RING-EXTENSIONS AND UNIVERSAL BIMODULES

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ABSTRACT. We consider A. Grothendiecks construction of the functors $\mathrm{Exan}_A(B, L)$ from from [2], prove existence, and elementary properties. As a special case we consider the module of differentials Ω^1_A and prove various general properties.

CONTENTS

INTRODUCTION

In the paper [2] the functors $\mathrm{Exan}_A(B, L)$ are introduced, parametrizing ring-extensions of an arbitrary rinng B with a bimodule L. These functors are representable by a universal bimodule $\mathfrak{I}/\mathfrak{I}^2$, generalizing to the case for an arbitrary ring the fact that the module of differentials represents the functor of derivations $\mathrm{Der}_k(A, -)$. In this note we prove following [2] basic properties of the $\mathrm{Exan}_A(-, -)$-construction, and relate it to the module of differentials.

1. EXTENSIONS OF RINGS BY BIMODULES

Let in the following A denote an arbitrary ring, not necessarily commutative or with unit. We are interested in the cateogry of A-algebras, ie the category of ring-homomorphisms $a : A \to B$ where B is an arbitrary ring, i.e B is an A-algebra. Morphisms in this category are morphisms of A-algebras, i.e morphisms commuting with the structure-morphism from A. Let $u : B \to C$ be a map of A-algebras, then the kernel $\mathfrak{I}$ is a 2-sided ideal of B. It is by definition an A-algebra, since we dont consider rings with unit. Beacuse $\mathfrak{I}$ is an ideal in B, it is a B-bimodule hence also an A-bimodule.

Let $f : E \to B$ be a surjective map of A-algebras, and let $\mathfrak{I}$ be the kernel of f. It follows that $\mathfrak{I}$ is an E-bimodule, and we have an exact sequence of rings

$$0 \to \mathfrak{I} \to E \to B \to 0$$

where we let $j : \mathfrak{I} \to E$ denote the inclusion-map. Note that the condition $j(\mathfrak{I})^2 = 0$ is equivalent to the following: $\mathfrak{I}$ is a B-bimodule and for all $x \in E$, $z \in \mathfrak{I}$ we have

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that $xj(z) = j(f(x)z)$ and $j(z)x = j(z(f(x))$. This is the same as saying $\mathfrak{I}$ is a B-bimodule and the map j is a map of E-bimodules.

Definition 1.1. An A-extension of an A-algebra B with a B-bimodule $\mathfrak{I}$ is an exact sequence of A-bimodules

$$0 \to \mathfrak{I} \to E \to B \to 0$$

where the map $f : E \to B$ is a surjective map of A-algebras, and the inclusion map $j : \mathfrak{I} \to E$ is a map of E-bimodules, i.e $j(\mathfrak{I})^2 = 0$. The set of all extensions of this form is denoted $\operatorname{exan}_A(B, L)$.

Given two extensions

$$0 \to \mathfrak{I} \to E \to B \to 0$$

and

$$0 \to \mathfrak{I}' \to E' \to B' \to 0$$

a map of extensions is a triple (w, u, v) giving commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I} & \xrightarrow{j} & E & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & \mathfrak{I}' & \xrightarrow{j'} & E' & \xrightarrow{f'} & B' & \longrightarrow & 0 \end{array}$$

where u, v are maps of A-algebras and w is a di-homomorphism, ie for all b in B and z in $\mathfrak{I}$ the following holds: $w(bz) = v(b)w(z)$ and $w(zb) = w(z)v(b)$. It follows that composition of maps of extensions is a map of extensions, hence we have a nice category of extensions.

Let $\underline{K}$ be the category of triples (A, B, L) where A is a ring, B is an A-algebra and L a B-bimodule. Morphisms from (A, B, L) to (A', B', L') in this category are triples of maps (w, u, v) where we get a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \quad & L \\ {\scriptstyle u}\uparrow & & {\scriptstyle v}\uparrow & & \downarrow{\scriptstyle w} \\ A' & \longrightarrow & B' & & L' \end{array} ,$$

and where $w : L \to L'$ is a *di-homomorphism*, ie $w(v(b')z) = b'w(z)$ and $w(zv(b')) = w(z)b'$. Any morphism in $\underline{K}$, $(w, u, v)_*$ can be factored as

$$(w, u, v)_* = (w, 1, 1)_* \circ (1, u, 1)_* \circ (1, 1, v)_*.$$

We want to define operations on extensions, pull-back with respect to A-algebra homomorphisms v, push-forward with respect to maps w of bimodules and change of ring-maps with respect to v. Let

$$0 \to \mathfrak{I}' \to^{j'} E' \to^{f'} B' \to 0$$

be an extension, and let $v : B \to B'$ be a map of A-algebras. Define $E' \times_{B'} B$ to be the set of all elements (x, y) in $E' \times B$ with $f'(x) = v(y)$. There exists a natural map $p_2 : E' \times_{B'} B \to B$ of sets which is surjective, since v is surjective. The set $E' \times_{B'} B$ has a product making it into a subring of the product ring $E' \times B$, and it is in a natural way an A-algebra, and the map p_2 is a map of A-algebras. There exists a map $\mathfrak{I}' \to E' \times_{B'} B$ sending z to $(j'(z), 0)$ and one easily sees that this defines an extension

$$0 \to \mathfrak{I}' \to E' \times_{B'} B \to B \to 0$$

giving an element of the set $\text{exan}_A(B, \mathfrak{I}')$, hence we get for each map v of A-algebras a map of sets $(v)_* : \text{exan}_A(B', \mathfrak{I}') \to \text{exan}_A(B, \mathfrak{I}')$. One shows that $(u \circ v)_* = (v)_* \circ (u)_*$ hence the exan-functor is contravariant with respect to A-algebra homomorphisms v.

Assume we have an extension

$$0 \to \mathfrak{I} \to^j E \to^f B \to 0$$

and let $w : \mathfrak{I} \to \mathfrak{I}'$ be a map of B-bimodules. We get a map $\theta : \mathfrak{I} \to E \times \mathfrak{I}'$ given by $\theta(z) = (j(z), -w(z))$. Define on $E \times \mathfrak{I}'$ the following product: $(x, s)(y, t) = (xy, xt + sy)$. It follows that $E \times \mathfrak{I}'$ is an A-algebra, and the natural map $E \times \mathfrak{I}' \to E$ is a map of A-algebras. The image $\theta(\mathfrak{I})$ is verified to be a 2-sided ideal in $E \times \mathfrak{I}'$, hence we get a well-defined ring $E \oplus_{\mathfrak{I}} \mathfrak{I}' = E \times_{\mathfrak{I}} \mathfrak{I}'/\theta(\mathfrak{I})$ with a natural map of rings $E \oplus_{\mathfrak{I}} \mathfrak{I}' \to B$, which is easily seen to be a map of A-algebras. There exists a natural map $\mathfrak{I}' \to E \oplus_{\mathfrak{I}} \mathfrak{I}'$, and we get an exact sequence of A-bimodules

$$0 \to \mathfrak{I}' \to^{j'} E \oplus_{\mathfrak{I}} \mathfrak{I}' \to B \to 0.$$

One verifies that $j'(\mathfrak{I}')^2 = 0$, hence the sequence is an element of $\text{exan}_A(B, \mathfrak{I}')$, hence we have a natural map of sets $(w)_* : \text{exan}_A(B, \mathfrak{I}) \to \text{exan}_A(B, \mathfrak{I}')$. We verify that $(w \circ w')_* = (w)_* \circ (w')_*$ hence the functor exan is covariant with respect to maps of B-bimodules.

Summing up we have defined the following: Given any extension of the A-algebra B' by a B'-bimodule $\mathfrak{I}'$

$$0 \to \mathfrak{I}' \to E' \to B' \to 0$$

and a map of A-algebras $u : B \to B'$, we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I}' & \longrightarrow & E' \times_{B'} B & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & \mathfrak{I}' & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & 0 \end{array}$$

where the exact sequence

$$0 \to \mathfrak{I}' \to E' \times_{B'} B \to B \to 0$$

is an extension of B by the B-bimodule (via u) $\mathfrak{I}'$. We have also defined for any extension

$$0 \to \mathfrak{I} \to E \to B \to 0$$

of the A-algebra B by the B-bimodule $\mathfrak{I}$, and any map of B-bimodules $w : \mathfrak{I} \to \mathfrak{I}'$ a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I} & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathfrak{I}' & \longrightarrow & E \oplus_{\mathfrak{I}} \mathfrak{I}' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where the sequence

$$0 \to \mathfrak{I}' \to E \oplus_{\mathfrak{I}} \mathfrak{I}' \to B \to 0$$

is an extension of B by the B-bimodule $\mathfrak{I}'$.

Also given any map of rings $u : A' \to A$ and any extension

$$(*) 0 \to L \to E \to B \to 0$$

of A-bimodules, the sequence $(*)$ is in particular an extension of A'-bimodules, hence we get a natural map $(u)_* : \operatorname{exan}_A(B,L) \to \operatorname{exan}_{A'}(B,L)$ of sets. One verifies that $(v \circ v')_* = (v)_* \circ (v')_*$ hence this construction is functorial. It follows that we have for any morphism $(w,u,v)_*$ defined a map of sets

$$(w,u,v)_* : \operatorname{exan}_A(B,L) \to \operatorname{exan}_{A'}(B',L'),$$

hence exan is a functor from $\underline{K}$ to the category of sets, contravariant with respect to A-algebra homomorphisms, covariant with respect to B-bimodule homomorphisms.

Let $A \to B$ be an A-algebra and L a B-bimodule, and consider a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{f} & & \downarrow{=} & & \\ 0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & B & \longrightarrow & 0 \end{array} .$$

We say that the two extensions are *equivalent* if the diagrams commute and the map f is an isomorphism of A-algebras. One verifies that this construction defines an equivalence-relation $\backsim$ on the set $\operatorname{exan}_A(B,L)$.

Definition 1.2. Let $\operatorname{Exan}_A(B,L)$ be the quotient of $\operatorname{exan}_A(B,L)$ by $\backsim$.

We have to prove that morphisms in $\underline{K}$ respects the equivalence-relation $\backsim$ in order to get a well-defined functor $\operatorname{Exan} : \underline{K} \to \underline{Sets}$. Assume we have a map of A-algebras $v : B' \to B$ and two equivalent extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{f} & & \downarrow{=} & & \\ 0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & B & \longrightarrow & 0 \end{array} ,$$

i.e the diagrams commute and f is an isomorphism of A-algebras. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E \times_B B' & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{f \times 1} & & \downarrow{=} & & \\ 0 & \longrightarrow & L & \longrightarrow & F \times_B B' & \longrightarrow & B' & \longrightarrow & 0 \end{array} .$$

where $f \times 1$ is an isomorphism og A-algebras, hence the pull-back of two equivalent extensions via an A-algebra homomorphism v are equivalent, and we get a well-defined map of sets

$$(v)_* : \operatorname{Exan}_A(B,L) \to \operatorname{Exan}_A(B'.L).$$

Assume $w : L \to L'$ is a map of B-bimodules and consider the two equivalent extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{f} & & \downarrow{=} & & \\ 0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & B & \longrightarrow & 0 \end{array} .$$

Pushing forward with respect to w we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & E \oplus_L L' & \longrightarrow & B & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow f\oplus 1 & & \Big\downarrow = & & \\ 0 & \longrightarrow & L' & \longrightarrow & F \oplus_L L' & \longrightarrow & B & \longrightarrow & 0 \end{array},$$

and the map $f \oplus 1$ is an ismorphism, hence we get a well-defined map of sets

$$(w)_* : \text{Exan}_A(B, L \to \text{Exan}_A(B, L')$$

. Clearly changing basering from A to A' via a map of rings $u : A' \to A$ respects the equivalence-relation, hence we get a functor

$$\text{Exan} : \underline{K} \to \underline{Sets}$$

sending a triple (A, B, L) to $\text{Exan}_A(B, L)$ as claimed.

Furthermore we want to prove that for all triples (A, B, L) in $\underline{K}$ there exists a structure of abelian group on $\text{Exan}_A(B, L)$ induced by the struture of abelian group on L. Let for now $T(L)$ denote $\text{Exan}_A(B, L)$. The structure of abelian group on L may be formulated in terms of a distinguished element $0 \in L$, maps $s : L \times L \to L$ - addition - and $t : L \to L$ - inversion - satisfying certain commutative diagrams. There exists a canonical isomorphism $T(L \times L) \cong T(L) \times T(L)$, hence we get natural maps $T(s) : T(L) \times T(L) \to T(L)$ and $T(t) : T(L) \to T(L)$, and commutative diagrams are transformed into commutative diagrams via T, hence we have indeed a structure of abelian group on $T(L) = \text{Exan}_A(B, L)$ for all objects (A, B, L) in $\underline{K}$. One may check that given any morphism (w, u, v) in $\underline{K}$ the induced map

$$(w, u, v)_* : \text{Exan}_A(B, L) \to \text{Exan}_{A'}(B', L')$$

is a homomorphism of groups hence we have in fact defined a functor

$$\text{Exan} : \underline{K} \to \underline{Groups}.$$

Note also that any element z in the centre of B defines a homothety h_Z of L, hence the map $(h_z)_*$ defines a homothety of the abliean group $\text{Exan}_A(B, L)$. This shows that the abelian group $\text{Exan}_A(B, L)$ is a module on the centre $Z(B)$ of B, hence in particular if B is commutative, $\text{Exan}_A(B, L)$ is a B-module.

Let B be an arbitrary ring and let $\mathfrak{I}$ in B be a 2-sided ideal. Let $C = B/\mathfrak{I}$ be the quotient ring, and consider the exact sequence of rings

$$0 \to \mathfrak{I}/\mathfrak{I}^2 \to B/\mathfrak{I}^2 \to C \to 0.$$

It follows that $\mathfrak{I}/\mathfrak{I}^2$ is a C-bimodule and that the sequence defined above is a sequence of B-bimodules, hence it is an element of $\text{Exan}_B(C, \mathfrak{I}/\mathfrak{I}^2)$. It has a certain universal property that we will describe below.

Theorem 1.3. *Let L be a C-bimodule. There exists an isomorphism of abelian groups*

$$\eta : \text{Hom}_C(\mathfrak{I}/\mathfrak{I}^2, L) \cong \text{Exan}_B(C, L).$$

Proof. We will prove this explicitly by giving an explicit isomorphism. Given a map $w : \mathfrak{I}/\mathfrak{I}^2 \to L$ of C-bimodules we get by the theory in this section an extension

$$0 \to L \to B/\mathfrak{I}^2 \oplus_{\mathfrak{I}/\mathfrak{I}^2} L \to C \to 0$$

denoted $\eta(w)$. We will prove that η has the required properties.

Surjectivity: Assume

$$0 \to L \to^j E \to^f C \to 0$$

is an element of $\mathrm{Exan}_B(C,L)$. Hence there exists a map of A-algebras $f : B \to E$ commuting with the natural map $B \to C$. It follows that the map f sends $\mathfrak{I}^2$ to zero, hence we get a well-defined map of A-algebras $B/\mathfrak{I}^2 \to E$ making a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{I}/\mathfrak{I}^2 & \to & B/\mathfrak{I}^2 & \to & C & \to & 0 \\ & & \downarrow w & & \downarrow u & & \downarrow 1 & & \\ 0 & \to & L & \to & E & \to & C & \to & 0 \end{array},$$

hence we have a map of extensions $(w,u,1)_*$. The map $(w,u,1)$ gives rise to a map of extensions

$$\begin{array}{ccccccccc} 0 & \to & L & \to & B/\mathfrak{I}^2 \oplus_{\mathfrak{I}/\mathfrak{I}^2} L & \to & C & \to & 0 \\ & & \downarrow 1 & & \downarrow f & & \downarrow 1 & & \\ 0 & \to & L & \xrightarrow{j} & E & \xrightarrow{p} & C & \to & 0 \end{array},$$

which we claim is an equivalence of extensions. Consider the map of B-algebras

$$f : B/\mathfrak{I}^2 \oplus_{\mathfrak{I}/\mathfrak{I}^2} L \to E.$$

Given an element x in E there exists an element y in B with $\overline{y} = p(x)$, hence $f(x) - x = j(z)$ for some z in L, hence $x = f(x) - j(z)$. It follows that the element $(x, -z)$ maps to x in E, hence the map is surjective. Injetivity: Pick an element (x,z) in $B/\mathfrak{I}^2 \oplus_{\mathfrak{I}/\mathfrak{I}^2} L$ mapping to zero, ie $f(x,z) = u(x) + j(z) = 0$. This means $pu(x) = 0$ ie x is an element of $\mathfrak{I}/\mathfrak{I}^2$, hence (x,z) is equivalent to $(0, z + w(x))$ in $B/\mathfrak{I}^2 \oplus_{\mathfrak{I}/\mathfrak{I}^2} L$. Since w is a map of C-bimodules, it follows that $z + w(x) = 0$, hence f is indeed injective, and it follows that the two extensions are equivalent, hence the map η is a surjective map of sets. We proceed to prove that η is a group-homomorphism. Consider the addition map $s : L \times L \to L$, which is a map of C-bimodules. We get a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_C(\mathfrak{I}/\mathfrak{I}^2, L \times L) & \xrightarrow{\eta_{L\times L}} & \mathrm{Exan}_B(C, L \times L) \\ \downarrow \mathrm{Hom}(1,s) & & \downarrow (h)_* \\ \mathrm{Hom}_C(\mathfrak{I}/\mathfrak{I}^2, L) & \xrightarrow{\eta_L} & \mathrm{Exan}_B(C,L) \end{array}$$

proving that η_L is a map of abelian groups. Finally we want to prove that η_L is injective. We prove that $ker(\eta_L) = 0$: Assume $w : \mathfrak{I}/\mathfrak{I}^2 \to L$ is C-bimodule homomorphism and consider the associated extension $\eta_L(w)$, which is equivalent to the trivial extension by hypothesis, ie we get an equivalence of extensions

$$\begin{array}{ccccccccc} 0 & \to & L & \to & B/\mathfrak{I}^2 \oplus_{\mathfrak{I}/\mathfrak{I}^2} L & \to & C & \to & 0 \\ & & \downarrow = & & \downarrow \phi & & \downarrow = & & \\ 0 & \to & L & \to & B/\mathfrak{I} \times L & \to & C & \to & 0 \end{array}.$$

A straight forwar verifiation shows that this implies that the map w has to be the zero-map. From this it follows that $ker(\eta_L) = 0$, and we have an isomorphism of abelian groups as claimed. □

2. Derivations and differentials

We define differentials and derivations for left modules on commutative rings and prove elementary properties on functoriality etc. following [1]. Let in the following section all rings be commutative with unit, and consider only unital maps of rings. All modules are left modules. Let $A \to B$ be a map of commutative rings, and let W be a B-module.

Definition 2.1. A map $w : B \to W$ of A-modules is a B-*derivation* if

$$w(xy) = xw(y) + yw(x)$$

for all x, y in B. Denote by $\mathrm{Der}_A(B, W)$ the set of all B-derivations.

Note that $\mathrm{Der}_A(B, W)$ is a left sub-B-module of $\mathrm{Hom}_A(B, W)$. Assume $f : W \to W'$ is a map of B-modules and let w be a derivation in $\mathrm{Der}_A(B, W)$. then $f_*(w) = f \circ w$ is an element of $\mathrm{Der}_A(B, W')$, and we get a map of B-modules

$$f_* : \mathrm{Der}_A(B, W) \to \mathrm{Der}_A(B, W'),$$

hence $\mathrm{Der}_A(B, -)$ is a covariant functor with respect to maps of B-modules. Consider the multiplication map $m : B \otimes_A B \to B$ defined by $m(x \otimes y) = xy$ and let $I = ker(m)$ be the kernel of m. It is an ideal in $B \otimes_A B$, hence a $\mathcal{B} \otimes_A B$-module. The ideal I^2 is a sub-module of I.

Definition 2.2. Let $\Omega^1_{B/A} = I/I^2$ be the *module of differentials* of B over A.

Note that $\Omega^1_{B/A}$ is a $B \otimes_A B$-module hence a B-module from the left and right. The ring B is isomorphic to $B \otimes_A B/I$ and $I(\Omega^1_{B/A}) = 0$, hence $\Omega^1_{B/A}$ is canonically a B-module, and from this it follows that the left and right B-module structures on $\Omega^1_{B/A}$ coincide. Define a map $d : B \to \Omega^1_{B/A}$ as follows: $d(x) = x \otimes 1 - 1 \otimes x$. The map d is clearly A-linear. We see that

$$d(xy) = xy \otimes 1 - 1 \otimes xy = xy \otimes 1 - x \otimes y + x \otimes x - 1 \otimes xy =$$

$$xd(y) + d(x)y = xd(y) + yd(x),$$

hence it follows that d is an A-derivation, called the *universal derivation.* One trivially verifies that $\Omega^1_{B/A}$ is generated by the set $\{d(x)\}_{x \in B}$ as a B-module. The pair $\{d, \Omega^1_{B/A}\}$ has a universal property which we will now describe: Consider a map of B-modules $f : \Omega^1_{B/A} \to W$ where W is a B-module. It follows that $d_*(f) = f \circ d$ is a derivation from B to W, hence we get a map of B-modules

$$d_* : \mathrm{Hom}_B(\Omega^1_{B/A}, W) \to \mathrm{Der}_A(B, W).$$

Proposition 2.3. *The map*

$$d_* : \mathrm{Hom}_B(\Omega^1_{B/A}, W) \to \mathrm{Der}_A(B, W)$$

is an isomorphism of B-modules.

Proof. Trivial. □

Definition 2.4. Let W be a left B-module and put $\Omega^1_{B/A}(W) = \Omega^1_{B/A} \otimes_B W$.

Consider a commutative diagram of ring-homomorphisms

$$\begin{array}{ccccc} A' & \longrightarrow & B' & & \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

and a map $w : W' \to W$ of C-modules. The maps of rings v and u induce a commutative diagram of maps of rings

$$\begin{array}{ccc} B' \otimes_{A'} B' & \xrightarrow{v \otimes v} & B \otimes_A B \\ \downarrow & & \downarrow \\ B' & \xrightarrow{v} & B \end{array}$$

proving the existence of a map of abelian groups $\Omega^1_{B'/A'} \to \Omega^1_{B/A}$ which is in fact a map og B'-modules. It induces a map of C-modules

$$\Omega(u, v, w) : \Omega^1_{B'/A'}(W') \to \Omega^1_{B/A}(W).$$

The map $\{u, v, w\}$ induces a map

$$\mathrm{Der}(u, v, w) : \mathrm{Der}_A(B, W') \to \mathrm{Der}_{A'}(B', W)$$

in the following way: A derivation $\delta : B \to W$ is mapped to $w \circ \delta \circ v$ which is easily seen to be a derivation. Hence the functor Der is covariant with respect to triples of maps $\{u, v, w\}$.

Lemma 2.5. *Let $A \to B \to C$ be maps of rings, and let $\{W_k\}_{k \in K}$ be a family of C-modules. There exists a tural isomorphism*

$$\mathrm{Der}_A(B, \prod W_k) \cong \prod \mathrm{Der}_A(B, W_k).$$

Proof. Trivial. □

Lemma 2.6. *Let B and C be A-algebras and let W be a $B \otimes_A C$-module. There exists isomorphisms*

$$\Omega^1_{B/A}(W) \cong \Omega^1_{B \otimes_A C/C}(W)$$

and

$$\mathrm{Der}_A(B, W) \cong \mathrm{Der}_C(B \otimes_A C, W)$$

of $B \otimes_A C$-modules.

Proof. We first prove the existence of an isomorphism

$$\mathrm{Hom}_C(B \otimes_A C, W) \cong \mathrm{Hom}_A(B, W).$$

Let $f : B \to W$ be an A-linear map. There exists a canonical map $p : B \to B \otimes_A C$ defining a map

$$p_* : \mathrm{Hom}_C(B \otimes_A C, W) \cong \mathrm{Hom}_A(B, W)$$

as follows: $p_*(f) = f \circ p$. It is trivial that $p_*(f)$ is A-linear, and injective. We prove it is surjective: Assume $g : B \to W$ is an A-linear map. Define a map $1 \otimes g : C \otimes_A B \to W$ in the obvious way. It is clear that $1 \otimes g$ is C-linear and also $p_*(1 \otimes g) = g$, hence p_* is surjective, and we have proved p_* is an isomorphism. We claim that the map p_* induces an isomorphism

$$p_* : \mathrm{Der}_C(B \otimes_A B, W) \to \mathrm{Der}_A(B, W) :$$

If $w : C \otimes_A B \to W$ is a C-derivation, it is clear that the map $p_*(w)$ is an A-derivation since p is a ring-homomorphism, hence the image of p_* is in $\mathrm{Der}_A(B, W)$. Assume $w : B \to W$ is an A-derivation. Then is it easy to show that the induced map $1 \otimes w : C \otimes_A B \to W$ is a C-derivation, and we have proved the claim. □

In particular we see that if S in A is a multiplicatively closed subset and W an $S^{-1}B$-module, we obtain a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ S^{-1}A & \longrightarrow & S^{-1}B \end{array}$$

of ring-homomorphisms, inducing an isomorphism

$$\mathrm{Der}_{S^{-1}A}(S^{-1}B, W) \cong \mathrm{Der}_A(B, W)$$

of B-modules.

Lemma 2.7. *Let B and C be A-algebras, W be a $C \otimes_A B$-module and N a B-module. There exists isomophisms*

$$\mathrm{Hom}_B(N, W) \cong \mathrm{Hom}_{C\otimes_A B}(C \otimes_A N, W),$$

$$(C \otimes_A N) \otimes_{C\otimes_A B} W \cong N \otimes_B W$$

of $C \otimes_A B$-modules.

Proof. We define a canonical map

$$p_* : \mathrm{Hom}_{C\otimes_A B}(N \otimes_A C, W) \to \mathrm{Hom}_B(N, W)$$

as follows: $p_*(f)(n) = f(n \otimes 1)$ where $1 \in C$ is the unit for the multiplication. Then it is an easy verifiacation to check that p_* induces the desired isomorphism. We prove the isomorphism

$$(C \otimes_A N) \otimes_{C\otimes_A B} W \cong N \otimes_B W :$$

Assume $f : N \times W \to$ is a B-bilinear map, where P is any B-module. Then we get a $C \otimes_A B$-bilinear map $\tilde{f} : C \otimes_A N \times W \to C \otimes_A P = \tilde{P}$ in the obvious way, and this sets up a 1-1 correspondence between B-bilinear maps from $N \times W$ to P, and $C \otimes_A B$-bilinear maps from $C \otimes_A N \times W$ to $\tilde{P}$ proving the desired isomorphism

$$(C \otimes_A N) \otimes_{C\otimes_A B} W \cong N \otimes_B W.$$

□

In a similar fashion we get an isomorphism $C \otimes_A \Omega^1_{B/A} \cong \Omega^1_{C\otimes_A B/C}$: We know that

$$\mathrm{Hom}_{C\otimes_A B}(C \otimes \Omega^1_{B/A}, W) \cong \mathrm{Hom}_B(\Omega^1_{B/A}, W) \cong \mathrm{Der}_A(B, W) \cong$$
$$\mathrm{Der}_C(C \otimes_A B, W) \cong \mathrm{Hom}_{C\otimes_A}(\Omega^1_{C\otimes_A B/C}, W),$$

hence by the universal property of the construction of the module of differentials it follows that there is an isomorphism

$$C \otimes_A \Omega^1_{B/A} \cong \Omega^1_{C\otimes_A B/C}.$$

Lemma 2.8. *There exists an isomorphism*

$$\Omega^1_{B/A} \otimes_B W \cong \Omega^1_{C\otimes_A B/C} \otimes_{C\otimes_A B} W.$$

Proof. By arguments above there exists isomorphisms

$$\Omega^1_{B/A} \otimes_B W \cong C \otimes_A \Omega^1_{B/A} \otimes_{C \otimes_A B} W \cong \Omega^1_{C \otimes_A B/C} \otimes_{C \otimes_A B} W,$$

and the lemma follows. □

Lemma 2.9. *Let B and C be A-algebras and let W be a $B \otimes_A C$-module. There exists isomorphisms*

$$\Omega^1_{B \otimes_A C/A}(W) \cong \Omega^1_{B/A}(W) \oplus \Omega^1_{C/A}(W)$$

and

$$\mathrm{Der}_A(B \otimes_A C, W) \cong \mathrm{Der}_A(B, W) \oplus \mathrm{Der}_A(C, W).$$

Proof. Trivial. □

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