MÖBIUS FUNCTIONS AND CHARACTERISTIC POLYNOMIALS FOR SUBSPACE ARRANGEMENTS EMBEDDED IN B_n

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July 5, 1996

ABSTRACT. In [BS], Björner and Sagan define certain subspace arrangements embedded in the Coxeter hyperplane arrangements of type $\mathcal{B}_n$ and $\mathcal{D}_n$. They give formulas for the exponential generating functions for the Möbius function and for the characteristic polynomial for a class of subspace arrangements embedded in $\mathcal{B}_n$. The purpose of this paper is to give simpler proofs of those formulas and also to generalize to a much wider class of subspace arrangements. As a corollary the exponential generating function for the Euler characteristic is also calculated.

1. INTRODUCTION

A *subspace arrangement* is a finite set

$$\mathcal{A} = \{K_1, \ldots, K_h\}$$

of linear subspaces of $\mathbb{R}^n$. We assume that there are no containments $K_i \subseteq K_j, i \neq j$. For general information on subspace arrangements, see the survey article of Björner [B]. We will deal with arrangements of hyperplanes perpendicular to the vectors in the root system of type A_n:

$$\mathcal{A}_n := \{x_i = x_j : 1 \leq i < j \leq n\},$$

of type B_n:

$$\mathcal{B}_n := \{x_i = \pm x_j : 1 \leq i < j \leq n\} \cup \{x_i = 0 : 1 \leq i \leq n\}$$

and of type D_n:

$$\mathcal{D}_n := \{x_i = \pm x_j : 1 \leq i < j \leq n\}.$$

The x_i are the coordinate functions in $\mathbb{R}^n$. Note that $\mathcal{A}_n \subseteq \mathcal{D}_n \subseteq \mathcal{B}_n$.

The *intersection lattice* $L_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ is the collection of all intersections of the subspaces in $\mathcal{A}$, ordered by reverse inclusion. This lattice has a unique minimal

1991 *Mathematics Subject Classification.* Primary:52B30;Secondary: 05A15, 06A07, 20F55.

Key words and phrases. characteristic polynomial, Euler characteristic, Möbius function, signed graph, subspace arrangement.

element $\hat{0} = \mathbb{R}^n$ and a unique maximal element $\hat{1} = \cap\mathcal{A} = K_1 \cap \cdots \cap K_h$. Let $\mathcal{A}$ be a hyperplane arrangement and let $\mathcal{A}'$ be a subspace arrangement. Then we will say that $\mathcal{A}'$ *is embedded in* $\mathcal{A}$ if each $K_i \in \mathcal{A}'$ is an element of $L_{\mathcal{A}}$. The subspace arrangements in this paper will be embedded in $\mathcal{A}_n$, $\mathcal{B}_n$ or $\mathcal{D}_n$.

We will work in a combinatorial setting that is more general than that of intersection lattices of subspace arrangement, i.e., we will deal with a class of posets of graphs that specialize to intersection lattices if certain conditions are fulfilled. In Section 3 we present such conditions.

In Section 4 formulas are given for the exponential generating function of both the Möbius function and the characteristic polynomial of subposets obtained by restricting the possible sizes of the components of the graphs; as a special case we obtain the generating function for the Euler charateristic of $\bigcup \mathcal{A} \cap S^{d-1}$ in those cases when the subposet is isomorphic to an intersection lattice of a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^d$. The proofs will be elementary manipulations of formal power series. It is also possible to derive the results via the exponential formula, see [F] or [St], but that would not simplify the presentation so the more direct approach is used here. There is also a method due to Blass and Sagan [BlS] to calculate the characteristic polynomials directly by counting integer points in $\mathbb{R}^n$ not on the arrangement.

2. Preliminaries

We will use Zaslavsky's theory of signed graphs [Z1][Z2] to describe the intersection lattices of these subspace arrangements. A *signed graph*, G, has vertex set $V(G) = [n]$ where $[n] = \{1, 2, \ldots, n\}$. The edges $E(G)$ of G can be of three types:

($\imath$) a positive edge between two vertices,

($\imath\imath$) a negative edge between two vertices,

($\imath\imath\imath$) a half-edge at a vertex.

The three different types of edges correspond to the three different types of equations in $\mathcal{B}_n$: $x_i = x_j$, $x_i = -x_j$ and $x_i = 0$. Now define $G_{\mathcal{A}_n}$ to be the graph on n vertices with all the positive edges; define $G_{\mathcal{D}_n}$ to be the graph with all the positive and all the negative edges; and define $G_{\mathcal{B}_n}$ to be the graph with all edges of the three types.

For any pair $U, W \subseteq [n]$ with $U \cap W = \emptyset$ (one of U or W may be empty) we define the *complete balanced graph* $K^b_{U,W}$ to be the graph consisting of all positive edges between vertices in U, all positive edges between vertices in W and all negative edges between a vertex in U and a vertex in W. It is called balanced because multiplying the signs along any cycle gives a positive sign. For any $U \in [n], U \neq \emptyset$ we define the *complete unbalanced graph* K^u_U to be the graph consisting of all edges of both signs between vertices in U; if $\mathcal{A} = \mathcal{B}_n$ then all possible half-edges of U are also included. It is called unbalanced because it contains cycles with negative edge product, where a single half-edge is also considered a negative cycle. Note that the vertices in a complete unbalanced component correspond to coordinates equal to zero.

We can now characterize $L_{\mathcal{A}}$ where $\mathcal{A} = \mathcal{A}_n$, $\mathcal{B}_n$ or $\mathcal{D}_n$ using these graphs.

Theorem 2.1. *(Zaslavsky*[Z2]*) Let $\mathcal{A} = \mathcal{A}_n$, $\mathcal{B}_n$ or $\mathcal{D}_n$. Then the lattice $L_{\mathcal{A}}$ is isomorphic to the lattice of subgraphs $G \subseteq G_{\mathcal{A}}$ such that*

1. every component of G is complete and

2. there is at most one unbalanced component

ordered by inclusion.

□

The isomorphism is obtained by sending each graph G to the subspace $\cap_{e\in G} K_e$ where K_e is the hyperplane corresponding to the edge e. With this isomorphism in mind, we will often talk about these lattices as if they were lattices of graphs.

We will now consider the special class of subposets obtained by restricting the possible component sizes. Given any sets $T, V \subseteq \mathbb{Z}_+ = \{1, 2, 3, \dots\}$ let $\Pi_{n,T,V}$ denote the subposet of all graphs in $L_{\mathcal{B}_n}$ with components K such that $|V(K)| \in T$ if K is balanced and $|V(K)| \in V$ if K is unbalanced. Similarly define $\Pi_{n,T}$ to be the subposet of $L_{\mathcal{A}_n}$ with component sizes in T. Note that these posets are not lattices in general. If $1 \in T$ then there will be a unique least element consisting of the graph with no edges. If $1 \notin T$, then we add the disrete partition as $\hat{0}$ to be able to talk about the Möbius function for lower intervals of the poset. If there is no unique largest element $\hat{1}$ in a poset P then we will use the convention that $\mu(P) = 0$. The posets $\Pi_{n,T}$ were introduced in [BL] and have been studied by several different authors recently, see [B][BW][L][Su]. The more general class of posets $\Pi_{n,T,V}$ appeared in [BS].

The posets $\Pi_{n,T}$ and $\Pi_{n,T,V}$ will not be intersection lattices of subspace arrangements in general. A natural question is: when do they correspond to a subspace arrangement embedded in $\mathcal{A}_n$,$\mathcal{B}_n$ or $\mathcal{D}_n$? An answer is given in Section 2.

Define the *reduced product* of two posets P and Q to be $((P\backslash\{\hat{0}\}) \times (Q\backslash\{\hat{0}\})) \cup \{\hat{0}\}$ and denote it $P\tilde{\times}Q$. We will frequently use the fact that $\mu(P \times Q) = \mu(P)\mu(Q)$ and that $\mu(P\tilde{\times}Q) = -\mu(P)\mu(Q)$. The last formula is proved as Lemma 1 in [L]. Let $\mu_T(n) = \mu_{\Pi_{n,T}}(\hat{0}, \hat{1})$ and let $\mu_{T,V}(n) = \mu_{\Pi_{n,T,V}}(\hat{0}, \hat{1})$. From our convention above we have that $\mu_T(n) = 0$ if $n \notin T$ and $\mu_{T,V}(n) = 0$ if $n \notin V \cup \{0\}$. However, let $\mu_{T,V}(0) = 1$ and $\mu_T(0) = 0$. By the *order ideal* of an element $\pi \in P$ we will mean $P_{\leq\pi} = \{\sigma|\sigma \leq \pi\}$. The *atoms* of a lattice are the elements covering $\hat{0}$, i.e., a is an atom if $\hat{0} < a$ and there is no element z such that $\hat{0} < z < a$.

3. Conditions on the posets to make them intersection lattices

In order to determine if $\Pi_{n,T}$ or $\Pi_{n,T,V}$ is an intersection lattice of a subspace arrangement embedded in $\mathcal{A}_n$, $\mathcal{B}_n$ or $\mathcal{D}_n$ we need to check that the poset is a lattice and that the join of two elements is the same as the join in $\Pi_{n,\mathbb{Z}_+}$ or in $\Pi_{n,\mathbb{Z}_+,\mathbb{Z}_+}$, since the intersection of the two corresponding subspaces is the same. We will call $\Pi_{n,T}$ or $\Pi_{n,T,V}$ *naturally isomorphic* to a subspace arrangement embedded in $\mathcal{A}_n, \mathcal{B}_n$ or $\mathcal{D}_n$ if it is isomorphic via the natural mapping explained in the previous section.

First consider arrangements embedded in $\mathcal{A}_n$. This is easy. Given two incomparable partitions which agree on all blocks except two, the join operator of $\Pi_{n,\mathbb{Z}_+}$ just merges these two blocks together. Hence we must demand additivity of T. In general join merges any number of blocks together, but there will be no problem with the block sizes since we could have merged them together two by two. Hence, if T is additive, then $\Pi_{n,T}$ is a lattice and is naturally isomorphic to the intersection lattice of the subspace arrangement consisting of the spaces corresponding to the atoms in $\Pi_{n,T}$. Hence we have proved:

Proposition 3.1. *For fixed $T \subseteq \mathbb{Z}_+$ the poset $\Pi_{n,T}$ is naturally isomorphic to the intersection lattice of a subspace arrangement embedded in $\mathcal{A}_n$ for every $n \in T$ if and only if*
A1. $a, b \in T,\ a > 1 \Rightarrow a + b \in T.$ □

For arrangements embedded in $\mathcal{B}_n$ the conditions on T and V are the following.

Proposition 3.2. *For fixed $T, V \subseteq \mathbb{Z}_+$ the poset $\Pi_{n,T,V}$ is naturally isomorphic to the intersection lattice of a subspace arrangement embedded in $\mathcal{B}_n$ for every $n \in V$ if and only if*
B1. $T\backslash\{1\} \subseteq V,$
B2. $a, b \in T,\ a > 1 \Rightarrow a + b \in T,$
B3. $a \in T,\ b \in V \Rightarrow a + b \in V.$

Proof In the $\mathcal{B}_n$ arrangement there are four different types of mergings possible. First, given a graph G of $\Pi_{n,T,V}$ with only balanced components we can get a new graph G' by changing some of the signs in a component of size $a > 1$. The join of G and G' in $\Pi_{n,T,V}$ should give a graph with a unbalanced component of size a. Hence $T\backslash\{1\} \subseteq V$ is a necessary condition. We can also merge two different balanced blocks of sizes a and b. The signs on the edges might agree and then this is asking for additivity of T, just as in the $\mathcal{A}_n$ case. The signs might disagree so we need to demand $a + b \in V$, but this follows from the two previous conditions. Finally we need to check what happens when a balanced and unbalanced block merge, this is $B3$. □

Proposition 3.3. *For fixed $T, V \subseteq \mathbb{Z}_+$ the poset $\Pi_{n,T,V}$ is naturally isomorphic to the intersection lattice of a subspace arrangement embedded in $\mathcal{D}_n$ for every $n \in V$ if and only if $1 \notin V$ and B1,B2 and B3 in Proposition 3.2 hold.*

Proof Since $\mathcal{D}_n \subseteq \mathcal{B}_n$, the conditions for $\mathcal{B}_n$ are necessary also for $\mathcal{D}_n$. The subspaces that are intersections of spaces in $\mathcal{B}_n$ but not intersections of spaces in $\mathcal{D}_n$

are those whose defining equation forces exactly one coordinate to be zero. This is equivalent with $1 \notin V$. □

For example, if $1 \in T$ then the only possibility is the so called k-equal arrangement where $T = \{1, k, k+1, k+2, \dots\}$ and $V = \{h, h+1, h+2, \dots\}$, $h \le k$, which is the case studied in detail in [BS]. Note that the propositions above deal with general n. For T and V that do not fulfill the requirements, $\Pi_{n,T,V}$ or $\Pi_{n,T}$ might still be an intersection lattice for some specific small values of n. As an example of this let $T = \mathbb{Z}_+ \backslash \{1, 5\}$, then T does not fulfill $A1$. But nevertheless $\Pi_{6,T}$ is naturally isomorphic to the intersection lattice of a subspace arrangement since $\Pi_{6,T} \cong \Pi_{6,\mathbb{Z}_+\backslash\{1\}}$.

4. Möbius function, characteristic polynomial and Euler characteristic

From now on we will consider the posets $\Pi_{n,T}$ and $\Pi_{n,T,V}$ in general, even though they do not always correspond to a subspace arrangement. First we need to study the structure of the interval $[\hat{0}, K^b_{U,W}]$ when $U \cup W = [n]$.

Lemma 4.1. *Given $V, T \subseteq \mathbb{Z}_+$ and $U, W \subseteq [n]$ such that $U \cup W = [n]$ and $U \cap W = \emptyset$, the interval $[\hat{0}, K^b_{U,W}]$ in $\Pi_{n,T,V}$ is isomorphic to $\Pi_{n,T}$. This means in particular that $\mu_{\Pi_{n,T,V}}(\hat{0}, K^b_{U,W}) = \mu_T(n)$.*

Proof The elements of $[\hat{0}, K^b_{U,W}]$ are graphs consisting of balanced complete components where every edge will have the same sign as in $K^b_{U,W}$. Hence there is a one to one correspondence between them and partitions of $[n]$. It is easily seen that the join and meet operators are the same in both cases. □

We can now prove the following basic recurrence formula.

Lemma 4.2. *If $n \in V$ we have:*

$$\mu_{T,V}(n) = -\sum_{\substack{u \in V \cup \{0\} \\ 0 \le u < n}} \left(\mu_{T,V}(u) \frac{1}{u!} \sum_{\sum i c_i = n-u} {\mu_T}^{c_1}(1) \dots \right.$$
$$\left. \dots {\mu_T}^{c_{n-u}}(n-u) \frac{n!}{\Pi (j!)^{c_j} c_j!} 2^{n-u-\sum c_i} \right), \qquad if\ 1 \in T;$$

$$\mu_{T,V}(n) = -\sum_{\substack{u\in V\\1\le u<n}} \left(\mu_{T,V}(u)\frac{1}{u!} \sum_{\sum ic_i=n-u} (-1)^{\sum c_i} {\mu_T}^{c_1}(1)\dots \right.$$
$$\left. \dots {\mu_T}^{c_{n-u}}(n-u)\,\frac{n!}{\Pi(j!)^{c_j}c_j!}2^{n-u-\sum c_i}\right) +$$
$$+\sum_{\sum ic_i=n}(-1)^{\sum c_i}{\mu_T}^{c_1}(1)\dots{\mu_T}^{c_n}(n)\frac{n!}{\Pi(j!)^{c_j}c_j!}2^{n-\sum c_i}-1, \qquad if\ 1\notin T.$$

PROOF When $1 \in T$ the order ideal defined by one element $\pi \in \Pi_{n,T,V}$ is just the product of the order ideals of the connected components of π. Hence the Möbius function of π is the product of the Möbius function of each component. So we sum over all possible subdivisions of $[n]$ with block sizes in V and T. For a given subdivision of $[n]$ into an unbalanced block of size $u \in V$ and c_i blocks of size i for $i \in T$ there are $\binom{n}{u}$ ways to choose the elements that belong to the unbalanced component. The number of partitions of type $(c_1, \dots, c_{n-u})$ is $\frac{(n-u)!}{\Pi(j!)^{c_j}c_j!}$ and for a block of size i there are 2^{i-1} different complete balanced graphs. By definition of the Möbius function we get the first formula.

When $1 \notin T$ we have instead that the order ideal defined by one element $\pi \in \Pi_{n,T,V}$ is the reduced product of the order ideals of the connected components. Hence the Möbius function for π is the product of the Möbius function for each component with a minus sign when the total number of components is even. This gives the second equation, where the -1 term is for $\hat{0} = (1)(2)\dots(n)$, which is not included in the sum. □

The next step is to calculate the exponential generating function for each specific T and V. Define

$$F_T(x) := \sum_{n=0}^{\infty} \mu_T(n)\frac{x^n}{n!}$$

and

$$F_{T,V}(x) := \sum_{n=0}^{\infty} \mu_{T,V}(n)\frac{x^n}{n!}$$

remembering that $\mu_T(n) = 0$ if $n \notin T$; $\mu_{T,V}(n) = 0$ if $n \in \mathbb{Z}_+ \backslash V$; but $\mu_{T,V}(0) = 1$. Define also for every positive integer n,

$$s_T(n) := \sum_{\sigma\in\Pi_{n,T}} \mu_T(\hat{0},\sigma)$$

and

$$s_{T,V}(n) := \sum_{\sigma \in \Pi_{n,T,V}} \mu_{T,V}(\hat{0}, \sigma).$$

By the definition of the Möbius function, we have $s_T(n) = 0$ for all $n \in T \backslash \{1\}$ and $s_{T,V}(n) = 0$ if $n \in V$, but $s_T(1) = 1$ regardless of whether 1 is in T or not.

Remark Note that if $n \notin V$, $s_{T,V}(n)$ gets the value that $-\mu_{T,V}(n)$ would have had if n had belonged to V. Hence if $n \notin V$ one can replace $\mu_{T,V}(n)$ by $-s_{T,V}(n)$ in Lemma 4.2. $\square$

In the case of $\Pi_{n,T}$ the exponential generating function $F_T(x)$ was described in [BL] when $1 \in T$, and a complete description for general T was given in [L], and independently by Sundaram in [Su]. Define

$$p_T(x) := \sum_{n \in \mathbb{Z}_+} s_T(n) \frac{x^n}{n!}.$$

Theorem 4.3. *The exponential generating function for $\Pi_{n,T}$ is given by:*

$$F_T(x) = \ln(1 + p_T(x)), \qquad if\ 1 \in T;$$

$$F_T(x) = -\ln(e^x - p_T(x)), \qquad if\ 1 \notin T. \qquad \square$$

There is a very pleasing duality between the cases $1 \in T$ and $1 \notin T$. This will continue to be the case. To state the next proposition we need to make the analogous definition

$$p_{T,V}(x) := \sum_{n \in \mathbb{Z}_+} s_{T,V}(n) \frac{x^n}{n!}.$$

Theorem 4.4 is a generalization of Theorem 3.6 in [BS], where the first formula is given.

Theorem 4.4. *The exponential generating function for $\Pi_{n,T,V}$ is given by:*

$$F_{T,V}(x) = (1 + p_T(2x))^{-1/2} (1 + p_{T,V}(x)) \qquad , if\ 1 \in T;$$

$$F_{T,V}(x) = 2 - \left(e^{2x} - p_T(2x)\right)^{-1/2} (e^x - p_{T,V}(x)) \qquad , if\ 1 \notin T.$$

Proof Case 1, $1 \in T$:
Using the recurrence formula in Lemma 4.2 and the result in Theorem 4.3 we get,

$$F_{T,V}(x)\left(1+p_T(2x)\right)^{1/2}=F_{T,V}(x)e^{\frac{1}{2}F_T(2x)}=$$
$$=F_{T,V}(x)\left(1+\frac{F_T(2x)}{2}+\frac{F_T^2(2x)}{2^2 2!}+\dots\right)=$$
$$=\sum_{u\in V\cup\{0\}}\mu_{T,V}(u)\frac{x^u}{u!}\left(\sum_{n>u}\sum_{\sum ic_i=n-u}\frac{(\mu_T(1)(2x)^1)^{c_1}}{2^{c_1}(1!)^{c_1}c_1!}\cdots\frac{(\mu_T(n-u)(2x)^{n-u})^{c_{n-u}}}{2^{c_{n-u}}(n-u)!^{c_{n-u}}c_{n-u}!}+1\right)=$$

$$=\sum_{n\in V}\left(\sum_{\substack{u\in V\\ u\leq n}}\mu_{T,V}(u)\frac{x^u}{u!}\sum_{\sum ic_i=n-u}\left(\frac{\mu_T(1)}{2}\right)^{c_1}\cdots\left(\frac{\mu_T(n-u)}{2}\right)^{c_{n-u}}\frac{(2x)^{n-u}}{\Pi(j!)^{c_j}c_j!}\right)+$$
$$+\sum_{n\notin V}\left(\sum_{\substack{u\in V\\ u\leq n}}\mu_{T,V}(u)\frac{x^u}{u!}\sum_{\sum ic_i=n-u}\left(\frac{\mu_T(1)}{2}\right)^{c_1}\cdots\left(\frac{\mu_T(n-u)}{2}\right)^{c_{n-u}}\frac{(2x)^{n-u}}{\Pi(j!)^{c_j}c_j!}\right)=$$
$$\stackrel{*}{=}\sum_{n\in V}0\frac{x^n}{n!}+\left(1+\sum_{n\in\mathbb{Z}_+\setminus V}s_{T,V}(n)\frac{x^n}{n!}\right)=1+p_{T,V}(x)$$

and the equation follows. The $*$ equality follows from the above Remark and from Lemma 4.2.

Case 2, $1\notin T$: A calculation similar to the $1\in T$ case gives the result. □

We now turn our attention to the *characteristic polynomial* for the poset. Let

$$\phi_{n,T,V}(t):=\sum_{\sigma\in\Pi_{n,T,V}}\mu_{T,V}(\sigma)t^{b(\sigma)}$$

denote the charateristic polynomial of $\Pi_{n,T,V}$ where $b(\sigma)$ is the number of balanced components in σ. Let also

$$G_{T,V}(x,t):=\sum_{n=0}^{\infty}\phi_{n,T,V}(t)\frac{x^n}{n!}$$

denote the exponential generating function of the characteristic polynomial. Just as is the case of the Möbius function there turn out to be nice formulas for the exponential generating function. The $1\in T$ case of the next theorem is a result in [BS, Theorem 3.6],which it thus generalizes.

Theorem 4.5. *The exponential generating function for the characteristic polynomial of* $\Pi_{n,T,V}$ *is given by:*

$$G_{T,V}(x,t) = (1+p_T(2x))^{(t-1)/2}\,(1+p_{T,V}(x)) \qquad , if\ 1\in T;$$

$$G_{T,V}(x,t) = 1+e^{tx} - \left(e^{2x}-p_T(2x)\right)^{(t-1)/2}(e^x - p_{T,V}(x)) \qquad , if\ 1\notin T.$$

Proof Let us do the more difficult $1\notin T$ case. The proof might look very messy, but it is not difficult to verify the calculations step by step.

$$\begin{aligned}
G_{T,V}(x,t) &= \sum_{n=0}^{\infty}\sum_{\sigma\in\Pi_{n,T,V}} \mu_{T,V}(\sigma)t^{b(\sigma)}\frac{x^n}{n!} = \\
&= \sum_{n=1}^{\infty}\Bigg(\sum_{u=1}^{n}\mu_{T,V}(u)\frac{x^u}{u!}\sum_{\sum ic_i=n-u}\left(-\frac{\mu_T(1)t}{2}\right)^{c_1}\cdots\left(-\frac{\mu_T(n-u)t}{2}\right)^{c_{n-u}}\frac{(2x)^{n-u}}{\Pi(j!)^{c_j}c_j!} - \\
&- \sum_{\sum ic_i=n-u}\left(-\frac{\mu_T(1)t}{2}\right)^{c_1}\cdots\left(-\frac{\mu_T(n)t}{2}\right)^{c_n}\frac{(2x)^n}{\Pi(j!)^{c_j}c_j!}\Bigg) + \sum_{n=0}^{\infty}\frac{t^nx^n}{n!} = \\
&= \sum_{u=1}^{\infty}\left(\mu_{T,V}(u)\frac{x^u}{u!}\sum_{n=u}^{\infty}\sum_{\sum ic_i=n-u}\frac{(\mu_T(1)t(2x)^1)^{c_1}}{2^{c_1}(1!)^{c_1}c_1!}\cdots\frac{(\mu_T(n-u)t(2x)^{n-u})^{c_{n-u}}}{2^{c_{n-u}}(n-u)!^{c_{n-u}}c_{n-u}!}\right) - \\
&- \sum_{n=0}^{\infty}\left(\sum_{\sum ic_i=n}\frac{(\mu_T(1)t(2x)^1)^{c_1}}{2^{c_1}(1!)^{c_1}c_1!}\cdots\frac{(\mu_T(n)(2x)^n)^{c_n}}{2^{c_n}(n)!^{c_n}c_n!}\right) + e^{tx} =
\end{aligned}$$

$$\begin{aligned}
&= \sum_{u=1}^{\infty}\mu_{T,V}(u)\frac{x^u}{u!}\prod_{j\geq 1}\left(1+\frac{\mu_T(j)t(2x)^j}{2\cdot j!}+\frac{{\mu_T}^2(j)t^2(2x)^{2j}}{(2\cdot j!)^2 2!}+\ldots\right) - \\
&- \prod_{j\geq 1}\left(1+\frac{\mu_T(j)t(2x)^j}{2\cdot j!}+\frac{{\mu_T}^2(j)t^2(2x)^{2j}}{(2\cdot j!)^2 2!}+\ldots\right) + 1 + e^{tx} = \\
&= (F_{T,V}(x)-1)e^{-\frac{t}{2}F_T(2x)} - e^{-\frac{t}{2}F_T(2x)} + 1 + e^{tx} = \text{[using Theorem 4.3 and 4.4]} \\
&= -\left(e^{2x}-p_T(2x)\right)^{(t-1)/2}(e^x-p_{T,V}(x)) + 1 + e^{tx}.
\end{aligned}$$

□

Now let $B_{n,T,V}$ be the union of the spaces in the arrangement corresponding to $\Pi_{n,T,V}$ where T and V satisfies Proposition 3.2. Let $B^o_{n,T,V} := B_{n,T,V}\cap S^{n-1}$ be the intersection with the unit sphere. Corollary 3.8 of [BL] says that

$$\chi(B^o_{n,T,V}) = 1 + \sum_{\substack{\sigma\in\Pi_{n,T,V}\\ \sigma>\hat{0}}} (-1)^{b(\sigma)}\mu(\hat{0},\sigma),$$

and hence $\tilde{\chi}(B^o_{n,T,V}) = \chi(B^o_{n,T,V}) - 1 = \phi_{n,T,V}(-1) - (-1)^n$. From this we see that the formula for the generating function for the reduced Euler characteristic is obtained by setting $t = -1$ in Theorem 4.5 and adding $\sum_{n=0}^{\infty}(-(-1)^n)\frac{x^n}{n!} = -e^{-x}$.

Corollary 4.6. *Given T and V satisfying Proposition* 3.2 *we have*

$$\sum_{n=0}^{\infty} \tilde{\chi}(B^o_{n,T,V})\frac{x^n}{n!} = (1+p_T(2x))^{-1}(1+p_{T,V}(x)) - e^{-x} \qquad ,if\ 1\in T;$$

$$\sum_{n=0}^{\infty} \chi(B^o_{n,T,V})\frac{x^n}{n!} = 1 - \left(e^{2x} - p_T(2x)\right)^{-1}(e^x - p_{T,V}(x)) \qquad ,if\ 1\notin T.$$

□

Acknowledgement: I thank Anders Björner and Bruce Sagan for suggesting the problem and for helpful comments on this manuscript.

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