COMPLETING A $k-1$ ASSIGNMENT

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ABSTRACT. We consider the distribution of the value of the optimal k-assignment in an $m \times n$-matrix, where the entries are independent exponential random variables with arbitrary rates.

We give closed formulas for both the Laplace transform of this random variable and for its expected value under the condition that there is a zero-cost $k-1$-assignment.

1. INTRODUCTION

Let M be an $m \times n$ matrix. A *k-assignment* is a collection of k entries in the matrix such that no two are in the same row or the same column. The *value* of a k-assignment is the sum of its entries. A k-assignment is called *optimal* if its value is no larger than the value of any other k-assignment. If the entries of the matrix M are random variables then so is the value of the optimal k-assignment, here denoted $\min_k(M)$.

The study of the optimal k-assignment has been pursued by researchers from different fields and with different random variables as entries in M. The main focus has been to estimate the size of the expected value of $\min_k(M)$. For references and more details on the history see [CS98] or [LW03].

In 1998 Giorgio Parisi [P98] conjectured that if M is a $k \times k$ matrix with independent exponential random variables with rate 1, exp(1), then the expected value of the optimal k-assignment is

$$\mathrm{E}[\min_k(M)] = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}.$$

Two very different proofs of this conjecture were announced simultaneously in March 2003, [LW03][NPS03]. The beautiful conjecture of Parisi inspired much work on exact formulas and many different generalizations where studied [AS02, BCR02, CS98, CS02, EES01, LW00, N02]. In [LW03] a formula for the expected value is given when the matrix entries are exp(1) or 0.

In this note we investigate the problem from a different extreme. We allow the rates of the exponential random variables to be arbitrary positive numbers. We include the infinity as a possible rate, which corresponds to the entry being constant zero. We prove exact formulas

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for $\mathrm{E}[\min_k(M)]$ and for the Laplace transform $L(\min_k(M))$ under the assumption that $\mathrm{E}[\min_{k-1}(M)] = 0$. Formulas for completing a $k-1$-assignment were considered previously in [CS02] in the case when all rates are equal to 1. The Laplace transform for some special cases when the rates are all equal to 1, have been determined in [AS02].

We will first prove the slightly easier result on expected value and then compute the Laplace transform for the entire distribution. In theory one should be able to deduce the first result from the second but it seems easier to compute them separately.

2. Preliminaries

As in [LW00, LW03] the concept of row and column covers of zeros will be important. The set of zeros will be denoted Z. We will consider sets of rows and columns in the $m \times n$-matrix. A set α of rows and columns is said to *cover* Z if every entry in Z is on either a row or on a column in α. A cover with $k-1$ rows and columns will be called a $k-1$-cover. For many readers it might be convenient to translate the matrix to a bipartite graph. In that setting our covers are so called vertex covers.

Given a set of rows and columns α let the *rectangle* $R(\alpha)$ be the part of the matrix not covered by α. If α is a $k-1$-cover of the zeros then the corresponding rectangle $R(\alpha)$ will be called *critical*. Let $Q_{k,Z}$ be the set of all critical rectangles in M. We define a partial ordering on $Q_{k,Z}$ by letting, for $R(\alpha), R(\beta) \in Q_{k,Z}$, $R(\alpha) \leq R(\beta)$ if the set of columns in α is a subset of the set of columns in β, and the set of rows in α is a superset of the set of rows in β.

Recall that a random variable $X \sim \exp(\alpha)$ if $P(X > t) = e^{-t\alpha}$. Then α is called the *rate* and $\mathrm{E}[X] = 1/\alpha$. If $X_1, \ldots, X_n$ are independent and $X_i \sim \exp(\alpha_i)$ then $\mathrm{E}[\min_i X_i] = 1/\sum_i \alpha_i$. Let the *rate* of a rectangle R be the sum of the rates of the individual entries. We denote it $I(R)$ and note that $0 < I(R) \leq \infty$.

From matching theory we have the following lemma that will be important.

Lemma 2.1. *Assume that Z contains $k-1$ zeros in independent position. Then $Q_{k,Z}$ is a lattice. In particular it has unique maximal and minimal elements.*

From Theorem 2.9 in [LW00] we cite the following useful fact.

Lemma 2.2. *Suppose that a row r belongs to every $k-1$-cover of the zeros Z in the matrix M. Then every optimal k-assignment contains a zero from row r. This means that we can remove row r from M and obtain a matrix M^r with the property that* $\mathrm{E}[\min_k(M)] = \mathrm{E}[\min_{k-1}(M^r)]$.

Recall that the *incidence algebra* over a poset is the algebra of functions defined on the intervals in the poset, see [S]. The product in the incidence algebra is defined by convolution. One way to visualize the incidence algebra over a poset Q is to let a function f be represented by a matrix F with rows and columns indexed by the elements in Q. The element on position (α, β) in F is $f(\alpha, \beta)$. The rows and columns must be arranged according to a linear extension of Q. Every such matrix is upper triangular and multiplication of functions corresponds to multiplication of matrices.

3. The expected value

Define a function in the incidence algebra of $Q_{k,Z}$ by $f(R(\alpha), R(\beta)) = I(R(\alpha) \cap R(\beta))$. Since $I(R(\alpha)) > 0$ for all critical rectangles, f is invertible in the incidence algebra.

Theorem 3.1. *Let M be an $m \times n$ matrix with all entries being either zero or independent exponential random variables with arbitrary positive rates. Assume that the set of zeros Z in M contains $k-1$ zeros in independent position, i.e.* $\mathrm{E}[\min_{k-1}(M)] = 0$.
Then

$$\mathrm{E}[\min_k(M)] = \sum_{R(\alpha) \leq R(\beta)} f^{-1}(R(\alpha), R(\beta)),$$

where the sum is over all intervals in $Q_{k,Z}$ and f^{-1} is the inverse of f in the incidence algebra. Equivalently we can write

$$\mathrm{E}[\min_k(M)] = \sum_{R_1 < R_2 < \cdots < R_s} (-1)^{(s-1)} \frac{I(R_1 \cap R_2) I(R_2 \cap R_3) \cdots I(R_{s-1} \cap R_s)}{I(R_1) I(R_2) \cdots I(R_s)},$$

where the sum is taken over all non-empty chains in $Q_{k,Z}$.

Remark 3.2. The second formula runs over all chains in $Q_{k,Z}$ which in the worst case has size of order $k!$. The first formula is a computational and conceptual improvement for large k. It involves taking the inverse of a matrix indexed by the elements of $Q_{k,Z}$ which is exponential in k in the worst case.

Proof. The equivalence follows from Lemma 5.1 below. The proof will be by induction over k. The theorem is certainly true for $k = 1$. Without loss of generality we may assume that the entries $(1,1), (2,2), \ldots (k-1, k-1)$ are zero entries. We may also assume that the maximal rectangle $R(\gamma)$ in $Q_{k,Z}$ corresponds to the cover γ consisting of columns $1, \ldots, k-1$. If this is not the case, then there is a row i that belongs to every $k-1$-cover. This implies by Lemma 2.2 that $\mathrm{E}[\min_k(M)] = \mathrm{E}[\min_{k-1}(M^i)]$, where M^i is obtained from M by completely removing row i. Since these matrices have the same $Q_{k,Z}$ the result is clear by induction.

Note that any chain in $Q_{k,Z}$ not ending with $R(\gamma)$ can be augmented with $R(\gamma)$ at the end.

We now use the same recursion procedure as in [AS02] and [LW00] corresponding to the optimal cover γ consisting of the first $k-1$ columns. That is, let X be the minimum of all the exponential random variables in $R(\gamma)$. Note that exactly one entry in $R(\gamma)$ will belong to an optimal k-assignment and that $\mathrm{E}[X] = 1/I(R(\gamma))$. Subtract X from all entries in $R(\gamma)$ and a new zero will occur at the position of the minimum. All other entries will be unchanged in distribution by the forgetfulness of the exponential distribution. Let $M_{i,j}$ be the matrix obtained when position (i,j) in M has been replaced with a zero.

Let K_i be the intersection of $R(\gamma)$ and row i. If the minimum is in K_i, $i > k-1$, then we get a zero k-assignment in $M_{i,j}$. If the minimum is in K_i, $i = 1, \ldots, k-1$, then row i has to be in every optimal cover of $M_{i,j}$ and as above we can remove row i and this case is done by induction. Let again M^i denote the matrix with row i removed from M. Also let R^i be the column maximal cover of M^i. The poset of $k-1$-covers of M^i will be the induced subposet of $Q_{k,Z}$ of elements $\leq R^i$.

The probability that the zero occurs in K_i is $\frac{I(K_i)}{I(R(\gamma))}$ and we get

$$\mathrm{E}[\min_k(M)] = \frac{1}{I(R(\gamma))} + \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))} \mathrm{E}[\min_{k-1}(M^i)].$$

Which by induction becomes

$$\frac{1}{I(R(\gamma))} + \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))} \sum_{R_1 < \cdots < R_s \leq R^i} (-1)^{(s-1)} \frac{I(R_1 \cap R_2) \cdots I(R_{s-1} \cap R_s)}{I(R_1) \cdots I(R_s)}.$$

Change the order of summation to get

$$\frac{1}{I(R\gamma)} + \sum_{R_1 < \cdots < R_s < R(\gamma)} (-1)^{(s-1)} \frac{I(R_1 \cap R_2) \cdots I(R_{s-1} \cap R_s)}{I(R_1) \cdots I(R_s)} \sum_{i \notin rowset(R_s)} \frac{I(K_i)}{I(R(\gamma))}.$$

Here $rowset(R)$ denotes the set of rows of M that intersect the rectangle R. Now

$$\sum_{i \notin rowset(R_s)} \frac{I(K_i)}{I(R(\gamma))} = \frac{I(R(\gamma) \backslash R_s)}{I(R(\gamma))} = 1 - \frac{I(R_s \cap R(\gamma))}{I(R(\gamma))},$$

since the entries are independent exponential random variables. The theorem follows. $\square$

4. The Laplace transform

We can in fact use the same proof technique to get the stronger result determining the Laplace transform of the distribution of $\min_k(M)$.

Recall that the Laplace transform of a random variable X is $L(X,t) = \mathrm{E}[e^{-tX}]$. It has the following well-known properties.

$$L(X+Y,t) = L(X,t)L(Y,t), \qquad \text{if } X \text{ and } Y \text{ are independent}$$

Given random variables $X_1,\dots,X_s$ and probabilities $p_1,\dots,p_s$, define the random variable I to take value i with probability p_i, independent of $X_1,\dots,X_s$. Then

$$L(X_I,t) = \sum_{i=1}^{s} p_i L(X_i,t).$$

In this situation we will need the special case $L(0,t)=1$.

For a critical rectangle R we will use the notation

$$\phi(R,t) = L(\min_1(R)) = \frac{I(R)}{I(R)+t}.$$

As for Theorem 3.1 we give two statements of the same formula using Lemma 5.1. Remark 3.2 applies also here.

Theorem 4.1. *Let M be an $m \times n$ matrix with all entries being either zero or exponential independent random variables with arbitrary positive rates. Assume that the set of zeros Z in M contains $k-1$ zeros in independent position, i.e.* $\mathrm{E}[\min_{k-1}(M)] = 0$.

Then we can write $L(\min_k(M)) = \sum_{R\in Q_{k,Z}} c_R(M)\phi(R,t)$. Furthermore

$c_R(M) = a_R \cdot b_R$, where

$$a_R = \sum_{R_1<R_2<\cdots<R_s=R} (-1)^s \prod_{i=1}^{s-1} \frac{I(R_i\cap R_{i+1}) - I(R)}{I(R_i)-I(R)}$$

and

$$b_R = \sum_{R=R_s<R_{s+1}<\cdots<R_u} (-1)^{(u-s+1)} \prod_{i=s+1}^{u} \frac{I(R_i\cap R_{i-1}) - I(R)}{I(R_i)-I(R)},$$

where the sums are taken over all chains containing R in $Q_{k,Z}$.

In the generic case rewrite this as

$$c_R(M) = \left(\sum_{R(\alpha)\le R} g_R^{-1}(R(\alpha),R)\right)\cdot\left(\sum_{R\le R(\beta)} g_R^{-1}(R,R(\beta))\right),$$

where the sums are over elements in $Q_{k,Z}$ and where g^{-1} is the inverse in the incidence algebra of

$$g_R(R_i, R_j) = \begin{cases} 1, & \text{if } R_i = R_j = R \\ 0, & \text{if } R_i \not\leq R_j \\ I(R_i \cap R_j) - I(R), & \text{otherwise.} \end{cases}$$

Proof. The proof is by induction over k. We use the same notations as in the proof of Theorem 3.1 and compute the Laplace transform using the same recursive step.

$$L(\min_k(M), t) = \phi(R(\gamma), t) \left(p + \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))} \cdot L(\min_k(M^i)) \right),$$

where p is the probability that the minimum is located so a zero cost k-assignment occurs. We now decompose this product by the method of partial fractions with respect to t. This proves the first claim and the uniqueness of the coefficients $c_R(M)$. The function g_R is invertible if $I(R) \neq I(R(\alpha))$ for all $R(\alpha) \in Q_{k,Z}$ such that $R \leq R(\alpha)$ or $R(\alpha) \leq R$. The equivalence of the two formulas for the coefficients then follows from Lemma 5.1.

After the decomposition by partial fractions follows an extraction of the terms involving $\phi(R, t)$ which gives

$$c_R(M) = \frac{I(R(\gamma))}{I(R(\gamma)) - I(R)} \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma))} \cdot c_R(M^i)$$

Assume that $R \neq R(\gamma)$ and we may inductively write

$$c_R(M) = \sum_{i=1}^{k-1} \frac{I(K_i)}{I(R(\gamma)) - I(R)} a_R(M) \cdot \sum_{R=R_s<R_{s+1}<\cdots<R_u} (-1)^{(u-s+1)} \prod_{i=s+1}^{u} \frac{I(R_i \cap R_{i-1}) - I(R)}{I(R_i) - I(R)},$$

where R_u does not intersect the row i. Change the order of summation to get

$$c_R(M) = a_R(M) \sum_{R=R_s<R_{s+1}<\cdots<R_u} (-1)^{(u-s+1)} \cdot \prod_{i=s+1}^{u} \frac{I(R_i \cap R_{i-1}) - I(R)}{I(R_i) - I(R)} \cdot \sum_{i \notin colset(R_u)} \frac{I(K_i)}{I(R(\gamma)) - I(R)},$$

and proceed as in the proof of Theorem 3.1. Here $colset(R)$ denotes the set of columns of M that intersect the rectangle R. Note that trivially $a_R(M) = a_R(M^i)$. If $R = R(\gamma)$ the recursive step uses the row maximal rectangle instead, that is the smallest element in $Q_{k,Z}$ to compute $a_R(M)$. $\square$

5. A lemma

Lemma 5.1. *Let f be a function in the incidence algebra over a poset P and let $\alpha \leq \beta \in P$ be arbitrary elements. The inverse of f can be written as*

$$f^{-1}(\alpha, \beta) = \sum_{\alpha=\gamma_1<\gamma_2<\cdots<\gamma_s=\beta} (-1)^{(s-1)} \frac{f(\gamma_1, \gamma_2) f(\gamma_2, \gamma_3) \cdots f(\gamma_{s-1}, \gamma_s)}{f(\gamma_1, \gamma_1) f(\gamma_2, \gamma_2) \cdots f(\gamma_s, \gamma_s)},$$

where the sum is over all chains in the interval $[\alpha, \beta]$ beginning in α and ending in β.

Proof. Let F be the upper triangular matrix corresponding to f as described in the preliminaries. Let D be the diagonal matrix with the values of $f(\gamma, \gamma)$ on the diagonal and zeroes elsewhere. Let N be the nilpotent matrix that agrees with F at all positions except on the diagonal, where N has zeros. We can then write $F = D + N$ and we can easily verify that

$$F^{-1} = D^{-1} - D^{-1}ND^{-1} + D^{-1}ND^{-1}ND^{-1} - \dots.$$

The matrix $D^{-1}N$ will have zeros on and below the diagonal. Thus the sum is finite. Since inverting f in the incidence algebra is the same as inverting the matrix F the lemma follows. $\square$

Remark 5.2. The main theorem in [LW03] when the exponential random variables all have rate 1 or infinity has a reformulation in [LW00] in terms of the Möbius function of a certain poset called P. That poset is different and much larger than $Q_{k,Z}$. To be more precise the atoms in P are the elements in $Q_{k,Z}$. All our efforts to join the two theorems to one for completely arbitrary rates have so far been fruitless.

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