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# Analytic nonlinearizable uniquely ergodic diffeomorphisms on $\mathbb{T}^2$

MARIA SAPRYKINA†

*Department of Mathematics, Royal Institute of Technology, S-10044, Sweden*
*(e-mail: masha@math.kth.se)*

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*Abstract*. In this paper we study the behavior of diffeomorphisms, contained in the closure $\overline{\mathcal{A}_\alpha}$ (in the inductive limit topology) of the set $\mathcal{A}_\alpha$ of real-analytic diffeomorphisms of the torus $\mathbb{T}^2$, which are conjugated to the rotation $R_\alpha : (x, y) \mapsto (x+\alpha, y)$ by an analytic measure-preserving transformation. We show that for a generic $\alpha \in [0, 1]$, $\overline{\mathcal{A}_\alpha}$ contains a dense set of uniquely ergodic diffeomorphisms. We also prove that $\overline{\mathcal{A}_\alpha}$ contains a dense set of diffeomorphisms that are minimal and non-ergodic.

## 0. *Introduction*

Consider the set $\text{Diff}(\mathbb{T}^2)$ of analytic diffeomorphisms of the torus homotopic to the identity (where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$). We provide $\text{Diff}(\mathbb{T}^2)$ with the inductive limit topology, induced by the supremum norms of analytic functions over the complex neighborhoods of $\mathbb{T}^2$ (see §1.1).

This paper is devoted to the study of the following subsets of $\text{Diff}(\mathbb{T}^2)$: for any $\alpha \in [0, 1]$, $\mathcal{A}_\alpha$ is the set of analytic diffeomorphisms $F : \mathbb{T}^2 \to \mathbb{T}^2$, which are analytically conjugated to the rotation $R_\alpha : (x, y) \mapsto (x + \alpha, y)$, i.e. such that there exists an analytic area preserving diffeomorphism $T$ of the torus verifying $F = T^{-1} \circ R_\alpha \circ T$.

For any $\alpha$, a generic small exact symplectic perturbation of a diffeomorphism from $\mathcal{A}_\alpha$ exhibits the whole spectrum of models of behavior: there is a set of large measure foliated by invariant circles with uniquely ergodic motion on each of them (this is described by the KAM-theorem), hyperbolic and elliptic periodic points, elliptic islands.

Here, we investigate a special type of such perturbations, namely those contained in the closure of $\mathcal{A}_\alpha$. The set $\overline{\mathcal{A}_\alpha}$ of such perturbations is 'small' in $\text{Diff}(\mathbb{T}^2)$. Indeed, all the diffeomorphisms $F \in \overline{\mathcal{A}_\alpha}$ are formally conjugated to the rotation in the following sense: there exists a formal Fourier series $T$ such that $T^{-1} \circ F \circ T = R_\alpha$. This is due to the fact that each of the Fourier coefficients of the normalizing transformation $T$ is a polynomial,

† Current address: Université Paris 7, Institut de Mathematiques, 'Géométrie et Dynamique', 175, rue du Chevaleret, 75013, Paris, France.

depending only on finitely many Fourier coefficients of $F$. At the same time, the formal normal form of an arbitrary exact symplectic perturbation of a diffeomorphism from $\mathcal{A}_\alpha$ is $(x, y) \mapsto (x + \alpha + f(y), y)$. This remark indicates that $\overline{\mathcal{A}_\alpha}$ has infinite codimension in $\mathrm{Diff}(\mathbb{T}^2)$.

Opposite to the case of generic perturbations in $\mathrm{Diff}(\mathbb{T}^2)$, it is probable that the properties of the sets $\overline{\mathcal{A}_\alpha}$ are different, depending on the arithmetical properties of $\alpha$. For a Diophantine (or Bruno) $\alpha$, one can believe that the analogy with the following result of Rüssmann [**Rus**] (see also [**B, E1**]) holds true. It claims, in particular, that if a real analytic Hamiltonian transformation in a neighborhood of an elliptic fixed point is formally conjugated to the *linear* normal form, then the normalizing transformation actually converges, provided that the eigenvalues satisfy a Diophantine (or Bruno) condition. In parallel to this result, it seems likely that, in the case of a Diophantine $\alpha$, for any $F \in \mathcal{A}_\alpha$ there exists a neighborhood $U(F)$ such that $U(F) \cap \overline{\mathcal{A}_\alpha} \subset \mathcal{A}_\alpha$.

On the other hand, we show that for a generic set of real numbers $\alpha$ (by 'generic' we mean 'containing a dense $G_\delta$-set'), namely for those $\alpha$ that are 'well enough approximated by rational numbers', there is a dense set of nonlinearizable diffeomorphisms in $\overline{\mathcal{A}_\alpha}$, i.e. such that there is no homeomorphism of the torus, conjugating it to a rotation. However, much more is indeed true. Let us recall that a transformation of a compact Hausdorff space is called uniquely ergodic if it leaves a unique probability measure on this space invariant.

THEOREM A. *For a generic set of real numbers $\alpha$, $\overline{\mathcal{A}_\alpha}$ contains a dense set of uniquely ergodic diffeomorphisms.*

Speaking of uniquely ergodic diffeomorphisms, one expects their genericity; but since the set of real-analytic diffeomorphisms with our topology is not a Baire space, genericity is not stronger than density in it. In the last section, we discuss the question of genericity of unique ergodicity in particular subspaces of $\overline{\mathcal{A}_\alpha}$.

Unique ergodicity with respect to the Lebesgue measure implies minimality, hence it follows from Theorem A that for a generic set of $\alpha$, $\overline{A_\alpha}$ contains a dense set of minimal diffeomorphisms. In addition to this, we prove the following result.

THEOREM B. *For a generic set of real numbers $\alpha$, $\overline{\mathcal{A}_\alpha}$ contains a dense set of diffeomorphisms that are minimal and non-ergodic.*

For a description of the set of numbers $\alpha$, appearing in the above theorems, see Lemma 7.2.

As a corollary, from Theorem A, we obtain the fact that $\overline{A_\alpha}$ contains a dense set of diffeomorphisms with zero topological entropy. The fact that on a two-dimensional manifold minimality implies vanishing of the topological entropy was proven by Katok in [**K**].

Our constructions are specifically adapted to the case of the torus. An interesting related question is what is the typical behavior of a real analytic measure preserving diffeomorphism near its closed periodic curve. Near an elliptic fixed point?

One of the motivations for the present work is the following. Anosov and Katok, in their well-known paper of 1970 [**AK**], presented a set of examples of nonlinearizable $C^\infty$-diffeomorphisms enjoying ergodic, weak mixing and other statistical properties.

The construction can be performed on an arbitrary smooth manifold $M$, supporting a periodic flow. Moreover, it is shown that in the closure of the set of diffeomorphisms of $M$, contained in smooth, measure preserving periodic flows on $M$, both ergodic and weakly mixing diffeomorphisms are generic. Their construction is based on the method of fast cyclic approximations by periodic diffeomorphisms, that relies on the existence of functions with compact support in $C^\infty$. Similar methods permit Fathi and Herman [**FH**] to prove the existence of smooth minimal and uniquely ergodic diffeomorphisms on smooth manifolds, supporting a periodic flow.

Work in the analytic category demands completely different techniques, and the corresponding set of examples does not yet exist in the whole generality. In this paper we present a part of the analogous treatment of the analytic case.

Let us briefly recall some of the related results in the analytic category. There is an example, due to Furstenberg [**M, KH**], of an analytic area-preserving diffeomorphism of $\mathbb{T}^2$, that is minimal, but not ergodic. Our method allows us to obtain this result as well (Theorem 1.2) (moreover, in Theorem B we present a dense set of such diffeomorphisms).

Uniquely ergodic examples constructed in the present work have the form of a skew-product with an irrational rotation in the base: $G(x, y) = (x + \alpha, y + g(x, y))$. In [**Fu**], Furstenberg studies this type of analytic skew-product and formulates a criterion of unique ergodicity for them. He also shows that for such diffeomorphisms ergodicity in the usual sense implies unique ergodicity. We give a constructive procedure for defining uniquely ergodic skew-products.

Ergodic properties of analytic *skew systems* were studied by Eliasson in [**E2**]. Namely, on $\mathbb{T}^d \times SO(3, \mathbb{R})$ he considers the systems of the form

$$\begin{cases} \dot{X} = F(x)X \\ \dot{x} = \omega, \end{cases}$$

where $F : \mathbb{T}^d \to o(3, \mathbb{R})$ is real analytic, and $\omega \in \mathbb{R}^d$ satisfies a Diophantine condition. Eliasson proves that, for a generic $F$ close to constant coefficients, this system is uniquely ergodic.

The weak mixing property for reparametrized irrational translation flows on the torus $\mathbb{T}^n$, $n \geq 2$, was investigated in its full generality by Fayad [**F1**]. In [**F2**], he presents a study of the mixing property for reparametrized irrational flows on the torus $\mathbb{T}^3$ (or any $\mathbb{T}^n$, $n \geq 3$). Mixing examples are obtained using analytic time changes in the irrational translation flow.

## 1. *Definitions and plan of the proof*

1.1. *Topology on $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha^r$.* The set $\mathcal{A}_\alpha$ under consideration lies in the set Diff($\mathbb{T}^2$) of analytic diffeomorphisms of the torus, homotopic to the identity. Let us discuss different topologies on Diff($\mathbb{T}^2$) and its subsets that we use.

For any $r > 0$, we define $C_r^\omega(\mathbb{T}^2)$ as the set of real analytic functions on $\mathbb{R}^2$, $\mathbb{Z}^2$-periodic, that can be extended to holomorphic functions on $A^r = \{|\mathrm{Im} x|, |\mathrm{Im} y| < r\}$. On this space, we use the uniform norm $|f|_r = \sup_{A^r} |f(x, y)|$.

Being homotopic to the identity, elements of Diff($\mathbb{T}^2$) have a lift of type $F(x, y) = (x + f_1(x, y), y + f_2(x, y))$ with $f_i$ analytic on $\mathbb{R}^2$ and $\mathbb{Z}^2$-periodic. For an arbitrary fixed $r$, consider the subspace $\mathrm{Diff}_r(\mathbb{T}^2)$ of $\mathrm{Diff}(\mathbb{T}^2)$, consisting of those diffeomorphisms, for whose lift it holds: $f_i \in C_r^\omega(\mathbb{T}^2)$, $i = 1, 2$. This gives a natural isomorphism between $\mathrm{Diff}_r(\mathbb{T}^2)$ and the Banach space $\mathcal{D}_r$ of pairs of functions in $C_r^\omega(\mathbb{T}^2)$, with the topology defined by the supremum norms. We endow $\mathrm{Diff}_r(\mathbb{T}^2)$ with the topology $\tau_r$, brought from $\mathcal{D}_r$ by this isomorphism.

For diffeomorphisms $F$, $G$ in $\mathrm{Diff}_r(\mathbb{T}^2)$ we shall use the distance, generating the same topology

$$|F - G|_r = \max_{i=1,2}\{\,|f_i - g_i|_r\}.$$

Up to the last section we shall study the following subset $\mathcal{A}_\alpha^r$ of $\mathcal{A}_\alpha \cap \mathrm{Diff}_r(\mathbb{T}^2)$ (for an arbitrary fixed $r > 0$):

$$\begin{aligned}\mathcal{A}_\alpha^r = \{F \in \mathcal{A}_\alpha \cap \mathrm{Diff}_r(\mathbb{T}^2) \mid F = T^{-1} \circ R_\alpha \circ T \text{ for some real analytic,}\\ \text{area preserving } T,\ \text{ such that } T^{-1} \text{ is analytic in a neighborhood of } \overline{R_\alpha \circ T(A^r)}\}\end{aligned}$$

with the topology induced by $\tau_r$. $\mathcal{A}_\alpha^r$ is a Baire space and we shall prove genericity of uniquely ergodic diffeomorphisms and density of minimal non-ergodic ones in $\overline{\mathcal{A}_\alpha^r}$ (Lemma 7.3). Note that $\mathcal{A}_\alpha^r \subsetneq \mathcal{A}_\alpha \cap \mathrm{Diff}_r(\mathbb{T}^2)$ and we do not know how to prove the same statement for this larger set.

The space $\mathrm{Diff}(\mathbb{T}^2)$ is isomorphic to the space $\mathcal{D} = \bigcup_r \mathcal{D}_r$ of pairs of analytic functions on the torus. The latter can be endowed with the inductive limit topology: a set $O \subset \mathcal{D}$ is open in $\mathcal{D}$ if and only if for any $r > 0$ the set $O \cap \mathcal{D}_r$ is open in $\mathcal{D}_r$. A good description of this topology can be found in [**L**, Appendix 2]. Finally, we define the topology $\tau$ on $\mathrm{Diff}(\mathbb{T}^2)$ bringing the above inductive limit topology to $\mathrm{Diff}(\mathbb{T}^2)$ by the natural isomorphism. We would like to stress that $\tau$ is non-metrizable; it does not make $\mathrm{Diff}(\mathbb{T}^2)$ into a Baire space (a countable intersection of open dense sets can be empty), hence we cannot, unfortunately, speak about genericity in it.

The set $\mathcal{A}_\alpha$ gets the subspace topology, generated by $\tau$. Moreover, $\mathcal{A}_\alpha$ equals $\bigcup_{r>0} \mathcal{A}_\alpha^r$ and, by construction of our topologies, a sequence converging in $\mathcal{A}_\alpha^r$ converges to the same limit in $\mathcal{A}_\alpha$. Hence, $\overline{\mathcal{A}_\alpha^r} \subset \overline{\mathcal{A}_\alpha}$ for any $r$ (closures are taken in the corresponding topologies) and the density of both minimal non-ergodic and uniquely ergodic diffeomorphisms in $\overline{\mathcal{A}_\alpha^r}$ for all $r$ implies their density in $\overline{\mathcal{A}_\alpha}$ with topology induced by $\tau$ (Lemma 7.1).

In what follows we shall not distinguish between a diffeomorphism of the torus and its lifts when this does not lead to confusion.

1.2. *Main construction.* The method is based on the following idea. In Lemma 7.1 we show that, provided that a real number $\alpha$ is 'well-enough approximated by rationals', for any $\varepsilon > 0$ and $r > 0$, we can find an area-preserving real-analytic transformation $T$, homotopic to the identity, such that the dynamics of $G := T^{-1} \circ R_\alpha \circ T$ are very different from those of $R_\alpha$ (in particular, the invariant curves are very far from circles), while the difference $|G - R_\alpha|_r$ is less than $\varepsilon$.

In the process of the construction we shall step-by-step produce a converging sequence of analytic diffeomorphisms $\{G_n\}$, each time conjugating $R_\alpha$ by 'wilder and wilder' transformations $T_n$: $G_n = T_n^{-1} \circ R_\alpha \circ T_n$, so that $|G_n - G_{n-1}|_r$ go to zero very fast. The desired ergodic diffeomorphism is defined as a limit

$$G = \lim_{n\to\infty} G_n.$$

To make this idea work, the value of $\alpha$ has to be chosen close to certain rational numbers $p_n/q_n$, $n \in \mathbb{N}$. (We always assume that $(p_n, q_n) = 1$, i.e. $p_n$ and $q_n$ are relatively prime.) We have chosen a way of presentation where we do not fix the value of $\alpha$ in advance, but construct it as a limit of the inductive procedure. Approximating diffeomorphisms, $G_n$ will thus depend on a real parameter $\alpha$. We express this dependence as $G_n(\alpha; x, y)$ or $G_n(\alpha)$. At the $n$th step we construct $G_n(\alpha)$ and choose a closed interval $I_n$, centered at $p_n/q_n$, $I_n \subset I_{n-1}$ such that the desired estimates for $G_n(\alpha)$ (in particular, $|G_n(\alpha) - G_{n-1}(\alpha)|_r$ small) hold for any $\alpha \in I_n$. The number $\hat{\alpha}$, providing all the necessary estimates, is

$$\hat{\alpha} = \bigcap_n I_n = \lim_n \frac{p_n}{q_n}.$$

If $|\alpha - (p_n/q_n)|$ decays fast with $n$, this number is irrational.

It is important to note that we do not need any lower bounds on $|I_n|$-s. In Lemma 7.2 we give a precise characterization of numbers $\hat{\alpha}$ that can be obtained as limits of the above construction. This will easily imply that such numbers form a $G_\delta$-set in $[0, 1]$.

The only parameters of the construction are a sequence of nested closed intervals $I_n \subset I_{n-1}$, centered at rational points $p_n/q_n$, $n \in \mathbb{N}$ with the corresponding lengths $|I_n|$, and a sequence of positive numbers $c_n$. The numbers $p_n$, $q_n$, $c_n$ and $|I_n| \in \mathbb{R}$ are chosen at the $n$th step of the construction.

Define $G_0 = R_\alpha$, let $I_0 \subset [0, 1]$ be an arbitrary closed interval and denote $r_n = 10^n$, $\varepsilon_n = 1/10^{n+1}$. Let us describe the $n$th step of the construction. Suppose that we have a nested sequence of closed intervals $I_j$, $j = 1, \dots, n-1$, centered at rational points $p_j/q_j$, a rational point $\frac{p_n}{q_n} \in I_{n-1}$, and a sequence of positive real numbers $c_j$, $j = 1, \dots, n$. Let $\Phi_n$ be the following map:

$$\Phi_n : (x, y) \mapsto (x,\ y + c_n \sin(2\pi q_n x)). \tag{1.1}$$

Note that $\Phi_n$ is area preserving. Let $\Phi_j$ be defined for all $j \le n$ by the above formula with $j$ instead of $n$. The conjugating transformation at the $n$th step has the form

$$T_n = \Phi_n \circ \Phi_{n-1} \circ \dots \circ \Phi_1,$$

and the $n$th approximation $G_n(\alpha)$ is defined as the composition

$$G_n(\alpha) := T_n^{-1} \circ R_\alpha \circ T_n. \tag{1.2}$$

We shall choose $I_n \subset I_{n-1}$ centered at $p_n/q_n$ so small that for any $\alpha \in I_n$

$$|G_n(\alpha) - G_{n-1}(\alpha)|_{r_n} < \varepsilon_n. \tag{1.3}$$

The possibility of this is proven in Proposition 2.1. We choose $p_{n+1}/q_{n+1} \in I_n$ with a large denominator, and repeat the procedure iteratively. Then for $\hat{\alpha} = \bigcap_n I_n$, the analytic limit $G(\hat{\alpha}) = \lim_{n\to\infty} G_n(\hat{\alpha})$ exists. Moreover,

$$|G(\hat{\alpha}) - R_{\hat{\alpha}}|_{10} \le |G_1(\hat{\alpha}) - R_{\hat{\alpha}}|_{r_1} + \sum_{n=2}^{\infty} |G_n(\hat{\alpha}) - G_{n-1}(\hat{\alpha})|_{r_n} < \frac{1}{10}.$$

We show that under certain conditions on the decay of $|I_n|$ and growth of $c_n$, the limit $G(\hat{\alpha})$ is analytic nonlinearizable (minimal non-ergodic, uniquely ergodic).

1.3. *Modification used to obtain density.* The above argument permits us, given $r$ and $\varepsilon > 0$, to find a number $\hat{\alpha}$ and a diffeomorphism $G(\hat{\alpha}) \in \overline{\mathcal{A}^r_{\hat{\alpha}}}$ with the required properties, such that $|G(\hat{\alpha}) - R_{\hat{\alpha}}|_r < \varepsilon$. In order to get a dense set of such examples in $\overline{\mathcal{A}^r_{\hat{\alpha}}}$ for an appropriate $\hat{\alpha}$, we modify the above construction a little. Namely, together with $\hat{\alpha}$, we construct not one, but an infinite sequence of diffeomorphisms $G_{(m)}(\hat{\alpha})$ with desired properties, pairwise conjugated by area-preserving analytic transformations (having, therefore, the same ergodic properties), such that

$$|G_{(m)}(\hat{\alpha}) - R_{\hat{\alpha}}|_{10^m} < \frac{1}{10^m}.$$

In the last section we show that this is enough to imply density in $\overline{\mathcal{A}^r_{\hat{\alpha}}}$.

Diffeomorphisms $G_{(m)}(\hat{\alpha})$ are obtained in the same way as $G(\hat{\alpha})$, but with $c_1 = \cdots = c_{m-1} = 0$. In other words, for any $n$ we consider

$$T_{m,n} = \Phi_n \circ \Phi_{n-1} \circ \ldots \circ \Phi_m, \quad \text{for } m \leq n \quad \text{and} \quad T_{n+1,n} = \mathrm{Id},$$

and

$$G_{m,n}(\alpha) = T^{-1}_{m,n} \circ R_\alpha \circ T_{m,n}, \quad \text{for } m \leq n \quad \text{and} \quad G_{n+1,n}(\alpha) = R_\alpha. \tag{1.4}$$

If the limit of $G_{m,n}(\alpha)$ exists, denote for any fixed $m$ ($n = m, \ldots,$)

$$\lim_{n\to\infty} G_{m,n}(\alpha) = G_{(m)}(\alpha).$$

In particular, $T_{1,n} = T_n$, $G_{1,n}(\alpha) = G_n(\alpha)$. For $G_{(1)}(\alpha)$ we keep the notation $G(\alpha)$. Now at the $n$th step of the construction we shall choose $I_n$ so small that, in addition to (1.3), for any $m \leq n$ we have

$$|G_{m,n}(\alpha) - G_{m,n-1}(\alpha)|_{r_n} < \varepsilon_n$$

for all $\alpha \in I_n$. The possibility of this choice follows from Proposition 2.1. Then for $\hat{\alpha} = \bigcap_n I_n$ we have, for any $m$, a sequence of diffeomorphisms $G_{m,n}(\alpha)$, converging to a nonlinearizable (minimal, ergodic) analytic limit $G_{(m)}$, and

$$|G_{(m)}(\hat{\alpha}) - R_{\hat{\alpha}}|_{10^m} \leq \sum_{n=m}^{\infty} |G_{m,n}(\hat{\alpha}) - G_{m,n-1}(\hat{\alpha})|_{r_n} < \frac{1}{10^m}.$$

*Remark 1.1.* Clearly, for any $n$ and $m \leq n$, $G_{m,n}(\alpha)$ is conjugated to $G_n(\alpha)$: $G_{m,n}(\alpha) = S_m^{-1} \circ G_n(\alpha) \circ S_m$, where $S_m^{-1} = \Phi_{m-1} \circ \ldots \circ \Phi_1$. Therefore, for a given $\alpha$ either for each $m$ the sequence $G_{m,n}(\alpha)$ converges with $n \to \infty$, or it does not converge for any $m$. If for some $\alpha$ the sequences do converge, then

$$G_{(m)}(\alpha) = S_m^{-1} \circ G(\alpha) \circ S_m.$$

Since $S_m$ is analytic and area preserving, the limits $G_{(m)}(\alpha)$ have the same ergodic properties for all $m$: they are either minimal (ergodic or uniquely ergodic) for all $m$, or not.

1.4. *Plan of the article.* Section 2 is dedicated to the preliminary technical work. In §3, with the help of the above construction, we produce nonlinearizable analytic diffeomorphism arbitrary close to the rotation. 'Nonlinearizable' means here that there is no homeomorphism of $\mathbb{T}^2$ such that $G(\alpha) = T^{-1} \circ R_\alpha \circ T$.

THEOREM 1.1. *Suppose that* $\sum_n c_n = \infty$. *Then there exists a sequence* $p_n/q_n$, *converging to an irrational number* $\alpha$, *such that for any* $r$ *and* $\varepsilon > 0$ *the above construction provides a nonlinearizable analytic diffeomorphism* $G_{r,\varepsilon} \in \overline{\mathcal{A}_\alpha^r}$ *satisfying* $|G_{r,\varepsilon} - R_\alpha|_r < \varepsilon$.

In §4 we chose the parameters to obtain minimal non-ergodic examples.

THEOREM 1.2. *Let* $c_n = 1/n$ *for all* $n \in \mathbb{N}$. *Then there exists a sequence* $p_n/q_n$, *converging to an irrational number* $\alpha$ *such that, for any* $r$ *and* $\varepsilon > 0$, *the above construction provides a minimal non-ergodic analytic diffeomorphism* $G_{r,\varepsilon} \in \overline{\mathcal{A}_\alpha^r}$ *satisfying* $|G_{r,\varepsilon} - R_\alpha|_r < \varepsilon$.

Sections 5 and 6 contain the proof of the following result.

THEOREM 1.3. *There exists an (increasing) sequence* $c_n$ *and a sequence* $p_n/q_n$, *converging to an irrational number* $\alpha$ *such that, for any* $r$ *and* $\varepsilon > 0$, *the above construction provides a uniquely ergodic analytic diffeomorphism* $G_{r,\varepsilon} \in \overline{\mathcal{A}_\alpha^r}$ *satisfying* $|G_{r,\varepsilon} - R_\alpha|_r < \varepsilon$.

The last section concerns the density of minimal non-ergodic and uniquely ergodic diffeomorphisms in $\overline{\mathcal{A}_\alpha}$ for an appropriate $\alpha$. We give a description of these 'appropriate' $\alpha$, and show that they are generic in $[0, 1]$. This will complete the proof of Theorems A and B.

At the end of the last section we discuss the genericity of uniquely ergodic diffeomorphisms in $\overline{\mathcal{A}_\alpha^r}$ for an appropriate $\alpha$.

## 2. *Important proposition*

In this section we show that the choice of a sufficiently small interval $I_n$ (centered at $p_n/q_n$) implies that the difference $G_{n-1}(\alpha) - G_n(\alpha)$ becomes small for all $\alpha \in I_n$. This statement follows from the continuity of the conjugation. Indeed, by (1.1) and (1.2), for $\alpha = p_n/q_n$ the difference $G_{n-1}(\alpha) - G_n(\alpha)$ equals zero. One would expect that $G_n(\alpha)$ is still close to $G_{n-1}(\alpha)$ for $\alpha$ sufficiently close to $p_n/q_n$.

PROPOSITION 2.1. *Suppose that* $T \in \bigcap_{r>0} \mathrm{Diff}_r$, *and let*

$$G(\alpha) = T^{-1} \circ R_\alpha \circ T.$$

*Then for any* $p_n/q_n \in [0, 1]$, $c_n$, $r \geq 0$ *and* $\varepsilon > 0$, *there exists an interval* $I = I(T, p_n, q_n, c_n, r, \varepsilon)$, *centered at* $p_n/q_n$, *such that, for any* $\alpha \in I$, *the diffeomorphism*

$$G_n(\alpha) = T_n^{-1} \circ R_\alpha \circ T_n, \quad T_n = \Phi_n \circ T \tag{2.1}$$

*with* $\Phi_n$ *defined by (1.1), satisfies*

$$|G_n(\alpha) - G(\alpha)|_r < \varepsilon.$$

COROLLARY 2.1. *Suppose that $G(\alpha)$ and $T$ are as in Proposition 2.1. Then for any $p_n/q_n \in [0,1]$, $c_n$, $r \geq 0$, $\varepsilon > 0$ and $\tau \in \mathbb{N}$, there exists an interval $I = I(T, p_n, q_n, c_n, r, \varepsilon, \tau)$, centered at $p_n/q_n$, such that, for any $\alpha \in I$, the diffeomorphism $G_n(\alpha)$, defined by (2.1), satisfies*

$$\max_{i=0,\dots,\tau} |G^i(\alpha) - G_n^i(\alpha)|_r < \varepsilon. \tag{2.2}$$

*Proof of Proposition 2.1.* Let us first estimate the difference

$$\Phi_n^{-1} \circ R_\alpha \circ \Phi_n - R_\alpha =: \hat{F}_n(\alpha).$$

Write down $\hat{F}_n(\alpha)$ explicitly (omitting the indices)

$$\begin{aligned}|\hat{F}(\alpha)|_r &= |(x+\alpha, y + c\sin 2\pi qx - c\sin 2\pi q(x+\alpha)) - (x+\alpha, y)|_r \\ &= c|\sin 2\pi qx - \sin 2\pi q(x+\alpha)|_r \leq ce^{2\pi qr}|1 - e^{2\pi iq\alpha}|.\end{aligned}$$

Let $\alpha = (p/q) + t|I|$, where $t \in (-1,1)$. Then

$$|1 - e^{2\pi iq\alpha}| = |1 - e^{2\pi i(q\alpha - p)}| \leq 2\pi|q\alpha - p| \leq 2\pi \left|q\left(\frac{p}{q} + t|I|\right) - p\right| \leq 2\pi q|I|.$$

Hence,

$$|\hat{F}(\alpha)|_r \leq 2\pi qce^{2\pi qr}|I|.$$

Returning to the initial problem, rewrite the definition (2.1) of $G_n(\alpha)$ in the following way:

$$G_n(\alpha) = T_n^{-1} \circ R_\alpha \circ T_n = T^{-1} \circ \Phi_n^{-1} \circ R_\alpha \circ \Phi_n \circ T = T^{-1} \circ (R_\alpha + \hat{F}_n(\alpha)) \circ T.$$

Denote

$$r_1 = \sup_{A^r} |\mathrm{Im}(T)|, \quad r_2 = r_1 + 1.$$

Consider a supplementary interval $J$, centered at $p/q$ of length $(2\pi qce^{2\pi qr_1})^{-1}$. Then $\max_{\alpha\in J} |\hat{F}(\alpha)|_{r_1} < 1$. For any $I \subset J$, for all $\alpha \in I$ we can write the following estimate:

$$\begin{aligned}|G_n(\alpha) - G(\alpha)|_r &= |T^{-1} \circ (R_\alpha + \hat{F}_n(\alpha)) \circ T - T^{-1} \circ R_\alpha \circ T|_r \\ &\leq |DT^{-1}|_{r_2} |\hat{F}_n(\alpha)|_{r_1} \leq |DT^{-1}|_{r_2} 2\pi qce^{2\pi qr_1}|I|.\end{aligned}$$

Finally, we take $I \subset J$ centered at the same point $p/q$ with the length

$$|I| \leq \varepsilon(|DT^{-1}|_{r_2} 2\pi qce^{2\pi qr_1})^{-1}.$$

Then

$$\max_{\alpha\in I} |G_n(\alpha) - G(\alpha)|_r < \varepsilon. \qquad \square$$

The proof of the corollary is analogous (it suffices to note that $G_n^i(\alpha) = G_n(i\alpha)$).

*Remark 2.1.* We would like to stress that the condition on the parameters, sufficient to imply the conclusion of Corollary 2.1, is

$$|I_n| \leq \varepsilon(\tau|DT^{-1}|_{r_2} 2\pi q_n c_n e^{2\pi q_n r_1})^{-1}, \tag{2.3}$$

where $T$, $r_1$ and $r_2$ do not depend on $q_n$. Suppose for a moment, that for some fixed $\delta > 0$, for any $n$ we define $I_n$ centered at $p_n/q_n$ by the following condition:

$$|I_n| = \exp(-q_n^{1+\delta}).$$

Then it is enough to take $q_n$ sufficiently large (in the above context, such that $\exp(-q_n^{1+\delta}) < \varepsilon(\tau|DT^{-1}|_{r_2} 2\pi q_n c_n e^{2\pi q_n r_1})^{-1}$) to imply the conclusion of Corollary 2.1. This remark shows that for any number $\alpha$ such that

$$|\alpha - p/q| < \exp(-q^{1+\delta}), \quad \delta > 0,$$

has an infinite number of solutions in natural numbers $p, q$ (with $p$ and $q$ relatively prime) one can find a $q_n$ (from the sequence of solutions) such that $|G_n(\alpha) - G(\alpha)|_r$ becomes as small as we want. As we shall see later (Lemma 7.2), for any such $\alpha$ all the constructions of this paper can be performed.

## 3. *Analytic nonlinearizable diffeomorphism*

*Proof of Theorem 1.1.* We begin by proving the *convergence of* $G_n(\alpha)$ for an appropriate $\alpha$. Denote $r_n = 10^n$, $\varepsilon_n = 1/10^{n+1}$, and let $I_0 \subset [0, 1]$ be an arbitrary closed interval, $G_0(\alpha) = R_\alpha$. Here we take $c_n > 0$ arbitrary, with the only condition that $\sum_n c_n = \infty$. Denote by $\beta_n$ the following often-used quantity:

$$\beta_n = 2\pi \sum_{i=1}^{n-1} \max\{1, c_i\} q_i.$$

Let us explain the choice of the parameters $p_n$ and $q_n$ at the $n$th step of the construction. Suppose that for $j = 1, \ldots, n-1$ the intervals $I_j \subset I_{j-1}$ centered at $p_j/q_j$ are chosen. Pick any $p_n/q_n \in I_{n-1}$ with a large denominator:

$$q_n > \beta_n/\varepsilon_n. \tag{3.1}$$

We shall choose $I_n$ centered at $p_n/q_n$ in the following way. For $m \leq n$, let $I_{m,n}$ denote an interval, given by Proposition 2.1 with $\varepsilon = \varepsilon_n$, $r = r_n$, $T = T_{m,n-1}$, $G(\alpha) = G_{m,n-1}(\alpha)$ (here $G_{n,n-1}(\alpha) = R_\alpha$) see (1.4).

Let $I_n^1$ denote an interval centered at $p_n/q_n$ given by Corollary 2.1 with $\varepsilon = \varepsilon_n$, $r = 0$, $T = T_{n-1}$, $G(\alpha) = G_{n-1}(\alpha)$ and $\tau = q_n$. Define

$$I_n = \left(\bigcap_{m=1}^{n} I_{m,n}\right) \cap I_n^1.$$

Assume also that the $I_n$ lies strictly inside $I_{n-1}$ and

$$|I_n| < 1/q_n^2 \tag{3.2}$$

(for large $n$ it follows from the conditions above, see (2.3)).

Suppose that for all $n$ the parameters are chosen as above and

$$\hat{\alpha} = \bigcap_n I_n = \lim_n \frac{p_n}{q_n}.$$

Then we have obtained an infinite number of converging sequences of diffeomorphisms $(G_{m,n}(\hat{\alpha}))_{n=m}^{\infty}$, $\lim_{n\to\infty} G_{m,n}(\hat{\alpha}) = G_{(m)}(\hat{\alpha})$, such that

$$|R_{\hat{\alpha}} - G_{(m)}(\hat{\alpha})|_{r_m} < \frac{1}{10^m}.$$

Indeed, for any fixed $n$, $\hat{\alpha}$ satisfies the conditions of Proposition 2.1 with $\varepsilon = \varepsilon_n$, $r = r_n$, $T = T_{m,n-1}$, $G(\hat{\alpha}) = G_{m,n-1}(\hat{\alpha})$ for any $m \leq n$. By this proposition, for all $m \leq n$, we have

$$|G_{m,n-1}(\hat{\alpha}) - G_{m,n}(\hat{\alpha})|_{r_n} < \varepsilon_n.$$

Fix an arbitrary $m$. For $n \geq m$, the diffeomorphisms $G_{m,n}(\hat{\alpha})$ form a Cauchy sequence in the metric $|\cdot|_{r_m}$ and the limit $G_{(m)}(\hat{\alpha}) \in \overline{\mathcal{A}_{\hat{\alpha}}^{r_m}}$ exists. Moreover,

$$\begin{aligned} |R_{\hat{\alpha}} - G_{(m)}(\hat{\alpha})|_r &\leq \sum_{j=-1}^{\infty} |G_{m,m+j}(\hat{\alpha}) - G_{m,m+j+1}(\hat{\alpha})|_{r_m} \\ &< \frac{1}{10^{m+1}} \sum_{j=0}^{\infty} \frac{1}{10^j} < \frac{1}{10^m}. \end{aligned}$$

*Nonlinearizability.* Now we show that for any $m$ the limit $G_{(m)}(\hat{\alpha})$ of the above construction is nonlinearizable. By Remark 1.1, it is enough to show it for $G(\hat{\alpha}) = G_{(1)}(\hat{\alpha})$. Suppose the contrary, i.e. that there exists a homeomorphism $T$ of the torus, such that $G(\hat{\alpha}) = T^{-1} \circ R_{\hat{\alpha}} \circ T$. Then any invariant curve $\Gamma = \Gamma(y_0)$ of $G(\hat{\alpha})$ has the form $T^{-1}\{y = y_0\}$ for some $y_0 \in [0, 1]$ and hence is a continuous closed curve, not dense on the torus. We shall come to the contradiction with this assumption by showing that $\Gamma$ is dense on the torus. (In fact, the arguments below imply even more: that the limit diffeomorphism is minimal.) In order to do this, for any fixed $n$ consider invariant curves $\Gamma_n(y_0)$ of $G_n(\hat{\alpha})$. Each of them has the form $T_n^{-1}\{y = y_0\}$, $y_0 \in [0, 1]$. Recalling that $T_n^{-1} = \Phi_1^{-1} \circ \cdots \circ \Phi_n^{-1}$, we compute

$$T_n^{-1}(x, y) = \left(x, y - \sum_{j=1}^{n} c_j \sin 2\pi q_j x\right). \tag{3.3}$$

Then the lift of $\Gamma_n(y_0)$ is the graph of the function

$$y = \Gamma_n(x) = y_0 - \sum_{j=1}^{n} c_j \sin 2\pi q_j x. \tag{3.4}$$

Note that

$$|\Gamma_n'|_0 \leq 2\pi \sum_{i=1}^{n} c_i q_i = \beta_{n+1}.$$

We split the proof in three steps. Namely, we show:

(a) for any $\varepsilon > 0$, there exists an $N_0$ such that for all $n \geq N_0$, the curve $\Gamma_n$ is $\varepsilon/4$-dense on $\mathbb{T}^2$;

(b) for these $\varepsilon$ and $n \geq N_0$, the set of points $\{((x + i\hat{\alpha}), \Gamma_n(x + i\hat{\alpha})) \mid i = 0, \dots, q_{n+1}\}$ is $\varepsilon/4$-dense on the curve $\Gamma_n$;

(c) in these points the curve $\Gamma$ is $\varepsilon/4$-close to $\Gamma_n$, in other words,

$$\max_{i=0,\dots,q_{n+1}} |G^i(\hat{\alpha}) - G_n^i(\hat{\alpha})|_0 < \varepsilon/4.$$

This will ensure that for an arbitrary $\varepsilon$ the curve $\Gamma$ is $\varepsilon$-dense on $\mathbb{T}^2$, contradicting the assumption.

To prove (a), note that $\Gamma_{m+1}(x) = \Gamma_m(x) - c_{m+1} \sin 2\pi q_{m+1} x$. It is easy to see that for any $m$ and any $x$ there exists a point $y$ such that $|x - y| < 1/q_{m+1}$ and $\Gamma_{m+1}(y) \geq \Gamma_m(x) + c_{m+1} - |\Gamma_m'|_0 |x - y|$. Since $|\Gamma_m'|_0 \leq \beta_{m+1}$, by (3.1) we can estimate:

$$\Gamma_{m+1}(y) \geq \Gamma_m(x) + c_{m+1} - \beta_{m+1} \frac{1}{q_{m+1}} \geq \Gamma_m(x) + c_{m+1} - \varepsilon_{m+1}.$$

By induction, for $n > m$ we can prove that for any $x$, there exists a point $z_1 = z_1(n)$,

$$|x - z_1| < \sum_{j=m+1}^{n} \frac{1}{q_j}$$

such that

$$\Gamma_n(z_1) \geq \Gamma_m(x) + \sum_{j=m+1}^{n} c_j - \sum_{j=m+1}^{n} \varepsilon_j.$$

Since the last term of this sum is bounded by $\varepsilon_m$ for any $n$, and $\sum_j c_j$ diverges, $\Gamma_n(z_1) > 1$ for all $n$ larger than some $N_1$. Moreover,

$$|x - z_1| < \sum_{j=m+1}^{n} \frac{1}{q_j} < \frac{1}{q_m}$$

for all $n$ (it follows from (3.1)).

By the same reasons, there exists a point $z_2$, $|x - z_2| < 1/q_m$, such that $\Gamma_n(z_2) < 1$ for all $n$ larger than some $N_2$. Now take an $m$ such that $1/q_m \leq \varepsilon/8$ and $N_0 = \max\{N_1, N_2\}$.

Let us prove (b). Take $n$ such that $\varepsilon_n < \varepsilon/4$. Since, in particular, $\hat{\alpha} \in I_{n+1}$, then

$$\left| \hat{\alpha} - \frac{p_{n+1}}{q_{n+1}} \right| < \frac{1}{q_{n+1}^2}$$

by assumption (3.2). Then the set $\{(x + i\hat{\alpha}) \mid i = 0, \dots, q_{n+1}\}$ is $2/q_{n+1}$-dense on $\mathbb{T}^1$. This implies that the set $\{((x + i\hat{\alpha}), \Gamma_n(x + i\hat{\alpha})) \mid i = 0, \dots, q_{n+1}\}$ is $\delta$-dense on the curve $\Gamma_n$ where

$$\delta \leq \frac{2}{q_{n+1}} |\Gamma_n'|_0 \leq \frac{2\beta_{n+1}}{q_{n+1}}.$$

By (3.1), $\delta \leq 2\varepsilon_{n+1} < \varepsilon_n < \varepsilon/4$. This proves (b).

Condition (c) holds since $\hat{\alpha} \in \bigcap_n I_n^1$. Indeed,

$$\max_{i=0,\dots,q_{n+1}} |G^i(\hat{\alpha}) - G_n^i(\hat{\alpha})|_0 \leq \sum_{j=n}^{\infty} \max_{i=0,\dots,q_{n+1}} |G_j^i(\hat{\alpha}) - G_{j+1}^i(\hat{\alpha})|_0 < \varepsilon_n < \varepsilon/4.$$

Hence, $\Gamma(\hat{\alpha})$ is dense on the torus, contradicting the assumed linearizability of $G(\hat{\alpha})$. □

## 4. *Minimal non-ergodic diffeomorphism*

*Proof of Theorem 1.2.* Recall that

$$T_n(x,y)=\left(x,y+\sum_{j=1}^{n}c_j\sin 2\pi q_j x\right),$$

$T_n^{-1}(x,y)$ has the form (3.3) and the diffeomorphism $G_n(\alpha)$, given by (1.2), has the form

$$G_n(\alpha;x,y)=(x+\alpha,y+g_n(x)-g_n(x+\alpha)),$$

where

$$g_n(x)=\sum_{j=1}^{n}c_j\sin 2\pi q_j x.$$

Suppose that $c_n=1/n$ for all $n\in\mathbb{N}$, and an irrational $\alpha=\lim p_n/q_n$ is chosen as in Theorem 1.1. Then the analytic limit $G(\alpha)=\lim_{n\to\infty}G_n(\alpha)$ exists and there exist diffeomorphisms with the same ergodic properties arbitrary close to $R_\alpha$ in any $r$-metric. Let us study the obtained diffeomorphisms closer.

If $c_n=1/n$, $g_n$ converge (in $L^2$-metric) to a 1-periodic $L^2$-function $g$ (whose Fourier series is $\tilde{g}=\sum_{j=1}^{\infty}\frac{1}{j}\sin 2\pi q_j x$). Then, almost everywhere,

$$G(\alpha;x,y)=(x+\alpha,y+g(x)-g(x+\alpha)).$$

It is evident that $G$ is not ergodic with respect to the Lebesgue measure. Indeed, it has an infinite number of ergodic components, supported almost everywhere on the 'strips' of the form

$$\{(x,y-g(x))\mid x\in\mathbb{T}^1,\ y\in[y_1,y_2]\subset\mathbb{T}^1\},$$

each having non-zero Lebesgue measure.

A short study of this type of diffeomorphisms can be found in [**KH**]. In particular, the following statement is proven.

LEMMA 4.1. *Consider an analytic mapping $f:(x,y)\mapsto(x+\alpha,y+\varphi(x))$, where $\varphi:S^1\to\mathbb{R}$. Then either $\varphi(x)=g(x)-g(x+\alpha)+r$ for some continuous function $g:S^1\to\mathbb{R}$ and $r\in\mathbb{R}$, or $f$ is minimal.*

Let us give another, more elegant proof of the minimality of the limit diffeomorphism (in addition to that contained in the proof of Theorem 1.1). Assume that $q_n$ grow faster than a geometric progression (this does not contradict the assumptions of Theorem 1.1; in fact, Theorem 1.1 requires much faster growth of $q_n$). Then the series $\tilde{g}$ is a so-called *lacunary series*. For such a series we have the following general statement [**Ka**].

LEMMA 4.2. *If a lacunary series $\sum_j c_j e^{2\pi i q_j x}$ is a Fourier series of a bounded function, then $\sum_j |c_j|<\infty$.*

This lemma implies that $g$ is unbounded and, therefore, non-continuous.

Suppose that $G(\alpha)$ is not minimal. Then, by Lemma 4.1, for the function $\varphi(x)=g(x)-g(x+\alpha)$ there exists a representation $\varphi(x)=h(x)-h(x+\alpha)+r$ with a continuous function $h=S^1\to\mathbb{R}$ and $r\in\mathbb{R}$. Integrating the equality $g(x)-g(x+\alpha)=h(x)-h(x+\alpha)+r$, we get that $r = 0$. Now denote $g - h$ by $f$. Then $f(x + \alpha) = f(x)$ for any $x$, implying that $f =$ constant by the unique ergodicity of the rotation by $\alpha$. This proves that $g = h +$ constant, which contradicts the continuity of $h$. Hence, $G(\alpha)$ is minimal.

Then, as remarked in the beginning of the proof, for any positive $r$ and $\varepsilon$ there exists a diffeomorphism $G_{r,\varepsilon}(\alpha)$ such that $|G_{r,\varepsilon}(\alpha) - R_\alpha|_r < \varepsilon$, and $G_{r,\varepsilon}(\alpha) = S^{-1} \circ G(\alpha) \circ S$ with an analytic area preserving diffeomorphism $S$. The latter implies that $G_{r,\varepsilon}(\alpha)$ is minimal and non-ergodic. This finishes the proof of Theorem 1.2. □

## 5. *Invariant measures*

Here we study the properties of the diffeomorphisms $G_n(\alpha)$, given by formula (1.2), namely, their ergodic invariant measures and the convergence of Birkhoff sums. The results of this section will be used in the proof of Theorem 1.3. Any invariant curve $\Gamma_n$ of $G_n(\alpha)$ is the graph of the function $y = \Gamma_n(x)$ (see (3.4)). For an arbitrary $n$, let us fix an invariant curve $\Gamma_n$. Suppose that $\alpha$ is irrational and consider the (unique) ergodic invariant probability measure $\mu_n$ of $G_n(\alpha)$, supported on this curve. We begin the section by discussing the dependence of the mean values over $\mu_n$ on the parameters of the construction. Let us denote $\int_{\mathbb{T}^2} f(z)\, d\mu_n$ by $\hat{f}_n$ and $\int_{\mathbb{T}^2} f(z)\, dz$ by $\hat{f}$. We prove the following result.

LEMMA 5.1. *Let $\Gamma_n$ be an invariant curve of $G_n(\alpha)$, $\alpha \in [0,1] \setminus \mathbb{Q}$ and consider the invariant probability measure $\mu_n$ of $G_n(\alpha)$, supported on this curve. Then for any fixed trigonometric polynomial $f$ and $\varepsilon > 0$, there exist numbers $\tilde{c}_n = \tilde{c}_n(\varepsilon, f)$ and $\tilde{q}_n = \tilde{q}_n(\varepsilon, f, c_1, \ldots, c_n, q_1, \ldots, q_{n-1})$ such that, if $c_n > \tilde{c}_n$ and $q_n > \tilde{q}_n$, then*

$$|\hat{f}_n - \hat{f}| = \left| \int_{\mathbb{T}^2} f(z)\, d\mu_n - \int_{\mathbb{T}^2} f(z)\, dz \right| < \varepsilon.$$

*In particular, $\tilde{c}_n$ and $\tilde{q}_n$ are independent of the choice of the invariant curve $\Gamma_n$.*

*Remark 5.1.* Although the diffeomorphism $G_n(\alpha)$ depends on $\alpha$, neither $\mu_n$ nor $\Gamma_n$ depend on it. Hence, $\tilde{c}_n$ and $\tilde{q}_n$ do not depend on $\alpha$.

*Proof of Lemma 5.1.* Fix an arbitrary $n$. It is sufficient to prove the statement for exponential functions, the result for trigonometric polynomials follows by linearity. By the definition of $\mu_n$, for any $k, l \in \mathbb{Z}$, we have

$$\int_{\mathbb{T}^2} e^{2\pi i(kx+ly)}\, d\mu_n = \int_0^1 e^{2\pi i(kx+l\Gamma_n(x))}\, dx.$$

For $k = l = 0$ the latter equals one, which coincides with the integral of this function over the Lebesgue measure. We shall show that for any non-zero pair $(k, l)$ the integral above is small (and hence close to the Lebesgue measure integral of the corresponding function). Let us consider $k = 0, l = 1$; the general situation can be studied in the same way. For any $\varepsilon > 0$, we shall show that

$$\left| \int_0^1 e^{2\pi i\Gamma_n(x)}\, dx \right| < \varepsilon, \tag{5.1}$$

provided that $c_n$ and $q_n$ are large enough. Rewrite formula (3.4) for the invariant curve of $G_n(\alpha)$ as

$$\Gamma_n(x) = -c_n \sin 2\pi q_n x + \Gamma_{n-1}(x). \tag{5.2}$$

In §3 we denoted $2\pi \sum_{j=1}^{n-1} c_j q_j$ by $\beta_n$. Then $|\Gamma'_{n-1}|_0 \leq \beta_n$ and $|\Gamma''_{n-1}|_0 \leq 2\pi q_n \beta_n$.

In order to prove the estimate (5.1), for each $n$, we approximate $\Gamma_n$ by a piecewise linear curve $y = L_n(x)$ in the following way. Suppose that $c_n \geq 1$ is chosen so large that there exists a natural number $s_n > e^2$ such that

$$2^4\sqrt{c_n/\varepsilon} < s_n, \quad \text{and} \quad s_n \ln s_n < c_n\varepsilon/2^4. \tag{5.3}$$

Suppose that

$$q_n > \tilde{q}_n := \beta_n s_n. \tag{5.4}$$

Divide the $x$-interval $[0, 1]$ into $4s_nq_n$ subintervals of equal length $\delta_n := 1/4s_nq_n$. We call the intervals $\Delta_j = [(j-1)\delta_n, j\delta_n]$, where $j = 1, \ldots, 4s_nq_n$. The graph of $L_n(x)$ consists of the line segments, connecting the points $((j-1)\delta_n, \Gamma_n((j-1)\delta_n))$ with $(j\delta_n, \Gamma_n(j\delta_n))$.

Let $\alpha_j$ be the inclination of $L_n$ on the interval $\Delta_j$ and let $\gamma_j = \Gamma_n((j-1)\delta_n)$, so that $L_n(x)$ has the form

$$L_n(x) = \alpha_j(x - (j-1)\delta_n) + \gamma_j, \quad x \in \Delta_j.$$

First we show that our approximation is 'good enough', i.e. the difference

$$\left| \int_0^1 e^{2\pi i\Gamma_n(x)}\, dx - \int_0^1 e^{2\pi i L_n(x)}\, dx \right| = \left| \int_0^1 e^{2\pi i L_n(x)}(e^{2\pi i(\Gamma_n(x) - L_n(x))} - 1)\, dx \right| \leq 2\pi|\Gamma_n - L_n|_0$$

is less than $\varepsilon/2$. It can be computed (applying the Mean Value Theorem on each of the subintervals $\Delta_j$ and using (5.4) and the first inequality of (5.3)) that

$$2\pi|\Gamma_n - L_n|_0 < 2\pi|\Gamma''_n|_0\delta_n^2 \leq \pi^3\left(\frac{c_n}{s_n^2} + \frac{\beta_n}{q_ns_n^2}\right) \leq 2\pi^3\frac{c_n}{s_n^2} < \varepsilon/2.$$

Now we estimate (5.1) with $L_n$ instead of $\Gamma_n$. Note that the inclinations $\alpha_j$ mostly depend on the first term of the sum (5.2). Let us first estimate $\alpha_j$ on the first quarter of the period of $\sin(2\pi q_nx)$, i.e. on $\Delta_j$ for $j = 1, \ldots, s_n$. By the Mean Value Theorem, for some $z_j \in \Delta_j$,

$$|\alpha_j| = |\Gamma'_n(z_j)| \geq 2\pi c_nq_n \min_{z\in\Delta_j} |\cos(2\pi q_nz)| - \beta_n.$$

Then for $j = 1, \ldots, (s_n - 1)$ the inclination $\alpha_j$ of $L_n$ on $\Delta_j$ satisfies the following:

$$\begin{aligned} |\alpha_j| &\geq 2\pi c_nq_n\cos(2\pi q_nj\delta_n) - \beta_n = 2\pi c_nq_n\cos\frac{\pi j}{2s_n} - \beta_n \\ &\geq 2\pi c_nq_n\left(1 - \frac{j}{s_n}\right) - \beta_n \geq c_nq_n\left(1 - \frac{j}{s_n}\right) := A_j. \end{aligned}$$

We used the fact that in this area $\cos x \geq 1 - (2/\pi)x$, and the estimate $\beta_n \leq c_nq_n(1 - (j/s_n))$ for all $j = 1, \ldots, (s_n - 1)$, which follows from (5.4) and $c_n \geq 1$.

The value of $\alpha_{s_n}$ has to be estimated separately. Using (5.4),

$$|\alpha_{s_n}| = \frac{|\Gamma_n(s_n\delta_n) - \Gamma_n((s_n-1)\delta_n)|}{\delta_n} \geq \frac{c_n(2\pi q_n\delta_n)^2}{2\delta_n} - \beta_n > \frac{c_n q_n}{s_n} := A_{s_n}.$$

It is not difficult to see that if we set

$$A_{2s_n-j+1} := A_j, \quad j = 1, \ldots, s_n,$$

then $|\alpha_j| \geq A_j$ for $j = 1, \ldots, 2s_n$.

Now we have got lower bounds on $\alpha_j$ on the interval $[0, 1/2q_n]$. Although the curve $L_n$ is not $1/2q_n$-periodic, the estimates of $|\alpha_j|$ can be extended $1/2q_n$-periodically to the whole interval $[0, 1]$. We set

$$A_{2s_nk+j} := A_j, \quad j = 1, \ldots, 2s_n,\ k = 1, \ldots, (2q_n - 1).$$

Then, on each $\Delta_j$ we have

$$|\alpha_j| \geq A_j.$$

The preceding estimates can be summarized as follows:

$$\int_0^1 e^{2\pi i L_n(x)}\,dx = \sum_{j=1}^{4s_nq_n} \int_{(j-1)\delta_n}^{j\delta_n} e^{2\pi i(\alpha_j(x-(j-1)\delta_n)+\gamma_j)}\,dx$$
$$= \sum_{j=1}^{4s_nq_n} e^{2\pi i\gamma_j} \frac{1}{2\pi i\alpha_j}(e^{2\pi i\alpha_j\delta_n} - 1).$$

The absolute value of the above expression is less than

$$\sum_{j=1}^{4s_nq_n} \frac{1}{|\alpha_j|} = \sum_{j=1}^{4s_nq_n} \frac{1}{A_j} = 4q_n \sum_{j=1}^{s_n} \frac{1}{A_j} \leq 4q_n\left(\frac{s_n}{c_nq_n} + \sum_{j=1}^{s_n-1} \frac{1}{c_nq_n(1-j/s_n)}\right)$$
$$= 4\frac{s_n}{c_n}\left(1 + \sum_{j=1}^{s_n-1} \frac{1}{s_n - j}\right) = 4\frac{s_n}{c_n}\left(1 + \sum_{j=1}^{s_n-1} \frac{1}{j}\right) \leq 4\frac{s_n}{c_n}(2 + \ln s_n) < \varepsilon/2.$$

The last inequality follows from the second inequality of (5.3). This finishes the proof of the lemma. □

We pass to the study of the convergence of Birkhoff sums of $G_n(\alpha)$ and its dependence on $\alpha$. Recall that the invariant curves of $G_n(\alpha)$ do not depend on $\alpha$. Since for an irrational $\alpha$ the restriction of $G_n(\alpha)$ to any of its continuous invariant curves $\Gamma_n$ is uniquely ergodic, for any fixed $f \in C^0$ the limit

$$\lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} f \circ G_n^i(\alpha; z) = \int_{\mathbb{T}^2} f(z)\,d\mu_n = \hat{f}_n(z) \tag{5.5}$$

exists for every $z \in \mathbb{T}^2$ (although the limit function is a constant on every invariant curve, it is not necessarily a global constant). The following statement shows that the limit in (5.5) is uniform in $z$ and $\alpha$ for all $\alpha$ in a small interval.

LEMMA 5.2. *Suppose that $G_n(\alpha)$ is defined by (1.2). Then for any $f \in C^\infty(\mathbb{T}^2)$, $\varepsilon > 0$, and any interval $I$, there exists a subinterval $I'_n \subset I$ and a natural number $\tau$—both depending on $G_n(\alpha)$, $f$, $\varepsilon$—such that, for any $\alpha \in I'_n$ and any $k \geq \tau$,*

$$\left|\frac{1}{k+1}\sum_{i=0}^{k} f \circ G_n^i(\alpha) - \hat{f}_n\right|_0 < \varepsilon.$$

*Moreover, for any irrational number $\alpha_0$, $I'_n$ can be chosen containing this number.*

*Proof.* Fix an arbitrary $\alpha_0 \in I \setminus \mathbb{Q}$. Then there exists a $K = K(\alpha_0) \in \mathbb{N}$ such that

$$\left|\frac{1}{k+1}\sum_{i=0}^{k} f \circ G_n^i(\alpha_0; z) - \hat{f}_n(z)\right| < \varepsilon/4, \quad \text{for all } k \geq K,\ z \in \mathbb{T}^2.$$

Indeed, for any particular $z$ such a number exists by the unique ergodicity of $G_n(\alpha)$ on any of its invariant curves; the common $K$ for all $z \in \mathbb{T}^2$ exists by compactness of $\mathbb{T}^2$. Now let

$$I'_n = \left\{\alpha \mid |\alpha - \alpha_0| < \frac{\varepsilon}{4(K+1)|\mathrm{D}(f \circ T_n^{-1})|_0}\right\},$$

and let $a \in \mathbb{N}$ be any number $a > 4|f|_0/\varepsilon$. We set $\tau = a(K+1)$ and show that these $I'_n$ and $\tau$ provide the conclusion of the lemma. Indeed, by the assumptions on $\alpha$ and $a$, for any $z$ and any $\alpha \in I'_n$ we have

$$\begin{aligned}
&\left|\sum_{i=0}^{K} f \circ G_n^i(\alpha; z) - (K+1)\hat{f}_n(z)\right| \\
&\quad \leq \left|\sum_{i=0}^{K}(f \circ G_n^i(\alpha; z) - f \circ G_n^i(\alpha_0; z))\right| + \left|\sum_{i=0}^{K}(f \circ G_n^i(\alpha_0; z) - (K+1)\hat{f}_n(z))\right| \\
&\quad < (K+1)^2|\mathrm{D}(f \circ T_n^{-1})|_0|\alpha - \alpha_0| + (K+1)\frac{\varepsilon}{4} < (K+1)\frac{\varepsilon}{2}.
\end{aligned}$$

Let $k \geq \tau$, i.e. $k = (K+1)b + j$ with some natural $b \geq a$ and $0 \leq j < (K+1)$. For an arbitrary $z$, denote $G_n^{l(K+1)}(\alpha_0; z)$ by $z_l$ for $l = 0, \dots, b$, and note that $\hat{f}_n(z_l) = \hat{f}_n(z)$ for all $l$. Then we have:

$$\begin{aligned}
&\left|\frac{1}{k+1}\sum_{i=0}^{k} f \circ G_n^i(\alpha; z) - \hat{f}_n(z)\right| \\
&\quad \leq \frac{1}{k+1}\left(\sum_{l=0}^{b-1}\left|\sum_{i=0}^{K} f \circ G_n^i(\alpha; z_l) - (K+1)\hat{f}_n(z)\right|\right. \\
&\qquad \left. + \left|\sum_{i=0}^{j} f \circ G_n^i(\alpha)\right|_0 + (j+1)|\hat{f}_n|_0\right) \\
&\quad \leq \frac{1}{(K+1)b}\left((K+1)b\frac{\varepsilon}{2} + 2(K+1)|f|_0\right) < \varepsilon. \qquad \square
\end{aligned}$$

## 6. *Unique ergodicity (proof of Theorem 1.3)*

Let $\mathcal{F} \subset C^\infty(\mathbb{T}^2)$ denote an ordered (countable, dense) set of trigonometric polynomials and let $\mathcal{F}_n$ be the finite set, consisting of the first $n$ elements of $\mathcal{F}$.

To prove the unique ergodicity of $G(\alpha)$ (for an appropriate $\alpha$), it is enough to show (see [**M**, Theorem 1.9.2]) that for any function $f \in \mathcal{F}$ and $z \in \mathbb{T}^2$,

$$\lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} f \circ G^i(\alpha; z) = \int_{\mathbb{T}^2} f(z)\, dz = \hat{f}. \tag{6.1}$$

Here we modify the construction of Theorem 1.1 in such a way that the nonlinearizable analytic limit $G(\alpha)$, existing by this theorem, satisfies the above property. Namely, we shall pay more attention to the choice of the growing sequences $c_n$ and $q_n$, and impose even sharper conditions on the decay of $|I_n|$ than is required in Theorem 1.1. Let us describe the step of the construction in detail.

ASSUMPTION 1. *For every n let us suppose that*

$$c_n \geq \max_{f\in\mathcal{F}_n} \tilde{c}_n \left(\frac{1}{10^{n+1}}, f\right),$$

*and*

$$q_n \geq \max_{f\in\mathcal{F}_n} \tilde{q}_n \left(\frac{1}{10^{n+1}}, f, c_1, \ldots, c_n, q_1, \ldots, q_{n-1}\right),$$

*where* $\tilde{q}_n$ *and* $\tilde{c}_n$ *are from Lemma 5.1. Then, by Lemma 5.1, for all n we have*

$$|\hat{f}_n - \hat{f}|_0 < \frac{1}{10^{n+1}}, \quad \textit{for all } f \in \mathcal{F}_n,\ \alpha \in [0,1]. \tag{6.2}$$

ASSUMPTION 2. *For any n, let the interval* $I'_n \subset I_n$ *and integer* $\tau_n$ *be such that*

$$\left|\frac{1}{k+1}\sum_{i=0}^{k} f \circ G_n^i(\alpha) - \hat{f}_n\right|_0 < \frac{1}{10^{n+1}}, \quad \textit{for all } f \in \mathcal{F}_n,\ \alpha \in I'_n,\ k \geq \tau_n. \tag{6.3}$$

This number and interval exist by Lemma 5.2.

ASSUMPTION 3. *Let* $I_{n+1} \subset I_n$ *be chosen in the following way. Take an arbitrary* $p_{n+1}/q_{n+1} \in I'_n$. *Fix an arbitrary natural* $a > 10^{n+1} \max_{f\in\mathcal{F}_{n+1}} |f|_0$. *We take* $I_{n+1} \subset I'_n$, *centered at* $p_{n+1}/q_{n+1}$, *such that, for all* $\alpha \in I_{n+1}$ *and* $f \in \mathcal{F}_n$,

$$\max_{i=0,\ldots,(\tau_n+1)a} |f \circ G_{n+1}^i(\alpha) - f \circ G_n^i(\alpha)|_0 < \frac{1}{10^{n+1}}. \tag{6.4}$$

Such an interval exists by Corollary 2.1. These are the assumptions on $c_n$, $p_n$, $q_n$ and $I_n$ that are sufficient to prove (6.1).

Assumption 3 guarantees that the first $(\tau_n+1)a$ iterates of $f \circ G_{n+1}^i(\alpha)$ are close to those of $f \circ G_n^i(\alpha)$. Let us show that, in fact, under the above assumptions they are always close 'in the mean'.

LEMMA 6.1. *Under Assumptions 1–3, for each n we have*

$$\frac{1}{k+1}\left|\sum_{i=0}^{k}(f \circ G_{n+1}^i(\alpha) - f \circ G_n^i(\alpha))\right|_0 < \frac{7}{10^{n+1}}, \quad \textit{for all } f \in \mathcal{F}_n,\ \alpha \in I_{n+1},\ k \in \mathbb{N}. \tag{6.5}$$

*Proof.* The proof is based on the fact that for large values of $i$, both $f \circ G_n^i(\alpha)$ and $f \circ G_{n+1}^i(\alpha)$ are close to the mean value of $f$.

Let $k = (\tau_n + 1)b + j$ with some natural $b \geq a$ and $0 \leq j < (\tau_n + 1)$. The left-hand side of (6.5) is less than

$$\left| \frac{1}{k+1} \sum_{i=0}^{k} f \circ G_{n+1}^i(\alpha) - \hat{f}_n \right|_0 + \left| \frac{1}{k+1} \sum_{i=0}^{k} f \circ G_n^i(\alpha) - \hat{f}_n \right|_0. \tag{6.6}$$

The second term is less than $1/10^{n+1}$ for $k \geq \tau_n$ by (6.3). Let us estimate the first. Note that for any $z$, we have

$$\begin{aligned} &\left| \sum_{i=0}^{\tau_n} f \circ G_{n+1}^i(\alpha; z) - (\tau_n + 1)\hat{f}_n(z) \right| \\ &\quad \leq \left| \sum_{i=0}^{\tau_n} f \circ G_{n+1}^i(\alpha; z) - \sum_{i=0}^{\tau_n} f \circ G_n^i(\alpha; z) \right| + \left| \sum_{i=0}^{\tau_n} f \circ G_n^i(\alpha; z) - (\tau_n + 1)\hat{f}_n(z) \right| \\ &\quad < \frac{2(\tau_n + 1)}{10^{n+1}}. \end{aligned}$$

This is true since each of the terms is less than $\tau_n + 1/10^{n+1}$ by assumption: the first one by (6.4), the second by (6.3).

Set $z_l = G_{n+1}^{l(\tau_n+1)}(\alpha; z)$ for $l = 0, \ldots, b$. Note that $|\hat{f}_n(z_l) - \hat{f}_n(z)|_0 < 2|\hat{f}_n - \hat{f}|_0 < 2/10^{n+1}$ by (6.2) for any $l$. Now the first term of (6.6) is less than

$$\begin{aligned} &\frac{1}{k+1} \left| \sum_{l=0}^{b-1} \sum_{i=0}^{\tau_n} f \circ G_{n+1}^i(\alpha; z_l) + \sum_{i=0}^{j} f \circ G_{n+1}^i(\alpha; z_b) - (k+1)\hat{f}_n(z) \right| \\ &\quad \leq \frac{1}{k+1} \left( \sum_{l=0}^{b-1} \left| \sum_{i=0}^{\tau_n} f \circ G_{n+1}^i(\alpha; z_l) - (\tau_n + 1)\hat{f}_n(z_l) \right| \right. \\ &\quad\quad \left. + \left| \sum_{i=0}^{j} f \circ G_{n+1}^i(\alpha) \right|_0 + (j+1)|\hat{f}_n|_0 \right) + 2|\hat{f}_n - \hat{f}|_0 \\ &\quad \leq \frac{1}{b(\tau_n + 1)} \left( \frac{2b(\tau_n + 1)}{10^{n+1}} + 2\tau_n |f|_0 \right) + \frac{2}{10^{n+1}} < \frac{6}{10^{n+1}}. \end{aligned}$$

Hence, (6.5) holds true. □

*Proof of Theorem 1.3.* Suppose that Assumptions 1–3 above hold true for all $n$ and $\hat{\alpha} = \bigcap_n I_n \in [0,1] \setminus \mathbb{Q}$. Then, in particular, the analytic nonlinearizable limit $G(\hat{\alpha}) = \lim_{n\to\infty} G_n(\hat{\alpha})$ exists. We conclude that for any $f \in \mathcal{F}$ and $\varepsilon$ there exists a $\tau = \tau(f, \varepsilon)$, such that, for any $k > \tau$,

$$\left| \frac{1}{k+1} \sum_{i=0}^{k} f \circ G^i(\hat{\alpha}) - \hat{f} \right|_0 < \varepsilon.$$

Indeed, given arbitrary $f \in \mathcal{F}$ and $\varepsilon > 0$, take $l$ such that $f \in \mathcal{F}_l$ and $1/10^l < \varepsilon$. For an arbitrary $k$ we can write

$$\left|\frac{1}{k+1}\sum_{i=0}^{k} f\circ G^i(\hat\alpha) - \hat f\right|_0$$
$$\leq \frac{1}{k+1}\left|\sum_{i=0}^{k}(f\circ G^i(\hat\alpha) - f\circ G_l^i(\hat\alpha))\right|_0 + \left|\frac{1}{k+1}\sum_{i=0}^{k} f\circ G_l^i(\hat\alpha) - \hat f_l\right|_0$$
$$+ |\hat f_l - \hat f|_0 = I + II + III.$$

Since $\hat\alpha = \bigcap_n I_n$, (6.5) holds for any $n$; hence, for any $k$ the first term of this sum estimates as follows:

$$\frac{1}{k+1}\left|\sum_{i=0}^{k}(f\circ G^i - f\circ G_l^i)\right|_0 \leq \sum_{j=l}^{\infty}\frac{1}{k+1}\left|\sum_{i=0}^{k}(f\circ G_j^i - f\circ G_{j+1}^i)\right|_0$$
$$< \frac{7}{10^{l+1}}\sum_{i=0}^{\infty}\frac{1}{10^i} < \frac{8\varepsilon}{10}.$$

Since $\hat\alpha \in I_l'$, then, by (6.3), there exists a $\tau_l$ such that for any $k \geq \tau_l$ the term $II$ is less than $1/10^{l+1} < \varepsilon/10$.

Finally, (6.2) guarantees that the term $III$ is less than $1/10^{l+1} < \varepsilon/10$.

Hence, we have proven that (6.1) holds for any function $f \in \mathcal{F}$ and any $z \in \mathbb{T}^2$, which implies the unique ergodicity of $G(\hat\alpha)$, as was mentioned at the beginning of the section. Remark 1.1 guarantees the existence, for any $m$, of a uniquely ergodic diffeomorphism $G_{(m)}(\hat\alpha)$, such that $|G_{(m)}(\hat\alpha) - R_\alpha|_{10^m} < 1/10^m$. This finishes the proof of Theorem 1.3. □

## 7. *Density and genericity*

LEMMA 7.1. *Suppose that $\alpha = \lim p_n/q_n$, where $|p_n/q_n - \alpha|$ decay sufficiently fast with $n$. Then minimal non-ergodic diffeomorphisms are dense in $\overline{\mathcal{A}_\alpha^r}$ for any $r > 0$, and in $\overline{\mathcal{A}_\alpha}$. The same is true for uniquely ergodic diffeomorphisms.*

*Proof.* It is enough to only give a proof for the first statement, the other is proved in exactly the same way. Suppose that $|p_n/q_n - \alpha|$ decay sufficiently fast to provide the conclusion of Theorem 1.2. First we show that in this case, for any $r > 0$, any $F \in \mathcal{A}_\alpha^r$ and $\varepsilon > 0$, there exists a minimal non-ergodic diffeomorphism $G \in \overline{\mathcal{A}_\alpha^r}$, such that $|F - G|_r < \varepsilon$.

By the definition of $\mathcal{A}_\alpha^r$, for any $F \in \mathcal{A}_\alpha^r$ there exists an analytic measure preserving diffeomorphism $S$ such that $F = S^{-1}\circ R_\alpha \circ S$ and $S^{-1}$ is analytic in some open neighborhood of $\overline{R_\alpha \circ S(A^r)}$. In particular, there exists a $\delta$ such that $S^{-1}$ is analytic in $H\circ S(A^r)$ for any analytic diffeomorphism $H$ such that $|(H - R_\alpha)\circ S|_r < \delta$.

Let us denote $r_1 = \sup_{A^r}|\mathrm{Im}(S)|$ (assume that $r_1 > \delta$) and set $M = |\mathrm{D}S^{-1}|_{2r_1}$, $\varepsilon_1 = \min\{\delta, \varepsilon/M\}$.

By Theorem 1.2, there exists a minimal non-ergodic analytic diffeomorphism $\tilde G \in \overline{\mathcal{A}_\alpha^{r_1}}$ such that $|\tilde G - R_\alpha|_{r_1} < \varepsilon_1$. Then $S^{-1}$ is well defined on $\tilde G \circ S(A^r)$ and we can set $G = S^{-1}\circ \tilde G \circ S$.

Evidently, $G \in \overline{\mathcal{A}_\alpha^r}$, and the desired estimate holds true:

$$|F - G|_r = |S^{-1}\circ R_\alpha \circ S - S^{-1}\circ \tilde G \circ S|_r \leq M|\tilde G - R_\alpha|_{r_1} < \varepsilon.$$

Hence, for an arbitrary $r$, minimal non-ergodic diffeomorphisms are dense in $\overline{\mathcal{A}^r_\alpha}$. The same statement for $\overline{\mathcal{A}_\alpha}$ follows from the definition of the topology $\tau$ (see §1.1). □

LEMMA 7.2. *The set of real numbers* $\alpha$*, satisfying the conditions of Theorem 1.3, is generic in* [0, 1]. *The same is true for Theorems 1.1 and 1.2.*

*Proof.* Consider the set of numbers $\alpha$, satisfying the following condition. For some $\delta > 0$,

$$|\alpha - p/q| < \exp(-q^{1+\delta}) \tag{7.1}$$

has an infinite number of solutions in natural numbers $p, q$ (with $p$ and $q$ relatively prime). It is well known that this set is generic in $\mathbb{R}$. We claim that for any such $\alpha$ the construction of Theorem 1.3 can be performed in $\overline{\mathcal{A}_\alpha}$.

To see this, let us fix an $\alpha$ satisfying the above condition and return to the proof of Theorem 1.3, namely to Assumptions 1–3 of §6. We let Assumption 1 be the same and reformulate Assumptions 2 and 3 in the following way.

ASSUMPTION 2′. *For any n, suppose that* $I_n' \subset I_n$, $I_n'$ *contains the fixed* $\alpha$ *and* $\tau_n$ *is such that*

$$\left|\frac{1}{k+1}\sum_{i=0}^{k} f \circ G_n^i(\alpha) - \hat{f}_n\right|_0 < \frac{1}{10^{n+1}} \quad \textit{for all } f \in \mathcal{F}_n,\ \alpha \in I_n',\ k \geq \tau_n.$$

This number and interval exist by Lemma 5.2.

ASSUMPTION 3′. *Suppose that for all n the interval* $I_n$ *is defined as*

$$I_n = [p_n/q_n - \exp(-q_n^{1+\delta})/2,\ p_n/q_n + \exp(-q_n^{1+\delta})/2].$$

*Fix an arbitrary natural* $a > 10^{n+1} \max_{f \in \mathcal{F}_{n+1}} |f|_0$.

*Suppose that for all n,* $p_{n+1}/q_{n+1} \in I_n'$ *are chosen from the set of solutions to (7.1) for the fixed* $\alpha$*, with* $q_{n+1}$ *sufficiently large to imply that for any* $\alpha \in I_{n+1}$

$$|G_{n+1}(\alpha) - G_n(\alpha)|_r < \varepsilon_n,$$

$$\max_{i=0,\dots,(\tau_n+1)a} |f \circ G_{n+1}^i(\alpha) - f \circ G_n^i(\alpha)|_0 < \varepsilon_{n+1}, \quad \textit{for all } f \in \mathcal{F}_n$$

*(such a choice is possible by Remark 2.1), and such that* $I_{n+1} \subset I_n'$.

The rest of the proof of Theorem 1.3 goes without changes and provides a uniquely ergodic diffeomorphism in $\overline{\mathcal{A}_\alpha}$, since $\lim_{n\to\infty} p_n/q_n = \alpha$ by definition.

The analogous reasoning (choosing a large $q_n$ from the sequence of solutions to (7.1) instead of taking a small $|I_n|$) works for Theorems 1.1 and 1.2. □

Theorems A and B follow from Lemmas 7.1 and 7.2.

Finally, we would like to discuss the genericity of the unique ergodicity. The following statement is proven in [**FH**].

LEMMA 7.3. *In the space of homeomorphisms of a compact metric space, uniquely ergodic homeomorphisms constitute a* $G_\delta$*-set.*

This implies that for any $r > 0$, in $\overline{\mathcal{A}^r_\alpha}$ endowed with the topology $\tau_r$, the set of uniquely ergodic diffeomorphisms is a $G_\delta$-set. Combined with Lemma 7.1, this provides the genericity of uniquely ergodic diffeomorphisms in any $\overline{\mathcal{A}^r_\alpha}$.

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