

The k -assignment polytope

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Abstract

In this paper we study the structure of the k -assignment polytope, whose vertices are the $m \times n$ (0,1)-matrices with exactly k 1:s and at most one 1 in each row and each column. This is a natural generalisation of the Birkhoff polytope and many of the known properties of the Birkhoff polytope are generalised. A representation of the faces by certain bipartite graphs is given. This tool is used to describe properties of the polytope, especially a complete description of the cover relation in the face poset of the polytope and an exact expression for the diameter. An ear decomposition of these bipartite graphs is constructed.

Key words: Birkhoff polytope, partial matching, face poset, ear decomposition, assignment polytope

1 Introduction

The Birkhoff polytope and its properties have been studied from different viewpoints, see e.g. [2,3,4,7]. The Birkhoff polytope B_n has the $n \times n$ permutation matrices as vertices and is known under many names, such as ‘The polytope of doubly stochastic matrices’ or ‘The assignment polytope’. A natural generalisation of permutation matrices occurring both in optimisation and in theoretical combinatorics is k -assignments. A k -assignment is k entries in

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a matrix that are required to be in different rows and columns. This can also be described as placing k non-attacking rooks on a chess-board.

Let $M(m, n, k)$ denote the polytope in $\mathbb{R}^{m \times n}$ whose vertices are the $m \times n$ (0,1)-matrices with exactly k 1:s and at most one 1 in each row and each column. It will be called ‘The k -assignment polytope’ and this paper is devoted to determine some of its combinatorial properties. The origin of our interest in the k -assignment polytope is the conjecture by G. Parisi on the so called Random Assignment Problem [16], which was immediately generalised by D. Coppersmith and G. Sorkin to k -assignments [6]. An interesting polytopal reformulation and extension of those conjectures were given in [5]. This inspired our study of the facial structure of $M(m, n, k)$ presented in this article. The main conjectures by Parisi and Coppersmith-Sorkin have however now been established by other means [11], [14]. In the first the generalisation to k -assignments was crucial to the proofs. We believe that an increased understanding of the structure of the polytope $M(m, n, k)$ could improve understanding of the behaviour of the optimal assignment and the corresponding network flow problems.

In Section 2 a description of the points in $M(m, n, k)$ in terms of inequalities and equalities is given, and the dimension and the facets of $M(m, n, k)$ are described. Also $M(m, n, k)$ is described as a facet of a transportation polytope, and as a projection of a network flow polytope. Optimization over $M(m, n, k)$ is also discussed.

In Section 3 the face poset of $M(m, n, k)$ is described, and a representation of the faces by bipartite graphs with a special property is given. These bipartite graphs will be called ‘doped elementary graphs’. Some properties following from this representation will be shown, for example the dimension of the faces and the number of one-dimensional faces of $M(m, n, k)$.

In Section 4 the diameter of $M(m, n, k)$ is studied, and an explicit formula for the diameter is given for all values on m , n and k in Theorem 20 and Theorem 21. The proofs of Theorem 19 and Theorem 21 are rather technical, and it is possible that the description of the k -assignment polytope $M(m, n, k)$ by a network flow polytope or the description of $M(m, n, k)$ as a face of a transportation polytope (given in section 2) can be used to simplify these proofs. We suggest this as further research.

In Section 5 an ear decomposition of the doped elementary graphs is constructed, and then the decomposition is used to compute the dimension of the faces of $M(m, n, k)$ in Theorem 30.

2 Some basic properties of the k -assignment polytope

Definition 1 *The k -assignment polytope is the polytope in $\mathbb{R}^{m \times n}$ whose vertices are the $m \times n$ $(0,1)$ -matrices with exactly k 1:s and at most one 1 in each row and each column. It will be denoted $M(m, n, k)$.*

The points in $M(m, n, k)$ are described by real $m \times n$ matrices $X = [x_{ij}]$. If V^1, \dots, V^T , where $V^r = [v_{ij}^r]$, are the vertices of $M(m, n, k)$ then

$$M(m, n, k) = \text{Conv}\{V^1, \dots, V^T\} = \left\{ \sum_{t=1}^T \lambda_t V^t ; \sum_{t=1}^T \lambda_t = 1, \lambda_t \geq 0 \text{ for all } t \right\}.$$

First an easy lemma, for which we omit the proof.

Lemma 2 *The polytope $M(m, n, k)$ has $\binom{m}{k} \cdot \binom{n}{k} \cdot k!$ vertices.*

It is also possible to describe the points in $M(m, n, k)$ with equalities and inequalities.

Theorem 3 *The points of $M(m, n, k)$ are precisely*

$$\{X \in \mathbb{R}_+^{m \times n}; \sum_{i,j} x_{ij} = k, \sum_{i=1}^m x_{ij} \leq 1 \text{ for all } j, \sum_{j=1}^n x_{ij} \leq 1 \text{ for all } i\}.$$

Proof. This could be proved in many different ways. The case for $m = n$ was proved by Mendelsohn and Dulmage in [13] and could be generalised directly. We will however instead deduce the theorem by describing $M(m, n, k)$ as a face of a transportation polytope.

Let $T(\mathbf{r}, \mathbf{c}) := \{X \in \mathbb{R}_+^{(m+1) \times (n+1)}; X\mathbf{1} = \mathbf{r}, \mathbf{1}^T X = \mathbf{c}^T\}$, where $\mathbf{r} := (1, \dots, 1, n-k) \in \mathbb{N}^{m+1}$ and $\mathbf{c} := (1, \dots, 1, m-k) \in \mathbb{N}^{n+1}$. Let also $P(m, n, k) := \{X \in \mathbb{R}_+^{m \times n}; \sum_{i,j} x_{ij} = k, \sum_{i=1}^m x_{ij} \leq 1 \text{ for all } j, \sum_{j=1}^n x_{ij} \leq 1 \text{ for all } i\}$.

Then the projection from $\mathbb{R}^{(m+1) \times (n+1)}$ to $\mathbb{R}^{m \times n}$ which erases the last row and column provides a linear bijection of the facet $F := \{X \in T(\mathbf{r}, \mathbf{c}); x_{m+1, n+1} = 0\}$ of $T(\mathbf{r}, \mathbf{c})$ onto $P(m, n, k)$.

By general theory of transportation polytopes, see e.g. Theorem 2.1 in [7, chapter 4], we know that all vertices of $T(\mathbf{r}, \mathbf{c})$ are integer valued since the defining matrix given by the row and column conditions is totally unimodular. That the defining matrix is totally unimodular can be seen using Theorem 4.1 in [7, chapter 4]. Thus also F and $P(m, n, k)$ are integral polytopes and this shows that $P(m, n, k) = M(m, n, k)$ as wanted. \square

We could also have deduced the previous theorem by extending to an $m +$

$n - k \times m + n - k$ matrix and then project down a face of the Birkhoff polytope B_{m+n-k} . This projection would however map several different vertices of B_{m+n-k} to the same vertex of $M(m, n, k)$. It is interesting to see that $M(m, n, k)$ falls into the class of so called $(1, 0)$ -truncated transportation polytopes, see Section 7.2 of [7]. We have not been able to use the generalities for such polytopes to prove the main theorems of the present paper.

Theorem 3 can be used to determine the dimension of $M(m, n, k)$ and the equations of the facets. It can also be used to describe all faces of $M(m, n, k)$ since the faces are obtained by replacing some of the inequalities by equalities.

Some inequalities in the description of $M(m, n, k)$ in Theorem 3 may be redundant. The facets of $M(m, n, k)$ are given by replacing one of the non-redundant inequalities with an equality, and the dimension is given by subtracting the number of non-redundant equalities from the dimension of the space (which is mn). By symmetry we can assume that $n \leq m$.

We omit the details and list the basic properties of $M(m, n, k)$ for all cases in the following table.

Case	Dimension	Number of facets	Comments
$1 = k \leq n \leq m$	$mn - 1$	mn	$(mn - 1)$ -simplex
$1 < k < n \leq m$	$mn - 1$	$mn + m + n$	
$1 < k = n < m$	$(m - 1)n$	$mn + m$	
$1 < k = n = m$	$(m - 1)(n - 1)$	mn	Birkhoff polytope

The k -assignment polytope $M(m, n, k)$ can also be described by a network flow polytope. Construct a directed bipartite graph from $K_{m,n} \cup NL \cup NR$ (NL and NR are nodes) by directing edges from the m -set L to the n -set R , and adding edges (NL, i) for all $i \in L$ and (j, NR) for all $j \in R$. Let all edges have capacity 1, and define the supply vector y by $y_{NL} = k$, $y_{NR} = -k$, and $y_v = 0$ for all other nodes. Since the capacities and the supply vector are integral we know that the corresponding network flow polytope has integral vertices. See e.g. Theorems 11.11 and 11.12 in [1]. Hence $M(m, n, k)$ is the projection of this polytope into the space of the variables corresponding to the edges (i, j) where $i \in L$, $j \in R$.

This description can for example be useful for linear optimization over $M(m, n, k)$, since there are several good algorithms for optimising over network flow polytopes. In [1, chapters 9 – 11], several pseudopolynomial-time algorithms, polynomial-time algorithms and network simplex algorithms are described. These general methods finds the optimum in time $O(nm^2)$ assuming $(n \leq m)$. Optimizing over a linear function over $M(m, n, k)$ is the same as finding the minimal k -

assignment for in the complete bipartite graph $K_{m,n}$. Efficient algorithms for this is known, in particular the wellknown primal-dual Hungarian method, see e.g [12,15]. Each stage in the Hungarian method takes at most $O(m^2)$ operations ($n \leq m$) and finds an optimal k -assignment given an optimal $k - 1$ -assignment. Thus we find the optimum in $M(m, n, k)$ in $O(km^2)$ which is better for small k .

3 Description of the face poset

There is a one-to-one correspondence between faces of the Birkhoff polytope B_n and bipartite graphs called elementary with $2n$ nodes, which is described in [2, Section 2].

Definition 4 [12, Chapter 4.1] *A bipartite graph G is said to be **elementary** if each edge of G lies in some perfect matching of G .*

The definition in [12] also requires G to be connected, which is not done here, nor in [2]. But each component of an elementary graph G will be elementary according to the original definition (see [3, Section 2]).

Every vertex P of B_n corresponds to a perfect matching where the edge (i, j) is in the matching if and only if $p_{ij} = 1$. A face of B_n corresponds to the elementary graph G that is the union of the perfect matchings corresponding to the vertices of the face. The face corresponding to an elementary graph G is denoted $F^B(G)$, and the vertices of $F^B(G)$ are exactly all perfect matchings P such that $P \subseteq G$. There is a similar correspondence between the faces of $M(m, n, k)$ and doped elementary bipartite graphs, which will be described in this section.

From now on only bipartite graphs $G = (V_1 \cup V_2, E)$ will be considered, but everything is easy to transform to $|V_1| \times |V_2|$ -matrices where the nodes in V_1 correspond to the rows in the matrix, the nodes in V_2 correspond to the columns, and the edges correspond to 1:s in the matrix. Note that as the terms are used in [3, Section 2] a matrix with total support corresponds to an elementary graph, an indecomposable matrix corresponds to a connected elementary graph, and a decomposable matrix corresponds to a not connected elementary graph.

The number of elements in a set B is denoted $|B|$. The set of edges in a graph G is denoted $E(G)$, and the set of nodes is denoted $V(G)$.

The vertices of $M(m, n, k)$ can be represented by k -matchings between L and R , where L is a set of m nodes and R is a set of n nodes. The k -matchings

can be extended to perfect matchings between $L \cup XR$ and $R \cup XL$ where $|XR| = n - k$, $|XL| = m - k$. Let F^M be a face of $M(m, n, k)$ with vertices Q_1, \dots, Q_t . Then the elementary graph G that is the union of all possible extensions of Q_1, \dots, Q_t corresponds to a face $F^B(G)$ of B_{m+n-k} . The face F^M is now a projection of $F^B(G)$ (follows easily from Remark 6 and Theorem 8) and it is possible to use known properties of $F^B(G)$ to examine the properties of F^M .

Definition 5 Let $G = (V_1 \cup V_2, E)$ be a bipartite graph where $|V_1| = |V_2| = m + n - k$. Let $V_1 = L \cup XR$ where L is the first m nodes in V_1 and XR is the last $n - k$ nodes, and let $V_2 = R \cup XL$ where R is the first n nodes in V_2 and XL is the last $m - k$ nodes. Then G is called **extended elementary** if it satisfies all of the following.

- G is elementary.
- There are no edges between nodes in XR and nodes in XL .
- Every node in L is adjacent to all or none of the nodes in XL .
- Every node in R is adjacent to all or none of the nodes in XR .

The number of nodes in L not adjacent to the nodes in XL will be denoted ℓ_0 , and the number of nodes in R not adjacent to the nodes in XR will be denoted r_0 . If $k = n$ and XR is empty, $r_0 = n$, and if $k = m$ and XL is empty, $\ell_0 = m$. An example of an extended elementary graph can be seen in Figure 1.

Figure 1. Extended elementary graph

Remark 6 The definition of an extended elementary graph G implies that if P is a perfect matching of G , then all perfect matchings of $K_{m+n-k, m+n-k}$ with the same k -matching between L and R as P are perfect matchings of G . There are exactly $\Psi := (m - k)! \cdot (n - k)!$ such perfect matchings for each k -matching.

We now construct a new class of graphs called doped elementary by identifying all nodes in XR and all nodes in XL respectively. It is easy to see that there is a one-to-one correspondence between extended elementary graphs and doped elementary graphs. For doped elementary graphs, ℓ_0 is the number of nodes in L not adjacent to NL and r_0 is the number of nodes in R not adjacent to NR .

Definition 7 Let $H = (V_1 \cup V_2, E)$ be a bipartite graph where $V_1 = L \cup \{NR\}$, $|L| = m$ and the node NR is present only if $n > k$, and $V_2 = R \cup \{NL\}$, $|R| = n$ and the node NL is present only if $m > k$. A **doped (m, n, k) -matching** consists of a k -matching of L and R , together with edges from NL and NR to all unmatched nodes in L and R respectively (there are $m - k$ unmatched nodes in L , and $n - k$ unmatched nodes in R). Note that the k -matching is enough to determine the doped (m, n, k) -matching. The graph H is said to be **doped elementary** if each edge of H lies in some doped (m, n, k) -matching of H . Figure 2 shows a doped elementary graph and a doped (m, n, k) -matching.

Figure 2. Doped elementary graph and doped (m, n, k) -matching

Remember that every vertex in $M(m, n, k)$ is a k -matching, and every doped (m, n, k) -matching is determined by a k -matching between L and R . So there is a one-to-one correspondence between vertices $Q = [q_{ij}]$ in $M(m, n, k)$ and doped (m, n, k) -matchings Q' , given by $q_{ij} = 1$ if and only if $(i, j) \in E(Q')$ ($1 \leq i \leq m$ and $1 \leq j \leq n$). This is exactly the same bijection as between the vertices of F and of $M(m, n, k)$ in the proof of Theorem 3.

Theorem 8 *There is a one-to-one correspondence between doped elementary graphs H and faces F^M of $M(m, n, k)$. The face corresponding to H is denoted $F^M(H)$, and its vertices are given by all doped (m, n, k) -matchings that are subsets of H .*

Proof. The empty face \emptyset of $M(m, n, k)$ corresponds to the graph with no edges. Let H be a bipartite graph on vertices $\{1, 2, \dots, m, NR\} \cup \{1, 2, \dots, n, NL\}$ and let Q^1, \dots, Q^t ($Q^\ell = [q_{ij}^\ell]$) be $m \times n$ $(0, 1)$ -matrices which satisfy the following conditions:

- a) $q_{ij}^\ell = 0$ for all ℓ if and only if $(i, j) \notin E(H)$, for $1 \leq i \leq m$, $1 \leq j \leq n$.
- b) $\sum_{j=1}^n q_{ij}^\ell = 1$ for all ℓ if and only if $(i, NL) \notin E(H)$, for $1 \leq i \leq m$.
- c) $\sum_{j=1}^n q_{ij}^\ell \leq 1$ for all ℓ if $(i, NL) \in E(H)$, for $1 \leq i \leq m$.
- d) $\sum_{i=1}^m q_{ij}^\ell = 1$ for all ℓ if and only if $(NR, j) \notin E(H)$, for $1 \leq j \leq n$.
- e) $\sum_{i=1}^m q_{ij}^\ell \leq 1$ for all ℓ if $(NR, j) \in E(H)$, for $1 \leq j \leq n$.
- f) $\sum_{i=1}^m \sum_{j=1}^n q_{ij}^\ell = k$ for all ℓ .

First suppose H is a doped elementary graph. Let Q^1, \dots, Q^t be all $m \times n$ $(0, 1)$ -matrices satisfying the above conditions. They are the vertices of a

face $F^M(H)$ of $M(m, n, k)$, since these conditions together with the condition $q_{ij}^\ell \geq 0$ (which is satisfied by all $(0, 1)$ -matrices) define a face of $M(m, n, k)$. Moreover, the corresponding doped (m, n, k) -matchings are subsets of H since $(i, j) \in E(H)$ if $q_{ij}^\ell = 1$ for any ℓ , $(i, NL) \in E(H)$ if $\sum_{j=1}^n q_{ij}^\ell = 0$ for any ℓ and $(NR, j) \in E(H)$ if $\sum_{i=1}^m q_{ij}^\ell = 0$ for any ℓ . If a doped (m, n, k) -matching $Q' \subseteq H$ it is easy to see that the corresponding vertex of $M(m, n, k)$ satisfies the above conditions, so the vertex is contained in $F^M(H)$.

Then suppose Q^1, \dots, Q^t are the vertices of a face F^M of $M(m, n, k)$. The conditions above together with $q_{ij}^\ell \geq 0$ are given by Q^1, \dots, Q^t and define F^M . Let H be the graph given by the conditions. It is to be shown that H is doped elementary. The doped (m, n, k) -matchings corresponding to the vertices are subsets of H by the same arguments as above. Suppose $(i, j) \in E(H)$. Then there is a vertex Q^ℓ where $q_{ij}^\ell = 1$, hence there is a doped (m, n, k) -matching $Q' \subseteq H$ such that $(i, j) \in E(Q')$. If $(i, NL) \in E(H)$ there is a vertex Q^ℓ where $\sum_{j=1}^n q_{ij}^\ell = 0$, hence there is a doped (m, n, k) -matching $Q' \subseteq H$ such that $(i, NL) \in E(Q')$. The same applies for $(NR, j) \in E(H)$. By the definition H is doped elementary.

Thus there is a one-to-one correspondence as described above between doped elementary graphs H and faces F^M of $M(m, n, k)$, where $Q' \subseteq H$ if and only if the corresponding vertex is a vertex of F^M . \square

Corollary 9 *If H_1 and H_2 are doped elementary graphs, and G_1 and G_2 are their corresponding extended elementary graphs, then*
 $H_1 \subset H_2 \iff F^M(H_1) \subset F^M(H_2) \iff F^B(G_1) \subset F^B(G_2)$.

Proof. The first equivalence follows easily from Definition 7 and Theorem 8, and the second then follows from the fact that $H_1 \subset H_2 \iff G_1 \subset G_2$ and the one-to-one correspondence between faces of B_{m+n-k} and elementary graphs with $2(m+n-k)$ nodes given in [2, Section 2]. \square

Corollary 10 *Let Q_1, \dots, Q_t be vertices of $M(m, n, k)$, and let Q'_1, \dots, Q'_t be the corresponding doped (m, n, k) -matchings. Let H be the (doped elementary) graph $\bigcup_{\ell=1}^t Q'_\ell$. Then $F^M(H)$ is the smallest face of $M(m, n, k)$ containing the vertices Q_1, \dots, Q_t .*

Theorem 11 *The face poset of $M(m, n, k)$ is isomorphic to the semi-lattice of all doped elementary pure subgraphs of $K_{m+1, n+1} \setminus (m+1, n+1)$ ordered by inclusion if $k < n \leq m$. In the case $k = n < m$ the same applies to $K_{m, n+1}$, and in the case $k = n = m$ it applies to $K_{m, n}$.*

Proof. It is easy to see that $M(m, n, k)$ is represented by the graph $K_{m+1, n+1} \setminus (m+1, n+1)$ (or $K_{m, n+1}$ or $K_{m, n}$). The graph without edges is doped elementary and corresponds to \emptyset . There is a one-to-one correspondence between doped elementary graphs and faces of $M(m, n, k)$, and by Corollary 9 the order

is preserved, so the two lattices are isomorphic. \square

Theorem 12 *Let H be a doped elementary graph with t connected components (each of which will be doped elementary graphs with other values on m , n , and k). Then $\dim F^M(H) = |E(H)| - |V(H)| + t$.*

Proof. The non-zero variables in $F^M(H)$ are represented by all edges between L and R in H , so let the edge (i, j) have weight x_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$. The following conditions define $F^M(H)$:

For every edge $(i, j) \in E(H)$, $x_{ij} \geq 0$.

For each node i in L , $\begin{cases} \sum_{(i,\ell) \in E(H)} x_{i\ell} = 1 \text{ if } (i, NL) \notin E(H) \\ \sum_{(i,\ell) \in E(H)} x_{i\ell} \leq 1 \text{ if } (i, NL) \in E(H) \end{cases}$.

For each node j in R , $\begin{cases} \sum_{(\ell,j) \in E(H)} x_{\ell j} = 1 \text{ if } (NR, j) \notin E(H) \\ \sum_{(\ell,j) \in E(H)} x_{\ell j} \leq 1 \text{ if } (NR, j) \in E(H) \end{cases}$.

If NL and NR belong to the same component K , then $\sum_{(i,j) \in E(K)} x_{ij} = k - m + m'$

where $m' = |L \cap V(K)|$.

The dimension of H is the number of variables minus the number of non-redundant equalities. No condition contains variables from more than one component. Hence we can look at each component separately. Take a component K with m' nodes from L and n' nodes from R . If NL and NR are not in K , then $m' = n'$ and there is one equality for each node in K . In that case the equality for one node r in K is redundant and can be removed. Otherwise no equality is redundant, which is shown below by finding an x for each equality such that all conditions except this equality are satisfied.

Let Q_1, \dots, Q_q be the vertices of $F^M(H)$, and let $x^{(0)} = \frac{1}{q} \sum_{\ell=1}^q Q_\ell \in F^M(H)$. Then $x_{ij}^{(0)} \geq \frac{1}{q}$, and for each node adjacent to NL or NR the sum of all incident weights is $\leq 1 - \frac{1}{q}$. Take $i' \in L \cap V(K)$ not adjacent to NL . Let P_1 be a path from i' to NL if possible, else from i' to NR , and otherwise from i' to r . If both NL and NR belong to K , let P_2 be a path from NL to NR . Construct $x^{(1)}$ and $x^{(2)}$ from $x^{(0)}$ by adding and subtracting $\frac{1}{q}$ to/from the weight of every other edge in P_1 and P_2 , respectively. Then $x^{(1)}$ satisfies all conditions except the equality for node i' , and $x^{(2)}$ satisfies all conditions except $\sum_{(i,j) \in E(K)} x_{ij} = k - m + m'$. The same can be done for $j' \in R \cap V(K)$ not adjacent to NR .

Case	# Variables	# Non-redundant equalities
$NL, NR \in V(K)$	$ E(K) - \deg(NL)$ $- \deg(NR)$	$ V(K) - 2 - \deg(NL)$ $- \deg(NR) + 1$
$NL \in V(K), NR \notin V(K)$	$ E(K) - \deg(NL)$	$ V(K) - 1 - \deg(NL)$
$NL, NR \notin V(K)$	$ E(K) $	$ V(K) - 1$

From the table above it is easily seen that the number of variables in K minus the number of non-redundant equalities concerning variables in K is $|E(K)| - |V(K)| + 1$. There were t components in H , so the dimension of $F^M(H)$ is $|E(H)| - |V(H)| + t$. \square

In [3, Corollary 2.11] it is described exactly when $F^B(G_2)$ is a facet of $F^B(G_1)$, given that G_1 is elementary and connected and that G_2 is elementary. This is easily generalised to a description of when $F^M(H_2)$ is a facet of $F^M(H_1)$, given that H_1 is extended elementary and connected and that H_2 is extended elementary (or both H_1 and H_2 are doped elementary). See [8, page 17].

Theorem 13 *Let H be a doped elementary graph. Then $F^M(H)$ is a one-dimensional face of $M(m, n, k) \iff H$ contains exactly one cycle.*

Proof. Let t be the number of components in H . By Theorem 12 follows that $\dim H = 1 \iff |E(H)| = |V(H)| - t + 1$, so H has one more edge than if each component in H were a tree. This is equivalent to that H contains exactly one cycle. \square

By using Theorem 13 and observing that all the vertices of $M(m, n, k)$ have the same degree the following is easy to obtain:

Corollary 14 *The number of one-dimensional faces of $M(m, n, k)$ is*

$$\binom{m}{k} \binom{n}{k} k! \cdot \frac{1}{2} \cdot \left(\sum_{r=2}^k \binom{k}{r} (r-1)! + (m+n-2k) \cdot \sum_{r=1}^k \binom{k}{r} r! + (m-k)(n-k) \cdot \sum_{r=1}^k r \binom{k}{r} r! \right)$$

4 The diameter of $M(m, n, k)$

The graph of a polytope is the graph whose nodes are the vertices of the polytope and whose edges are the one-dimensional faces of the polytope. The

diameter of the polytope is the diameter of its graph, which is the smallest number δ such that between any two nodes in the graph there is a path with at most δ edges. The diameter of a polytope is an important characteristic since it gives a lower bound on the maximum number of steps necessary to solve a linear programming problem on the polytope.

In this section the diameter of $M(m, n, k)$, which is denoted $\delta(M(m, n, k))$, will be computed. The algorithm given in the proofs of Theorem 19 and Theorem 21 can be used to find a path with at most $\delta(M(m, n, k))$ edges between two given vertices of $M(m, n, k)$.

Definition 15 *Let H_1 and H_2 be doped (m, n, k) -matchings. Let $b_L(H_1, H_2)$ be the number of nodes in L adjacent to NL in H_1 but not in H_2 , and let $b_R(H_1, H_2)$ be the number of nodes in R adjacent to NR in H_1 but not in H_2 . If $b = \max(b_L, b_R)$, then b is called the **difference** of H_1 and H_2 . Note that b_L and b_R are well defined.*

Theorem 16 *Let H_1 and H_2 be doped (m, n, k) -matchings, i.e. vertices of $M(m, n, k)$. If $H = H_1 \cup H_2$ contains exactly one cycle, then the difference of the matchings is at most 1.*

Proof. Let H_1 and H_2 be doped (m, n, k) -matchings and $H = H_1 \cup H_2$. Note that in a union of two doped matchings, every node except NL and NR has degree at most 2. Suppose $b_L(H_1, H_2) \geq 2$ (b_R is treated analogously). In H then at least four nodes in L have degree 2. It is easy to see that each of these four nodes has to be contained in a cycle or in a path from NL to NR (no node in L with degree 1 is adjacent to a node in R with degree 2 and vice versa).

Since there are edges from each of these four nodes to NL , at most two of them can be contained in one single cycle, and two paths from NR to NL form a cycle. Hence there are at least two cycles in H . Thus if H contains exactly one cycle, then the difference of H_1 and H_2 is at most 1. \square

Theorem 13 implies the following corollary.

Corollary 17 *If the difference of two doped (m, n, k) -matchings is b , then each shortest path between the two corresponding vertices of $M(m, n, k)$ has at least b edges.*

Lemma 18 *If two doped matchings H_1 and H_2 have the same edges between L and R except in one path of even length between L and R or if they have the same edges except in two odd paths of odd length between L and R in $H_1 \cup H_2$, then $H_1 \cup H_2$ contains exactly one cycle.*

Proof. If H_1 and H_2 have the same edges between L and R except in a path

of even length, then this path is connected with two edges to NL or NR so it is contained in one cycle. Elsewhere H_1 and H_2 are identical, so $H_1 \cup H_2$ contains exactly one cycle.

If H_1 and H_2 have the same edges between L and R except in two paths of odd length, then these paths are contained in two paths from NL to NR . Elsewhere H_1 and H_2 are identical, so $H_1 \cup H_2$ contains exactly one cycle. \square

Theorem 19 *If $k \geq 1$ and $\max(m, n) \leq k + 2$, then $\delta(M(m, n, k)) \leq 2$.*

Proof. For $k = 1$ it is trivial, since $M(m, n, 1)$ is a simplex with diameter 1.

Now suppose $k \geq 2$. Since $\max(m, n) \leq k + 2$ we can write $m = k + a_1$ and $n = k + a_2$, where $0 \leq a_1, a_2 \leq 2$. Let H_0 and H_2 be doped $(k + a_1, k + a_2, k)$ -matchings corresponding to two arbitrary vertices of $M(m, n, k)$. By Theorem 11 and Theorem 13 it suffices to show that there is a $(k + a_1, k + a_2, k)$ -matching H_1 such that the doped elementary graphs $H_0 \cup H_1$ and $H_1 \cup H_2$ contain at most one cycle each. The doped matching H_1 will in this case be called an **intermediate matching** for H_0 and H_2 .

To show the above we will use induction over k . We will construct $(k - 1 + a_1, k - 1 + a_2, k - 1)$ -matchings H'_0 and H'_2 from H_0 and H_2 . By induction there is an intermediate matching H'_1 for H'_0 and H'_2 . Then a matching H_1 will be constructed from H'_1 such that the cycles in $H_i \cup H_1$, $i = 0, 2$, correspond to cycles in $H'_i \cup H'_1$, so H_1 is an intermediate matching for H_0 and H_2 . Suppose that an intermediate matching can be found for all pairs of doped $(k + a_1, k + a_2, k)$ -matchings when $k < p$. Let $k = p$, and let $H = H_0 \cup H_2$. We now treat four different cases separately.

Case I: If the two doped matchings H_0 and H_2 share an edge e between L and R , then we can delete this edge and obtain two doped matchings H'_0 and H'_2 for which $k = p - 1$. By induction, there is an intermediate matching H'_1 for H'_0 and H'_2 , and by adding the edge e we obtain an intermediate matching H_1 for H_0 and H_2 .

Case II: Else if there is a path ℓ_2, r_2, ℓ_3, r_3 of length 3 between L and R in H that is not contained in a cycle of length 4, then we can proceed as follows, see also Figure 3. We may assume $\ell_2 r_2, \ell_3 r_3 \in E(H_0)$ and thus $r_2 \ell_2 \in E(H_2)$. Now, remove the nodes r_2 and ℓ_3 and the three edges in the path, and add the edge $\ell_2 r_3$. Thus, we obtain two doped matchings H'_0 and H'_2 with $k = p - 1$, with $E(H'_0) = E(H_0) \setminus \{\ell_2 r_2, \ell_3, r_3\} \cup \{\ell_2 r_3\}$. By induction there is an intermediate matching H'_1 for H'_0 and H'_2 . Then we construct H_1 as follows. If H'_1 contains the edge $\ell_2 r_3$ then we replace this edge in H_1 by the edges $\ell_2 r_2$ and $\ell_3 r_3$, otherwise we just add the edge $\ell_3 r_2$ to H_1 . It is easy to check that the cycles in $H_i \cup H_1$, $i = 0, 2$, correspond to cycles in $H'_i \cup H'_1$, so H_1 is the desired intermediate matching.

Figure 3. The first case possible to reduce in the induction

Case III: If there are two cycles of length 4 between L and R in H , then we can proceed as shown in Figure 4. We remove the nodes ℓ_1, ℓ_2, r_2 , and r_3 , and all edges in the two cycles. Then we add edges to get one cycle of length 4. Thus we obtain two doped matchings H'_0 and H'_2 with $k = p - 2$. By induction there is an intermediate matching H'_1 . The doped matching H_1 is constructed from H'_1 as shown in Figure 4. There are 4 different cases depending on the edges in H'_1 . The difference of the third case and the fourth case is that in the third case $H'_0 \cup H'_1$ contains exactly one cycle, and in the fourth case $H'_0 \cup H'_1$ contains no cycle (so $H'_0 = H'_1$).

Figure 4. The second case possible to reduce in the induction

Case IV: The remaining case is when H contains 0 or 1 cycle of length 4

between L and R , and paths of length 1 or 2 between L and R where the paths of length 1 are contained in paths from NL to NR . Then there must be an even number of paths of length 1 between L and R . Also, each paths endpoints must be adjacent to NL or NR , and there are at most 4 edges incident to each of NL and NR . If $2c_1$ is the number of paths of length 1 and c_2 is the number of paths of length 2, then $2c_1 + c_2 \leq 4$. Now $k = c_1 + c_2 (+2) \geq 2$ ($+2$ if there is a cycle of length 4).

Case IVa: If there is one cycle of length 4 between L and R , then $(c_1, c_2) \in \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), (1, 2), (0, 3), (0, 4)\}$, and intermediate matchings for H_0 and H_2 in these cases are shown in Figure 5. When there is an alternative for the matching H_2 in the figures, the alternative matching is obtained by altering the edges in the small cycle made of edges from H_2 and its alternative edges.

Figure 5. Base cases with one cycle of length 4

Case IVb: If there is no cycle of length 4 between L and R , then $(c_1, c_2) \in \{(2, 0), (1, 1), (0, 2), (1, 2), (0, 3), (0, 4)\}$, and intermediate matchings for H_0 and H_2 in these cases are shown in Figure 6. When $k = 2$ in Figure 6 only a part of each matching is sketched, but Lemma 18 implies that the given matching is intermediate.

Now it is shown that if H_0 and H_2 are two arbitrary doped (m, n, k) -matchings where $\max(m, n) \leq k+2$, then there is an intermediate matching H_1 such that $H_0 \cup H_1$ and $H_1 \cup H_2$ contain at most one cycle each. Thus $\delta(M(m, n, k)) \leq 2$ when $\max(m, n) \leq k + 2$. \square

Figure 6. Base cases with no cycle of length 4

Theorem 20 *The diameter of $M(m, n, k)$ when $\max(m, n) < k + 2$ is 1 if $(m + n - k) \leq 3$ and 2 if $(m + n - k) \geq 4$.*

Proof. Let H be the union of two doped (m, n, k) -matchings where $\max(m, n) < k + 2$. Since in this case the nodes in H have at most degree 2, H has to have at least 8 edges if there are two cycles in H . There are at most $2(m + n - k)$ edges in H . By Theorem 13 follows that $M(m, n, k)$ have diameter 1 if $m + n - k \leq 3$.

The number of cycles in a doped elementary graph does not decrease when adding one new node in L and one new node in R together with an edge between the new nodes. This increases m , n , k , and $m + n - k$ by 1. Now Theorem 13 and Theorem 19 implies that $\delta(M(m+1, n+1, k+1)) \geq \delta(M(m, n, k))$ if $\max(m, n) < k + 2$.

Figure 7. Matchings corresponding to vertices in $M(4, 4, 4)$, $M(4, 3, 3)$, and $M(3, 3, 2)$

The above is sufficient for completing the proof if $M(4, 4, 4)$, $M(4, 3, 3)$ and $M(3, 3, 2)$ have diameter 2, which is shown by Theorem 13 and Figure 4. \square

Theorem 21 *If $\max(m, n) \geq k + 2$, then $\delta(M(m, n, k)) = \min(\max(m, n) - k, k)$.*

Proof. If $k = 1$ then $M(m, n, k)$ is a simplex and hence it has a complete 1-skeleton, so $\delta(M(m, n, 1)) = 1 = \min(\max(m, n) - 1, 1)$. Now suppose $k \geq 2$. Let G be the doped elementary graph corresponding to $M(m, n, k)$. Take two arbitrary vertices v_0 and v_f . They correspond to two doped (m, n, k) -matchings H_0 and H_f . Remember that a doped (m, n, k) -matching is determined by its

k edges between L and R .

We can assume that H_0 has edges from node j in L to node j in R , for $1 \leq j \leq k$. We can also assume that if $m > 2k$ and/or $n > 2k$ then both H_0 and H_f has edges from the last $m - 2k$ nodes in L to NL and edges from the last $n - 2k$ nodes in R to NR . Let $m' = \min(m, k + 2)$ and $n' = \min(n, k + 2)$. Now denote the first m' nodes in L and the first n' nodes in R by LU and RU respectively, denote the following $\min(m - m', k - 2)$ nodes in L and the following $\min(n - n', k - 2)$ nodes in R by LC and RC respectively, and denote the last $\max(m - 2k, 0)$ nodes in L and the last $\max(n - 2k, 0)$ nodes in R by LD and RD respectively (see Figure 8).

Figure 8. Construction of H_2 from H_0 and H_f

A new doped (m, n, k) -matching H_2 is defined as follows: Let E_1 be the edges of H_f between LU and RU . Let LU_{RC} be all nodes in LU adjacent to nodes in RC in H_f , and let RU_{LC} be all nodes in RU adjacent to nodes in LC in H_f . We can without loss of generality assume that $|LU_{RC}| \leq |RU_{LC}|$. Put $t := |LU_{RC}|$. Let E_2 be a t -matching between LU_{RC} and RU_{LC} . Let E_3 be $k - |E_1| - t$ edges between LU and RU such that $E_1 \cup E_2 \cup E_3$ is a k -matching between LU and RU . Note that H_f has $|E_3|$ edges between LC and R , so $|E_3| \geq t$. Let H_2 be the doped matching containing $E_1 \cup E_2 \cup E_3$. There are examples of H_0 , H_f and H_2 in Figure 8.

Let G' be the subgraph of G with nodes NL , LU , RU and NR and all edges between them in G . Note that H_0 and H_2 are identical outside G' . Then the restrictions H'_0 and H'_2 of H_0 and H_2 to G' are doped (m', n', k) -matchings. Since $\max(m', n') \leq k + 2$ Theorem 13 and Theorem 19 imply that there is a doped (m', n', k) -matching H'_1 in G' such that $H'_0 \cup H'_1$ and $H'_2 \cup H'_1$ have at most one cycle each. This matching H'_1 can be extended to a doped (m, n, k) -matching H_1 in G by adding the edges of H_0 outside G' , so $H_0 \cup H_1$ and $H_2 \cup H_1$ have at most one cycle each.

For $i = 1, \dots, t$, construct the doped (m, n, k) -matching H_{i+2} from H_{i+1} in the following way: Remove edge number i in E_2 and edge number i in E_3 from H_{i+1} , and then add the two edges in H_f adjacent to edge number i in E_2 . Then Lemma 18 implies that $H_{i+2} \cup H_{i+1}$ has exactly one cycle for all i . An example is given in Figure 9. Now H_{t+2} contains all edges of H_f between L and R except $|E_3| - t$ edges incident with nodes in LC .

For $i = t+1, \dots, |E_3|$, construct the doped (m, n, k) -matching H_{i+2} from H_{i+1} by adding one edge between LC and R belonging to H_f but not to H_{i+1} and removing one of the last $|E_3| - t$ edges in E_3 from H_{i+1} , if possible should the removed edge be adjacent to the added edge. By Lemma 18 $H_{i+2} \cup H_{i+1}$ has exactly one cycle for all i , and $H_{t+s+2} = H_f$. An example is given in Figure 9.

Figure 9. Example of H_{i+2} and H_{i+1}

Since $H_i \cup H_{i-1}$ has at most one cycle for $i = 1, \dots, |E_3| + 2$ Theorem 13 implies that there is a path between v_0 and v_f of at most length $|E_3| + 2 \leq \max(|LC|, |RC|) + 2 = \min(\max(m, n) - k - 2, k - 2) + 2 = \min(\max(m, n) - k, k)$ since $\max(m, n) \geq k + 2$. The two vertices v_0 and v_f were arbitrary, hence $\delta(M(m, n, k)) \leq \min(\max(m, n) - k, k)$.

Let H_0 be the same doped (m, n, k) -matching as before, and let H_f be the doped (m, n, k) -matching with edges between the last k nodes in L and R . Then the difference of H_0 and H_f is $\min(\max(m, n) - k, k)$ and hence Corollary 17 implies that $\delta(M(m, n, k)) \geq \min(\max(m, n) - k, k)$.

Thus $\delta(M(m, n, k)) = \min(\max(m, n) - k, k)$. □

5 Ear decomposition

Ear decompositions of bipartite graphs are described in [12]. They were introduced in [9]. In this section we will generalise this and apply the decomposition to compute the dimension of faces of $M(m, n, k)$ in Theorem 30.

Definition 22 [12, Chapter 4.1] *Let x be an edge. Join its endpoints by a path E_1 of odd length (the first ear). Then a sequence of bipartite graphs can be constructed as follows: If $G_{s-1} = x + E_1 + \dots + E_{s-1}$ has already been constructed, add a new ear E_s by picking any two nodes that are connected by an odd path in G_{s-1} and joining them by an odd path ($= E_s$) having no other node in common with G_{s-1} . The decomposition $G_s = x + E_1 + \dots + E_s$ will be called an **ear decomposition** of G_s , and E_i will be called an **ear** ($i = 1, \dots, s$).*

Theorem 23 [12, Theorem 4.1.6] *A bipartite graph G is elementary if and only if each component of G has an ear decomposition.*

Theorem 24 [2, page 6] *If G is an elementary bipartite graph, then the total number of ears in ear decompositions of all the components of G is equal to the dimension of $F^B(G)$.*

Since doped elementary graphs are not elementary graphs, a slightly different kind of ear decomposition is more convenient to use here.

Definition 25 *In each step of an ear decomposition a new ear is added. When new nodes are added because of the new ear they are said to be **activated**. This means that an ear begins and ends in already activated nodes, and has no other already activated nodes.*

Definition 26 *Let G be a connected extended elementary graph. Suppose there is an ear decomposition. An ear that has $2(m - k) - 1$ edges between XL and L or $2(n - k) - 1$ edges between R and XR , and no other edges, is called an **extended ear**. See Figure 10. This means that an extended ear has $m - k - 1$ non-activated nodes in L , or $n - k - 1$ non-activated nodes in R .*

Definition 27 *Let H be a connected doped elementary graph. A **doped ear** is a set of $m - k - 1$ non-activated nodes in L and all edges between them and NL given that NL is already activated, or a set of $n - k - 1$ non-activated nodes in R and all edges between them and NR given that NR is already activated. See Figure 10.*

*A **doped ear decomposition** is a modified ear decomposition that except normal ears with nodes in L and R , has one (if $k = m$ or $k = n$) or two doped ears.*

Figure 10. Extended ear and doped ear

Theorem 28 *A bipartite graph $H = (V_1 \cup V_2, E)$ where $V_1 = L \cup NR$, $V_2 = NL \cup R$, $|NL| = |NR| = 1$, and where there is no edge between NL and NR , is doped elementary if and only if every component not containing NR or NL has an ear decomposition and every component containing at least one of NR and NL has a doped ear decomposition.*

Proof. Suppose there is such a graph H where every component has an ear decomposition or a doped ear decomposition. Construct a graph G by replacing the node NL with $m - k$ nodes in a set XL and letting every node in XL be adjacent to the same nodes in L as NL , and by replacing the node NR with $n - k$ nodes in a set XR in the same manner.

Consider the components K_H and K_G in H and G containing NL and XL respectively. Then $K_H = x + E_1 + \cdots + E_s$, where E_j is the doped ear. Then $K_G = x + E_1 + \cdots + E'_j + \cdots + E_s + E_{s+1} + \cdots + E_{s+\ell}$, where the first node in XL replaces NL , E'_j is an extended ear which begins in the first node in XL and ends in a previously activated node in L (there is such a node since the first node in XL is activated) and has the same other nodes in L as E_j , and $E_{s+1}, \dots, E_{s+\ell}$ are ears consisting of one edge each between XL and L . An example is seen in Figure 11, where $E_{s+1}, \dots, E_{s+\ell}$ are omitted. The same can

Figure 11. Extension of ear decomposition

be done with the components in H and G containing NR and XR respectively. Theorem 23 now implies that G is elementary, and the construction of G and its ear decomposition implies that G is an extended elementary graph, and that H is a doped elementary graph.

Suppose H is doped elementary. Let G be the corresponding extended elementary graph. Then the components of G have ear decompositions. The ear decomposition of the component K_G containing XL can be rearranged into an ear decomposition with an extended ear according to Lemma 29 below (the same applies to the component containing XR). Now $K_G = x + E_1 + \cdots + E'_j + \cdots + E_s + E_{s+1} + \cdots + E_{s+\ell}$, where E'_j is an extended ear and $E_{s+1}, \dots, E_{s+\ell}$ are all ears consisting of one edge between XL and L not incident with the first node in XL . Then $K_H = x + E_1 + \cdots + E_j + \cdots + E_s$, where E_j is a doped ear corresponding to the extended ear E'_j . The same can be done with the components containing XR and NR . The components of H containing NR and NL now have doped ear decompositions, and the other components can keep the same ear decompositions as in G . \square

The following lemma is proven in Appendix A.

Lemma 29 *Let G be an extended elementary graph. An ear decomposition for a component in G containing XL or XR can always be changed into an ear decomposition with an extended ear containing XL or XR respectively.*

Theorem 30 *Let H be a doped elementary graph, corresponding to the face $F^M(H)$ of $M(m, n, k)$. Then the total number of ears not being doped ears in doped ear decompositions of all the components of H is equal to the dimension of $F^M(H)$.*

Proof. Let G be the extended elementary graph corresponding to H . If NL belongs to component K_1^H in H , then the ear decomposition of K_1^H can be extended to an ear decomposition for the corresponding component K_1^G in G . This extension is described in the proof of Theorem 28. There are $m - \ell_0$ edges between NL and L , and there are $(m - \ell_0)(m - k)$ edges between XL and L . The doped ear in K_1^H is changed to an extended ear in K_1^G , and the extended ear contains $m - k$ more edges than the doped ears. All other ears remain as they are. Now there are $(m - k - 1)(m - \ell_0 - 1) - 1$ edges between XL and L that are not part of any ear, and since all vertices in XL and R are previously activated, each of these edges will be a new ear. If the doped ear is not counted, K_1^G will have $(m - k - 1)(m - \ell_0 - 1)$ more ears than K_1^H , and if there is a component K_2^G containing XR , it will have $(n - k - 1)(n - r_0 - 1)$ more ears than the corresponding component K_2^H .

By Theorem 12, $\dim F^B(G) - \dim F^M(H) = |E(G)| - |V(G)| - t - |E(H)| + |V(H)| + t = (m - k - 1)(m - \ell_0) + (n - k - 1)(n - r_0) - \max(m - k - 1, 0) - \max(n - k - 1, 0)$. So the difference between the number of ears in G and H is exactly $\dim F^B(G) - \dim F^M(H)$. The result now follows from Theorem 24. \square

Corollary 31 *The only two-dimensional faces of $M(m, n, k)$ are triangles and rectangles.*

Proof. Follows easily from Theorem 30. The graphs corresponding to two-dimensional faces has either one non-trivial component with two ordinary ears or two non-trivial components with one ordinary ear each. It is then easy to see that the number of vertices is three or four, and not so hard to see that if there are four vertices then the face is a rectangle (in that case there are two cycles). \square

APPENDIX: Proof of Lemma 29

Lemma 29 stated the following: If G is an extended elementary graph, then an ear decomposition for a component in G containing XL or XR can always be changed into an ear decomposition with an extended ear containing XL or XR respectively.

Proof. Suppose there is an ear decomposition for the component containing XL (the component containing XR can be treated analogously). Note that ears consisting of one edge each can be placed anywhere in the ear decomposition after the activation of its endpoints. This ear decomposition will be changed to an ear decomposition with an extended ear containing XL . Three types of paths in ears and two types of ears will be characterised, and then it will be described how to replace the ears containing these types of paths with other ears so that we get an extended ear.

In the figures a node in a circle is an earlier activated node, and the relative order of the ears in the figures are given.

Paths of type 1, 2, and 3: Paths of *type 1* begin with an edge from an activated node in XL to L followed by an even number (> 0) of edges between L and XL , and then an edge from L to R follows.

Paths of *type 2* begin with an edge from L to XL followed by an odd number (> 1) of edges between XL and L , and then an edge from L to R , and are not contained in longer paths of type 2 or paths of type 1.

Paths of *type 3* begin with an edge from L to XL followed by an edge to L and an edge to R , and are not contained in paths of type 2 or type 1.

Paths of type 1, 2, and 3 are shown in Figure 12.

Figure 12. Paths of type 1, 2, and 3, and ear of type B

Ears of type A and B Ears of *type A* are ears consisting of one edge each

between XL and L , except the starting edge x for the ear decomposition. Ears of *type B* are ears that have at least 3 edges, and only have edges between XL and L . An ear of type B is shown in Figure 12.

Ears containing paths of type 1, 2 or 3 will, with the help of convenient ears of type A, be replaced by ears with paths not of type 1, 2 or 3. First we replace all ears with paths of type 2, then we replace all ears with paths of type 3 (except the first one in the ear decomposition if x is not between XL and L), and at last we replace all ears with paths of type 1. How to do this is shown in Figure 13. In the case ' $3 + A \rightarrow 1/B + \text{other}$ ', the ear of type A is attached to the node in L which is followed by an odd number of edges in the ear containing the path of type 3.

Figure 13.

Now there is no ear containing a path of type 1 or 2, and at most one ear containing paths of type 3. If there is an ear containing more than one path of type 3, then it can be replaced by an ear containing only one path of type 3 and no path of type 1 or 2. Finally all ears of type B will be replaced by one ear of type B and many ears of type A. These replacements are described in Figure 14.

The edge x or an ear with one path of type 3 activates the first activated node in XL . After that, only ears of type B can activate nodes in XL . Since only one ear of type B is left, this ear has to begin in the first activated node in XL , contain the remaining $m - k - 1$ nodes in XL , and end in a node in L . It follows that the ear of type B is an extended ear.

Now the ear decomposition has been changed to an ear decomposition with an extended ear containing XL . \square

Figure 14.

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