n)$ via the isomorphism $\alpha \mapsto \alpha \cup \{r\}$, and $i_{\alpha\cup\{r\}} = i_\alpha + 1$. Hence $F_{k-1,Z\setminus r}(m-1,n)$ equals (11).

We can assume that the $k-1$-assignment in Z is contained in the first $k-1$ rows and the first $k-1$ columns. Actually, we can assume that all elements of Z are contained in this $k-1$ by $k-1$ square, since if there is an element of Z in another column, say, then the row containing this element must be used in every $k-1$-covering of Z, and we can use induction on k.

We now subtract the minimum of the elements in columns $k,\dots,n$, and sum over the m possibilities for the row r containing the new zero, using induction:

$$F_{k,Z}(m,n) = \frac{1}{m(n-k+1)} + \frac{1}{m}\sum_r \sum_{\substack{\alpha \in P_{k,Z}(m,n)\setminus \hat{0} \\ r\in\alpha}} \frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}.$$

Now we change the order of summation, and use the fact that the number of rows r for which $r \in \alpha$ is i_α.

$$F_{k,Z}(m,n) = \frac{1}{m(n-k+1)} + \frac{1}{m}\sum_{\alpha \in P_{k,Z}(m,n)\setminus \hat{0}} \frac{i_\alpha g(\alpha)}{(m-i_\alpha)(n-j_\alpha)} = \tag{12}$$

$$= \frac{1}{m(n-k+1)} + \sum_{\alpha \in P_{k,Z}(m,n)\setminus \hat{0}} \frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)} - \sum_{\alpha \in P_{k,Z}(m,n)\setminus \hat{0}} \frac{g(\alpha)}{m(n-j_\alpha)}.$$

We have to show that the first term and the last sum cancel, that is, that

$$\sum_{\alpha \in P_{k,Z}(m,n)\setminus \hat{0}} \frac{g(\alpha)}{(n-j_\alpha)} = \frac{1}{n-k+1}.$$

When α is the set of the first $k-1$ columns, we get a term which equals the right hand side. Hence we are going to show that everything else cancels. It clearly suffices to show that if J is a proper subset of the set of the first $k-1$ columns, then

$$\sum_{\substack{\alpha\in P_{k,Z}(m,n)\\ column-set(\alpha)=J}} g(\alpha)=0.$$

If J is nonempty, then the induced sub-poset of $P_{k,Z}(m,n)$ consisting of those elements that contain J is isomorphic to $P_{k-|J|,Z\setminus J}(m,n-|J|)$ via the isomorphism $\alpha\mapsto\alpha\setminus J$. Hence the claim follows by induction on k.

If on the other hand $J=\emptyset$, we have

$$\sum_{\substack{\alpha\in P_{k,Z}(m,n)\\ column-set(\alpha)=\emptyset}} g(\alpha) = \sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}} g(\alpha) - \sum_{\substack{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}\\ column-set(\alpha)\neq\emptyset}} g(\alpha). \tag{13}$$

By what we already know, the second sum equals 1, since everything cancels except the term where α is the set of the first $k-1$ columns. If we let α_0 be the intersection of all coverings of Z with $k-1$ rows and columns, then

$$\sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}} g(\alpha)=\sum_{\alpha\leq\alpha_0} g(\alpha)=1,$$

by the definition of g. Hence

$$\sum_{\substack{\alpha\in P_{k,Z}(m,n)\\ column-set(\alpha)=\emptyset}} g(\alpha)=1-1=0,$$

which proves the theorem. □

As a special case Theorem 3.7 enables us to compute the expected cost for completing a zero cost $k-1$-assignment. This question was considered in [CS98].

Corollary 3.8. Let $Z_k=\{(1,1),(2,2),\dots,(k-1,k-1)\}$. Then

$$F_{k,Z_k}(m,n)=\sum_{i+j<k}\binom{k-1}{i,j,k-1-i-j}\frac{(-1)^{k-1-i-j}}{(m-i)(n-j)}.$$

In particular,

$$F_{k,Z_k}(k,n)=\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{n-j},$$

and

$$F_{k,Z_k}(k,k)=\frac{1}{k}\left(1+1/2+\cdots+1/k\right)\sim\frac{\log k}{k}.$$

Proof. The $k-1$-coverings of Z_k are all obtained by choosing, for each zero, whether to cover it with a row or with a column. It is easily seen that all subsets of $k-1$-coverings (that is, all partial $k-1$-coverings) can be obtained as intersections of $k-1$-coverings. The elements of the poset P_{k,Z_k} are therefore obtained by choosing, for every zero, whether to cover it with a row, with a column, or not to cover it.

For any partial $k-1$-coverings $\hat{0} \neq \alpha \leq \beta$, the interval $[\alpha, \beta] \subset P_{k,Z_k}$ is isomorphic to the Boolean lattice on $|\alpha| - |\beta|$ elements. We get

$$\mu(\hat{0}, \hat{1}) = -\sum_{\alpha > \hat{0}} \mu(\alpha, \hat{1}) = -\sum_{\alpha > \hat{0}} (-1)^{|\alpha|} = \tag{14}$$
$$= -\sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} 2^l = -(1-2)^{k-1} = (-1)^k.$$

Any interval $[\hat{0}, \alpha] \subset P_{k,Z_k}$, $\alpha \neq \hat{0}$ is isomorphic to P_{l,Z_l}, where $l = k - |\alpha|$. Hence $\mu(\hat{0}, \alpha) = (-1)^{k-|\alpha|}$, for all $\alpha \in P_{k,Z_k}$. The number of partial $k-1$-coverings with i rows and j columns is given by the trinomial coefficient $\binom{k-1}{i,j,k-1-i-j}$. This proves the first statement.

For the second statement, observe that we can rewrite $F_{k,Z_k}(m,n)$ by

$$\sum_{j=0}^{k-1} \sum_{i=0}^{k-1-j} \binom{k-1}{i, j, k-1-i-j} \frac{(-1)^{k-1-i-j}}{(m-i)(n-j)} = \tag{15}$$
$$\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{1}{n-j} \sum_{i=0}^{k-1-j} \binom{k-1-j}{i} \frac{(-1)^{k-1-i-j}}{m-i} =$$
$$[\text{by Lemma 3.4 }] = \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}}{\binom{m-1}{k-1-j}} \cdot \frac{1}{m(n-j)}.$$

In particular, when $k = m$, we have

$$F_{k,Z_k}(k,n) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{n-j}.$$

□

4. Some consequences of the main conjecture

A consequence of the Main Conjecture is the following.

Theorem 4.1. *Let $k \leq m, n$ be positive integers and Z the set of zero entries. If Conjecture 1.2 is true, then there exist integers $b_{i,j}$ (independent of m and n) such that*

$$F_{k,Z}(m,n) = \sum_{i+j<k} \frac{b_{i,j}}{(m-i)(n-j)}, \tag{16}$$

Moreover, if all the zeros in Z are in an $m' \times n'$ sub-array A', then $b_{i,j} = 1$ if $i \geq m'$ or $j \geq n'$.

Proof. Assume Conjecture 1.2. Rewrite $F_{k,Z}(m,n)$ as

$$\sum_{x+y<k} \frac{1 - q_{x,y}}{(m-x)(n-y)},$$

where $q_{x,y}$ is the probability that a random set with x rows and y columns (chosen uniformly) is a bad set. Now refine the counting of bad sets by letting s and t be the number of rows resp. columns that intersect $A' = [1, m'] \times [1, n']$. Then

$$\frac{q_{x,y}}{(m-x)(n-y)} = \frac{\sum_{s=0}^{x}\sum_{t=0}^{y}\binom{m-m'}{x-s}\binom{n-n'}{y-t} d_{s,t}^{k-1-y-x}(A')}{\binom{m}{x}(m-x)\binom{n}{y}(n-y)}, \tag{17}$$

where $d_{s,t}^{r}(A')$ is the number of sets of s rows and t columns that are not partial $s+t+r$-coverings.

We use partial fraction decomposition with respect to m to get

$$\frac{\binom{m-m'}{x-s}}{\binom{m}{x}(m-x)} = \sum_{i=0}^{\min\{x, m'-1\}} \frac{(-1)^{s-i}\binom{x}{i}\binom{m'-i+x-s-1}{m'-i-1}}{m-i}. \tag{18}$$

Inserting (18) and the corresponding identity for $\frac{\binom{n-n'}{y-t}}{\binom{n}{y}(n-y)}$ into (17) we get that the contribution of $\frac{q_{x,y}}{(m-x)(n-y)}$ to $b_{i,j}$ is the integer

$$(-1)^{i+j}\binom{x}{i}\binom{y}{j}\sum_{s=0}^{x}\sum_{t=0}^{y}(-1)^{s+t}\binom{m'-i+x-s-1}{m'-i-1}\binom{n'-j+y-t-1}{n'-j-1} d_{s,t}^{k-1-y-x}(A'),$$

if $i \le \min\{x, m'-1\}$ and $j \le \min\{y, n'-1\}$. Otherwise the contribution is zero and the theorem follows. □

Theorem 4.1 enables us to write the value of $F_{k,Z}$ as a triangular matrix of $b_{i,j}$'s. This is the notation used in the Appendix. For example the resulting expected value in Example 3.3 can be written as

$$\begin{array}{ccc} -2 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{array} \tag{19}$$

The equation in the preceding proof can be used to give exact formulas for $b_{i,j}$. Let $r = x + y$ and sum over all x, y, then

$$b_{i,j} = 1 - (-1)^{i+j}\sum_{r=i+j}^{k-1}\sum_{s=0}^{m'-k+r}\sum_{t=0}^{n'-k+r}\Bigg((-1)^{s+t} d_{s,t}^{k-1-r}(A') \cdot \\ \cdot \sum_{x=\max\{i,s\}}^{r-\max\{j,t\}}\binom{x}{i}\binom{r-x}{j}\binom{m'-i+x-s-1}{m'-i-1}\binom{n'-j+r-x-t-1}{n'-j-1}\Bigg). \tag{20}$$

Here we use the fact that $d_{s,t}^{k-1-r}(A') = 0$ if $s+k-1-r \ge m'$ or $t+k-1-r \ge n'$.

4.1. **A conjecture about the coefficients** $b_{i,j}$**.** By inspecting the coefficients $b_{i,j}$, one discovers certain patterns. Given a set of zeros Z, let $Z_{row}(i)$, be the number of zeros in row i and $Z_{col}(j)$ the number of zeros in column j. Ordering the nonzero $Z_{row}(i)$, $1 \le i \le m$ by size, we get an integer partition of $|Z|$, call it $\lambda_{row}(Z)$. Similarly we define $\lambda_{col}(Z)$.

Conjecture 4.2. *Given k and Z, write*

$$F_{k,Z}(m,n) = \sum_{i+j<k}\frac{b_{i,j}}{(m-i)(n-j)}.$$

Then the values of $b_{0,0}, b_{1,0}, \ldots, b_{k-1,0}$ *only depend on* $\lambda_{row}(Z)$ *and* $b_{0,0}, b_{0,1}, \ldots, b_{0,k-1}$ *only depend on* $\lambda_{col}(Z)$.

4.2. **The probability that the smallest element in a row is used.** In [O92], B. Olin conjectured, on the basis of experimental data, that in the case $k = m = n$, the probability that the smallest element in a row is used in the optimal k-assignment tends to $1/2$, as $n \to \infty$ (Olin was considering uniform distribution in $[0, 1]$ instead of exponential distribution, but this does not seem to matter in the limit). Olin's original conjecture has recently been proved [A01]. We now show that Conjecture 1.2 implies a more precise result.

Note that the probability that the smallest element in a row is used is equal to the probability that the zero is used in a matrix with just one zero. This in turn is equal to the probability that the smallest element in a matrix without zeros is used. Assuming Conjecture 1.2, we can compute this probability.

Theorem 4.3. *Let A be an $m \times n$-matrix with one zero. Let k be a positive integer. Assuming Conjecture 1.2, the probability that the zero is used in the optimal k-assignment is*

$$1 - \frac{\binom{k}{2}}{mn}.$$

Proof. We can assume that the zero is $a_{1,1}$, so $Z = \{(1, 1)\}$. Note that one element from the first row is always used in the optimal k-assignment. We therefore compute the probability that another element in the first row, say $a_{1,2}$, is used. Let $Z' = \{(1, 1), (1, 2)\}$. Let $p_{i,j}$ and $p'_{i,j}$ be the corresponding probabilities for partial $k-1$-coverings with i rows and j columns. We are interested in the difference $p_{i,j} - p'_{i,j}$. If $i+j < k-1$, then $p_{i,j} = p'_{i,j} = 1$. If $i+j = k-1$, then $p_{i,j} - p'_{i,j}$ is the probability that, if i rows and j columns are chosen randomly, position $(1, 1)$ is covered, but position $(1, 2)$ is not. This is the case if and only if column 1 is used, but not row 1 or column 2. The probability for this is

$$\frac{m-i}{m} \cdot \frac{n-j}{n} \cdot \frac{j}{n-1} = \frac{j(m-i)(n-j)}{mn(n-1)}.$$

We have

$$Pr[a_{1,2} \text{ is used}] = \sum_{i+j<k} \frac{p_{i,j} - p'_{i,j}}{(m-i)(n-j)} = \sum_{i+j=k-1} \frac{j}{mn(n-1)} = \frac{\binom{k}{2}}{mn(n-1)}.$$

Hence the probability that one of the elements $a_{1,2}, a_{1,3}, \ldots, a_{1,n}$ is used, is

$$\frac{\binom{k}{2}}{mn}.$$

□

5. Asymptotic consequences

In this section, we show that certain limits can be computed from Conjecture 1.2. We assume throughout this section that Conjecture 1.2 holds.

Let K be a subset of the unit square $[0, 1] \times [0, 1]$. Let n be a positive integer. Later, we will let n tend to infinity. We divide the unit square into squares $s_{i,j}$ of side $1/n$, such that

$$s_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right].$$

Let Z_n be the set of all (i,j) such that $s_{i,j}$ intersects K. We show that under certain circumstances, we can compute the limit of $F_n = F_{n,Z_n}(n,n)$, as $n \to \infty$.

For a fixed n we have

$$F_n = \sum_{i,j} \frac{p_{i,j}}{(n-i)(n-j)},$$

where $p_{i,j}$ is the probability that a set of i rows and j columns is a partial $k-1$-covering.

By the substitutions $x = i/n$, $y = j/n$, we can rewrite this as

$$\int_0^1 \int_0^1 \frac{p_n(x,y)}{(1-x)(1-y)}\, dx\, dy,$$

where $p_n(x,y) = p_{\lfloor xn \rfloor, \lfloor yn \rfloor}$.

If S is a subset of $\mathbb{R}^2$, we let $M_{a,b}(S)$ be the infimum of $a\lambda(S_1)+b\lambda(S_2)$ (Lebesgue-measure), taken over all sets S_1, S_2 of real numbers such that $S \subseteq (S_1 \times \mathbb{R}) \cup (\mathbb{R} \times S_2)$.

Lemma 5.1. *Suppose that K is compact. Let x and y be such that $M_{1-x,1-y}(K) < 1-x-y$. Then $p_n(x,y) \to 1$, as $n \to \infty$.*

Proof. We can find S_1, S_2 and δ such that $K \subseteq (S_1 \times \mathbb{R}) \cup (\mathbb{R} \times S_2)$, and $(1-x)\lambda(S_1) + (1-y)\lambda(S_2) < 1-x-y-\delta$. By the compactness of K, we can assume that S_1 and S_2 are finite unions of intervals.

Let n be a (large) integer. Then the number of intervals of the form $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ which intersect S_1 is at most $n \cdot \lambda(S_1) + constant$, and similarly for the number of such intervals which intersect S_2. Hence we can find a covering α of Z_n using at most $n \cdot (\lambda(S_1) + \lambda(S_2)) + constant$ rows and columns.

Now choose a set α' consisting of $\lfloor nx \rfloor$ rows and $\lfloor ny \rfloor$ columns at random. The average number of rows and columns in α which are not in α' is at most

$$n\left((1-x)\lambda(S_1) + (1-y)\lambda(S_2)\right) + constant < n(1-x-y-\delta) + constant.$$

If n is large, then by the weak law of large numbers, the probability that $|\alpha - \alpha'|$ deviates from its mean value by more than $n \cdot \delta$ tends to zero. With high probability, $|\alpha \bigcup \alpha'| < n$. Hence with high probability, α' is a partial $n-1$-covering. ☐

We say that a covering of Z_n is *minimal*, if no proper subset is a covering of Z_n.

Lemma 5.2. *Suppose that the number of minimal coverings of Z_n grows slower than $(1+\epsilon)^n$ for every ϵ, as $n \to \infty$. Let x and y be such that $M_{1-x,1-y}(K) > 1-x-y$. Then $p_n(x,y) \to 0$, as $k \to \infty$.*

Proof. Let $T(n)$ be the number of minimal coverings of Z_n. Let α be a minimal covering. Since $M_{1-x,1-y}(K) > 1-x-y+\delta$, for a suitably chosen $\delta > 0$, we have

$$(1-x)i_\alpha + (1-y)j_\alpha > n \cdot (1-x-y+\delta).$$

If we let $i = \lfloor n \cdot x \rfloor$ and $j = \lfloor n \cdot y \rfloor$, we certainly have

$$(1-x) \cdot i_\alpha + (1-y) \cdot j_\alpha > n - i - j + 2\delta_1 \cdot n,$$

for some δ_1 slightly smaller than $\delta/2$.

Let α' be a random set constructed by letting each row belong to α' with probability x, and each column with probability y. Since with probability at least $1/4$, α' will contain at least i rows and j columns, in order to prove that $p_n(x,y)$ tends to 0, it will suffice to show that the probability that one can cover Z_n by adding $n-i-j-1$ rows and columns to α' tends to 0.

Let $B(p)$ denote the random variable that is 1 with probability p and 0 with probability $1-p$. The number of rows in $\alpha \setminus \alpha'$ is a sum of i_α independent $B(1-x)$-distributed random variables, and similarly, the number of columns in $\alpha \setminus \alpha'$ is a sum of j_α independent $B(1-y)$-variables. In order that the size of $\alpha \setminus \alpha'$ be at most $n-i-j-1$, one of these sums must deviate by at least $n \cdot \delta_1$ from its mean value. By standard estimates in probability theory, this is at most $e^{-c \cdot n}$, for some constant c depending only on δ_1.

Hence the probability that one can obtain a covering of Z_n by adding at most $n-i-j-1$ rows and columns to α' is at most $T(n) \cdot e^{-c \cdot n}$, which, by assumption, tends to 0 as $n \to \infty$. $\square$

Corollary 5.3. If K is compact, and the number of minimal coverings of Z_n grows slower than $(1+\epsilon)^n$ for every ϵ, as $n \to \infty$, then

$$\lim_{n\to\infty} F_n = \int_D \frac{dx\,dy}{(1-x)(1-y)},$$

where D is the region in which $M_{1-x,1-y}(K) < 1-x-y$.

5.1. **An example: Zeros outside a quarter of a circle.** Let K be the region $x^2+y^2 \geq 1$, $0 \leq x, y \leq 1$. To find $M_{a,b}(K)$, for given a and b, we have to find the point where the slope of the curve $x^2+y^2=1$ is $-a/b$. The part of the curve which is above this point should be covered with a horizontal strip, and the part which is to the right of this point should be covered with a vertical strip. This point is

$$\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

Hence

$$M_{a,b}(K) = a\left(1 - \frac{a}{\sqrt{a^2+b^2}}\right) + b\left(1 - \frac{b}{\sqrt{a^2+b^2}}\right),$$

which simplifies to

$$a+b-\sqrt{a^2+b^2}.$$

The number $T(n)$ of minimal coverings of Z_n is at most $n+1$, since any minimal covering consists of, for some i, $0 \leq i \leq n$, columns $i, \dots, n$, and all rows that contain some element of Z_n not covered by the columns.

The region D is given by the inequality $M(K_{1-x,1-y}) < 1-x-y$, which becomes

$$(1-x)^2 + (1-y)^2 > 1.$$

In other words, D is the part of the unit square which is outside the circle of radius 1 centered in the point $(1,1)$. By Corollary 5.3, we have

$$\lim_{n\to\infty} F_n = \int_D \frac{dx\,dy}{(1-x)(1-y)} = \frac{\pi^2}{24}.$$

5.2. **Generalizing the example.** Fix $p > 1$. Let K be the region given by $x^p + y^p \geq 1$, $0 \leq x, y \leq 1$. We wish to compute $M_{a,b}(K)$. To find the point on the curve $x^p + y^p = 1$ where the slope is $-a/b$, we compute:

$$\frac{dy}{dx} = -x^{p-1}(1-x^p)^{1/p-1} = -(x/y)^{p-1}.$$

It is not difficult to see that the point is

$$\left(\frac{a^{\frac{1}{p-1}}}{\left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right)^{1/p}}, \frac{b^{\frac{1}{p-1}}}{\left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right)^{1/p}} \right).$$

We have

$$M_{a,b}(K) = a\left(1 - \frac{a^{\frac{1}{p-1}}}{\left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right)^{1/p}}\right) + b\left(1 - \frac{b^{\frac{1}{p-1}}}{\left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right)^{1/p}}\right) = \tag{21}$$

$$= a - \frac{a^{\frac{p}{p-1}}}{\left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right)^{1/p}} + b - \frac{b^{\frac{p}{p-1}}}{\left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right)^{1/p}} =$$

$$= a + b - \left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Hence the inequality $M_{1-x,1-y}(K) < 1 - x - y$ becomes

$$\left((1-x)^{\frac{p}{p-1}} + (1-y)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} > 1.$$

Again the number of minimal coverings of Z_n is at most $n+1$. To find the limit of F_n, we now by Corollary 5.3 wish to compute the integral

$$\int_{x=0}^{1} \int_{y=(1-x^u)^{1/u}}^{1} \frac{dx\,dy}{xy},$$

where $u = p/(p-1)$. We eliminate the inner integral, and obtain

$$\int_0^1 \frac{-\log\left((1-x^u)^{1/u}\right)}{x}\,dx = -\frac{1}{u}\int_0^1 \frac{\log(1-x^u)}{x}\,dx.$$

We make the substitution $t = x^u$, which gives $dx = \frac{dt}{ux^{u-1}}$. We get

$$-\frac{1}{u^2}\int_0^1 \frac{\log(1-t)}{t}\,dt = -\frac{1}{u^2}\int_0^1 \left(\frac{-t - t^2/2 - t^3/3 - \dots}{t}\right)dt = \tag{22}$$

$$= \frac{1}{u^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \dots\right) = \frac{1}{u^2}\cdot\frac{\pi^2}{6}.$$

Substituting back $p/(p-1)$ for u,

$$\lim_{n\to\infty} F_n = \left(1 - \frac{1}{p}\right)^2 \cdot \frac{\pi^2}{6}.$$

Acknowledgement: We thank Mireille Bousquet-Mélou and Gilles Schaeffer for describing the problem to us.

References

[A92] Aldous, D., *Asymptotics in the random assignment problem*, Pr. Th. Related Fields, **93** (1992) 507–534.

[A01] Aldous, D., *The $\zeta(2)$ limit in the random assignment problem*, Random Structures Algorithms **18** (2001), no 4. 381–418.

[AS99] Alm, Sven-Erick and Sorkin, Gregory B, *Exact expectations and distributions in the random assignment problem*, Combin. Probab. Comput. **11** (2002), no. 3, 217–248.

[BFM98] Beveridge Andrew, Frieze, Alan M. and McDarmid Colin J.H., *Random minimum length spanning trees in regular graphs*, Tech Report, Carnegie Mellon Univ., Pittsburgh PA, February (1998).

[BCR02] Buck, M. W., Chan, C. S., Robbins, D. P., *On the expected value of the minimum assignment*, Random Structures Algorithms 21 (2002), no. 1, 33–58.

[CS98] Coppersmith, D., Sorkin, G. B., *Constructive Bounds and Exact Expectations For the Random Assignment Problem*, IBM Research Report, (1998).

[FM89] Frieze, Alan M. and McDarmid Colin J.H., *On random minimum spanning trees*, Combinatorica **9** (1989).

[F85] Frieze, Alan M., *On the value of a random minimum spanning tree problem*, Disc Applied Math. **10** 47–56 (1985).

[MP85] Mézard and Parisi, *Replicas and optimization*, J. Physiques Lettres **46** 771–778, (1985).

[O92] Olin, Birgitta, *Asymptotic properties of the random assignment problem*, Ph.D. thesis, Kungl Tekniska Högskolan, Stockholm, Sweden, (1992).

[P98] Parisi, Giorgio, *A conjecture on random bipartite matching*, Physics e-Print archive, http://xxx.lanl.gov/ps/cond-mat/9801176, January (1998).

6. Appendix: Computer generated verification of Conjecture 1.1 for $k \leq 5$

Our algorithm for calculating $F_{k,Z}(m,n)$ is essentially a formalization of the method outlined in Section 2. It was implemented in Maple to take advantage of Maple's built in functions for simplifying rational expressions.

Let us immediately state that we do *not* have a proof that this algorithm always returns a result. However, if a result is returned, we know that this result is correct. The difficulty lies in that this algorithm does not use fixed values for m and n, but tries to compute $F_k(m,n)$ as a rational function of m and n. For fixed numerical values of m, n and k, it is in principle always possible to calculate $F_k(m,n)$.

The algorithm uses a data-structure which we will here call an *array*. An array is a matrix whose elements are finite sums of the form

$$\sum_i r_i X_i,$$

where $X_1, X_2, X_3, \dots$ are independent exp(1)-distributed random variables, and r_i is a rational function in the variables m and n. An array M is thought of as representing a window of an m by n matrix A, all of whose entries outside the region visible in M are exp(1)-variables, independent among themselves, and independent of all entries in M. In this way we can for example represent a pattern Z of zeros, independently of m and n, and compute $F_{k,Z}(m,n)$ as a rational function in m and n.

Below are the cases computed to calculate $F_5(m,n)$ using Theorem 2.1. The same computer program has been used to verify Conjecture 1.1 also for $k = 6$, but this computation needed about 10,000 cases, and is not written out here. In each case, M is an array, and an optimal covering of the zeros of M is computed. The other cases that are used for a particular case are listed. The expected value of the optimal k-assignment is denoted F. The value of F is given as a matrix. This matrix represents the coefficients $b_{i,j}$. Hence the entry in position $(i+1, j+1)$ of the matrix is the coefficient of $1/((m-i)(n-j))$ in the expression for F. Sometimes the value of F cannot be written as a linear combination of such terms. Then the extra terms are written out explicitly.

6.1. The case $k = m = n = 7$.

In case $k = m = n$, we can take advantage of the fact that every k-assignment contains one element from every row and every column. This means that we can start by subtracting the minimal element from every row, and then continue by subtracting the minimal element from every column that does not already contain a zero. Hence we need only take into account the cases where Z contains at least one element in every row and every column. It turns out that in the calculation of $F_7(7,7)$, all the cases that occur can easily be reduced to cases that have already occurred in the calculation of $F_k(m,n)$ for $k < 7$. In this way we have verified that $F_7(7,7) = 1 + 1/4 + 1/9 + \cdots + 1/49$, as conjectured.

Case 1

$$k = 0,\ M = [],\ F = 0$$

Case 2

$$k = 1,\ M = [],\ F = \begin{bmatrix} 1 \end{bmatrix}$$

using case 1
Case 3

$$k = 2,\ M = \begin{bmatrix} 0 \end{bmatrix},\ F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

covering row 1
using cases 1 2
Case 4

$$k = 2,\ M = [],\ F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

using case 3
Case 5

$$k = 3,\ M = \begin{bmatrix} 0 & 0 \\ 0 & X_1 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
using cases 1 2 3
Case 6

$$k = 3,\ M = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ F = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

covering column 1
using cases 3 5
Case 7

$$k = 3,\ M = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix},\ F = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
using cases 1 3
Case 8

$$k = 3,\ M = \begin{bmatrix} 0 \end{bmatrix},\ F = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

covering row 1
using cases 6 7
Case 9

$$k = 3,\ M = [],\ F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

using case 8
Case 10

$$k = 4,\ M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & X_2 & X_3 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 1 2 3 5
Case 11

$$k = 4,\ M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X_1 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering row 1
covering column 1
using cases 3 5 10
Case 12

$$k = 4,\ M = \begin{bmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & -2 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 1 3 7
Case 13

$$k = 4,\ M = \begin{bmatrix} 0 & 0 \\ 0 & X_1 \\ 0 & X_2 \end{bmatrix},\ F = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2
using cases 7 11 12
Case 14

$$k = 4,\ M = \begin{bmatrix} 0 & 0 & 0 \\ X_1 & 0 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -2 & 1 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 1 2 5
Case 15

$$k = 4,\ M = \begin{bmatrix} 0 & 0 \\ X_1 & 0 \\ 0 & X_2 \end{bmatrix},\ F = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & -4 & 1 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2
using cases 3 10 14
Case 16

$$k = 4,\ M = \begin{bmatrix} 0 & X_1 & 0 \\ X_2 & 0 & X_3 \\ 0 & X_4 & X_5 \end{bmatrix},\ F = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 1 & -3 & 2 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 1 3 5 7
Case 17

$$k = 4,\ M = \begin{bmatrix} 0 & 0 \\ 0 & X_1 \end{bmatrix},\ F = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
using cases 13 15 16
Case 18

$$k = 4,\ M = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \\ 0 & X_3 \end{bmatrix},\ F = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 0 \\ -3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2
using cases 7 11 16
Case 19

$$k = 4,\ M = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ F = \begin{bmatrix} 3 & 1 & 1 & 1 \\ -5 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering column 1
using cases 17 18
Case 20

$$k=4,\ M=\begin{bmatrix} X_1 & X_2 & 0 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F=\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 1 7
Case 21

$$k=4,\ M=\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix},\ F=\begin{bmatrix} 3 & -5 & 1 & 1 \\ -5 & 4 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
using cases 18 20
Case 22

$$k=4,\ M=\begin{bmatrix} 0 \end{bmatrix},\ F=\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

covering row 1
using cases 19 21
Case 23

$$k=4,\ M=[],\ F=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

using case 22
Case 24

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & X_2 & X_3 \\ \frac{X_4}{(m-2)\,n} & X_5 & X_6 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}+\frac{1}{(m-2)\,(2\,m\,n-5\,n-1)}$$

covering rows 1 2 3
using cases 1 2 3 5
Case 25

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ \frac{X_2}{(m-2)\,(n-1)} & 0 & X_3 & X_4 \\ 0 & X_5 & X_6 & X_7 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\frac{1}{2}\,\frac{2\,m-5}{(m^2-4\,m+4)\,(2\,m\,n-2\,m-5\,n+4)}-\frac{1}{2}\,\frac{1}{(m^2-4\,m+4)\,n}$$

covering rows 1 2 3 4
using cases 1 2 3 10 24
Case 26

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ 0 & 0 & X_2 & X_3 \\ 0 & X_4 & X_5 & X_6 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 2 3 5 10
Case 27

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ 0 & 0 & X_2 & X_3 \\ 0 & \frac{X_4}{(m-2)\,(n-1)} & X_5 & X_6 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\frac{1}{2}\,\frac{2\,m-5}{(m^2-4\,m+4)\,(2\,m\,n-2\,m-5\,n+4)}-\frac{1}{2}\,\frac{1}{(m^2-4\,m+4)\,n}$$

covering rows 1 2 3 4
using cases 1 2 3 5 24
Case 28

$$k=3,\ M=\begin{bmatrix} 0 & 0 \\ 0 & X_1 \\ \dfrac{X_2}{(m-1)\,n} & X_3 \end{bmatrix},\ F=\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\frac{1}{(m-1)\,(2\,m\,n-3\,n-1)}$$

covering rows 1 2
using cases 1 2 3
Case 29

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ \dfrac{X_1}{(m-1)\,(n-1)} & 0 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F=$$

$$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}+\frac{1}{2}\,\frac{2\,m-3}{(m^2-2\,m+1)\,(2\,m\,n-2\,m-3\,n+2)}-\frac{1}{2}\,\frac{1}{(m^2-2\,m+1)\,n}$$

covering rows 1 2 3
using cases 28 1 2 5
Case 30

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & \dfrac{X_2}{(m-1)\,(n-1)} & X_3 \end{bmatrix},\ F=$$

$$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}-\frac{1}{2}\,\frac{1}{(m^2-2\,m+1)\,n}+\frac{1}{2}\,\frac{2\,m-3}{(m^2-2\,m+1)\,(2\,m\,n-2\,m-3\,n+2)}$$

covering rows 1 2 3
using cases 28 1 2 3
Case 31

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ \dfrac{X_3}{(m-2)\,(n-1)} & 0 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+\frac{2\,m-5}{(m^2-4\,m+4)\,(m-1)\,(2\,m\,n-2\,m-5\,n+4)}+\frac{m-3}{(m^2-4\,m+4)\,n}$$

covering row 1
covering columns 1 2
using cases 29 30 3 5 10 25 26 27
Case 32

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & X_2 \\ \dfrac{X_3}{(m-2)\,(n-1)} & 0 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+\frac{1}{2}\,\frac{2\,m-5}{(m^2-4\,m+4)\,(2\,m\,n-2\,m-5\,n+4)}-\frac{1}{2}\,\frac{1}{(m^2-4\,m+4)\,n}$$

covering rows 1 2 3 4
using cases 1 2 5 14 24
Case 33

$$k=4,\ M=\begin{bmatrix} 0 & X_1 & 0 \\ \dfrac{X_2}{(m-1)\,(n-1)} & 0 & X_3 \\ 0 & X_4 & X_5 \end{bmatrix},\ F=$$

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & -2 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}+\frac{1}{2}\,\frac{2\,m-3}{(m^2-2\,m+1)\,(2\,m\,n-2\,m-3\,n+2)}-\frac{1}{2}\,\frac{1}{(m^2-2\,m+1)\,n}$$

covering rows 1 2 3
using cases 28 1 3 7

Case 34

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \dfrac{X_1}{(m-2)(n-1)} & 0 & X_2 & X_3 \\ X_4 & 0 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \\ 0 & X_{10} & X_{11} & X_{12} \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + 2\,\frac{1}{(2\,m\,n - 2\,m - 5\,n + 4)\,(m-1)\,(m-2)}$$

covering row 1
covering columns 1 2
using cases 28 3 5 10 24
Case 35

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ 0 & X_3 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
covering column 1
using cases 31 32 33 34 7 14
Case 36

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & X_2 \\ 0 & 0 & X_3 & X_4 \\ 0 & X_5 & X_6 & X_7 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 3 7 12
Case 37

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & X_2 \\ 0 & X_3 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 2 & -4 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
covering column 1
using cases 36 3 5 10 12 26
Case 38

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & X_2 & X_3 \\ 0 & X_4 & 0 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 3 5 7 16
Case 39

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & X_2 & X_3 \\ 0 & X_4 & X_5 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 35 37 38 16
Case 40

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ X_2 & 0 & X_3 & X_4 \\ 0 & X_5 & X_6 & X_7 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 2 3 10

Case 41

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & X_1 & X_2\\ X_3 & 0 & X_4 & X_5\\ 0 & X_6 & X_7 & X_8\end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1\\ 0 & 2 & 1 & 1 & 0\\ 1 & -4 & 1 & 0 & 0\\ -2 & 2 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}$$

covering row 1
covering columns 1 2
using cases 40 3 5 10 14 26
Case 42

$$k=3,\ M=\begin{bmatrix} 0 & 0\\ 0 & X_1\\ \frac{X_2}{(m-1)(n+1)} & X_3\end{bmatrix},\ F=\begin{bmatrix} 0 & -1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}+\frac{1}{(m-1)(2mn-3n-2+m)}$$

covering rows 1 2
using cases 1 2 3
Case 43

$$k=4,\ M=\begin{bmatrix} 0 & X_1 & 0\\ \frac{X_2}{(m-1)n} & 0 & X_3\\ 0 & X_4 & X_5\end{bmatrix},$$

$$F=\begin{bmatrix} 0 & 1 & -2 & 1\\ 0 & -2 & 2 & 0\\ -1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}+\frac{2m-3}{(m^2-2m+1)(2mn-m-3n+1)}-\frac{1}{(m^2-2m+1)n}$$

covering rows 1 2 3
using cases 42 1 3 7
Case 44

$$k=5,\ M=\begin{bmatrix} 0 & 0 & X_1 & 0\\ 0 & \frac{X_2}{(m-1)(n-1)} & 0 & X_3\\ X_4 & 0 & X_5 & X_6\\ 0 & X_7 & X_8 & X_9\end{bmatrix},F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1\\ 0 & 0 & -2 & 2 & 0\\ 1 & -2 & 1 & 0 & 0\\ -2 & 2 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}$$
$$-\frac{1}{(m^2-2m+1)(n-1)}+\frac{2m-3}{(m^2-2m+1)(2mn-3m-3n+4)}$$

covering rows 1 2 3 4
using cases 42 43 1 3 12
Case 45

$$k=5,\ M=\begin{bmatrix} 0 & 0 & X_1 & 0\\ 0 & 0 & X_2 & X_3\\ 0 & \frac{X_4}{(m-1)(n-1)} & 0 & X_5\\ 0 & X_6 & X_7 & X_8\end{bmatrix},F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1\\ 0 & 0 & -2 & 2 & 0\\ 0 & -1 & 1 & 0 & 0\\ -1 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}-\frac{1}{(m^2-2m+1)(n-1)}$$
$$+\frac{2m-3}{(m^2-2m+1)(2mn-3m-3n+4)}$$

covering rows 1 2 3 4
using cases 42 43 1 3 7
Case 46

$$k=5,\ M=\begin{bmatrix} 0 & \frac{X_1}{(m-1)(n-1)} & 0\\ 0 & 0 & X_2\\ X_3 & 0 & X_4\\ 0 & X_5 & X_6\end{bmatrix},F=\begin{bmatrix} 0 & -2 & -2 & 1 & 1\\ 0 & 2 & 1 & 1 & 0\\ 1 & -4 & 1 & 0 & 0\\ -2 & 2 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}$$
$$+2\,\frac{2m-3}{m(m^2-2m+1)(2mn-3m-3n+4)}+2\,\frac{m-2}{(m^2-2m+1)(n-1)}$$

covering columns 1 2 3
using cases 36 41 44 45 7
Case 47

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ X_3 & X_4 & 0 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 2 5 10 14
Case 48

$$k = 4,\ M = \begin{bmatrix} 0 & 0 & 0 \\ X_1 & 0 & X_2 \\ 0 & \dfrac{X_3}{(m-1)\,n} & X_4 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \frac{2\,m-3}{(m^2-2\,m+1)\,(2\,m\,n-m-3\,n+1)} - \frac{1}{(m^2-2\,m+1)\,n}$$

covering rows 1 2 3
using cases 42 1 2 5
Case 49

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{X_1}{(m-1)\,(n-1)} & X_2 \\ X_3 & X_4 & 0 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \frac{2\,m-3}{(m^2-2\,m+1)\,(2\,m\,n-3\,m-3\,n+4)} - \frac{1}{(m^2-2\,m+1)\,(n-1)}$$

covering rows 1 2 3 4
using cases 42 48 1 2 10
Case 50

$$k = 4,\ M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & \dfrac{X_2}{(m-1)\,n} & X_3 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{(m^2-2\,m+1)\,n} + \frac{2\,m-3}{(m^2-2\,m+1)\,(2\,m\,n-m-3\,n+1)}$$

covering rows 1 2 3
using cases 42 1 2 3
Case 51

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ 0 & 0 & \dfrac{X_2}{(m-1)\,(n-1)} & X_3 \\ 0 & X_4 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \frac{2\,m-3}{(m^2-2\,m+1)\,(2\,m\,n-3\,m-3\,n+4)} - \frac{1}{(m^2-2\,m+1)\,(n-1)}$$

covering rows 1 2 3 4
using cases 42 50 1 2 3
Case 52

$$k = 5,\ M = \begin{bmatrix} \dfrac{X_1}{(m-1)\,(n-1)} & 0 & 0 \\ 0 & 0 & X_2 \\ X_3 & X_4 & 0 \\ 0 & X_5 & X_6 \end{bmatrix},\ F =$$

$$\begin{bmatrix} 0 & -1 & 2 & 1 & 1 \\ 0 & 0 & -4 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + 2\,\frac{m-2}{(m-1)^2\,(n-1)} + 2\,\frac{2\,m-3}{m\,(m-1)^2\,(2\,m\,n-3\,m-3\,n+4)}$$

covering columns 1 2 3
using cases 47 49 51 3 26
Case 53

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & X_2 & X_3 \\ X_4 & X_5 & 0 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 3 7 10 16
Case 54

$$k = 5,\ M = \begin{bmatrix} \frac{X_1}{(m-1)(n-1)} & 0 & 0 \\ 0 & 0 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F = \begin{bmatrix} 0 & -4 & -1 & 1 & 1 \\ -4 & 3 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + 9\,\frac{1}{(3m-4)\,n}$$
$$-\,4\,\frac{1}{(m^2-2m+1)(n-1)}$$
$$+\,4\,\frac{(2m-3)^2}{(m^2-2m+1)\,m\,(3m-4)(2mn-3m-3n+4)}$$

covering rows 1 2 3
using cases 39 46 52 53
Case 55

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & X_2 & X_3 \\ \frac{X_4}{(m-2)(n-1)} & X_5 & 0 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\,\frac{1}{2}\,\frac{2m-5}{(m^2-4m+4)(2mn-2m-5n+4)} - \frac{1}{2}\,\frac{1}{(m^2-4m+4)\,n}$$

covering rows 1 2 3 4
using cases 1 3 7 16 24
Case 56

$$k = 5,\ M = \begin{bmatrix} 0 & X_1 & 0 & 0 \\ 0 & 0 & X_2 & X_3 \\ 0 & X_4 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ 0 & -1 & -1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
covering column 1
using cases 31 33 34 55 7 16
Case 57

$$k = 5,\ M = \begin{bmatrix} \frac{X_1}{(m-1)(n-1)} & 0 & X_2 & 0 \\ 0 & X_3 & 0 & X_4 \\ X_5 & 0 & X_6 & X_7 \\ 0 & X_8 & X_9 & X_{10} \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$-\,\frac{1}{(m^2-2m+1)(n-1)} + \frac{2m-3}{(m^2-2m+1)(2mn-3m-3n+4)}$$

covering rows 1 2 3 4
using cases 42 43 1 5 16
Case 58

$$k = 5,\ M = \begin{bmatrix} \frac{X_1}{(m-1)(n-1)} & X_2 & 0 & 0 \\ 0 & 0 & X_3 & X_4 \\ 0 & X_5 & X_6 & X_7 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ -3 & 2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\,2\,\frac{(2m-3)^2}{(3m-4)(m-1)^2(2mn-3m-3n+4)} - 2\,\frac{m}{(m-1)^2(n-1)}$$
$$+\,6\,\frac{1}{(3m-4)\,n}$$

covering rows 1 2 3
using cases 41 56 57 16
Case 59

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & X_2 & X_3 \\ X_4+\dfrac{X_5}{(m-2)\,n} & X_6 & X_7 \end{bmatrix},$$
$$F=\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}+\frac{2\,m-5}{(6-5\,m+m^2)\,(2\,m\,n-5\,n-1)}$$

covering rows 1 2 3
using cases 1 2 3 5 24
Case 60

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ X_2+\dfrac{X_3}{(m-2)\,(n-1)} & 0 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\frac{1}{2}\,\frac{(2\,m-5)^2}{(16\,m-7\,m^2+m^3-12)\,(2\,m\,n-2\,m-5\,n+4)}+\frac{1}{2}\,\frac{m-3}{(m^2-4\,m+4)\,n}$$
$$-\frac{3}{2}\,\frac{1}{(m-3)\,n}$$

covering rows 1 2 3 4
using cases 59 1 2 3 10
Case 61

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ 0 & 0 & X_2 & X_3 \\ 0 & X_4+\dfrac{X_5}{(m-2)\,(n-1)} & X_6 & X_7 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}-\frac{1}{2}\,\frac{1}{(m-3)\,n}$$
$$+\frac{1}{2}\,\frac{m-3}{(m^2-4\,m+4)\,n}+\frac{1}{2}\,\frac{(2\,m-5)^2}{(m^2-4\,m+4)\,(m-3)\,(2\,m\,n-2\,m-5\,n+4)}$$

covering rows 1 2 3 4
using cases 59 1 2 3 5
Case 62

$$k=3,\ M=\begin{bmatrix} 0 & 0 \\ 0 & X_1 \\ X_2+\dfrac{X_3}{(m-1)\,n} & X_4 \end{bmatrix},\ F=\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\frac{2\,m-3}{(2\,m\,n-3\,n-1)\,(m-1)\,(m-2)}$$

covering rows 1 2
using cases 28 1 2 3
Case 63

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ X_1+\dfrac{X_2}{(m-1)\,(n-1)} & 0 & X_3 \\ 0 & X_4 & X_5 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}-\frac{3}{2}\,\frac{1}{(m-2)\,n}$$
$$+\frac{1}{2}\,\frac{(2\,m-3)^2}{(m^2-2\,m+1)\,(m-2)\,(2\,m\,n-2\,m-3\,n+2)}+\frac{1}{2}\,\frac{m-2}{(m^2-2\,m+1)\,n}$$

covering rows 1 2 3
using cases 62 1 2 5
Case 64

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & X_2+\dfrac{X_3}{(m-1)\,(n-1)} & X_4 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}-\frac{1}{2}\,\frac{1}{(m-2)\,n}$$
$$+\frac{1}{2}\,\frac{m-2}{(m^2-2\,m+1)\,n}+\frac{1}{2}\,\frac{(2\,m-3)^2}{(m-2)\,(m^2-2\,m+1)\,(2\,m\,n-2\,m-3\,n+2)}$$

covering rows 1 2 3
using cases 62 1 2 3
Case 65

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ X_3+\dfrac{X_4}{(m-2)(n-1)} & 0 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} - \frac{3}{2}\,\frac{1}{(m-3)\,n}$$
$$-\frac{1}{2}\,\frac{1}{(m-1)\,n}+\frac{2\,m-5}{(m^2-4\,m+4)\,n}$$
$$+\frac{(2\,m-5)^2}{(m^2-4\,m+4)\,(m-1)\,(m-3)\,(2\,m\,n-2\,m-5\,n+4)}$$

covering row 1
covering columns 1 2
using cases 60 61 63 64 3 5 10 26
Case 66

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & X_2 \\ X_3+\dfrac{X_4}{(m-2)(n-1)} & 0 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\frac{1}{2}\,\frac{(2\,m-5)^2}{(16\,m-7\,m^2+m^3-12)\,(2\,m\,n-2\,m-5\,n+4)}+\frac{1}{2}\,\frac{m-3}{(m^2-4\,m+4)\,n}$$
$$-\frac{3}{2}\,\frac{1}{(m-3)\,n}$$

covering rows 1 2 3 4
using cases 59 1 2 5 14
Case 67

$$k=4,\ M=\begin{bmatrix} 0 & X_1 & 0 \\ X_2+\dfrac{X_3}{(m-1)(n-1)} & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F=\begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$+\frac{1}{2}\,\frac{(2\,m-3)^2}{(m-2)\,(m^2-2\,m+1)\,(2\,m\,n-2\,m-3\,n+2)}+\frac{1}{2}\,\frac{m-2}{(m^2-2\,m+1)\,n}$$
$$-\frac{3}{2}\,\frac{1}{(m-2)\,n}$$

covering rows 1 2 3
using cases 62 1 3 7
Case 68

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ X_1+\dfrac{X_2}{(m-2)(n-1)} & 0 & X_3 & X_4 \\ X_5 & 0 & X_6 & X_7 \\ 0 & X_8 & X_9 & X_{10} \\ 0 & X_{11} & X_{12} & X_{13} \end{bmatrix},$$
$$F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}+2\,\frac{2\,m-5}{(m-2)\,(m-1)\,(m-3)\,(2\,m\,n-2\,m-5\,n+4)}$$

covering row 1
covering columns 1 2
using cases 59 62 3 5 10
Case 69

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ X_1 & 0 & X_2 & X_3 \\ 0 & X_4 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
covering column 1
using cases 65 66 67 68 7 14
Case 70

$$k=5,\ M=\begin{bmatrix} \dfrac{X_1}{(m-1)(n-1)} & 0 & 0 \\ X_2 & 0 & X_3 \\ 0 & X_4 & X_5 \\ 0 & X_6 & X_7 \end{bmatrix},\ F=\begin{bmatrix} 0 & -3 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + 2\,\frac{m-2}{(m^2-2m+1)(n-1)}$$
$$+\,2\,\frac{2m-3}{m(m^2-2m+1)(2mn-3m-3n+4)}$$

covering columns 1 2 3
using cases 37 57 69 16
Case 71

$$k=5,\ M=\begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & X_2 & X_3 \\ X_4+\dfrac{X_5}{(m-2)(n-1)} & X_6 & 0 & X_7 \\ 0 & X_8 & X_9 & X_{10} \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} - \frac{3}{2}\,\frac{1}{(m-3)n}$$
$$+\,\frac{1}{2}\,\frac{m-3}{(m^2-4m+4)n} + \frac{1}{2}\,\frac{(2m-5)^2}{(m^2-4m+4)(m-3)(2mn-2m-5n+4)}$$

covering rows 1 2 3 4
using cases 59 1 3 7 16
Case 72

$$k=5,\ M=\begin{bmatrix} 0 & X_1 & 0 & 0 \\ X_2 & 0 & X_3 & X_4 \\ 0 & X_5 & X_6 & X_7 \\ 0 & X_8 & X_9 & X_{10} \end{bmatrix},\ F=\begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ -1 & 0 & -1 & 2 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
covering column 1
using cases 65 67 68 71 7 16
Case 73

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 \\ 0 & X_1 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F=\begin{bmatrix} 0 & 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & 1 & 0 \\ -5 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering row 1
covering column 1
using cases 54 58 70 72
Case 74

$$k=5,\ M=\begin{bmatrix} 0 & X_1 & X_2 & 0 \\ 0 & X_3 & 0 & X_4 \\ 0 & 0 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F=\begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ 0 & 3 & -6 & 3 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 7 20
Case 75

$$k=5,\ M=\begin{bmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ 0 & X_3 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F=\begin{bmatrix} 0 & 3 & -5 & 1 & 1 \\ 0 & -5 & 4 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 56 74 20
Case 76

$$k=5,\ M=\begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & X_2 & 0 & X_3 \\ X_4 & 0 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 3 5 12 16
Case 77

$$k = 5,\ M = \begin{bmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 1 & 0 \\ 1 & -5 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 36 38 41 76 7
Case 78

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & X_2 & 0 & X_3 \\ 0 & 0 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 3 5 7 16
Case 79

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 \\ 0 & X_2 & 0 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 1 & 0 \\ 1 & -5 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 36 41 76 78 7
Case 80

$$k = 5,\ M = \begin{bmatrix} 0 & X_1 & X_2 & 0 \\ 0 & X_3 & 0 & X_4 \\ X_5 & 0 & X_6 & X_7 \\ 0 & X_8 & X_9 & X_{10} \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ 0 & 3 & -6 & 3 & 0 \\ 1 & -4 & 3 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 7 12 20
Case 81

$$k = 5,\ M = \begin{bmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F = \begin{bmatrix} 0 & 3 & -5 & 1 & 1 \\ -2 & -3 & 4 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 75 77 79 80
Case 82

$$k = 5,\ M = \begin{bmatrix} 0 & X_1 & 0 \\ X_2 & 0 & X_3 \\ 0 & X_4 & X_5 \\ 0 & X_6 & X_7 \end{bmatrix},\ F = \begin{bmatrix} 0 & 3 & -5 & 1 & 1 \\ -1 & -4 & 4 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 56 72 80 20
Case 83

$$k = 5,\ M = \begin{bmatrix} 0 & 0 \\ 0 & X_1 \\ 0 & X_2 \end{bmatrix},\ F = \begin{bmatrix} 1 & -4 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2
using cases 73 81 82
Case 84

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 \\ X_1 & 0 & X_2 \\ 0 & X_3 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 37 69 76 16

Case 85

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ X_1 & X_2 & 0 & X_3 \\ X_4 & 0 & X_5 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ -1 & 3 & -3 & 1 & 0 \\ 3 & -6 & 3 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 2 14
Case 86

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 \\ X_1 & X_2 & 0 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 3 & 1 & 1 \\ -1 & 3 & -6 & 1 & 0 \\ 3 & -6 & 3 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 40 85 3
Case 87

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ X_2 & X_3 & 0 & X_4 \\ X_5 & 0 & X_6 & X_7 \\ 0 & X_8 & X_9 & X_{10} \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ -1 & 3 & -4 & 2 & 0 \\ 3 & -6 & 3 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 3 14 16
Case 88

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 \\ X_1 & 0 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F = \begin{bmatrix} 0 & -2 & 0 & 1 & 1 \\ 3 & -3 & -1 & 1 & 0 \\ -5 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 84 86 87
Case 89

$$k = 5,\ M = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & 0 & X_2 \\ X_3 & X_4 & 0 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & -6 & 1 & 0 \\ 1 & -4 & 3 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 47 3 26
Case 90

$$k = 5,\ M = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & 0 & X_2 \\ 0 & X_3 & X_4 \end{bmatrix},\ F = \begin{bmatrix} 0 & -2 & 0 & 1 & 1 \\ -2 & 3 & -2 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 39 53 77 89
Case 91

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & X_2 \\ X_3 & 0 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 2 5 10 14
Case 92

$$k = 5,\ M = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & -6 & 1 & 0 \\ 1 & -4 & 3 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3

using cases 91 3 26
Case 93

$$k=5,\ M=\begin{bmatrix} 0 & 0 & X_1 \\ 0 & X_2 & 0 \\ X_3 & 0 & X_4 \end{bmatrix},\ F=\begin{bmatrix} 0 & -2 & 0 & 1 & 1 \\ -2 & 3 & -2 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 39 53 79 92
Case 94

$$k=5,\ M=\begin{bmatrix} 0 & 0 & X_1 \\ X_2 & X_3 & 0 \\ X_4 & 0 & X_5 \\ 0 & X_6 & X_7 \end{bmatrix},\ F=\begin{bmatrix} 0 & -2 & 0 & 1 & 1 \\ -1 & 7 & -3 & 1 & 0 \\ 3 & -8 & 3 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 41 53 87 7
Case 95

$$k=5,\ M=\begin{bmatrix} 0 & 0 \\ X_1 & 0 \\ 0 & X_2 \end{bmatrix},\ F=\begin{bmatrix} 1 & 6 & 1 & 1 & 1 \\ 1 & -11 & 1 & 1 & 0 \\ -4 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2
using cases 88 90 93 94
Case 96

$$k=5,\ M=\begin{bmatrix} X_1 & 0 & X_2 & 0 \\ 0 & X_3 & 0 & X_4 \\ X_5 & 0 & X_6 & X_7 \\ 0 & X_8 & X_9 & X_{10} \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 2 & -4 & 2 & 0 \\ 1 & -4 & 3 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 5 16
Case 97

$$k=5,\ M=\begin{bmatrix} X_1 & 0 & 0 \\ X_2 & 0 & X_3 \\ 0 & X_4 & X_5 \\ 0 & X_6 & X_7 \end{bmatrix},\ F=\begin{bmatrix} 0 & -2 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 & 0 \\ 3 & -2 & 2 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 37 69 16 96
Case 98

$$k=5,\ M=\begin{bmatrix} 0 & X_1 & X_2 & 0 \\ X_3 & X_4 & 0 & X_5 \\ X_6 & 0 & X_7 & X_8 \\ 0 & X_9 & X_{10} & X_{11} \end{bmatrix},\ F=\begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ -1 & 5 & -7 & 3 & 0 \\ 3 & -7 & 4 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 7 16 20
Case 99

$$k=5,\ M=\begin{bmatrix} 0 & X_1 & 0 \\ X_2 & 0 & X_3 \\ 0 & X_4 & X_5 \end{bmatrix},\ F=\begin{bmatrix} 1 & 1 & -4 & 1 & 1 \\ 1 & -5 & 3 & 1 & 0 \\ -4 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 82 94 97 98
Case 100

$$k=5,\ M=\begin{bmatrix} 0 & 0 \\ 0 & X_1 \end{bmatrix},\ F=\begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
using cases 83 95 99
Case 101

$$k = 3,\ M = \begin{bmatrix} & 0 & 0 \\ & 0 & X_1 \\ X_2 + \dfrac{X_3}{(m-1)(n+1)} & & X_4 \end{bmatrix},$$
$$F = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \frac{2m-3}{(m^3-5m^2+8m-4)(2mn-3n-2+m)} - \frac{m-3}{(m^2-4m+4)n}$$

covering rows 1 2
using cases 42 1 2 3
Case 102

$$k = 4,\ M = \begin{bmatrix} 0 & X_1 & 0 \\ X_2 + \dfrac{X_3}{(m-1)n} & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & -2 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$- \frac{(2m-3)^2}{(m^4-6m^3+13m^2-12m+4)(2mn-m-3n+1)} - \frac{1}{(m^2-2m+1)n}$$
$$+ \frac{1}{(m^2-4m+4)(n-1)}$$

covering rows 1 2 3
using cases 1 3 7 101
Case 103

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & X_2 + \dfrac{X_3}{(m-1)(n-1)} & 0 & X_4 \\ X_5 & 0 & X_6 & X_7 \\ 0 & X_8 & X_9 & X_{10} \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 2 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$- \frac{(2m-3)^2}{(m-2)^2(m-1)^2(2mn-3m-3n+4)} - \frac{1}{(m-1)^2(n-1)} + \frac{1}{(m-2)^2(n-2)}$$

covering rows 1 2 3 4
using cases 1 3 12 101 102
Case 104

$$k = 5,\ M = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & X_2 & X_3 \\ 0 & X_4 + \dfrac{X_5}{(m-1)(n-1)} & 0 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$- \frac{(2m-3)^2}{(m^2-2m+1)(m-2)^2(2mn-3m-3n+4)} - \frac{1}{(m^2-2m+1)(n-1)}$$
$$+ \frac{1}{(m-2)^2(n-2)}$$

covering rows 1 2 3 4
using cases 1 3 7 101 102
Case 105

$$k = 5,\ M = \begin{bmatrix} 0 & X_1 + \dfrac{X_2}{(m-1)(n-1)} & 0 \\ 0 & 0 & X_3 \\ X_4 & 0 & X_5 \\ 0 & X_6 & X_7 \end{bmatrix},\ F = \begin{bmatrix} 0 & -2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{2}\,\frac{m-4}{(m-2)^2(n-2)}$$
$$+ 2\,\frac{m-2}{(m^2-2m+1)(n-1)} - 2\,\frac{(2m-3)^2}{(m-2)^2(m^2-2m+1)\,m\,(2mn-3m-3n+4)}$$
$$- \frac{3}{2}\,\frac{1}{m(n-2)}$$

covering columns 1 2 3
using cases 36 41 7 103 104
Case 106

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ X_1 & 0 & X_2 \\ 0 & X_3+\dfrac{X_4}{(m-1)\,n} & X_5 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$-\frac{(2\,m-3)^2}{(m-2)^2\,(m^2-2\,m+1)\,(2\,m\,n-m-3\,n+1)}-\frac{1}{(m^2-2\,m+1)\,n}$$
$$+\frac{1}{(m-2)^2\,(n-1)}$$

covering rows 1 2 3
using cases 1 2 5 101
Case 107

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & X_1+\dfrac{X_2}{(m-1)\,(n-1)} & X_3 \\ X_4 & X_5 & 0 & X_6 \\ 0 & X_7 & X_8 & X_9 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\frac{1}{(m^2-4\,m+4)\,(n-2)}-\frac{1}{(m-1)^2\,(n-1)}$$
$$-\frac{(2\,m-3)^2}{(m-1)^2\,(m^2-4\,m+4)\,(2\,m\,n-3\,m-3\,n+4)}$$

covering rows 1 2 3 4
using cases 1 2 10 101 106
Case 108

$$k=4,\ M=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X_1 \\ 0 & X_2+\dfrac{X_3}{(m-1)\,n} & X_4 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$-\frac{(2\,m-3)^2}{(m^4-6\,m^3+13\,m^2-12\,m+4)\,(2\,m\,n-m-3\,n+1)}-\frac{1}{(m^2-2\,m+1)\,n}$$
$$+\frac{1}{(m^2-4\,m+4)\,(n-1)}$$

covering rows 1 2 3
using cases 1 2 3 101
Case 109

$$k=5,\ M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 \\ 0 & 0 & X_2+\dfrac{X_3}{(m-1)\,(n-1)} & X_4 \\ 0 & X_5 & X_6 & X_7 \end{bmatrix},\ F=\begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$-\frac{(2\,m-3)^2}{(m-1)^2\,(m-2)^2\,(2\,m\,n-3\,m-3\,n+4)}-\frac{1}{(m-1)^2\,(n-1)}+\frac{1}{(m-2)^2\,(n-2)}$$

covering rows 1 2 3 4
using cases 1 2 3 101 108
Case 110

$$k=5,\ M=\begin{bmatrix} X_1+\dfrac{X_2}{(m-1)\,(n-1)} & 0 & 0 \\ 0 & 0 & X_3 \\ X_4 & X_5 & 0 \\ 0 & X_6 & X_7 \end{bmatrix},\ F=\begin{bmatrix} 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -4 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}-\frac{1}{2}\,\frac{m-4}{(m-2)^2\,(n-2)}$$
$$+2\,\frac{m-2}{(m^2-2\,m+1)\,(n-1)}-2\,\frac{(2\,m-3)^2}{(m^2-2\,m+1)\,m\,(m-2)^2\,(2\,m\,n-3\,m-3\,n+4)}$$
$$+\frac{5}{2}\,\frac{1}{m\,(n-2)}$$

covering columns 1 2 3
using cases 47 3 26 107 109
Case 111

$$k = 5,\ M = \begin{bmatrix} X_1 + \dfrac{X_2}{(m-1)(n-1)} & 0 & 0 \\ 0 & 0 & X_3 \\ 0 & X_4 & X_5 \end{bmatrix},\ F = \begin{bmatrix} 0 & -4 & 0 & 1 & 1 \\ -4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{(m-2)^2 (n-2)}$$
$$+ \frac{27}{4} \frac{1}{(3m-4)n} - 4 \frac{1}{(m^2-2m+1)(n-1)}$$
$$- 4 \frac{(4m^2-12m+9)(2m-3)}{(m^2-2m+1)(m-2)^2 m (3m-4)(2mn-3m-3n+4)}$$
$$+ \frac{1}{2} \frac{1}{(m-2)(n-2)} - \frac{1}{2} \frac{1}{(m-2)n} + \frac{1}{4} \frac{1}{mn} - \frac{1}{2} \frac{1}{m(n-2)}$$

covering rows 1 2 3
using cases 39 53 105 110
Case 112

$$k = 5,\ M = \begin{bmatrix} X_1 + \dfrac{X_2}{(m-1)(n-1)} & 0 & X_3 & 0 \\ 0 & X_4 & 0 & X_5 \\ X_6 & 0 & X_7 & X_8 \\ 0 & X_9 & X_{10} & X_{11} \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \frac{1}{(m^2-4m+4)(n-2)} - \frac{1}{(m^2-2m+1)(n-1)}$$
$$- \frac{(2m-3)^2}{(m^2-2m+1)(m^2-4m+4)(2mn-3m-3n+4)}$$

covering rows 1 2 3 4
using cases 1 5 16 101 102
Case 113

$$k = 5,\ M = \begin{bmatrix} X_1 + \dfrac{X_2}{(m-1)(n-1)} & X_3 & 0 & 0 \\ 0 & 0 & X_4 & X_5 \\ 0 & X_6 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ -3 & 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \frac{1}{2} \frac{1}{(m-2)(n-2)} - \frac{1}{2} \frac{1}{(m-2)n} + \frac{1}{2} \frac{m}{(m-2)^2 (n-2)} - 2 \frac{m}{(m-1)^2 (n-1)}$$
$$+ \frac{9}{2} \frac{1}{(3m-4)n} - 2 \frac{(4m^2-12m+9)(2m-3)}{(3m-4)(m-1)^2 (m-2)^2 (2mn-3m-3n+4)}$$

covering rows 1 2 3
using cases 41 56 16 112
Case 114

$$k = 5,\ M = \begin{bmatrix} X_1 + \dfrac{X_2}{(m-1)(n-1)} & 0 & 0 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \\ 0 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 0 & -3 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$- \frac{1}{2} \frac{m-4}{(m^2-4m+4)(n-2)} + 2 \frac{m-2}{(m-1)^2 (n-1)}$$
$$- 2 \frac{(2m-3)^2}{(m^2-4m+4)(m-1)^2 m (2mn-3m-3n+4)} - \frac{1}{2} \frac{1}{m(n-2)}$$

covering columns 1 2 3
using cases 37 69 16 112
Case 115

$$k = 5,\ M = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & X_3 \\ 0 & X_4 & X_5 \end{bmatrix},\ F = \begin{bmatrix} 1 & 1 & -4 & 1 & 1 \\ 1 & 5 & -1 & 1 & 0 \\ -4 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering row 1
covering column 1
using cases 72 111 113 114
Case 116

$$k = 5,\ M = \begin{bmatrix} X_1 & X_2 & 0 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \\ 0 & X_7 & X_8 \end{bmatrix},\ F = \begin{bmatrix} 1 & 1 & -4 & 1 & 1 \\ -4 & 1 & 2 & 1 & 0 \\ 6 & -5 & 2 & 0 & 0 \\ -4 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2 3
using cases 72 20 98
Case 117

$$k = 5,\ M = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \\ 0 & X_3 \end{bmatrix},\ F = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 11 & -7 & 1 & 1 & 0 \\ -9 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering columns 1 2
using cases 99 115 116
Case 118

$$k = 5,\ M = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ F = \begin{bmatrix} 6 & 1 & 1 & 1 & 1 \\ -9 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering column 1
using cases 100 117
Case 119

$$k = 5,\ M = \begin{bmatrix} X_1 & X_2 & X_3 & 0 \\ X_4 & X_5 & 0 & X_6 \\ X_7 & 0 & X_8 & X_9 \\ 0 & X_{10} & X_{11} & X_{12} \end{bmatrix},\ F = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3 4
using cases 1 20
Case 120

$$k = 5,\ M = \begin{bmatrix} X_1 & X_2 & 0 \\ X_3 & 0 & X_4 \\ 0 & X_5 & X_6 \end{bmatrix},\ F = \begin{bmatrix} -4 & 11 & -9 & 1 & 1 \\ 11 & -21 & 9 & 1 & 0 \\ -9 & 9 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2 3
using cases 116 119
Case 121

$$k = 5,\ M = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix},\ F = \begin{bmatrix} 6 & -9 & 1 & 1 & 1 \\ -9 & 7 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering rows 1 2
using cases 117 120
Case 122

$$k = 5,\ M = \begin{bmatrix} 0 \end{bmatrix},\ F = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

covering row 1
using cases 118 121
Case 123

$$k = 5,\ M = [],\ F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

using case 122

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