# A COUNTER-INTUITIVE CORRELATION IN A RANDOM TOURNAMENT

SVEN ERICK ALM AND SVANTE LINUSSON

Abstract. Consider a randomly oriented graph $G = (V, E)$ and let $a$, $s$ and $b$ be three distinct vertices in $V$. We study the correlation between the events $\{a \to s\}$ and $\{s \to b\}$. We show that, counter-intuitively, when $G$ is the complete graph $K_n$, $n \geq 5$, then the correlation is positive. (It is negative for $n = 3$ and zero for $n = 4$.) We briefly discuss and pose problems for the same question on other graphs.

## 1. Introduction

Given a graph $G = (V, E)$ we orient each edge with equal probability for the two possible directions and independent of all other edges. This model has been studied previously in for instance [5, 9, 10]. Let $a$, $s$ and $b$ be three distinct vertices in $V$. The object of this paper is to study the correlation of the two events $\{a \to s\}$, that there exists a directed path from $a$ to $s$, and $\{s \to b\}$. One might intuitively guess that they are always negatively correlated, i.e. that $\mathbb{P}(a \to s, s \to b) < \mathbb{P}(a \to s) \cdot \mathbb{P}(s \to b)$. This is however not true in general.

In turns out to be easier to study the complementary events, $A := \{a \nrightarrow s\}$, that there does not exist a directed path from $a$ to $s$, and $B := \{s \nrightarrow b\}$ that have the same covariance. In Theorem 2.1 we prove that for $K_n$, $n$ large, the two dominating obstructions for a path from $a$ to $s$ are the event that all edges incident to $a$ are directed towards $a$ and the event that all edges incident to $s$ are directed away from $s$. This fact might even strengthen ones belief that the events $A$ and $B$ are negatively correlated. However, in section 2 we prove that for the complete graph, $K_n$, the events are positively correlated for $n \geq 5$, and we show that $\mathbb{P}(B|A)/\mathbb{P}(B)$ converges to $3/2$ as $n \to \infty$.

In Section 3 we give exact recursions for the probabilities $\mathbb{P}(A)$ and $\mathbb{P}(A \cap B)$. For completeness, in Section 4 we show that the events are negatively correlated when $G$ is a cycle and negatively correlated or independent when $G$ is a tree and give an example of a graph where the correlation is negative or positive depending on the choice of $a, b, s$. We end with stating a number of conjectures and open problems.

In a coming paper, [2], we study this problem when $G$ is the random graph $G(n, p)$.

Having proved something that at first surprised us, we have tried in hindsight to formulate a new hueristic understanding. One way to understand this result is that the event $a \nrightarrow s$ in $K_n$ really has only two cases, of equal probability, we need to consider.

---

*Date*: April 20, 2010.

Svante Linusson is a Royal Swedish Academy of Sciences Research Fellow supported by a grant from the Knut and Alice Wallenberg Foundation.

This research was conducted when both authors visited the Institut Mittag-Leffler (Djursholm, Sweden).

In the first case the outdegree of $a$ is zero which means that we can restrict to the graph on $[n]\backslash\{a\}$, where the event $s \nrightarrow b$ is twice as likely heuristically speaking, see Theorem 2.1. In the second case the indegree of $s$ is zero which makes the event $s \nrightarrow b$ less likely (50% less in fact since only the case of indegree zero for $b$ remains) but it is still larger than zero so together it gives an increase.

The question studied here was posed in [9]. There it was proved that under this model for any vertices $a, b, s, t \in V$ the events $\{s \to a\}$ and $\{s \to b\}$ are never negatively correlated. This was shown to be true also if we first conditioned on $\{s \nrightarrow t\}$, i.e. $\mathbb{P}(s \to a, s \to b | s \nrightarrow t) \geq \mathbb{P}(s \to a | s \nrightarrow t) \cdot \mathbb{P}(s \to b | s \nrightarrow t)$. As a sort of converse it was also proved that $\mathbb{P}(s \to a, b \to t | s \nrightarrow t) \leq \mathbb{P}(s \to a | s \nrightarrow t) \cdot \mathbb{P}(b \to t | s \nrightarrow t)$. The proofs in [9] relied heavily on the results in [3] and [4], where similar statements were proved for edge percolation on a given graph and a result from [10] that relates the random orientation to edge percolation. This cluster of questions on correlation of paths have been inspired by an interesting conjecture due to Kasteleyn, named the Bunkbed conjecture by Häggström [7], see also [8] and Remark 5 in [3].

**Acknowledgment:** We thank Svante Janson, Stanislav Volkov and Johan Wästlund for fruitful discussions. We also want to thank an anonymous referee for comments that has improved the exposition.

## 2. The complete graph $K_n$

For three different vertices, $a$, $s$ and $b$, of $K_n$ we want to know if the event $\{a \to s\}$ and the event $\{s \to b\}$ are positively or negatively correlated. As mentioned in the introduction it turns out to be easier to study the correlation of the complementary events, i.e. $A := \{a \nrightarrow s\}$, that there does not exist a directed path from $a$ to $s$, and the event $B := \{s \nrightarrow b\}$, which have the same correlation.

Think of the vertices of $K_n$ as $[n] := \{1, \dots, n\}$.

**Theorem 2.1.** *For all $n \geq 2$,*

$$\left(\frac{1}{2}\right)^{n-2} \left(1 - \left(\frac{1}{2}\right)^{n-1}\right) \leq \mathbb{P}(A) \leq \left(\frac{1}{2}\right)^{n-2} \left(1 + 3.2 \cdot \left(\frac{7}{8}\right)^{n-1}\right).$$

*In particular,*

$$\lim_{n\to\infty} 2^{n-2} \cdot \mathbb{P}(A) = 1.$$

*Proof.* For a vertex $x \in [n]$, let $O_x$ denote the sets of vertices in $[n] \setminus \{x\}$ that can be reached from $x$ in one step. Similarly, let $I_x$ denote the sets of vertices in $[n] \setminus \{x\}$ that can reach $x$ in one step. We want to prove that $O_a = \emptyset$ and $I_s = \emptyset$ are the two main obstructions for a path from $a$ to $s$. The lower bound is easy,

$$\mathbb{P}(A) \geq \mathbb{P}((O_a = \emptyset) \cup (I_s = \emptyset)) = \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n-3}.$$

For the upper bound, we estimate $\mathbb{P}(A)$ with the probability that there is no path from $a$ to $s$ of length 1,2 or 3. This is equivalent to $a \in O_s$, $O_a \subset O_s \backslash \{a\}$ and the event that the vertices in $O_a$ have no directed edges to vertices in $I_s$, respectively. Note that, if $k = |O_a|$ and $m = |I_s|$, then $k + m \leq n - 2$ is necessary, and there are $k \cdot m$ edges between $O_a$ and $I_s$. We split out the two terms $k = 0$ and $m = 0$ and show that these are dominant.

$$\begin{aligned}
\mathbb{P}(A) &\le \frac{1}{2} \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} \left(\frac{1}{2}\right)^{n-2} \sum_{m=0}^{n-2-k} \binom{n-2-k}{m} \left(\frac{1}{2}\right)^{n-2} \cdot \left(\frac{1}{2}\right)^{k\cdot m} \\
&= \left(\frac{1}{2}\right)^{2n-3} \sum_{m=0}^{n-2} \binom{n-2}{m} + \left(\frac{1}{2}\right)^{2n-3} \sum_{k=1}^{n-2} \binom{n-2}{k} \\
&\quad + \left(\frac{1}{2}\right)^{2n-3} \sum_{k=1}^{n-2} \binom{n-2}{k} \sum_{m=1}^{n-2-k} \binom{n-2-k}{m} \left(\frac{1}{2}\right)^{k\cdot m} \\
&\le \left(\frac{1}{2}\right)^{n-2} + \left(\frac{1}{2}\right)^{2n-3} \sum_{k=1}^{n-3} \binom{n-2}{k} \sum_{m=1}^{n-2-k} \binom{n-2-k}{m} \left(\frac{1}{2}\right)^{k\cdot m}.
\end{aligned}$$

By Lemma 2.2 below, $a(n) := \sum_{k=1}^{n-1} \binom{n}{k} \sum_{m=1}^{n-k} \binom{n-k}{m} \left(\frac{1}{2}\right)^{km} \le 5.6 \cdot \left(\frac{7}{4}\right)^n$, so

$$\mathbb{P}(A) \le \left(\frac{1}{2}\right)^{n-2} + 5.6 \cdot \left(\frac{1}{2}\right)^{2n-3} \left(\frac{7}{4}\right)^{n-2} \le \left(\frac{1}{2}\right)^{n-2} \left(1 + 3.2 \cdot \left(\frac{7}{8}\right)^{n-1}\right).$$

□

**Lemma 2.2.** *For all $n \ge 0$,*

$$a(n) := \sum_{k=1}^{n-1} \binom{n}{k} \sum_{m=1}^{n-k} \binom{n-k}{m} \left(\frac{1}{2}\right)^{km} \le 5.6 \cdot \left(\frac{7}{4}\right)^n.$$

*Proof.* As $a(0) = a(1) = 0$, we will assume that $n \ge 2$. We will use that $(\frac{1}{2})^{km} = (\frac{1}{4})^m \cdot (\frac{1}{2})^{(k-2)m} \le (\frac{1}{4})^m \cdot (\frac{1}{2})^{k-2} = 4 \cdot (\frac{1}{2})^k \cdot (\frac{1}{4})^m$, if $m \ge 1$ and $k \ge 2$, and split the sum into two parts, $k = 1$ and $k \ge 2$.

$$\begin{aligned}
a(n) &= n \cdot \sum_{m=1}^{n-1} \binom{n-1}{m} \left(\frac{1}{2}\right)^m + \sum_{k=2}^{n-1} \binom{n}{k} \sum_{m=1}^{n-k} \binom{n-k}{m} \left(\frac{1}{2}\right)^{km} \\
&\le n \cdot \left(\frac{3}{2}\right)^{n-1} + 4 \cdot \sum_{k=2}^{n-1} \binom{n}{k} \left(\frac{1}{2}\right)^k \sum_{m=1}^{n-k} \binom{n-k}{m} \left(\frac{1}{4}\right)^m \\
&\le n \cdot \left(\frac{3}{2}\right)^{n-1} + 4 \cdot \sum_{k=2}^{n-1} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{5}{4}\right)^{n-k} \le n \cdot \left(\frac{3}{2}\right)^{n-1} + 4 \cdot \left(\frac{7}{4}\right)^n.
\end{aligned}$$

The lemma follows by showing that $n \cdot (\frac{3}{2})^{n-1} \le 1.6 \cdot (\frac{7}{4})^n$ holds for all $n \ge 2$. □

**Remark 2.3.** *As $k = 1$ contributes $n \cdot (\frac{3}{2})^{n-1}$, this gives a lower bound for $a(n)$. This is in fact the dominating term, so that $a(n)$ asymptotically is of order $p(n) \cdot (\frac{3}{2})^n$, where $p(n)$ is a polynomial in $n$. By splitting the sum into three parts, we can, for example, show that $a(n) \le 13.6 \cdot (\frac{13}{8})^n$ for all $n \ge 0$.*

To estimate $\mathbb{P}(A \cap B)$, the following lemma, in combination with Lemma 2.2, is useful.

**Lemma 2.4.** *For all $n \geq 0$,*

$$b(n) := \sum_{k=1}^{n-1} \binom{n}{k} \sum_{i=1}^{n-1-k} \binom{n-k}{i} \left(\frac{1}{2}\right)^{ki} \sum_{m=1}^{k} \binom{k}{m} \left(\frac{1}{2}\right)^{m(n-k-i)} \leq 4 \cdot \left(\frac{7}{4}\right)^n.$$

*Proof.* Note first that $b(0) = b(1) = b(2) = 0$, so that we may assume that $n \geq 3$. Note also that for $k, i \geq 1$, $ki = (k-1)(i-1) + k + i - 1 \geq k + i - 1$, so that $(\frac{1}{2})^{ki} \leq 2(\frac{1}{2})^k(\frac{1}{2})^i$. This gives

$$\begin{aligned} b(n) &\leq 4 \cdot \sum_{k=1}^{n-1} \binom{n}{k} \cdot \left(\frac{1}{2}\right)^k \sum_{i=1}^{n-1-k} \binom{n-k}{i} \cdot \left(\frac{1}{2}\right)^i \cdot \left(\frac{1}{2}\right)^{n-k-i} \sum_{m=1}^{k} \binom{k}{m} \cdot \left(\frac{1}{2}\right)^m \\ &\leq 4 \cdot \sum_{k=1}^{n-1} \binom{n}{k} \cdot \left(\frac{1}{2}\right)^k \cdot 1 \cdot \left(\frac{3}{2}\right)^k = 4 \cdot \sum_{k=1}^{n-1} \binom{n}{k} \cdot \left(\frac{3}{4}\right)^k \leq 4 \cdot \left(\frac{7}{4}\right)^n. \end{aligned}$$

$\square$

**Theorem 2.5.** *For all $n \geq 3$,*

$$\left(\frac{1}{2}\right)^{2n-3} \left(3 - 2\left(\frac{1}{2}\right)^{n-3}\right) \leq \mathbb{P}(A \cap B) \leq \left(\frac{1}{2}\right)^{2n-3} \left(3 + 20.8 \cdot \left(\frac{7}{8}\right)^{n-3}\right).$$

*In particular,*

$$\lim_{n\to\infty} 2^{2n-3} \cdot \mathbb{P}(A \cap B) = 3.$$

*Proof.* A necessary condition for $A \cap B$ is that the edge between $a$ and $s$ is directed from $s$ to $a$, that the edge between $s$ and $b$ is directed from $b$ to $s$ and that the edge between $a$ and $b$ is directed from $b$ to $a$. Let $E$, with $\mathbb{P}(E) = 1/8$, denote this event.

Let $O_a$, $O_s$ and $O_b$ denote the sets of vertices in $[n] \setminus \{a, s, b\}$ that can be reached from $a$, $s$ and $b$ respectively in one step. Similarly, let $I_a$, $I_s$ and $I_b$ denote the sets of vertices in $[n] \setminus \{a, s, b\}$ that can reach $a$, $s$ and $b$ respectively in one step. For the lower bound, note that $E \cap (O_a = O_s = \emptyset) \Rightarrow A \cap B$, $E \cap (O_a = I_b = \emptyset) \Rightarrow A \cap B$ and $E \cap (I_s = I_b = \emptyset) \Rightarrow A \cap B$, so that

$$\begin{aligned} \mathbb{P}(A \cap B) &\geq \mathbb{P}((O_a = O_s = \emptyset) \cup (O_a = I_b = \emptyset) \cup (I_s = I_b = \emptyset))/8 \\ &= \left(\frac{1}{2}\right)^3 \left(3\left(\frac{1}{2}\right)^{2n-6} - 2\left(\frac{1}{2}\right)^{3n-9}\right) = \left(\frac{1}{2}\right)^{2n-3} \left(3 - 2\left(\frac{1}{2}\right)^{n-3}\right), \end{aligned}$$

as $(O_a = O_s = \emptyset) \cap (I_s = I_b = \emptyset) = \emptyset$.

For the upper bound, we note that $A \cap B \Rightarrow E \cap (O_a \subset O_s \subset O_b) \cap F$, where $F$ denotes the event that the vertices in $O_a$ have no directed edges to vertices in $I_s$ and the vertices in $O_s$ have no directed edges to vertices in $I_b$, so that $\mathbb{P}(A \cap B) \leq \mathbb{P}((O_a \subset O_s \subset O_b) \cap F)/8$. Let $k = |O_s|$, $i = |I_b|$ and $m = |O_a|$. Then $0 \leq k \leq n-3$, $0 \leq i \leq n-3-k$ and $0 \leq m \leq k$ and the direction of all $3(n-3)+3 = 3n-6$ edges connected to $a$, $s$ and $b$ are determined. Further, the event $F$ determines the direction of $ki + m(n-3-k-i)$ edges, so that

$$\begin{aligned} \mathbb{P}(A \cap B) &\leq \left(\frac{1}{2}\right)^{3n-6} \sum_{k=0}^{n-3} \binom{n-3}{k} \sum_{i=0}^{n-3-k} \binom{n-3-k}{i} \left(\frac{1}{2}\right)^{ki} \sum_{m=0}^{k} \binom{k}{m} \left(\frac{1}{2}\right)^{m(n-3-k-i)} \\ &= \left(\frac{1}{2}\right)^{3n-6} \cdot (S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7), \end{aligned}$$

where the triple sum is split into seven parts:
1: $k = 0 \Rightarrow m = 0$, 2: $k = n-3 \Rightarrow i = 0$, 3: $i = m = 0, 1 \le k \le n-4$,
4: $i = 0, m \ge 1, 1 \le k \le n-4$, 5: $m = 0, i \ge 1, 1 \le k \le n-4$,
6: $i = n-3-k, m \ge 1, 1 \le k \le n-4$,
7: $1 \le k \le n-4, 1 \le i \le n-4-k, 1 \le m \le k$.
The first three cases correspond to the three cases of the lower bound,

$$S_1 = S_2 = \sum_{i=0}^{n-3} \binom{n-3}{i} = 2^{n-3},$$

$$S_3 = \sum_{k=1}^{n-4} \binom{n-3}{k} = 2^{n-3} - 2 \le 2^{n-3}.$$

The next three can be expressed by the function $a$ of Lemma 2.2,

$$S_4 = \sum_{k=1}^{n-4} \binom{n-3}{k} \sum_{m=1}^{k} \binom{k}{m} \cdot \left(\frac{1}{2}\right)^{m(n-3-k)} = \sum_{j=1}^{n-4} \binom{n-3}{j} \sum_{m=1}^{n-3-j} \binom{n-3-j}{m} \cdot \left(\frac{1}{2}\right)^{mj}$$
$$= a(n-3) \le 5.6 \cdot \left(\frac{7}{4}\right)^{n-3},$$
$$S_5 = \sum_{k=1}^{n-4} \binom{n-3}{k} \sum_{i=0}^{n-3-k} \binom{n-3-k}{i} \cdot \left(\frac{1}{2}\right)^{ki} = a(n-3) \le 5.6 \cdot \left(\frac{7}{4}\right)^{n-3},$$
$$S_6 = \sum_{k=1}^{n-4} \binom{n-3}{k} \cdot \left(\frac{1}{2}\right)^{k(n-3-k)} \sum_{m=1}^{k} \binom{k}{m} \le \sum_{k=1}^{n-4} \binom{n-3}{k} \sum_{m=1}^{k} \binom{k}{m} \cdot \left(\frac{1}{2}\right)^{m(n-3-k)}$$
$$= \sum_{i=1}^{n-4} \binom{n-3}{i} \sum_{m=1}^{n-3-i} \binom{n-3-i}{m} \cdot \left(\frac{1}{2}\right)^{im} = a(n-3) \le 5.6 \cdot \left(\frac{7}{4}\right)^{n-3},$$

and the last by the function $b$ of Lemma 2.4,

$$S_7 = \sum_{k=1}^{n-4} \binom{n-3}{k} \sum_{i=1}^{n-4-k} \binom{n-3-k}{i} \cdot \left(\frac{1}{2}\right)^{ki} \sum_{m=1}^{k} \binom{k}{m} \cdot \left(\frac{1}{2}\right)^{m(n-3-k-i)}$$
$$= b(n-3) \le 4 \cdot \left(\frac{7}{4}\right)^{n-3}.$$

Collecting the estimates gives the theorem. □

**Theorem 2.6.**
$$\lim_{n\to\infty} \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A) \cdot \mathbb{P}(B)}{\mathbb{P}(A \cap B)} = \frac{1}{3}.$$

*Proof.* Follows immediately from Theorems 2.1 and 2.5 as $\mathbb{P}(B) = \mathbb{P}(A)$. □

**Remark 2.7.** *Note that* $\frac{\mathbb{P}(A\cap B)-\mathbb{P}(A)\cdot\mathbb{P}(B)}{\mathbb{P}(A\cap B)} = 1 - \frac{\mathbb{P}(A)\cdot\mathbb{P}(B)}{\mathbb{P}(A)\cdot\mathbb{P}(B \mid A)} = 1 - \frac{\mathbb{P}(B)}{\mathbb{P}(B \mid A)}$, *so that Theorem 2.6 can be formulated as*

$$\lim_{n\to\infty} \frac{\mathbb{P}(B \mid A)}{\mathbb{P}(B)} = \frac{3}{2}.$$

Theorem 2.6 shows that the events $A = \{a \not\twoheadrightarrow s\}$ and $B = \{s \not\twoheadrightarrow b\}$ are positively correlated for sufficiently large $n$. From this follows that the complementary events $\{a \to s\}$ and $\{s \to b\}$ also are positively correlated for sufficiently large $n$. It is in fact true for all $n \geq 5$ as the next theorem shows.

**Theorem 2.8.** *The events $A = \{a \not\twoheadrightarrow s\}$ and $B = \{s \not\twoheadrightarrow b\}$ are negatively correlated for $n = 3$, independent for $n = 4$ and positively correlated for $n \geq 5$.*

*Proof.* From Theorems 2.1 and 2.5 we get

$$\begin{aligned}
\mathbb{P}(A \cap B) - \mathbb{P}(A) \cdot \mathbb{P}(B) &= \mathbb{P}(A \cap B) - (\mathbb{P}(A))^2 \\
&\geq \left(\frac{1}{2}\right)^{2n-3} \cdot \left(3 - 2\left(\frac{1}{2}\right)^{n-3}\right) - \left\{\left(\frac{1}{2}\right)^{n-2} \cdot \left(1 + 3.2 \cdot \left(\frac{7}{8}\right)^{n-1}\right)\right\}^2 \\
&\geq \left(\frac{1}{2}\right)^{2n-4} \cdot \left(6 - 4 \cdot \left(\frac{1}{2}\right)^{n-3} - 1 - 6.4 \cdot \left(\frac{7}{8}\right)^{n-1} - 10.24 \cdot \left(\frac{49}{64}\right)^{n-1}\right) \\
&= \left(\frac{1}{2}\right)^{2n-4} \cdot (5 - c(n)),
\end{aligned}$$

where $c(n)$ is a decreasing function of $n$, with $c(8) < 5$, so that the theorem holds for $n \geq 8$. The remaining cases, $3 \leq n \leq 7$, are proved using the recursion formulas in Lemmas 3.1 and 3.2 in the next section. □

## 3. Exact recursions

For $n \geq 2$, $s \in [n]$ and $K \subset [n] \setminus \{s\}$, let $\{K \not\twoheadrightarrow s\}$ denote the event $\{a \not\twoheadrightarrow s$ for every $a \in K\}$ in $K_n$. With $|K| = k$ define

$$f(n,k) := \mathbb{P}_n(K \not\twoheadrightarrow s),$$

where in particular $f(n, 0) = 1$. Also set $f(1, 0) = 1$ for convenience.

For $n \geq 3$ and $s, b \in [n], K \subset [n] \setminus \{s, b\}$, $s \neq b$ and $|K| = k$ define:

$$g(n,k) := \mathbb{P}_n(K \not\twoheadrightarrow s, s \not\twoheadrightarrow b),$$

where in particular $g(n, 0) = f(n, 1)$. Also let $g(2, 0) := f(2, 1) = 1/2$.

**Lemma 3.1.** *For $n \geq k + 1 \geq 2$ we have*

$$f(n,k) = \sum_{i=0}^{n-k-1} \binom{n-k-1}{i} \frac{(2^k - 1)^i}{2^{k(n-k)}} f(n-k, i).$$

**Lemma 3.2.** *For $n \geq k + 2 \geq 3$ we have*

$$g(n,k) = \sum_{i=0}^{n-k-2} \binom{n-k-2}{i} \frac{(2^k - 1)^i}{2^{k(n-k)}} g(n-k, i).$$

We trust the reader to be able to deduce the correctness of these recursions. Proofs can be found in the arXiv version of this paper, [1]. Using the lemmas, we can recursively compute the desired probabilities $\mathbb{P}(A) = f(n, 1)$ and $\mathbb{P}(A \cap B) = g(n, 1)$ in $K_n$.

## 4. Other graphs

**Proposition 4.1.** *When $G$ is the cyclic graph with $n$ vertices, $C_n$, the covariance between the events $\{a \to s\}$ and $\{s \to b\}$ is at most $-\left(\frac{1}{2}\right)^{2n}$, with equality if and only if the vertices $a$ and $b$ are adjacent to $s$.*

*Proof.* Let $c$, $d$ and $n-c-d$ denote the distances (number of edges) between $a$ and $s$, $s$ and $b$, and $b$ and $a$, respectively. We assume the three vertices to be distinct, so that $c, d, n-c-d \geq 1$. Then,

$$\mathbb{P}(a \to s) = \left(\frac{1}{2}\right)^{c} + \left(\frac{1}{2}\right)^{n-c} - \left(\frac{1}{2}\right)^{n},$$

$$\mathbb{P}(s \to b) = \left(\frac{1}{2}\right)^{d} + \left(\frac{1}{2}\right)^{n-d} - \left(\frac{1}{2}\right)^{n},$$

$$\mathbb{P}(\{a \to s\} \cap \{s \to b\}) = \left(\frac{1}{2}\right)^{c} \cdot \left(\frac{1}{2}\right)^{d} + \left(\frac{1}{2}\right)^{n},$$

where the last term corresponds to the case when there is a directed path $s \to a \to b \to s$. The proposition follows. □

For trees the situation is even simpler, as there are no cycles so that there is a unique path between any two vertices. This implies that when $G$ is a tree (or a forest), the events $\{a \to s\}$ and $\{s \to b\}$ are either independent or mutually exclusive.

It is not difficult to find graphs where the sign of the correlation depends on which vertices are chosen as $a, b, s$. The smallest example is $K_4$ with one edge removed. If the vertices $a, b, s$ are chosen so that $\{a, b\}$ is the missing edge, then the correlation is $\mathbb{P}(\{a \to s\} \cap \{s \to b\}) - \mathbb{P}(a \to s) \cdot \mathbb{P}(s \to b) = \frac{7}{16} - \left(\frac{21}{32}\right)^2 = \frac{7}{1024} > 0$. If not, then the correlation is negative.

## 5. Open problems and conjectures

From the observations in Section 4, we make the following conjecture.

**Conjecture 5.1.** *For any connected graph $G = (V, E)$ and three distinct vertices $a$, $s$ and $b$ in $V$; if $s$ has degree at most two, then the events $\{a \to s\}$ and $\{s \to b\}$ are independent or negatively correlated.*

Any connected simple graph $G = (V, E), |V| \geq 3$ belongs to (at least) one of the following classes.

- I For any three distinct vertices $a, b, s \in V(G)$, the events $\{a \to s\}$ and $\{s \to b\}$ are non-positively correlated.
- II The events $\{a \to s\}$ and $\{s \to b\}$ are independent for some distinct $a, b, s \in V(G)$ or they are negatively or positively correlated depending on the choice of $a, b, s \in V(G)$.
- III For any three distinct vertices $a, b, s \in V(G)$, the events $\{a \to s\}$ and $\{s \to b\}$ are non-negatively correlated.

We have seen that trees and cycles belong to Class I, $K_n, n \geq 5$ belongs to Class III and $K_4$ minus one edge belongs to Class II. Note that due to independent events there is some overlap between the classes, in particular $K_4$ belongs to all three classes.

**Conjecture 5.2.** *For large $n$ most graphs will belong to Class II.*

In fact we guess that for $n$ large enough, the graphs in Class I are joins (in some vague sense) of cycles and trees. It would be interesting if it was possible to characterize the graphs in Class I or Class III. We formulate the following more specific questions. Recall that outerplanar graphs are the graphs that do not have $K_4$ or $K_{2,3}$ as minors.

**Problems 5.3.** *We pose the following problems:*
*1) Are all graphs in Class I (with $|V(G)| \geq 5$) outerplanar?*
*2) For a given $n$, what is the smallest number $k$ such that there exist $k$ edges whose removal from $K_n$ gives a graph not in Class III?*
*3) Is it true that if $G$ belongs to Class I, but not to Class II, then so does any connected subgraph obtained by removing one edge?*
*4) Similarly, is it true that if $G$ belongs to Class III, but not to Class II, then so does $G$ plus any new edge?*
*5)The results in* [2] *seem to suggest that Class III is larger than Class I. Is this true?*

## References


[1] Sven Erick Alm and Svante Linusson, A counter-intuitive correlation in a random tournament, *early version of the present paper.* arXiv:0906.0240.

[2] Sven Erick Alm and Svante Linusson, Correlations for paths in random orientations of $G(n, p)$, *Preprint 2009.* arXiv:0906.0720.

[3] Jacob van den Berg and Jeff Kahn, A correlation inequality for connection events in percolation, *Annals of Probability* **29** No. 1 (2001), 123–126.

[4] Jacob van den Berg, Olle Häggström and Jeff Kahn, Some conditional correlation inequalities for percolation and related processes, *Rand. Structures Algorithms* **29** (2006), 417–435.

[5] Geoffrey R. Grimmett, Infinite Paths in Randomly Oriented Lattices, *Random Structures and Algorithms* **18**, Issue 3, (2001), 257 – 266.

[6] Geoffrey R. Grimmett, *Percolation*, Springer-Verlag, Berlin, (1999).

[7] Olle Häggström, Probability on Bunkbed Graphs, *Proceedings of FPSAC'03, Formal Power Series and Algebraic Combinatorics* Linköping, Sweden 2003. Available at http://www.fpsac.org/FPSAC03/ARTICLES/42.pdf

[8] Svante Linusson, On percolation and the bunkbed conjecture, accepted for publication in *Combinatorics, Probability and Computing.*

[9] Svante Linusson, A note on correlations in randomly oriented graphs, *Preprint 2009.* arXiv:0905.2881.

[10] Colin McDiarmid, General percolation and random graphs, *Adv. in Appl. Probab.* **13**, 40–60 (1981).



Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06, Uppsala, Sweden.
*E-mail address*: sea@math.uu.se

Department of Mathematics, KTH-Royal Institute of Technology, SE-100 44, Stockholm, Sweden.
*E-mail address*: linusson@math.kth.se