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Universal asymptotics in hyperbolicity breakdown

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Abstract

We study a scenario for the disappearance of hyperbolicity of invariant tori in a class of quasi-periodic systems. In this scenario, the system loses hyperbolicity because two invariant directions come close to each other, losing their regularity. In a recent paper, based on numerical results, Haro and de la Llave (2006 *Chaos* **16** 013120) discovered a quantitative universality in this scenario, namely, that the minimal angle between the two invariant directions has a power law dependence on the parameters and the exponents of the power law are universal. We present an analytic proof of this result.

Mathematics Subject Classification: 37C55, 37C60, 37D20, 37D25

1. Introduction

In a recent publication [4], Haro and de la Llave have found spectacular asymptotics when numerically investigating the disappearance of normally hyperbolic invariant tori in quasi-periodically forced systems (see also [5] for a more detailed exposition). The purpose of this paper is to provide analytic proofs of the existence of these asymptotics in a class of systems. We will focus on one concrete model, but our method should be applicable to other systems also.

The model we consider in this paper is a quasi-periodically forced Hénon map $H : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) of the form

$$\theta \mapsto \theta + \omega,$$

$$(x, y) \mapsto h(x, y) + \varepsilon(x - x_0)V(\theta),$$

where $h(x, y) = (1 + y - ax^2, bx)$ is the Hénon map. Here V is a function $V : \mathbb{T} \to \mathbb{R}$ which is at least continuous and ω is an irrational number. The map $h(x, y)$ has a fixed point (x_0, bx_0) where $x_0 = (b - 1 + \sqrt{(b - 1)^2 + 4a})/(2a)$. Note that the perturbation in our case is chosen so that the torus (θ, x_0, bx_0) is H-invariant for all ε.

We shall study the linearization of the above map along the invariant torus (θ, x_0, bx_0). This dynamics is given by the cocycle

$$\mathbb{T} \times \mathbb{R}^2 \ni (\theta, u) \mapsto (\theta + \omega, M_\varepsilon(\theta)u) \in \mathbb{T} \times \mathbb{R}^2, \tag{1.1}$$

where M_ε is the matrix

$$M_\varepsilon(\theta) = \begin{pmatrix} -2ax_0 + \varepsilon V(\theta) & 1 \\ b & 0 \end{pmatrix}. \tag{1.2}$$

The time evolution of this linearized map shall be denoted by

$$M_\varepsilon^n(\theta) = \begin{cases} M_\varepsilon(\theta + (n-1)\omega) \cdots M_\varepsilon(\theta), & n > 0, \\ Id, & n = 0, \\ M_\varepsilon(\theta + n\omega)^{-1} \cdots M(\theta - \omega)^{-1}, & n < 0. \end{cases}$$

We say that the cocycle (ω, M_ε) is uniformly hyperbolic if there are constants $C > 0$ and $\delta^- < 0 < \delta^+$ and a splitting $\mathbb{R}^2 = E_\varepsilon^-(\theta) \oplus E_\varepsilon^+(\theta)$ such that

$$\begin{aligned} |M_\varepsilon^n(\theta)u| &\leqslant C\mathrm{e}^{\delta^- n}|u| \qquad \text{for all } u \in E_\varepsilon^-(\theta), \quad n > 0, \\ |M_\varepsilon^{-n}(\theta)u| &\leqslant C\mathrm{e}^{-\delta^+ n}|u| \qquad \text{for all } u \in E_\varepsilon^+(\theta), \quad n > 0. \end{aligned} \tag{1.3}$$

It is well known that such a splitting, if it exists, is invariant, $M_\varepsilon(\theta)E_\varepsilon^\pm(\theta) = E_\varepsilon^\pm(\theta + \omega)$, and that the subspaces $E_\varepsilon^\pm(\theta)$, as functions of θ, are as smooth as M (see [7,8]). That the subspaces vary smoothly with θ implies that the angles between them are uniformly bounded away from zero. We define

$$\Delta(\varepsilon) = \min_{\theta \in \mathbb{T}} \angle(E_\varepsilon^+(\theta), E_\varepsilon^-(\theta)). \tag{1.4}$$

This quantity is the main object of interest in this paper. It is related to the constant C in (1.3). Roughly speaking, if the angle between $E_\varepsilon^\pm(\theta)$ is very small for some θ, then, by continuity, the two vectors $M_\varepsilon^n(\theta)u^\pm$ ($u^\pm \in E_\varepsilon^\pm(\theta)$, $|u^\pm| = 1$) will be close for a long time and hence the constant C must be large.

Furthermore, we define the Lyapunov exponents as

$$\Lambda^+(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|M_\varepsilon^n(\theta)\| \,\mathrm{d}\theta,$$

$$\Lambda^-(\varepsilon) = \lim_{n\to-\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|M_\varepsilon^n(\theta)\| \,\mathrm{d}\theta.$$

It is a general fact that, for systems of the form (1.2), $\Lambda^+ + \Lambda^- = \log|b| = \log|\det(M_\varepsilon(\theta))|$. In the situation when the cocycle (ω, M_ε) is uniformly hyperbolic, it is well known that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log|M_\varepsilon^n(\theta)v| = \Lambda^+ \qquad \text{for all } \theta \in \mathbb{T}, \quad v \in E_\varepsilon^+(\theta) \setminus \{0\},$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log|M_\varepsilon^n(\theta)v| = \Lambda^- \qquad \text{for all } \theta \in \mathbb{T}, \quad v \in E_\varepsilon^-(\theta) \setminus \{0\}.$$

Moreover, the convergence is uniform in θ [3]. This implies that $\Lambda^\pm$ are the optimal values for $\delta^\pm$ in (1.3). We also recall that we can have $\Lambda^-(\varepsilon) < 0 < \Lambda^+(\varepsilon)$ but the cocycle (1.1) fails to be uniformly hyperbolic. In this case it follows from the Oseledets theorem that the subspaces $E^\pm(\theta)$ exist for a.e. θ and that they vary measurably. They cannot be continuous, since it is a classical fact that non-zero Lyapunov exponents and continuous subspaces $E^\pm(\theta)$ imply uniform hyperbolicity.

In [4] it was studied numerically how the uniform hyperbolicity breaks down as ε is increased (it is assumed that M_0 is hyperbolic). The scenario studied there was that $\Delta(\varepsilon) \to 0$,

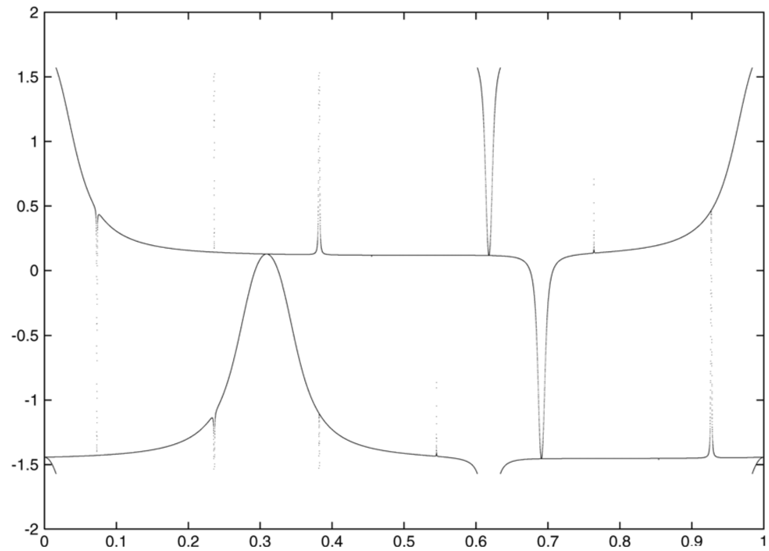

Figure 1. The curves $\gamma^{\pm}(\theta)$ when $\omega = (\sqrt{5}-1)/4$, $b = 1$, $2ax_0 = 10$, $\varepsilon = 10.256\,785$ and V as in (1.5), with $\lambda = 5$.

but the Lyapunov exponents $\Lambda^{\pm}(\varepsilon)$ are uniformly separated from each other. The striking asymptotics observed in [4] are the following. Denoting by $\varepsilon_c > 0$ the smallest $\varepsilon > 0$ for which the cocycle (ω, M_ε) fails to be uniformly hyperbolic (note that uniform hyperbolicity is an open condition), it was found that there are constants α, β such that

$$\Delta(\varepsilon) \sim \alpha(\varepsilon_c - \varepsilon)^{\beta}$$

for all $\varepsilon < \varepsilon_c$ close to ε_c. Moreover,

$$\Lambda^{+}(\varepsilon) \sim \Lambda^{+}(\varepsilon_c) + A(\varepsilon_c - \varepsilon)^{B},$$

for some A, B. In both models they had $\Lambda^{+}(\varepsilon_c) > 0$, and it was found that $\beta = 1$. We will establish the same asymptotic rate in our model (see the main theorem)

In figure 1 we have plotted the graphs of the projections of the subspaces $E^{\pm}_{\varepsilon}(\theta)$ in the case when $\omega = (\sqrt{5}-1)/4$, $b = 1$, $2ax_0 = 10$, $\varepsilon = 10.256\,785$ and V is as in (1.5), with $\lambda = 5$. In our model, for any fixed θ, ε, each $E^{\pm}_{\varepsilon}(\theta)$ is a line in $\mathbb{R}^2$, passing through the origin. In the figure it is represented by the corresponding angle in $(-\pi/2, \pi/2]$. For a fixed $\varepsilon < \varepsilon_c$ we get two smooth curves $\gamma^{\pm}_{\varepsilon}$ corresponding to E^{+}_{ε} and E^{-}_{ε}, respectively. When ε approaches ε_c from below, the smallest distance between the two curves goes to zero. Still, if the Lyapunov exponent $\Lambda^{+}(\varepsilon_c)$ is positive, then the curves $\gamma^{\pm}(\theta)$ for ε must be well separated for 'most' θ, each curve supporting its Lyapunov exponent. This forces the curves to 'fractalize' as $\varepsilon \to \varepsilon_c$. The presence of this fractalization process is one of the difficulties in estimating the asymptotics of $\Delta(\varepsilon)$.

Results. In order to make the presentation of our proof as transparent as possible, we have chosen $b = 1$ and a such that $2ax_0 = 10$. The only thing we actually need is that the unperturbed matrix M_0 is of saddle type. What is important, though, is the shape of the forcing function V. It has to be 'flat' with a single sharp 'spike'. We have taken it to be

$$V(\theta) = \frac{1}{1 + \lambda \sin^2(\pi\theta)}. \tag{1.5}$$

Here the constant λ should be extremely large. For our method it is enough that V is C^2; it is the spike shape that is important. We also need that the frequency ω satisfies the Diophantine condition

$$(DC)_{\kappa,\tau} \qquad \inf_{p\in\mathbb{Z}} |q\omega - p| > \frac{\kappa}{|q|^\tau} \qquad \text{for all } q \in \mathbb{Z}\setminus\{0\} \tag{1.6}$$

for some constants $\kappa > 0$, $\tau \geqslant 1$.

Main theorem. Let

$$M_\varepsilon(\theta) = \begin{pmatrix} \varepsilon V(\theta) - 10 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{where } V(\theta) = \frac{1}{1 + \lambda \sin^2(\pi\theta)},$$

and assume that $\omega \in \mathbb{T}$ satisfies the Diophantine condition $(DC)_{\kappa,\tau}$ for some $\kappa > 0$, $\tau \geqslant 1$. For all $\lambda > 0$ sufficiently large (depending on κ and τ) there is an $\varepsilon_c > 0$ (close to 10) such that the cocycle (ω, M_ε) is uniformly hyperbolic for all $0 \leqslant \varepsilon < \varepsilon_c$ and the minimal angle $\Delta(\varepsilon)$, defined in (1.4), satisfies

$$\lim_{\varepsilon\to\varepsilon_c^-} \frac{\Delta(\varepsilon)}{\varepsilon_c - \varepsilon} = \alpha$$

for some constant $\alpha > 0$. Moreover, the Lyapunov exponent $\Lambda^+(\varepsilon_c) > 0.5 \log 5$ for all $\varepsilon \in [0, \varepsilon_c]$.

Remark 1.

(i) Since $b = 1$, we get that $\Lambda^-(\varepsilon) = -\Lambda^+(\varepsilon)$.
(ii) Note that the asymptotics does not depend on the Diophantine class; we always get $\beta = 1$.
(iii) One can use Herman's subharmonic trick [6] to show that for all sufficiently large $\lambda > 0$, the following holds: $\Lambda^+(\varepsilon) > \log 10$ for all ε and any irrational ω. See the appendix for the details.
(iv) The methods of proof of the main theorem permit us to obtain the same result in the 'multi-frequency case', i.e. in the case when $V(\theta)$ is a flat function with a single sharp spike, defined on $\mathbb{T}^d$, and $\omega \in \mathbb{T}^d$ is a Diophantine vector, see [1].
(v) In this paper we were unable to estimate the asymptotics of the Lyapunov exponent when $\varepsilon \to \varepsilon_c$. We believe that $\Lambda^+(\varepsilon) \sim \Lambda^+(\varepsilon_c) + A(\varepsilon_c - \varepsilon)^{1/2}$.

The proof of the main theorem is based on a technique developed in [1]. The general strategy follows the same lines, but the details differ almost everywhere. Therefore we have chosen to present all the details in this paper, without referring to analogous parts in [1].

The rest of this paper is organized as follows. In section 2 we describe the projective coordinates and the projectivization of the cocycle which we shall work with, and in section 3 we introduce some notation and important definitions. Section 4 includes basic estimates. In section 5 we give a brief explanation of the key ideas behind the proof of the main theorem. To control the geometry of the projective curves $\gamma_\varepsilon^\pm$, which we shall construct, we will need certain formulae. These formulae are derived in section 6. Section 7 contains abstract help lemmas which are included in order to keep the proof of the main inductive lemma (section 8) to a reasonable size. In section 8 we prove the inductive lemma, which is the heart of the proof of the main theorem. Finally, in the last section, section 9, we put everything together and derive the statements of the main theorem. In the appendix we show how to apply Herman's subharmonic trick to our model.

We close this section with a discussion of the Schrödinger cocycle:

$$(\theta, u) \mapsto (\theta + \omega, S(\theta)u), \qquad S(\theta) = \begin{pmatrix} \varepsilon V(\theta) - E & -1 \\ 1 & 0 \end{pmatrix}.$$

This cocycle has been widely studied in the literature (see, e.g., [1] and references therein). The same result as in the main theorem also holds in this case when V is as above and $E = 10$.

We believe that the asymptotic of $\Delta(\varepsilon)$ depends on whether $\Lambda^+(\varepsilon_c)$ is positive or not. We have performed computer simulations in the case when $E = 2.1$ and $V(\theta) = \cos(2\pi\theta)$. In this case it is well known that if $\varepsilon \in [0, 2]$ and the cocycle is not uniformly hyperbolic, then $\Lambda^+(\varepsilon) = 0$ (see, e.g., [2]). The numerical results we got are that the cocycle is uniformly hyperbolic for $0 \leqslant \varepsilon < \varepsilon_c$ (ε_c is close to 0.7455), $\Lambda^+(\varepsilon_c) = 0$ and $\Delta(\varepsilon) \sim \alpha(\varepsilon_c - \varepsilon)^{1/2}$. Thus, in this case $\beta = 1/2$. The same asymptotic is also found in the following (highly degenerate) example. Let $E = 3$ and $V(\theta) = 1$. Then an easy computation shows that the cocycle is uniformly hyperbolic for $0 \leqslant \varepsilon < 1$, $\Lambda^+(1) = 0$ and $\Delta(\varepsilon) = \arctan(\psi^+) - \arctan(\psi^-) \sim \sqrt{1-\varepsilon}$, where $\psi^{\pm} = (\varepsilon - 3 \pm \sqrt{(\varepsilon - 1)(\varepsilon - 5)})/2$.

2. Projective dynamics

The way we are going to investigate cocycle (1.1) is to study its action on the projective space (the space of lines in $\mathbb{R}^2$ passing through $(0, 0)$). We will think of the projective space as $\mathbb{R} \cup \{\infty\}$. Since

$$\begin{pmatrix} \varepsilon V(\theta) - 10 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix} = r \begin{pmatrix} \varepsilon V(\theta) - 10 + 1/r \\ 1 \end{pmatrix}$$

we see that the projective map can be expressed as

$$\Phi_\varepsilon(\theta, r) = (\theta + \omega, \varepsilon V(\theta) - 10 + 1/r), \tag{2.1}$$

where $\theta \in \mathbb{T}$ and $r \in \mathbb{R} \cup \{\infty\}$.

We will use the notation

$$(\theta_k, r_k) = \Phi_\varepsilon^k(\theta_0, r_0).$$

In our estimates we will often use expressions such as $r_0 r_1 \cdots r_k$. This can be well defined if $r_0 \neq \infty$. Indeed, if $r_j = \infty$ for some $j > 0$, then we must have $r_{j-1} = 0$, and we get $r_{j-1} r_j = r_{j-1}(\varepsilon V(\theta) - 10 + 1/r_{j-1}) = 1$. Note that

$$M_\varepsilon^n(\theta) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} = r_0 \cdots r_{n-1} \begin{pmatrix} r_n \\ 1 \end{pmatrix}.$$

Thus the product $r_0 \cdots r_{n-1}$ is directly related to the Lyapunov exponents.

3. Notation and basic definitions.

- For an interval $I \in \mathbb{T}$, let $|I|$ denote its length.
- We fix ω satisfying the Diophantine condition $(\mathrm{DC})_{\kappa,\tau}$ for some $\kappa > 0$, $\tau \geqslant 1$ (we shall also express this as $\omega \in (\mathrm{DC})_{\kappa,\tau}$):

$$\inf_{p \in \mathbb{Z}} |q\omega - p| > \kappa/|q|^\tau \qquad \text{for all } q \in \mathbb{Z} \setminus \{0\}.$$

- For our construction to go through we need $V(\theta)$ to have a special form: we want it to be 'close' to zero outside a 'small' interval I_0, on this interval $V(\theta)$ will have a unique non-degenerate maximum equal to 1. To be precise, we fix

$$V(\theta) = V(\theta, \lambda) = \frac{1}{1 + \lambda \sin^2(\pi\theta)},$$

where λ should be thought of as a sufficiently large constant. We have the bounds

$$\|V\|_{C^1} \leqslant (5\pi/2)\sqrt{\lambda}; \qquad \|V\|_{C^2} = |V''(0)| = 2\pi^2\lambda. \tag{3.1}$$

- Define the first *critical interval*

$$I_0 = [-\lambda^{-1/6}, \lambda^{-1/6}]. \tag{3.2}$$

By this choice $|V|$, $|V'|$ and $|V''|$ are small outside I_0 for large λ:

$$\max_{\theta\in\mathbb{T}\setminus I_0} |V(\theta)| \leqslant \lambda^{-1/2}, \qquad \max_{\theta\in\mathbb{T}\setminus I_0} |V'(\theta)| \leqslant \lambda^{-1/3}, \qquad \max_{\theta\in\mathbb{T}\setminus I_0} |V''(\theta)| \leqslant \lambda^{-1/4}. \tag{3.3}$$

- Let

$$I_0' = \{\theta : V(\theta) \geqslant 0.88\}. \tag{3.4}$$

Then

$$|I_0'| = c_2\lambda^{-1/2}, \qquad 1/20 < c_2 < 1/8,$$

and $I_0' \subset I_0$. This interval is introduced because $|V''(\theta)|$ is large on it:

$$\max_{I_0'} V''(\theta) < -\tfrac{1}{3}\|V\|_{C^2}. \tag{3.5}$$

- We shall consider the values of ε lying in the interval

$$\mathcal{E}_{-1} = \left[10\tfrac{2}{100}, 10\tfrac{2}{5}\right].$$

Inductively we will show that the critical value ε_c mentioned in the main theorem lies in this interval.
- The diffeomorphism $\Phi_\varepsilon(\theta, r)$ is defined in (2.1). Note that

$$\Phi_\varepsilon^{-1}(\theta, r) = \left(\theta - \omega, \frac{1}{r - \varepsilon V(\theta - \omega) + 10}\right).$$

- If $I \subset \mathbb{T}$ is an interval centred at c, we denote by kI the interval centred at c of length $k|I|$.
- Define the projections

$$\pi_\theta(\theta, r) = \theta, \qquad \pi_r(\theta, r) = r.$$

- Given θ_0 and r_0, denote

$$(\theta_k, r_k) = \Phi_\varepsilon^k(\theta_0, r_0), \qquad k \in \mathbb{Z}.$$

- Let

$$R_u = [-100, -5], \qquad R_s = [1/100, 1/5].$$

The notation reflects the fact that the strip $\mathbb{T} \times R_u$ is contracted by the forward (and $\mathbb{T} \times R_s$ by the backward) iterates of Φ while iterating outside I_0 (respectively, outside $I_0 + \omega$).

- Given integers $0 < M_0 < \ldots < M_n$ and intervals $I_0 \supset I_1 \supset \ldots \supset I_n$, define the following sets:

$$\Sigma_n^F = \bigcup_{j=0}^{n} \bigcup_{m=1}^{M_j+1} (I_j + m\omega), \qquad \Sigma_n^B = \bigcup_{j=0}^{n} \bigcup_{m=0}^{M_j+1} (I_j - m\omega)$$

and

$$\Theta_n = \mathbb{T} \setminus (\Sigma_n^F \cup \Sigma_n^B).$$

On each scale, Σ_j^F and Σ_j^B are the sets of θ, for which the behaviour of the system is 'irregular'; these sets should be thought of as 'small'. Θ_n are the sets of 'good' parameters θ. Note that $\Theta_0 \supset \Theta_2 \supset \ldots \supset \Theta_n$. Actually, later in the proof the sets Σ_n^B and Σ_n^F will be seen to be disjoint. This will be assured by the Diophantine condition on ω and the choice of M_j. The fact that these sets are disjoint will be very important for controlling where the minimum angle can be located.
- The building blocks of our construction are the following 'boxes':

$$\tilde{A}_j(\varepsilon) = \Phi_\varepsilon^{M_j+1}(A_j), \qquad \text{where } A_j = \{(\theta, r) \mid \theta \in I_j - M_j\omega, r \in R_u\}, \tag{3.6}$$

$$\tilde{B}_j(\varepsilon) = \Phi_\varepsilon^{-M_j+1}(B_j), \qquad \text{where } B_j = \{(\theta, r) \mid \theta \in I_j + M_j\omega, r \in R_s\}, \tag{3.7}$$

for $j = 0, \ldots n$. We shall see that the sets $\tilde{A}_j$ and $\tilde{B}_j$ are very thin curvilinear rectangles placed over $I_j + \omega$. By construction, they contain pieces of stable and unstable manifolds of Φ, respectively. These sets should be thought of as jth approximations to the stable and unstable manifolds.

4. Preparatory lemmas

This section contains the necessary estimates for the mappings V and Φ. The first two lemmas assert that the sets $R_u \times \mathbb{T}$ and $R_s \times \mathbb{T}$, defined above, attract forward and backward iterates, respectively.

Lemma 4.1. *Let $\varepsilon \in \mathcal{E}_{-1}$ and assume that $\lambda > 0$ is large. Suppose $r_0 \in [-101, -4]$ and $\theta_0 \notin I_0$. Then $r_1 \in R_u$. Moreover, if $r_0 \notin R_s$, and $\theta_0 \notin I_0$, $\theta_0 + \omega \notin I_0$, then $r_2 \in R_u$.*

The proof is an easy verification, using estimates (3.3). The corresponding lemma, with R_u replaced by R_s, is true for backward iterations.

Lemma 4.2. *Let $\varepsilon \in \mathcal{E}_{-1}$ and assume that $\lambda > 0$ is large. Suppose $r_0 \in [1/1000, 1/4]$ and $\theta_0 - \omega \notin I_0$. Then $r_{-1} \in R_s$. Moreover, if $r_0 \notin R_u$, $\theta_0 - \omega \notin I_0$ and $\theta_0 - 2\omega \notin I_0$, then $r_{-2} \in R_s$.*

Lemma 4.3. *For any $\theta \in \mathbb{T}$ and $\varepsilon \in \mathcal{E}_{-1}$ we have*

$$|r_0| \geqslant \frac{1}{11} \Rightarrow |r_{-k} \ldots r_0| \geqslant \left(\frac{1}{11}\right)^{k+1} \qquad \text{for all } k \geqslant 0 \tag{4.1}$$

and

$$|r_0| \leqslant 11 \Rightarrow |r_0 \ldots r_k| \leqslant 11^{k+1} \qquad \text{for all } k \geqslant 0. \tag{4.2}$$

Proof. We shall prove the first of these statements; the second one can be verified in the same way. Suppose that $|r_l| < 1/11$. Then $l < 0$ by assumption, and we have

$$|r_l r_{l+1}| = \left| r_l \left(\varepsilon V - 10 + \frac{1}{r_l} \right) \right| > 1 - \frac{10}{11} = \frac{1}{11} > \frac{1}{11^2},$$

since $|\varepsilon V - 10| < 10$ for $\varepsilon \in \mathcal{E}_{-1}$. The result follows by induction using the fact that $|r_0| \geqslant \frac{1}{11}$. □

Since the frequency vector $\omega \in \mathbb{T}$ satisfies the Diophantine condition $(\mathrm{DC})_{\kappa,\tau}$, one can get a lower bound for the return time into a small ball under the rotation by ω.

Lemma 4.4. *Suppose $\omega \in DC(\kappa, \tau)$, and let I be an interval of length Δ. Then for any $N \leqslant [\kappa/\Delta]^{1/\tau}$ all the intervals $I + j\omega$, $|j| = 0, 1, \ldots, N$, are disjoint.*

Proof. If $x \in I$ and $x + m\omega \in I$, then

$$\frac{\kappa}{|m|^\tau} \leqslant \inf_{p \in \mathbb{Z}} |m\omega - p| < \Delta.$$

Therefore, $|m| > \left[(\kappa/\Delta)^{1/\tau} \right]$. □

5. A brief sketch of the proof

Since the proof of the main theorem is a quite lengthy inductive argument, we will briefly discuss the idea behind it, at least on the first scale. There will be some overlap here with results of the following sections, but we hope that this discussion will help the reader to better understand the inductive assumptions in section 8 and to have an idea of where we are heading.

We stress that the parameter λ in V should always be thought of as being extremely large.

What we are going to do is construct the invariant curves $\Gamma_\varepsilon^\pm(\theta)$, which are the projectivizations of the subspaces $E_\varepsilon^\pm(\theta)$. In figure 1 the subspaces were represented by angles in $(-\pi/2, \pi/2]$; here they are represented by their slopes, i.e. by the tangent of the angle. Our construction will give us such good estimates that we will know where the minimal angle between the subspaces is located and how the minimum changes with ε.

The interval I_0 defined in section 3 is of length $2\lambda^{-1/6}$. Thus, by lemma 4.4, we know that a point θ starting in I_0 will not return to I_0 (under forward and backward translation by ω) for at least $N_0 = \mathrm{const}\, \lambda^{1/(6\tau)}$ steps. We let $M_0 = \sqrt{N_0}$ and define A_0, $\tilde{A}_0$, B_0, $\tilde{B}_0$ as in (3.2) and (3.3). The sets $\tilde{A}_0$ and $\tilde{B}_0$ will be the first approximations of $\Gamma_\varepsilon^+(\theta)$ and $\Gamma_\varepsilon^-(\theta)$ over $I_0 + \omega$, respectively. We will show that the minimal angle is attained for $\theta \in I_0 + \omega$.

Using lemmas (4.1) and (4.2) repeatedly we get the following statements, provided that $\varepsilon \in \mathcal{E}_{-1}$

(a) If $\theta_0 \in \mathbb{T}$ and $r_0 \in R_u$, let $N \geqslant 0$ be the smallest integer such that $\theta_N \in I_0$. Then $r_k \in R_u$ for $k = 0, 1, \ldots, N$.
(b) If $\theta_0 \in \mathbb{T}$ and $r_0 \in R_s$, let $N \geqslant 0$ be the smallest integer such that $\theta_{-N} \in I_0 + \omega$. Then $r_{-k} \in R_u$ for $k = 0, 1, \ldots, N$.

From the definition of M_0, we in particular have that

$$(I_0 + m\omega) \cap I_0 = \emptyset$$

for $0 < |m| \leqslant M_0$. Thus the above statements imply

$$\tilde{B}_0 \subset (I_0 + \omega) \times R_s$$

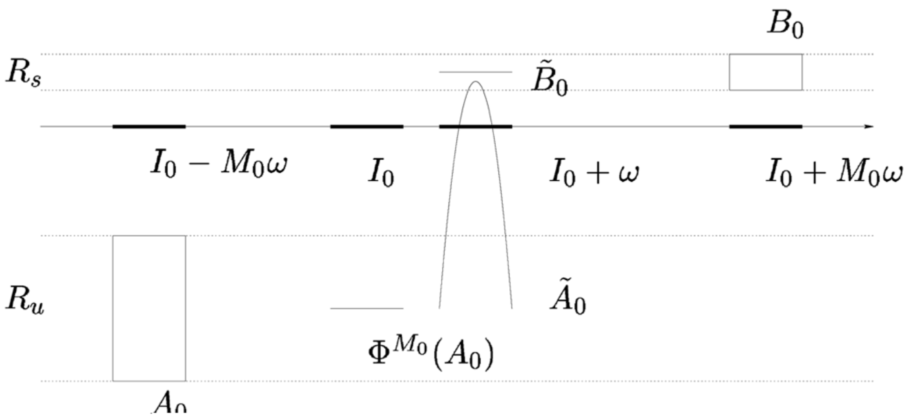

Figure 2. Approximating the curves $\Gamma^{\pm}$.

and

$$\Phi^{M_0}(A_0) \subset I_0 \times R_u.$$

Given (θ_0, r_0) and (θ_0, s_0) (that is, r_0 and s_0 are both over θ_0), then a trivial computation shows that

$$|r_1 - s_1| = \frac{|r_0 - s_0|}{|r_0 s_0|} \qquad \text{and} \qquad |r_{-1} - s_{-1}| = |r_{-1}s_{-1}(r_0 - s_0)|.$$

Thus points in R_u (R_s) are contracted by at least a factor of 25 when iterated forwards (backwards). Therefore the sets $\tilde{B}_0$ and $\Phi^{M_0}(A_0)$ will be very thin (the thickness is smaller than 25^{-M_0}). We will also show that they are almost horizontal (using the formulae in section 7) and vary very little with ε. The reason why they are flat is that we have iterated outside I_0, and there V is as flat as we like (by taking λ huge).

To get $\tilde{A}_0$, we have to apply Φ to $\Phi^{M_0}(A_0)$. Since $\Phi^{M_0}(A_0)$ lies over I_0, and is almost horizontal, we will get

$$\tilde{A}_0 = \{(\theta, r) : \theta \in I_0 + \omega, \varphi^-(\theta) \leqslant r \leqslant \varphi^+(\theta)\},$$

where

$$\varphi^{\pm}(\theta) = \varepsilon V(\theta - \omega) - 10 + \text{error}^{\pm},$$

so $\tilde{A}_0$ looks almost like $\varepsilon V - 10$ over I_0, that is, it looks like the peak of εV, see figure 2. Since $V(0) = 1$, we see that the 'peak' of $\tilde{A}_0$ moves linearly with ε. The set $\tilde{B}_0$ remains (almost) constant, as we will show. Writing $\mathcal{E}_{-1} = [\varepsilon_{-1}^-, \varepsilon_{-1}^+]$, we will see that there is an $\varepsilon_0^- \in \mathcal{E}_{-1}$ such that $\tilde{B}_0 \cap \tilde{A}_0 = \emptyset$ for $\varepsilon \in [\varepsilon_{-1}^-, \varepsilon_0^-)$ and $\tilde{B}_0 \cap \tilde{A}_0 \neq \emptyset$ for $\varepsilon \in [\varepsilon_0^-, \varepsilon_{-1}^+]$. The reason is just that $\tilde{A}_0$ moves up linearly with ε and $\tilde{B}_0$ is almost still.

If $\tilde{A}_0 \cap \tilde{B}_0 = \emptyset$, the cocycle (ω, M_ε) is uniformly hyperbolic. To see this, we proceed as follows.

Let Σ_0^F, Σ_0^B and Θ_0 be as in section 3. Take a $\theta_0 \in \Theta_0$, and let $N > 0$ be the smallest integer such that $\theta_N \in I_0$. Then $N > M_0$ by the definition of Θ_0. By (a) above we get

$$\theta_k \in R_u \qquad \text{for all } k \in [0, N].$$

In particular we have $r_{N-M_0} \in R_u$, i.e. $(\theta_{N-M_0}, r_{N-M_0}) \in A_0$. Thus $(\theta_{N+1}, r_{N+1}) \in \tilde{A}_0$. From the assumption $\tilde{A}_0 \cap \tilde{B}_0 = \emptyset$, we know that $r_{N+M_0} \notin B_0$, i.e. $r_{N+M_0} \notin R_s$. Since $\theta_N \in I_0$, we know that $\theta_k \notin I_0$ for $k \in [N+1, N+N_0]$. Recall that $N_0 \approx M_0^2$, so in particular $\theta_{N+M_0}, \theta_{N+M_0+1} \notin I_0$, and by lemma 4.2 we thus have $r_{N+M_0+2} \in R_u$, i.e. we are back in the 'good' region. We now let $N' \geqslant 0$ be the smallest integer such that $\theta_{N+M_0+1+N'} \in I_0$. Then, since $\theta_N \in I_0$, we must have $N' \geqslant N_0 - M_0$. Hence, using condition (a) above, we have $r_k \in R_u$ for $k \in [N+M_0+1, N+M_0+1+N']$. During the passage from $k = N+1$ to $k = N+M_0+1$, we can use lemma 4.4 to get a worst estimate for the product $|r_{N+1} \cdots r_{N+1+M_0}|$. It is only during this passage that the r_k can be outside R_u. Note also that $\theta_{N+1+N'} \in I_0 - M_0\omega$ and $r_{N+1+N'} \in R_u$, i.e. the point $(\theta_{N+1+N'}, r_{N+1+N'}) \in A_0$. Therefore we can repeat the argument forever. Since N_0 is so much larger than M_0, we can obtain the following.

Take $\theta_0 \in \Theta_0$ and $r_0 \in R_u$, and let $0 < T_1 < T_2 < \cdots$ be the times when $\theta_{T_i} \in I_0$. Then for all $i \geqslant 1$

$$|r_k \cdots r_{T_i}| > 5^{(T_i+1-k)/2} \qquad \text{for all } k \in [0, T_i].$$

Moreover, for all $k \geqslant 0$

$$r_k \notin R_u \Rightarrow \theta_k \in \Sigma_0^F.$$

The first condition shows that the Lyapunov exponent $\Lambda^+(\varepsilon) \geqslant 0.5 \log 5$, since the measure of Θ_0 is positive (recall the discussion in section 2). Note also that if $\theta_0 \in \Theta_0$ and $r_0, s_0 \in R_u$, and if the T_i are as above, then we get, using the formulae for contraction,

$$|r_{T_i+1} - s_{T_i+1}| \leqslant 5^{-(T_i+1)}|r_0 - s_0|. \tag{5.1}$$

The second condition gives us good control when iterates can be outside the 'cone' R_u. This will be important several times, for example, when we control the location of the minimal angle.

Analogously, when we consider the backward iterations, we get the following.

Take $\theta_0 \in \Theta_0$ and $r_0 \in R_s$, and let $0 < T_1 < T_2 < \cdots$ be the times when $\theta_{-T_i} \in I_0 + \omega$. Then for all $i \geqslant 1$

$$|r_{-T_i} \cdots r_{-k}| > 5^{-(T_i+1-k)/2} \qquad \text{for all } k \in [0, T_i].$$

Moreover, for all $k \geqslant 0$

$$r_{-k} \notin R_s \quad \Rightarrow \quad \theta_{-k} \in \Sigma_0^B.$$

By taking bigger and bigger N such that $I_0 - N\omega \subset \Theta_0$, and studying the set $\Phi^{N+1}((I_0 - N\omega) \times R_u) \subset \tilde{A}_0$, we can use the above estimates to obtain better and better approximations of the curve $\Gamma^+(\theta)$ over $I_0 + \omega$ (recall expression (5.1)). In the limit we get a piece of the curve Γ^+ lying over $I_0 + \omega$, which will be continuous by uniform convergence. By iterating this piece under Φ, we get the whole invariant curve Γ^+. Similarly we obtain Γ^-. This shows that the cocycle (ω, M_ε) is uniformly hyperbolic. Thus, by general results (see, e.g., [7, 8]), the curves $\Gamma^\pm$ must be as smooth as M_ε.

The curves will have the following properties.

$$\Gamma^+(\theta) \notin R_u \Rightarrow \theta \in \Sigma_0^F, \qquad \Gamma^-(\theta) \notin R_s \Rightarrow \theta \in \Sigma_0^B. \tag{5.2}$$

Moreover, $\Gamma^+(\theta) = \varepsilon V(\theta - \omega) - 10 + \text{error}$ over $I_0 + \omega$ and $\Gamma^-(\theta)$ will be almost horizontal.

We now explain why the minimal distance between the two curves must be attained over $I_0 + \omega$. Note that the sets Σ_0^F and Σ_0^B are disjoint (this is important). This means that if $\Gamma^+(\theta)$ and $\Gamma^-(\theta)$ are very close for some θ, then either $\Gamma^+(\theta) \in R_u$ and $\Gamma^-(\theta)$ is in (or very close to) R_u or $\Gamma^-(\theta) \in R_s$ and $\Gamma^+(\theta)$ is in (or very close to) R_s. Assume that the minimum distance between Γ^+ and Γ^- was attained for a θ^* outside I_0 and $I_0 + \omega$. If $\Gamma^+(\theta^*), \Gamma^-(\theta^*)$ is close to R_u, iterate the points $(\theta^*, \Gamma^\pm(\theta^*))$ one step forward. Then they are contracted at least by a factor of 20, contradicting the assumption on θ^*. Similarly, if $\Gamma^+(\theta^*)$ and $\Gamma^-(\theta^*)$ are close to R_s, iterate one step backward. If the minimal distance was located over $\theta_0 \in I_0$, then, since $\Gamma^+(\theta) \in R_u$ for $\theta \in I_0$, we would have $\Gamma^{\pm}(\theta_0) < -4$. By iterating the two points $(\theta_0, \Gamma^{\pm}(\theta_0))$ one step forward, using the above formula for contraction, we would get an even smaller distance between $(\theta_0 + \omega, \Gamma^{\pm}(\theta_0 + \omega))$, which is a contradiction.

In the proof later on, we should be able to treat the case $\tilde{A}_0 \cap \tilde{B}_0 \neq \emptyset$. Then we have to use a multi-scale analysis to 'zoom in' near $I_0 + \omega$. The philosophy is the same as in this first step, but the technicality becomes a bit more involved.

6. Important formulae

This section contains important expressions that will be used throughout the proofs. The formulae below give us control of the derivatives, once we have a control on products $|r_0 \cdots r_k|$ of the iterates. This is what we will do in this paper: estimate these products.

Let $r_0^{\pm} = r_0^{\pm}(\theta, \varepsilon)$ be given and define

$$r_k^{\pm} = \pi_r(\Phi_\varepsilon^k(\theta, r_0^{\pm})), \qquad k \in \mathbb{Z}.$$

Forward ($k > 0$). In particular, skipping the $\pm$,

$$r_1(\theta, \varepsilon) = \varepsilon V(\theta) - 10 + \frac{1}{r_0(\theta, \varepsilon)}.$$

We shall write r_i instead of $r_i(\theta, \varepsilon)$. Calculating the different derivatives, we get

$$\partial_\theta r_1 = \varepsilon V'(\theta) - \frac{\partial_\theta r_0}{r_0^2}, \qquad \partial_\varepsilon r_1 = V(\theta) - \frac{\partial_\varepsilon r_0}{r_0^2},$$

$$\partial_{\theta\theta} r_1 = \varepsilon V''(\theta) - \frac{\partial_{\theta\theta} r_0}{r_0^2} + 2\frac{(\partial_\theta r_0)^2}{r_0^3}, \qquad \partial_{\varepsilon\varepsilon} r_1 = -\frac{\partial_{\varepsilon\varepsilon} r_0}{r_0^2} + 2\frac{(\partial_\varepsilon r_0)^2}{r_0^3}.$$

For the contraction/expansion we have the formula

$$|r_1^+ - r_1^-| \leqslant \frac{|r_0^+ - r_0^-|}{|r_0^+ r_0^-|}. \tag{6.1}$$

Hence, by induction, we get the expressions

$$\partial_\theta r_k = \varepsilon V'(\theta + (k-1)\omega) + \varepsilon \sum_{j=1}^{k-1} (-1)^{k-j} \frac{V'(\theta + (j-1)\omega)}{r_{k-1}^2 \ldots r_j^2} + (-1)^k \frac{\partial_\theta r_0}{r_0^2 \ldots r_{k-1}^2}, \tag{6.2}$$

$$\partial_\varepsilon r_k = V(\theta + (k-1)\omega) + \sum_{j=1}^{k-1} (-1)^{k-j} \frac{V(\theta + (j-1)\omega)}{r_{k-1}^2 \ldots r_j^2} + (-1)^k \frac{\partial_\varepsilon r_0}{r_0^2 \ldots r_{k-1}^2}, \tag{6.3}$$

$$\begin{aligned}\partial_{\theta\theta} r_k = {} & \varepsilon V''(\theta + (k-1)\omega) + \varepsilon \sum_{j=1}^{k-1} (-1)^{k-j} \frac{V''(\theta + (j-1)\omega)}{r_{k-1}^2 \ldots r_j^2} + (-1)^k \frac{\partial_{\theta\theta} r_0}{r_0^2 \ldots r_{k-1}^2} \\ & - 2 \sum_{j=1}^{k-1} (-1)^{k-j} \frac{(\partial_\theta r_j)^2}{r_{k-1}^2 \ldots r_j^2 r_j},\end{aligned} \tag{6.4}$$

$$\partial_{\varepsilon\varepsilon} r_k = (-1)^k \frac{\partial_{\varepsilon\varepsilon} r_0}{r_0^2 \ldots r_{k-1}^2} - 2 \sum_{j=1}^{k-1} (-1)^{k-j} \frac{(\partial_\varepsilon r_j)^2}{r_{k-1}^2 \ldots r_j^2 r_j} \tag{6.5}$$

and

$$|r_k^+ - r_k^-| \leqslant \frac{|r_0^+ - r_0^-|}{|(r_0^+ \dots r_{k-1}^+)(r_0^- \dots r_{k-1}^-)|}. \tag{6.6}$$

Backward ($k < 0$)**.** Similarly,

$$r_{-1}(\theta, \varepsilon) = \frac{1}{r_0 - \varepsilon V(\theta - \omega) + 10}$$

and

$$\partial_\theta r_{-1} = (\varepsilon V'(\theta - \omega) - \partial_\theta r_0) r_{-1}^2, \qquad \partial_\varepsilon r_{-1} = (V(\theta - \omega) - \partial_\varepsilon r_0) r_{-1}^2,$$

$$\partial_{\theta\theta} r_{-1} = (\varepsilon V''(\theta - \omega) - \partial_{\theta\theta} r_0) r_{-1}^2 + 2\frac{(\partial_\theta r_{-1})^2}{r_{-1}}, \qquad \partial_{\varepsilon\varepsilon} r_{-1} = -\partial_{\theta\theta} r_0 r_{-1}^2 + 2\frac{(\partial_\theta r_{-1})^2}{r_{-1}}.$$

By induction, we prove

$$\partial_\theta r_{-k} = \varepsilon \sum_{j=1}^{k} (-1)^{k-j} V'(\theta - j\omega) r_{-k}^2 \dots r_{-j}^2 + (-1)^k (\partial_\theta r_0) r_{-k}^2 \dots r_{-1}^2,$$

$$\partial_\varepsilon r_{-k} = \sum_{j=1}^{k} (-1)^{k-j} V(\theta - j\omega) r_{-k}^2 \dots r_{-j}^2 + (-1)^k (\partial_\varepsilon r_0) r_{-k}^2 \dots r_{-1}^2,$$

$$\begin{aligned}\partial_{\theta\theta} r_{-k} = {}& \varepsilon \sum_{j=1}^{k} (-1)^{k-j} V(\theta - j\omega) r_{-k}^2 \dots r_{-j}^2 + (-1)^k (\partial_{\theta\theta} r_0) r_{-k}^2 \dots r_{-1}^2 \\ & + 2\sum_{j=1}^{k-1} (-1)^{k-j} \frac{(\partial_\theta r_{-j})^2}{r_{-j}} r_{-k}^2 \dots r_{-j+1}^2 + 2\frac{(\partial_\theta r_{-k})^2}{r_{-k}},\end{aligned}$$

$$\partial_{\varepsilon\varepsilon} r_{-k} = (-1)^k (\partial_{\varepsilon\varepsilon} r_0) r_{-k}^2 \dots r_{-1}^2 + 2\sum_{j=1}^{k-1} (-1)^{k-j} \frac{(\partial_\varepsilon r_{-j})^2}{r_{-j}} r_{-k}^2 \dots r_{-j+1}^2 + 2\frac{(\partial_\varepsilon r_{-k})^2}{r_{-k}},$$

$$|r_{-k}^+ - r_{-k}^-| \leqslant (r_0^+ - r_0^-)(r_{-k}^+ \dots r_{-1}^+)(r_{-k}^- \dots r_{-1}^-). \tag{6.7}$$

7. Basic lemmas

Here we show how to derive geometry control by using the formulae in the previous section, together with certain estimates on products $r_0 r_1 \cdots r_k$. The setting in this section is abstract and self-contained, but it is exactly this setting we will have in the inductive construction in section 8.

The geometric picture behind the first lemma is that a box $A = (I - M\omega) \times R_u$ ($I \subset I_0$) is mapped by Φ^{M+1} into a very thin strip $\Phi^{M+1}(A)$, which looks like the graph of the function $(\varepsilon V(\theta - \omega) - 10)$ over $I_0 + \omega$. The second lemma shows that a box $B = (I + M\omega) \times R_s$ is mapped by Φ^{-M+1} into a very thin, almost horizontal strip over $I_0 + \omega$. Recall the picture in figure 2.

Lemma 7.1. *There exists λ_0 such that for $\lambda > \lambda_0$ the following holds. Let $\varepsilon \in \mathcal{E}_{-1}$ and suppose that an interval $I \subset I_0$ and an integer $M > 100$ satisfy the following properties: for any point $(\theta_0, r_0) \in (I - M\omega) \times R_u$ we have for all $k = 0, \dots M$*

$$|r_k \dots r_M| \geqslant 5^{(M-k)/2+1} \quad \text{and}$$
$$|r_k^2 \cdots r_{p-1}^2 r_p \cdots r_M| \geqslant 5^{(M-k)/2+1} \qquad \text{for } k \leqslant p - 1 \leqslant M, \quad \text{if } |r_{p-1}| \geqslant 1/11. \tag{7.1}$$

Denote

$$A = \{(\theta, r) \mid \theta \in I - M\omega,\ r \in R_u\}.$$

Then we have

$$\tilde{A} = \Phi_\varepsilon^{M+1}(A) = \{(\theta, r) \mid \theta \in I + \omega,\ \varphi^-(\theta, \varepsilon) \leqslant r \leqslant \varphi^+(\theta, \varepsilon)\},$$

where

$$\varphi^\pm(\theta, \varepsilon) = \varepsilon V(\theta - \omega) - 10 + \phi^\pm(\theta, \varepsilon), \tag{7.2}$$

and the functions $\phi^\pm$ satisfy the following estimates:

$$|\phi^\pm(\theta, \varepsilon)| \leqslant \tfrac{1}{5}, \tag{7.3}$$

$$0 < \phi^+(\theta, \varepsilon) - \phi^-(\theta, \varepsilon) \leqslant \frac{1}{5^{M-1}}, \tag{7.4}$$

$$|\partial_\theta \phi^\pm(\theta, \varepsilon)| \leqslant 2\lambda^{-1/3}, \tag{7.5}$$

$$|\partial_\varepsilon \phi^\pm(\theta, \varepsilon)| \leqslant \tfrac{1}{20}, \tag{7.6}$$

$$|\partial_{\theta\theta} \phi^\pm(\theta, \varepsilon)| \leqslant \|V\|_{C^2}, \tag{7.7}$$

$$|\partial_{\varepsilon\varepsilon} \phi^\pm(\theta, \varepsilon)| \leqslant 1. \tag{7.8}$$

Proof. Here, as before, we use the notation

$$r_k(\theta, \varepsilon) = \pi_r(\Phi_\varepsilon^k(\theta, r_0(\theta, \varepsilon))) \qquad \text{for } \theta \in (I_n - M\omega), \quad \varepsilon \in \mathcal{E}_{n-1}.$$

Let $r_0^-(\theta, \varepsilon)$ and $r_0^+(\theta, \varepsilon)$, defined for $\theta \in I_n - M\omega$ and $\varepsilon \in \mathcal{E}_{n-1}$, be the horizontal boundaries of the set A. The signs '+' and '−' are chosen so that the horizontal boundaries of the set $\tilde{A}$ satisfy

$$\varphi^-(\theta, \varepsilon) < \varphi^+(\theta, \varepsilon), \qquad \varphi^\pm(\theta, \varepsilon) = r_{M+1}^\pm(\theta - (M+1)\omega, \varepsilon), \quad \theta \in I_n + \omega, \ \ \varepsilon \in \mathcal{E}_{n-1}.$$

One of r_0^+ and r_0^- equals identically -100, the other one equals -5.

Since $r_{M+1} = \varepsilon V - 10 + 1/r_M$, we can write

$$\varphi^\pm(\theta, \varepsilon) = \varepsilon V(\theta) - 10 + \phi^\pm(\theta, \varepsilon),$$

where

$$\phi^\pm(\theta, \varepsilon) = \frac{1}{r_M^\pm(\theta - (M+1)\omega, \varepsilon)}, \qquad \theta \in I_n + \omega, \quad \varepsilon \in \mathcal{E}_{n-1}.$$

From (7.1) with $k = M$ we get (7.3).

Using (7.1), (6.3) and the fact that $r_0^\pm(\theta, \varepsilon)$ is a constant, we obtain (7.6):

$$|\partial_\varepsilon \phi(\theta, \varepsilon)| = \sum_{k=1}^{M} \frac{1}{|r_M^2 \cdots r_k^2|} \leqslant \sum_{k=0}^{\infty} \frac{1}{5^{k+2}} = \frac{1}{20}. \tag{7.9}$$

In order to estimate $|\partial_\theta \phi(\theta, \varepsilon)|$, we write (using (6.2) and the fact that r_0 is a constant):

$$|\partial_\theta \phi(\theta, \varepsilon)| = \varepsilon \sum_{k=1}^{M} \frac{|V'(\theta - (k-1)\omega)|}{|r_M^2 \cdots r_k^2|}. \tag{7.10}$$

Estimating this sum needs a certain care, because $|V'(\theta)|$ can become large (of order $\lambda^{1/2}$) when $\theta \in I_0$.

Recall the definition of I_0 from (3.2). Let λ be so large that $-\omega \notin I_0$ and take $\theta \in I_0$. Then, by lemma 4.4, $\theta - j\omega \notin I_0$ for $j = 1, 2, \ldots N$, where $N = c_1\lambda^{1/(6\tau)}$ and $c_1 = c_1(\kappa, \tau)$. Let N_i, $i = 0, \ldots, m$, be such that $0 = N_0$, $N_i < N_{i+1}$ and

$$(\theta - (M - N_i)\omega) \in I_0, \qquad i = 0, \ldots, m.$$

Then

$$N_i \geqslant \lambda^{1/(6\tau)} \qquad \text{for } i = 1, \ldots, m. \tag{7.11}$$

We shall split the sum in (7.10) into sub-sums corresponding to the following 'periods': let the ith period be $k = (M - N_{i+1} + 1), \ldots, (M - N_i)$ for $i = 0, \ldots, m-1$, and the mth period be $k = 1, \ldots, (M - N_m)$. The periods are characterized by the fact that in each period there is at most one value of k such that $(\theta + (k-1)\omega) \in I_0$. Namely, for the first element of periods with number $i = 0, \ldots, m-1$ we have $(\theta + (k-1)\omega) = (\theta + (M - N_{i+1})\omega) \in I_0$, so the best estimate for $|V'|$ is $\|V\|_{C^1} \leqslant (5\pi/2)\sqrt{\lambda}$.

For all the other elements we have

$$|V'(\theta + (k-1)\omega)| \leqslant \lambda^{-1/3}.$$

In the mth period all the elements satisfy the latter estimate. Therefore, the part of sum (7.10) over all k from the ith period can be estimated as

$$\varepsilon \sum_{k=M-N_{i+1}}^{M-N_i} \frac{|V'(\theta - (k-1)\omega)|}{|r_M^2 \cdots r_k^2|} + \varepsilon \frac{|V'(\theta - (M - N_{i+1})\omega)|}{|r_M^2 \cdots r_{M-N_{i+1}+1}^2|}$$
$$\leqslant 2\varepsilon\lambda^{-1/3}5^{-N_i-2} + \frac{5\pi}{2}\sqrt{\lambda}5^{-N_{i+1}-1} < \lambda^{-1/3}5^{-i}.$$

The last inequality follows from (7.11). Therefore,

$$|\partial_\theta \phi(\theta, \varepsilon)| \leqslant \lambda^{-1/3} \sum_{k=0}^{m} 5^{-i} < 2\lambda^{-1/3}. \tag{7.12}$$

From formulae (7.1) and (6.6) we obtain

$$0 < |\phi^+ - \phi^-| = |\varphi^+ - \varphi^-| = \frac{|r_0^+ - r_0^-|}{|r_0^+ \cdots r_M^+||r_0^- \cdots r_M^-|} < \frac{100}{5^{M+2}} < \frac{1}{5^{M-1}}. \tag{7.13}$$

Now let us estimate the *second derivative* in θ. The argument in this part of the proof is rather technical. The complications are due to our choice of coordinates: we are working with the tangents of angles, whose range includes infinity. The general form of the second derivative is given by (6.4). Since r_0 is a constant,

$$\partial_{\theta\theta}\phi = \varepsilon \sum_{k=1}^{M} (-1)^{M-k+1} \frac{V''(\theta + (k-1)\omega)}{r_M^2 \ldots r_k^2} - 2\sum_{k=1}^{M} (-1)^{M-k+1} \frac{(\partial_\theta r_k)^2}{r_M^2 \ldots r_k^2 r_k} := I + II.$$

First estimate a part of I corresponding to $k = 1, \ldots M - 10$:

$$\varepsilon \sum_{k=1}^{M-10} \frac{|V''(\theta + (k-1)\omega)|}{r_M^2 \cdots r_k^2} \leqslant \varepsilon \sum_{k=10}^{\infty} \frac{\|V\|_{C^2}}{5^{k+2}} \leqslant \varepsilon 5^{-12}\|V\|_{C^2} \sum_{k=0}^{\infty} \frac{1}{5^k} \leqslant 5^{-10}\|V\|_{C^2}. \tag{7.14}$$

Now consider $k = M - 9, \ldots, M$. Since $\theta + M\omega \in I_0$, and the return time to I_0 is large, we know that $\theta + (M - j)\omega \notin I_0$ (at least) for $j = 1, \ldots, 20$. Hence, $(\theta + (k-1)\omega) \notin I_0$ for all $k = M - 9, \ldots, M$, and therefore $|V''(\theta - (k-1)\omega)| < \lambda^{-1/4}$ by (3.3). Hence,

$$|I| = \varepsilon \sum_{k=1}^{M-10} \frac{|V''|}{r_M^2 \cdots r_k^2} + \varepsilon \sum_{k=M-9}^{M} \frac{|V''|}{r_M^2 \cdots r_k^2}$$
$$\leqslant 5^{-10}\|V\|_{C^2} + \varepsilon\lambda^{-1/4} \sum_{k=M-9}^{M} \frac{1}{r_M^2 \cdots r_k^2} \leqslant 5^{-10}\|V\|_{C^2} + 10\varepsilon\lambda^{-1/4} < \frac{1}{2}\|V\|_{C^2}. \tag{7.15}$$

The latter inequality holds since λ is assumed to be large. The above argument will be used several times during the proof of this lemma.

Estimating II requires more work. Denote the kth term of II by A_k. Let j_i be a subset of indices, $1 \leqslant j_i \leqslant M$ such that

$$|r_{j_i}| < 1/11, \qquad i = 1, \dots I.$$

Note that, by (7.1), $|r_M| \geqslant 5$, hence in our case $1 \leqslant j_i < M$. By definition of $\Phi_\varepsilon(\theta, r)$, $|r_{j_i+1}| \geqslant 1$. Now we have two cases

(a) If $k \neq j_i$ and $k \neq j_i + 1$ for all $i = 1, \dots I$, then we estimate $|A_k|$ separately.
(b) If $k = j_i$ for some $i = 1, \dots I$, then we shall estimate the corresponding *pair* of terms, i.e.

$$|A_k + A_{k+1}| = \left| \frac{(\partial_\theta r_k)^2}{r_M^2 \cdots r_{k+1}^2 r_k^2 r_k} - \frac{(\partial_\theta r_{k+1})^2}{r_M^2 \cdots r_{k+1}^2 r_{k+1}} \right|. \tag{7.16}$$

First consider case (a). By using (6.2), and the fact that $\partial_\theta r_0 = 0$, we can rewrite $|A_k|$ in the following way:

$$\begin{aligned}
|A_k| = \frac{(\partial_\theta r_k)^2}{r_M^2 \cdots r_k^2 |r_k|} &= \frac{\left(\varepsilon V'(\theta + (k-1)\omega) + \varepsilon \sum_{p=1}^{k-1} (-1)^{k-p} \dfrac{V'(\theta + (p-1)\omega)}{r_{k-1}^2 \dots r_p^2}\right)^2}{r_M^2 \cdots r_k^2 |r_k|} \\
&= \varepsilon^2 \left(\frac{V'(\theta + (k-1)\omega)}{r_M \cdots r_k \cdot |r_k|^{1/2}} + \sum_{p=1}^{k-1} \frac{(-1)^{k-p} V'(\theta + (p-1)\omega)}{r_M \cdots r_k |r_k|^{1/2} r_{k-1}^2 \cdots r_p^2} \right)^2.
\end{aligned} \tag{7.17}$$

The assumption of this case says that both $|r_k| > 1/11$ and $|r_{k-1}| > 1/11$. This permits us to estimate $|r_k|^{-1/2} < 4$ and apply (7.1) for the denominators. Now we use the same argument as in estimating I. For $1 \leqslant k \leqslant M - 10$,

$$\begin{aligned}
|A_k| &\leqslant 16\varepsilon^2 \|V\|_{C^1}^2 \left(\frac{1}{r_M \cdots r_k} + \sum_{p=1}^{k-1} \frac{1}{r_M \cdots r_k r_{k-1}^2 \cdots r_p^2} \right)^2 \\
&= \frac{16\varepsilon^2 \|V\|_{C^1}^2}{5^{(M-k)+2}} \sum_{j=0}^{\infty} \frac{1}{5^j} \leqslant \frac{2\varepsilon^2 \|V\|_{C^1}^2}{5^{(M-k)}}.
\end{aligned}$$

For each $k = M - 9, \dots, M$, we split the estimate in the same way as above: for $p = 1, \dots, k-10$, we estimate $|V'| \leqslant \|V\|_{C^1}$; for $p = k-9, \dots, k-1$ the point $(\theta + (p-1)\omega) \notin I_0$, and, hence, $V'(\theta + (p-1)\omega) \leqslant \lambda^{-1/3}$. Thus,

$$|A_k| \leqslant 16\varepsilon^2 \left(\frac{\lambda^{-1/3}}{r_M \cdots r_k} + \sum_{p=k-9}^{k-1} \frac{\lambda^{-1/3}}{r_M \cdots r_k r_{k-1}^2 \cdots r_p^2} + \sum_{p=1}^{k-10} \frac{\|V\|_{C^1}}{r_M \cdots r_k r_{k-1}^2 \cdots r_p^2} \right)^2 \leqslant \frac{\|V\|_{C^1}^2}{5^7}.$$

Consider case (b). Here we assumed that $k = j_i$ for some $i = 1, \dots, I$, so that $|r_k| < 1/11$. Recall that in this case $k < M$. An easy calculation with the definition of $\Phi_\varepsilon(\theta, r)$ shows that in this case

$$|r_{k+1}| \geqslant 1 \qquad \text{and} \qquad 1/11 \leqslant |r_k r_{k+1}| \leqslant 2.$$

Using the formula $\partial_\theta r_{k+1} = \varepsilon V'(\theta + k\omega) - \frac{\partial_\theta r_k}{r_k^2}$, we estimate the value in (7.16) by

$$\begin{aligned}
|A_k - A_{k+1}| &\leqslant \left| \frac{(\varepsilon V'(\theta + k\omega))^2}{r_M^2 \cdots r_{k+1}^2 r_{k+1}} \right| + \left| \frac{2\varepsilon V'(\theta + k\omega) \partial_\theta r_k}{r_M^2 \cdots r_{k+1}^2 r_{k+1} r_k^2} \right| + \left| \frac{(\partial_\theta r_k)^2}{r_M^2 \cdots r_{k+2}^2} \left(\frac{1}{r_{k+1}^3 r_k^4} - \frac{1}{r_{k+1}^2 r_k^3} \right) \right| \\
&:= E_k^1 + E_k^2 + E_k^3.
\end{aligned}$$

Since $|r_{k+1}| \geqslant 1$, we can write

$$E_k^1 \leqslant \left|\frac{(\varepsilon V')^2}{r_M^2 \cdots r_{k+1}^2}\right| \leqslant \frac{\varepsilon^2 \|V\|_{C_1}^2}{5^{M-k+2}} \qquad \text{for } k = 1, \ldots M-10 \qquad \text{and}$$

$$E_k^1 \leqslant \varepsilon^2 \lambda^{-1/3} \qquad \text{for } k = M-9, \ldots, M-1.$$

Using $|r_{k+1}| > 1$, (6.2) and the fact that $\partial_\theta r_0 = 0$, we estimate

$$E_k^2 \leqslant 2\varepsilon^2 |V'(\theta + k\omega)| \left| \frac{V'(\theta + (k-1)\omega)}{r_M^2 \cdots r_k^2} + \sum_{p=1}^{k-1} \frac{(-1)^{k-p} V'(\theta + (p-1)\omega)}{r_M^2 \cdots r_p^2} \right|.$$

The latter is estimated in the same way as in (a):

$$E_k^2 \leqslant \frac{\varepsilon^2 \|V\|_{C_1}^2}{5^{M-k}} \qquad \text{for } k = 1, \ldots M-10, \qquad \text{and}$$

$$E_k^2 \leqslant \frac{\|V\|_{C^1}^2}{5^7} \qquad \text{for } k = M-9, \ldots, M-1.$$

Note that $1 - r_{k+1} r_k = r_k(\varepsilon V - 10)$. The reason for distinguishing case (b) is the following cancellation:

$$E_k^3 = \frac{(\partial_\theta r_k)^2}{|r_M^2 \cdots r_{k+2}^2|} \frac{|\varepsilon V - 10|}{|r_{k+1}^3 r_k^3|} \leqslant 20 \frac{(\partial_\theta r_k)^2}{|r_M^2 \cdots r_{k+2}^2 r_{k+1}^4 r_k^4|}$$

Again, by (6.2) and $\partial_\theta r_0 = 0$, we get

$$E_k^3 \leqslant 20\varepsilon^2 \left(\frac{V'(\theta + (k-1)\omega)}{r_M \cdots r_{k+2} r_{k+1}^2 r_k^2} + \sum_{p=1}^{k-1} \frac{(-1)^{k-p} V'(\theta + (p-1)\omega)}{r_M \cdots r_{k+2} r_{k+1}^2 r_k^2 \cdots r_p^2} \right)^2$$

Since $|r_{k+1}| \geqslant 1$, we can use (7.1) for the denominators. Then

$$E_k^3 \leqslant \frac{2\varepsilon^2 \|V\|_{C_1}^2}{5^{M-k}} \qquad \text{for } k = 1, \ldots M-10 \qquad \text{and}$$

$$E_k^3 \leqslant \frac{\|V\|_{C^1}^2}{5^7} \qquad \text{for } k = M-9, \ldots, M-1.$$

We have obtained that, for k as in (b).

$$|A_k + A_{k+1}| \leqslant \frac{6\varepsilon^2 \|V\|_{C_1}^2}{5^{M-k}} \qquad \text{for } k = 1, \ldots M-10 \qquad \text{and}$$

$$|A_k + A_{k+1}| \leqslant \frac{3\|V\|_{C^1}^2}{5^7} \qquad \text{for } k = M-9, \ldots, M-1.$$

Since we have $\varepsilon < 11$,

$$II \leqslant \sum_{k=10}^{\infty} \frac{6\varepsilon^2 \|V\|_{C_1}^2}{5^{M-k}} + 30 \frac{\|V\|_{C^1}^2}{5^7} < \frac{1}{2} \|V\|_{C^2}.$$

Collecting the above estimates,

$$|\partial_{\theta\theta} \phi(\theta, \varepsilon)| < \|V\|_{C^2}. \tag{7.18}$$

The second derivative in ε is estimated by 1 in a similar way. This finishes the proof of the lemma. $\square$

The next lemma gives analogous estimates for the backward iterates.

Lemma 7.2. *There exists λ_0 such that for $\lambda > \lambda_0$ the following hold. Let $\varepsilon \in \mathcal{E}_{-1}$ and suppose that an interval $I \subset I_0$ and an integer $M > 100$ satisfy the following properties: for any point $(\theta_0, r_0) \in (I + M\omega) \times R_s$ we have for all $k = 0, \ldots M$*

$$|r_{-M} \ldots r_{-k}| \leqslant 5^{-(M-k)/2-1} \quad \text{and} \quad |r_{-M} \cdots r_{-p} r_{-p+1}^2 \cdots r_{-k}^2| \leqslant 5^{-(N-k)/2-1}$$
$$\text{for } -M \leqslant -p+1 \leqslant -k, \quad \text{if } |r_{-p+1}| \leqslant 11. \tag{7.19}$$

Denote

$$B = \{(\theta, r) \mid \theta \in I + M\omega,\ r \in R_s\}.$$

Then we have

$$\tilde{B} = \Phi_\varepsilon^{-M+1}(B) = \{(\theta, r) \mid \theta \in I + \omega,\ \psi^-(\theta, \varepsilon) \leqslant r \leqslant \psi^+(\theta, \varepsilon)\},$$

where the functions $\psi^\pm$ satisfy the following estimates:

$$\begin{aligned}
&|\psi^\pm(\theta, \varepsilon)| \leqslant \tfrac{1}{5}, \\
&0 < \psi^+(\theta, \varepsilon) - \psi^-(\theta, \varepsilon) \leqslant \frac{1}{5^{M-1}}, \\
&|\partial_\varepsilon \psi^\pm(\theta, \varepsilon)| \leqslant \tfrac{1}{20}, \\
&|\partial_\theta \psi^\pm(\theta, \varepsilon)| \leqslant 2\lambda^{-1/3}, \\
&|\partial_{\theta\theta} \psi^\pm(\theta, \varepsilon)| \leqslant \|V\|_{C^2}, \qquad |\partial_{\varepsilon\varepsilon} \psi^\pm(\theta, \varepsilon)| \leqslant 1.
\end{aligned} \tag{7.20}$$

8. Induction

This section contains the inductive procedure on which the proof of the main theorem is based.

8.1. Conditions $(\mathcal{C}1)_0$, $(\mathcal{C}2)_0$ and $(\mathcal{C}3)_0$.

In order to keep the formulation of the inductive lemma below (lemma 8.2) more compact, we introduce the following notation.

Suppose that integers $M_0 < \ldots < M_n$, non-empty closed intervals $I_0 \supset I_1 \supset \ldots \supset I_n$ and $\mathcal{E}_{-1} \supset \mathcal{E}_0 \supset \ldots \supset \mathcal{E}_n$ are chosen. Let the sets Σ_j, Θ_j, $\tilde{A}_j(\varepsilon)$ and $\tilde{B}_j(\varepsilon)$ be defined as in section 3.

Condition $(\mathcal{C}1)_n^F$. Assume that $\theta_0 \in \Theta_{n-1}$, $r_0 \in R_u$ and $\varepsilon \in \mathcal{E}_{n-1}$. Let $N > 0$ be the smallest natural number such that $\theta_N \in I_n$. Then for any $k = 0, \ldots, N$

$$|r_k \ldots r_N| \geqslant 5^{(1/2+1/2^{n+1})(N-k)+1}, \quad \text{and} \quad |r_k^2 \cdots r_{p-1}^2 r_p \cdots r_N| \geqslant 5^{(1/2+1/2^{n+1})(N-k)+1}$$
$$\text{for } k \leqslant p-1 \leqslant N, \qquad \text{if } |r_{p-1}| \geqslant 1/11; \tag{8.1}$$

$$r_k \notin R_u \Rightarrow \theta_k \in \Sigma_{n-1}^F = \bigcup_{i=0}^{n-1} \bigcup_{m=1}^{M_i+1} (I_i + m\omega). \tag{8.2}$$

Condition $(\mathcal{C}1)_n^B$. Assume that $\theta_0 \in \Theta_{n-1}$, $r_0 \in R_s$ and $\varepsilon \in \mathcal{E}_{n-1}$. Let $N > 0$ be the smallest natural number such that $\theta_{-N} \in I_n$. Then for any $k = 0, \ldots, N$

$$|r_{-N} \ldots r_{-k}| \leqslant 5^{-(1/2+1/2^{n+1})(N-k)-1}, \quad \text{and} \quad |r_{-N} \cdots r_{-p} r_{-p+1}^2 \cdots r_{-k}^2| \leqslant 5^{-(1/2+1/2^{n+1})(N-k)-1}$$
$$\text{for } -N \leqslant -p+1 \leqslant -k, \quad \text{if } |r_{-p+1}| \leqslant 11; \tag{8.3}$$

$$r_k \notin R_s \Rightarrow \theta_k \in \Sigma_{n-1}^B = \bigcup_{i=0}^{n-1} \bigcup_{m=0}^{M_i+1} (I_i - m\omega). \tag{8.4}$$

Condition $(\mathcal{C}2)_n$. For $j = 0, 1, 2, 3$, $I_n \pm (M_n + j)\omega$ lie in Θ_{n-1}.
Condition $(\mathcal{C}3)_n$. Define the functions $\varphi_n^\pm(\theta, \varepsilon)$, $\psi_n^\pm(\theta, \varepsilon) : (I_n + \omega) \times \mathcal{E}_{n-1} \to \mathbb{R}$:

$$\begin{aligned}\tilde{A}_n(\varepsilon) &= \{(\theta, r) \mid \theta \in I_n + \omega, \varphi_n^-(\theta, \varepsilon) \leqslant r \leqslant \varphi_n^+(\theta, \varepsilon)\},\\ \tilde{B}_n(\varepsilon) &= \{(\theta, r) \mid \theta \in I_n + \omega, \psi_n^-(\theta, \varepsilon) \leqslant r \leqslant \psi_n^+(\theta, \varepsilon)\}.\end{aligned} \tag{8.5}$$

Then these functions are C^2, and for all (θ, ε) they satisfy the following conditions:

$$\varphi_n^\pm(\theta, \varepsilon) = \varepsilon V(\theta - \omega) + 10 + \phi_n^\pm(\theta, \varepsilon), \tag{8.6}$$

where $\phi_n^\pm : (I_n + \omega) \times \mathcal{E}_{n-1} \to \mathbb{R}$, and

$$0 < \phi_n^+ - \phi_n^- < 5^{-M_n+1}, \qquad 0 < \psi_n^+ - \psi_n^- < 5^{-M_n+1}, \tag{8.7}$$

$$|\partial_\theta \phi_n^\pm(\theta, \varepsilon)| \leqslant 2\lambda^{-1/3}, \qquad |\partial_\theta \psi_n^\pm(\theta, \varepsilon)| \leqslant 2\lambda^{-1/3}, \tag{8.8}$$

$$|\partial_\varepsilon \phi_n^\pm(\theta, \varepsilon)| \leqslant \tfrac{1}{20}, \qquad |\partial_\varepsilon \psi_n^\pm(\theta, \varepsilon)| \leqslant \tfrac{1}{20}, \tag{8.9}$$

$$|\partial_{\theta\theta} \phi_n^\pm(\theta, \varepsilon)| \leqslant \|V\|_{C^2}, \qquad |\partial_{\theta\theta} \psi_n^\pm(\theta, \varepsilon)| \leqslant \|V\|_{C^2}, \tag{8.10}$$

$$|\partial_{\varepsilon\varepsilon} \phi_n^\pm(\theta, \varepsilon)| \leqslant 1, \qquad |\partial_{\varepsilon\varepsilon} \psi_n^\pm(\theta, \varepsilon)| \leqslant 1. \tag{8.11}$$

Moreover,

$$\varphi_n^+(\theta, \varepsilon) < \psi_n^-(\theta, \varepsilon) \qquad \text{for all } \theta \in (I_n + \omega) \setminus \left(\tfrac{1}{3} I_n + \omega\right), \quad \varepsilon \in \mathcal{E}_{n-1}. \tag{8.12}$$

Finally, writing $\mathcal{E}_n = [\varepsilon_n^-, \varepsilon_n^+]$ we have

$$\tilde{A}_n \cap \tilde{B}_n = \emptyset \qquad \text{for } \varepsilon \in [\varepsilon_{n-1}^-, \varepsilon_n^-), \tag{8.13}$$

$$\text{there is a unique point } \theta^* \in \tfrac{1}{3} I_n + \omega \quad \text{s.t. } \varphi_n^+(\theta^*, \varepsilon_n^-) = \psi_n^-(\theta^*, \varepsilon_n^-), \tag{8.14}$$

$$\text{there is a unique point } \theta^{**} \in \tfrac{1}{3} I_n + \omega \quad \text{s.t. } \varphi_n^-(\theta^{**}, \varepsilon_n^+) = \psi_n^+(\theta^{**}, \varepsilon_n^+). \tag{8.15}$$

8.2. Basic step

Recall that $|I_0| = 2\lambda^{-1/6}$. By lemma 4.4, we know that for any $\theta_0 \in I_0$ we have $\theta_k \notin I_0$ for all $|k| = 1, 2, \dots N$, where $N = [c_1 \lambda^{1/(6\tau)}]$, $\quad c_1 = (\kappa/2)^{1/\tau}$. We define

$$M_0 = [\lambda^{1/(12\tau)}]. \tag{8.16}$$

Then M_0 is of order $\sqrt{N}$.

Lemma 8.1 (Basic step). *Let λ in the definition of V be sufficiently large. Then there exists an interval $\mathcal{E}_0 = [\varepsilon_0^-, \varepsilon_0^+] \subset \mathcal{E}_{-1}$ such that conditions $(\mathcal{C}1)_0$, $(\mathcal{C}2)_0$ and $(\mathcal{C}3)_0$ hold.*

Proof. Assume that λ is sufficiently large depending on κ, τ and V. Condition $(\mathcal{C}1)_0$ follows from lemmas 4.1 and 4.2. Condition $(\mathcal{C}2)_0$ is trivial, since $\Theta_{-1} = \mathbb{T}$.

Now we shall choose $\mathcal{E}_0 = [\varepsilon_0^-, \varepsilon_0^+] \subset \mathcal{E}_{-1}$ in such a way that $(\mathcal{C}3)_0$ holds. Define $\tilde{A}_0$ and $\tilde{B}_0$ as in $(3.6)_0$, $(3.7)_0$, and let $\varphi_0^\pm(\theta, \varepsilon)$ and $\psi_0^\pm(\theta, \varepsilon)$ be as in $(8.5)_0$.

Estimates $(8.7)_0$–$(8.11)_0$ for the functions $\varphi_0^\pm$ and $\psi_0^\pm$ follow from lemmas 7.1 and 7.2.

It can be verified by a calculation that $(8.12)_0$ holds for all $\varepsilon \in \mathcal{E}_{-1}$. In fact, it is easy to prove a stronger estimate (recall that $V(\theta) < 0.88$ outside I_0'):

$$\varphi_0^+(\theta, \varepsilon) < -1 \qquad \text{for all } \theta \in (I_0 + \omega) \setminus (I_0' + \omega), \quad \varepsilon \in \mathcal{E}_{-1}.$$

Let us verify $(8.14)_0$ and $(8.15)_0$. Note that $\tilde{B}_0(\varepsilon) \subset (I_0+\omega)\times R_s$ and $\Phi_\varepsilon^{-1}(\tilde{A}_0(\varepsilon)) \subset (I_0 \times R_u)$. Let $\tilde{A}(\varepsilon) = \Phi_\varepsilon(I_0 \times R_u)$ and denote its 'upper' and 'lower' boundaries by $\varphi^+(\theta,\varepsilon)$ and $\varphi^-(\theta,\varepsilon)$, respectively:

$$\tilde{A}(\varepsilon) = \{(\theta, r) \mid \theta \in I_0+\omega, \varphi^-(\theta,\varepsilon) \leqslant r \leqslant \varphi^+(\theta,\varepsilon)\}.$$

Then $\tilde{A}_0(\varepsilon) \subset \tilde{A}(\varepsilon)$ or, in other words,

$$\varphi^-(\theta,\varepsilon) \leqslant \varphi_0^-(\theta,\varepsilon) < \varphi_0^+(\theta,\varepsilon) \leqslant \varphi^+(\theta,\varepsilon).$$

One can verify by computation the following two statements. For $\varepsilon < \varepsilon_{-1}^- = 10\frac{2}{100}$, $\tilde{A}(\varepsilon)\cap(I_0+\omega)\times R_s = \emptyset$, which implies that $\varphi_0^+(\theta,\varepsilon_{-1}^-) \leqslant \psi_0^-(\theta,\varepsilon_{-1}^-)$ for $\theta \in I_0+\omega$. On the other hand, for $\varepsilon_{-1}^+ = 10\frac{2}{5}$, $\varphi^-(0,\varepsilon_{-1}^+) = 1/5$, which implies that $\varphi_0^-(0,\varepsilon_{-1}^+) \geqslant \psi_0^+(0,\varepsilon_{-1}^+)$.

By (7.2), (7.6) and (7.20), for $\theta \in I_0'+\omega$ and all $\varepsilon \in \mathcal{E}_{-1}$ we have

$$\partial_\varepsilon(\varphi_0^\pm(\theta,\varepsilon) - \psi_0^\pm(\theta,\varepsilon)) > 0.88 - 1/10 > 1/2.$$

In particular, this implies that in the interval $\mathcal{E}_{-1} = [\varepsilon_{-1}^-, \varepsilon_{-1}^+]$ there is ε_0^- such that $\varphi_0^+(\theta,\varepsilon) < \psi_0^-(\theta,\varepsilon)$ for all $\theta \in I_0+\omega$ and $\varepsilon < \varepsilon_0^-$, and for any $\varepsilon > \varepsilon_0^-$ there is $\theta \in (I_0'+\omega)$ such that $\varphi_0^+(\theta,\varepsilon) = \psi_0^-(\theta,\varepsilon)$. This gives $(8.13)_0$. To see that there is a unique point θ^* such that $\varphi_0^+(\theta^*,\varepsilon_0^-) = \psi_0^-(\theta_0^*,\varepsilon_0^-)$, it is enough to show that the function

$$h(\theta,\varepsilon) = \varphi_0^+(\theta,\varepsilon) - \psi_0^-(\theta,\varepsilon)$$

has a unique non-degenerate maximum. Recall the definition of I_0' from (3.4). For any fixed $\theta \in (I_0+\omega)\setminus(I_0'+\omega)$ and all $\varepsilon \in \mathcal{E}_{-1}$, we have $h(\theta,\varepsilon) < -1$. Therefore, it is enough to verify that $\partial_{\theta\theta}h(\theta,\varepsilon)$ is negative for all $\theta \in I_0'$. By (3.5) and $(8.10)_0$,

$$\partial_{\theta\theta}h(\theta) < \varepsilon_0^-(-\tfrac{1}{3}\|V\|_{C^2}) + 2\|V\|_{C^2} < 0.$$

This proves $(8.14)_0$. Similarly we find ε_0^+ and θ^{**} such that $(8.15)_0$ holds. Finally, define

$$\mathcal{E}_0 = [\varepsilon_0^-, \varepsilon_0^+]. \qquad \square$$

8.3. Induction step

The following lemma contains the induction step as well as some extra information that is needed for the proof of the main theorem.

Lemma 8.2. *There exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the following holds. Suppose that M_0, $\mathcal{E}_0$ and I_0 are as above, and we have chosen integers $0 < M_0 < \dots M_n$, non-empty closed intervals $I_0' \supset I_1 \supset \dots \supset I_n$ and $\mathcal{E}_0 \supset \mathcal{E}_1 \supset \dots \supset \mathcal{E}_n$ ($n \geqslant 0$) with the following properties:*

$$M_0 = [\lambda^{1/12\tau}], \qquad 5^{M_{j-1}/4\tau} \leqslant M_j \leqslant 2\cdot 5^{M_{j-1}/4\tau}, \quad j = 1,\dots,n, \tag{8.17}$$

$$|I_0| = l_0 = 2\lambda^{-1/6}, \qquad |I_j| = l_0 5^{-M_{j-1}/2}, \quad j = 1,\dots,n, \tag{8.18}$$

and $(\mathcal{C}1)_n$, $(\mathcal{C}2)_n$ and $(\mathcal{C}3)_n$ hold.

Then there exists an integer M_{n+1} satisfying (8.17) with $j = (n+1)$, a non-empty closed interval $I_{n+1} \subset I_n$ ($I_1 \subset I_0'$ if $n = 0$) satisfying (8.18) and a non-empty closed interval $\emptyset \neq \mathcal{E}_{n+1} = [\varepsilon_{n+1}^-, \varepsilon_{n+1}^+] \subset \mathcal{E}_n$ such that $(\mathcal{C}1)_{n+1}$, $(\mathcal{C}2)_{n+1}$ and $(\mathcal{C}3)_{n+1}$ hold. Moreover,

$$\text{if } \theta \in (I_n \setminus I_{n+1})+\omega \text{ and } (\theta,r) \in \tilde{A}_n, (\theta,s) \in \tilde{B}_n, \text{ then } |r-s| > 4\cdot 5^{-M_n+1}, \tag{8.19}$$

$$\tilde{A}_{n+1} \subset \tilde{A}_n, \qquad \tilde{B}_{n+1} \subset \tilde{B}_n \tag{8.20}$$

and

$$\tilde{A}_{n+1}\cap\tilde{B}_{n+1} \neq \emptyset \quad \text{for } \varepsilon \in \mathcal{E}_{n+1}, \qquad \tilde{A}_{n+1}\cap\tilde{B}_{n+1} = \emptyset \quad \text{for } \varepsilon \in [\varepsilon_n^-, \varepsilon_{n+1}^-). \tag{8.21}$$

Furthermore, suppose that $\tilde{A}_n \cap \tilde{B}_n = \emptyset$ for some $\varepsilon \in \mathcal{E}_{n-1}$. Then the following extension of $(\mathcal{C}1)_{n+1}$, call it condition $\widetilde{(\mathcal{C}1)}_{n+1}$, holds: assume that $\theta_0 \in \Theta_n$, $r_0 \in R_u$. Let $N > 0$ be any *integer such that* $\theta_N \in I_n$. *Then for any* $k = 0, \ldots, N$

$$|r_k \ldots r_N| \geqslant 5^{(1/2+1/2^{n+2})(N-k)+1} \quad \text{and}$$
$$|r_k^2 \cdots r_{p-1}^2 r_p \cdots r_N| \geqslant 5^{(1/2+1/2^{n+2})(N-k)+1} \quad \text{for } k \leqslant p-1 \leqslant N, \text{ if } |r_{p-1}| \geqslant 1/11; \tag{8.22}$$

$$r_k \notin R_u \Rightarrow \theta_k \in \Sigma_n^F = \bigcup_{i=0}^{n} \bigcup_{m=1}^{M_i+1} (I_i + m\omega). \tag{8.23}$$

Let $N_1 > 0$ *be any integer such that* $\theta_{-N_1} \in I_n$. *Then for any* $k = 0, \ldots, N_1$

$$|r_{-N_1} \ldots r_{-k}| \leqslant 5^{-(1/2+1/2^{n+2})(N_1-k)-1}, \quad \text{and } |r_{-N_1} \cdots r_{-p} r_{-p+1}^2 \cdots r_{-k}^2| \leqslant 5^{-(1/2+1/2^{n+2})(N_1-k)-1}$$
$$\text{for } -N_1 \leqslant -(p-1) \leqslant -k, \quad \text{if } |r_{-p+1}| \leqslant 11;$$

$$r_k \notin R_s \Rightarrow \theta_k \in \Sigma_n^B = \bigcup_{i=0}^{n} \bigcup_{m=0}^{M_i+1} (I_i - m\omega).$$

The 'Furthermore...' part of the lemma differs from $(\mathcal{C}1)_{n+1}$ by the fact that in the former we do not assume $N > 0$ to be the *smallest* natural number such that $\theta_N \in I_{n+1}$; we only need that $\theta_N \in I_n$.

Proof. We assume that λ_0 is sufficiently large, depending on κ, τ and V, to make sure that the statements below hold true. Since the proof of this lemma is quite long, we split it into a number of steps.

Basics. It follows from the choice of $|I_n|$ and M_n by lemma 4.4 that

$$(I_n - M_n\omega) \cap \bigcup_{m=0}^{M_n} (I_n + m\omega) = \emptyset \qquad \text{and} \qquad (I_n + M_n\omega) \cap \bigcup_{m=0}^{M_n} (I_n - m\omega) = \emptyset, \tag{8.24}$$

$$\text{if } \theta \in I_n - M_n\omega, \text{ then } k = M_n \text{ is the smallest positive integer such that } \theta + k\omega \in I_n, \tag{8.25}$$

$$\text{if } \theta \in (I_n + M_n\omega),$$
$$\text{then } k = M_n - 1 \text{ is the smallest positive integer such that } \theta - k\omega \in (I_n + \omega). \tag{8.26}$$

Step 1. Here we define the critical set I_{n+1}. The idea is that the projection onto the base $\mathbb{T}$ of the intersection $\tilde{A}_n(\varepsilon) \cap \tilde{B}_n(\varepsilon)$ should be in $(I_{n+1} + \omega)$ for all $\varepsilon \in \mathcal{E}_n$. For $\theta \in I_n + \omega$ consider the functions

$$\varphi_n^{\pm}(\theta, \varepsilon) = \varepsilon V(\theta - \omega) - 10 + \phi_n^{\pm}(\theta, \varepsilon), \qquad \text{and} \qquad \psi_n^{\pm}(\theta, \varepsilon),$$

defined in (8.5).

If $n = 0$ we note the following. By definition we have $V(\theta) < 0.88$ for $\theta \notin I_0'$. Since $|\phi_0^{\pm}| \leqslant 1/5$ and $|\psi_0^{\pm}| \leqslant 1/5$, it thus follows that

$$\psi_0^-(\theta, \varepsilon) - \varphi_0^+(\theta, \varepsilon) > 1 \qquad \text{for } \theta \in I_0 \setminus I_0', \quad \varepsilon \in \mathcal{E}_{-1}. \tag{8.27}$$

In other words, $\tilde{A}_0(\varepsilon)$ and $\tilde{B}_n(\varepsilon)$ are 'far away' outside $I_0' + \omega$.

For $\varepsilon \in \mathcal{E}_n$, we need to estimate the length of the sets:

$$K(\varepsilon) = \{\theta \in I_n + \omega \mid \varphi_n^+(\theta, \varepsilon) \geqslant \psi_n^-(\theta, \varepsilon)\}.$$

Recall that, by (8.7), $\phi_n^+(\theta, \varepsilon) - \phi_n^-(\theta, \varepsilon) \leqslant 5^{-M_n+1}$ and $\psi_n^+(\theta, \varepsilon) - \psi_n^-(\theta, \varepsilon) \leqslant 5^{-M_n+1}$. It is easier to estimate the length of slightly larger sets:

$$K(\varepsilon) \subset K'(\varepsilon) = \{\theta \in I_n + \omega \mid \varphi_n^-(\theta, \varepsilon) + 5^{-M_n+1} \geqslant \psi_n^-(\theta, \varepsilon) - 5^{-M_n+1}\}.$$

By the above argument we must have $K'(\varepsilon) \subset I_0' + \omega$ if $n = 0$. If $n > 0$ we have this by assumption, since $I_n \subset I_0'$. In order to estimate the length of $K'(\varepsilon)$, consider

$$h(\theta, \varepsilon) = \varphi_n^-(\theta, \varepsilon) + 5^{-M_n+1} - (\psi_n^+(\theta, \varepsilon) - 5^{-M_n+1}).$$

If $n = 0$, we think of h as being defined for $\theta \in I_0' + \omega$. By (8.9), for each fixed $\theta \in I_n + \omega$ we have

$$\partial_\varepsilon h(\theta, \varepsilon) \geqslant \min_{\theta \in I_0'} |V(\theta)| - \|\phi_n\|_{C_1} - \|\psi_n\|_{C_1} = 0.88 - 1/10 > 1/2,$$

i.e. $h(\theta, \varepsilon)$ grows with ε for each fixed θ. Therefore, $K'(\varepsilon) \subset K'(\varepsilon_n^+)$ for all $\varepsilon \in \mathcal{E}_n$. It is left to estimate the length of $K'(\varepsilon_n^+)$. Let us fix $\varepsilon = \varepsilon_n^+$ and omit the dependence on ε.

By (8.12), (8.15) and by the choice of $\mathcal{E}_n$, $h(\theta) \leqslant 2 \cdot 5^{-M_n+1}$ for all $\theta \in I_n + \omega$ with the equality only for $\theta = \theta^{**}$. Therefore, $h(\theta)$ has a maximum at the point θ^{**} (this is the reason why we estimate $K'(\varepsilon)$ rather than $K(\varepsilon)$). Recall that θ^{**} is in $(1/3)I_n + \omega$, so it lies well inside $I_n + \omega$ (when $n = 0$, it is clear from (8.27) that θ^{**} must be in $I_0' + \omega$). In order to show that this maximum is non-degenerate (quadratic), as well as to estimate the length of K', we shall verify that the maximum of $\partial_{\theta\theta} h(\theta)$ over $\theta \in K'$ is negative and large in absolute value if λ is large. By (8.10), (3.5) and (3.1),

$$\partial_{\theta\theta} h(\theta) < \varepsilon_n^+(-\tfrac{1}{3}\|V\|_{C^2}) + 2\|V\|_{C^2}) < -4\lambda.$$

Hence, $h(\theta)$ has a unique quadratic maximum at θ^{**}. The above estimate also permits us to prove that $K'(\varepsilon_n^+)$ is contained in an interval $J_{n+1} + \omega$, centred at θ^{**}, of length $\frac{l_0}{10} 5^{-M_n/2}$ (we estimate the set of θ for which $h(\theta) \geqslant 0$). Recall that $l_0 = 2\lambda^{-1/6}$. Moreover, by estimating the set of θ for which $h(\theta, \varepsilon) \geqslant -4 \cdot 5^{-M_n+1}$, we get

$$\psi_n^-(\theta, \varepsilon) - \varphi_n^+(\theta, \varepsilon) > 4 \cdot 5^{-M_n+1} \qquad \text{for } \theta \in (I_n \setminus J_{n+1}) + \omega, \quad \varepsilon \in \mathcal{E}_n. \tag{8.28}$$

Recall (8.27) in the case when $n = 0$. In particular we have

$$\pi_\theta(\tilde{A}_n(\varepsilon) \cap \tilde{B}_n(\varepsilon)) \in J_{n+1} + \omega \qquad \text{for all } \ \varepsilon \in \mathcal{E}_n.$$

Let I_{n+1} be an interval of length $l_0 5^{-M_n/2}$ centred at θ^{**}. Since $J_{n+1} \subset I_{n+1}$, we see that (8.28) gives $(8.19)_{n+1}$. For later use we stress that we have

$$\varphi_n^+ < \psi_n^- \qquad \text{for all } \ \theta \in (I_{n+1} + \omega) \setminus (J_{n+1} + \omega), \quad \varepsilon \in \mathcal{E}_n, \tag{8.29}$$

and

$$J_{n+1} = (1/10) I_{n+1}. \tag{8.30}$$

Step 2. Here we verify that condition $(\mathcal{C}1)_{n+1}$ holds. We shall start by considering forward iterations. Let $\theta_0 \in \Theta_n$, $r_0 \in R_u$, $\varepsilon \in \mathcal{E}_n$, and let $N > 0$ be the smallest positive integer such that $\theta_N \in I_n$. For an integer T denote $(\mathcal{C}1)[T]$ the condition that for any $k = 0, \ldots, T$

$$|r_k \ldots r_T| \geqslant 5^{(1/2+1/2^{n+2})(T-k)+1} \quad \text{and} \quad |r_k^2 \ldots r_{p-1}^2 r_p \cdots r_T| \geqslant 5^{(1/2+1/2^{n+2})(T-k)+1},$$

$$\text{for } k \leqslant p-1 \leqslant T \quad \text{if } |r_{p-1}| \geqslant 1/11; \tag{8.31}$$

$$r_k \notin R_u \Rightarrow \theta_k \in \Sigma_n^F. \tag{8.32}$$

If I_{n+1} is defined, then $(\mathcal{C}1)[N]$ coincides with $(\mathcal{C}1)_{n+1}^F$.

Let $0 < T_0 < T_1 < \cdots < T_p = N$ be the times such that $\theta_{T_j} \in I_n$. Since $|I_n| = l_0 5^{-M_{n-1}/2}$ (by assumption) and since $(DC)_{\kappa,\tau}$ holds, it follows from lemma 4.4 that

$$T_{j+1} - T_j \geqslant c 5^{M_{n-1}/(2\tau)}, \qquad c = c(\kappa, \tau, l_0). \tag{8.33}$$

Moreover, since $\theta_0 \in \Theta_n$ we also have

$$T_0 > M_n. \tag{8.34}$$

Condition $(\mathcal{C}1)[T_0]$ follows from the assumed $(\mathcal{C}1)_n^F$, which we can apply because $\Theta_n \subset \Theta_{n-1}$ and $\mathcal{E}_n \subset \mathcal{E}_{n-1}$.

Assume now that we have proved that $(\mathcal{C}1)[T_j]$ holds for some $j < p$. We shall prove $(\mathcal{C}1)[T_{j+1}]$. First we show that $(\theta_{T_j+1}, r_{T_j+1}) \in \tilde{A}_n$. By the definition of T_j we have

$$(\theta_{T_j-M_n}, r_{T_j-M_n}) \in I_n - M_n\omega.$$

By $(\mathcal{C}2)_n$, $I_n - M_n\omega \subset \Theta_{n-1}$. Moreover, from (8.24),

$$(I_n - M_n\omega) \cap \bigcup_{n=1}^{M_n}(I_n + m\omega) = \emptyset.$$

Hence,

$$(I_n - M_n\omega) \cap \bigcup_{j=0}^{n}\bigcup_{m=1}^{M_j}(I_j + m\omega) = \emptyset.$$

By (8.34), $T_j - M_n > 0$. Hence, we can apply (8.32), which gives us $r_{T_j-M_n} \in R_u$. Then $(\theta_{T_j-M_n}, r_{T_j-M_n}) \in A_n$ and

$$(\theta_{T_j+1}, r_{T_j+1}) \in \tilde{A}_n. \tag{8.35}$$

Since $\theta_{T_j} \notin I_{n+1}$ (by assumption),

$$(\theta_{T_j+1}, r_{T_j+1}) \notin \tilde{B}_n. \tag{8.36}$$

Therefore, $(\theta_{T_j+M_n}, r_{T_j+M_n}) \notin B_n$, i.e. $r_{T_j+M_n} \notin R_s$. By $(\mathcal{C}2)_n$, neither $\theta_{T_j+M_n}$ nor $\theta_{T_j+M_n+1}$ lie in I_0. By lemma 4.1 we get

$$r_{T_j+M_n+2} \in R_u.$$

Then we can apply lemma 4.3, which gives that for any $k \in [T_j+1, T_j+M_n+2]$ (note that $11^2 < 5^3$)

$$|r_k \ldots r_{T_j+M_n+2}| \geqslant (1/11)^{T_j+M_n-k+3} > (1/5)^{3(T_j+M_n-k+3)} \quad \text{and}$$
$$|r_k^2 \ldots r_{p-1}^2 r_p \cdots r_{T_j+M_n+2}| = |r_k^2 \ldots r_{p-1}^2||r_p \cdots r_{T_j+M_n+2}| \geqslant (1/11)^{2(p-k)}(1/11)^{T_j+M_n+3-p}$$
$$\geqslant (1/5)^{3(T_j+M_n-k+3)} \qquad \text{for } k \leqslant p-1 \leqslant T_j+M_n+2, \quad \text{if } |r_{p-1}| \geqslant 1/11.$$

From $(\mathcal{C}2)_n$ we have $\theta_{T_j+M_n+3} \in I_n + (M_n+3)\omega \subset \Theta_{n-1}$. Now we apply inductive assumption $(\mathcal{C}1)_n^F$ to the point $(\theta_{T_j+M_n+3}, r_{T_j+M_n+3})$ and conclude that for each $k = T_j+M_n+3, \ldots, T_{j+1}$ we have the following estimate (here $N = T_{j+1} - (T_j+M_n+3)$ is the smallest positive integer such that $\theta_{(T_j+M_n+3)+N} \in I_n$):

$$|r_k \ldots r_{T_{j+1}}| \geqslant 5^{(1/2+1/2^{n+1})(T_{j+1}-k)+1}; \quad \text{and}$$
$$|r_k^2 \cdots r_{p-1}^2 r_p \cdots r_{T_{j+1}}| \geqslant 5^{(1/2+1/2^{n+1})(T_{j+1}-k)+1} \quad \text{for } k \leqslant p-1 \leqslant T_{j+1}, \text{ if } |r_{p-1}| \geqslant 1/11; \tag{8.37}$$

$$r_k \notin R_u \quad \Rightarrow \quad \theta_k \in \Sigma_{n-1}^F = \bigcup_{j=0}^{n-1}\bigcup_{m=1}^{M_j+1}(I_j + m\omega). \tag{8.38}$$

Combining the above estimates, we get for any $k \in [T_j+1, T_j+M_n+2]$

$$|r_k \cdots r_{T_{j+1}}| = \left|r_k \cdots r_{T_j+M_n+2}\right| \left|r_{T_j+M_n+3} \cdots r_{T_{j+1}}\right| > 5^{-3(T_j+M_n-k+3)} 5^{\left(\frac{1}{2}+\frac{1}{2^{n+1}}\right)(T_{j+1}-T_j-M_n-3))+1}$$

and

$$|r_k^2 \cdots r_{p-1}^2 r_p \cdots r_{T_{j+1}}| \geqslant 5^{-3(T_j+M_n-k+3)} 5^{\left(\frac{1}{2}+\frac{1}{2^{n+1}}\right)(T_{j+1}-T_j-M_n-3))+1}$$

for $k \leqslant p-1 \leqslant T_{j+1}$, if $|r_{p-1}| \geqslant 1/11$. One can verify that

$$\left(\frac{1}{2}+\frac{1}{2^{n+1}}\right)(T_{j+1}-T_j-M_n-3))+1-3(T_j+M_n-k+3)$$
$$> \left(\frac{1}{2}+\frac{1}{2^{n+2}}\right)(T_{j+1}-k))+1.$$

Indeed, this inequality follows from a stronger one:

$$\frac{T_{j+1}-T_j}{2^{n+2}}-5M_n>0.$$

The latter follows from (8.33) and (8.17) for any n, provided that λ is sufficiently large. Thus, for any $k=T_j+1,\dots T_j+M_n+2$ we have

$$|r_k\cdots r_{T_{j+1}}| \geqslant \lambda^{\left(\frac{1}{2}+\frac{1}{2^{n+2}}\right)(T_{j+1}-k)+1}, \quad \text{and}$$
$$|r_k^2\cdots r_{p-1}^2 r_p\cdots r_{T_{j+1}}| \geqslant \lambda^{\left(\frac{1}{2}+\frac{1}{2^{n+2}}\right)(T_{j+1}-k)+1} \quad \text{for } k\leqslant p-1\leqslant T_{j+1}, \quad \text{if } |r_{p-1}|\geqslant 1/11. \tag{8.39}$$

Now $(8.31)_{T_{j+1}}$ follows from $(8.31)_{T_j}$, (8.37) and (8.39). Thus $(\mathcal{C}1)[T_{j+1}]$ holds. By induction we see that $(\mathcal{C}1)[N]$, i.e., $(\mathcal{C}1)^F_{n+1}$, holds.

Note that if $\tilde{A}_n\cap\tilde{B}_n=\emptyset$ for some $\varepsilon\in\mathcal{E}_{n-1}$, then (8.35) implies (8.36) directly. In this case, to make the proof work we do not need to know that $\theta_{T_j}\notin I_{n+1}$. Therefore, we can prove estimates (8.31) and (8.32) for any $N>0$ such that $\theta_N\in I_n$. This is the content of the 'Furthermore..' part of lemma 8.2.

Note that (8.35) implies the first assertion of (8.20): $\tilde{A}_{n+1}\subset\tilde{A}_n$.

The verification of $(\mathcal{C}1)^B_{n+1}$, as well as the second assertion of (8.20), $\tilde{B}_{n+1}\subset\tilde{B}_n$, is very similar.

Step 3. Here we chose the number M_{n+1} and verify that $(\mathcal{C}2)_{n+1}$ holds. The argument at this step is an exact repetition of the corresponding argument in [1], but we decided to include it here for completeness. For each $j=0,1,\dots,n$, let N_j be the positive integer given by lemma 4.4 when it is applied to $I=3I_j$. By the inductive estimates (8.18), and the definition of I_0, we get

$$N_j=\left[\left(\frac{\kappa}{3|I_j|}\right)^{1/\tau}\right]=\left[\left(\frac{\kappa}{3l_0}\right)^{1/\tau}5^{M_{j-1}/(2\tau)}\right], \qquad j=1,\dots,n. \tag{8.40}$$

We thus have

$$(3I_j)\cap\bigcup_{0<|m|\leqslant N_j}((3I_j)+m\omega)=\emptyset, \qquad \text{for } j=0,1,\dots,n,$$

Note that M_j, given by (8.17), is of the size $\sqrt{N_j}$. From this it is easy to deduce that the following is true for each $j=0,\dots,n$: Given any $k\in\mathbb{Z}$, we can have

$$(I_j+p\omega)\cap\bigcup_{m=-2M_j}^{2M_j}(I_j+m\omega)\neq\emptyset$$

for at most $4M_j+1$ integers p in the interval $[k,k+N_j]$. Similarly,

$$(I_j-q\omega)\cap\bigcup_{m=-2M_j}^{2M_j}(I_j+m\omega)\neq\emptyset$$

for at most $4M_j+1$ integers q in $[k, k+N_j]$. Using this, we see that in the interval $[k, k+N_n]$, there are at most

$$s := 2\left((4M_n+1)+(4M_{n-1}+1)\left(\left[\frac{N_n}{N_{n-1}}\right]+1\right)+\cdots+(4M_0+1)\left(\left[\frac{N_n}{N_0}\right]+1\right)\right)$$

integers p such that

$$(I_n \pm p\omega) \cap \bigcup_{j=0}^{n} \bigcup_{m=-2M_j}^{2M_j} (I_j + m\omega) \neq \emptyset.$$

Since

$$s < 100\left(M_n + M_{n-1}\frac{N_n}{N_{n-1}} + \ldots + M_0\frac{N_n}{N_0}\right) = 100N_n\left(\frac{M_n}{N_n} + \frac{M_{n-1}}{N_{n-1}} + \ldots + \frac{M_0}{N_0}\right),$$

it follows from estimates (8.17) and (8.40) that $s \ll N_n$ for all large λ, independently of n. Hence we have proved that for any k there is an integer a in the interval $[k, k+N_n]$ such that

$$(I_n \pm (a+p)\omega) \cap \bigcup_{j=0}^{n} \bigcup_{m=-2M_j}^{2M_j} (I_j + m\omega) = \emptyset \qquad \text{for } p = 0, 1, 2, 3.$$

By definition of Θ_n, the latter implies that

$$(I_n \pm (a+p)\omega) \subset \Theta_n \qquad \text{for } p = 0, 1, 2, 3.$$

Take $k = 5^{M_n/(4\tau)}$, and take an integer with the above property in the interval $[5^{M_n/(4\tau)}, 5^{M_n/(4\tau)} + N_n]$. Call this integer M_{n+1}. Since $I_{n+1} \subset I_n$, the above expression of course implies the weaker condition

$$I_{n+1} \pm (M_{n+1}+p)\omega \subset \Theta_n \qquad \text{for } j = 0, 1, 2, 3. \tag{8.41}$$

Hence $(\mathcal{C}2)_{n+1}$ holds. Moreover, since $N_n \ll 5^{M_n/(4\tau)}$, we have

$$5^{M_n/(4\tau)} \leqslant M_{n+1} \leqslant 2 \times 5^{M_n/(4\tau)}, \tag{8.42}$$

as required.

From now on M_{n+1} is fixed (this is the M_{n+1} in the statement of lemma 8.2).

Step 4. Here we choose the interval $\mathcal{E}_{n+1}$ and verify that $(\mathcal{C}3)_{n+1}$ and (8.21) hold true.

Since $(\mathcal{C}1)_{n+1}$ holds, it follows from lemmas 7.1 and 7.2 that the functions $\varphi^{\pm}_{n+1}$ and $\psi^{\pm}_{n+1}$ satisfy $((8.7)–(8.11))_{n+1}$. Recall that we have

$$I_{n+1} \cap \bigcup_{0<|m|\leqslant M_{n+1}} (I_{n+1} + m\omega) = \emptyset,$$

so we can indeed apply lemmas 7.1 and 7.2. Moreover, (8.20) implies the inequalities

$$\varphi^-_n(\theta,\varepsilon) \leqslant \varphi^-_{n+1}(\theta,\varepsilon) < \varphi^+_{n+1}(\theta,\varepsilon) \leqslant \varphi^+_n(\theta,\varepsilon), \qquad \theta \in I_{n+1}+\omega, \tag{8.43}$$

and

$$\psi^-_n(\theta,\varepsilon) \leqslant \psi^-_{n+1}(\theta,\varepsilon) < \psi^+_{n+1}(\theta,\varepsilon) \leqslant \psi^+_n(\theta,\varepsilon), \qquad \theta \in I_{n+1}+\omega, \tag{8.44}$$

for all $\varepsilon \in \mathcal{E}_n$. Recall the definition of the interval J_{n+1} in step 1. The above two inequalities, combined with (8.29), give

$$\varphi^+_{n+1}(\theta) < \psi^-_{n+1}(\theta) \qquad \text{for all } \theta \in (I_{n+1}+\omega) \setminus (J_{n+1}+\omega). \tag{8.45}$$

By (8.30), this gives $(8.12)_{n+1}$.

Since $((8.14)–(8.15))_n$ hold, and since we have the inclusions (8.43) and (8.44) and the derivative estimates on φ_{n+1}, ψ_{n+1}, we can proceed as in the proof of lemma 8.1 to find a non-degenerate interval $\mathcal{E}_{n+1} \subset \mathcal{E}_n$ such that $((8.14)–(8.15))_{n+1}$ and (8.21) hold.

The induction step is complete.

9. Proof of the main theorem

We are now ready to prove the main theorem. As before, assume that λ is sufficiently large. From the inductive construction in the previous section, we get a nested sequence of intervals $\mathcal{E}_n = [\varepsilon_n^-, \varepsilon_n^+]$. Since $\varepsilon_n^- \leqslant \varepsilon_{n+1}^-$ for each n, there is an ε_∞ such that $\varepsilon_n^- \to \varepsilon_\infty$ as $n \to \infty$. From the estimates in section 8 it follows that the cocycle (ω, M_ε) is uniformly hyperbolic on each interval $[\varepsilon_{n-1}^-, \varepsilon_n^-)$ and therefore on $[0, \varepsilon_\infty)$. Indeed, if $\tilde{A}_n \cap \tilde{B}_n = \emptyset$ for some n, then the cocycle is uniformly hyperbolic as we shall see below. We shall also see that the cocycle is not uniformly hyperbolic for $\varepsilon = \varepsilon_\infty$, and thus ε_∞ is the value ε_c in the statement of the main theorem.

Fix an arbitrary $n > 0$, and take any $\varepsilon \in [\varepsilon_{n-1}^-, \varepsilon_n^-)$. For this ε, we shall construct the stable and unstable invariant curves, $\Gamma^+(\theta, \varepsilon)$ and $\Gamma^-(\theta, \varepsilon)$, and prove that the minimal pointwise distance between these curves is attained for θ in the interval $(I_n + \omega)$. We shall also estimate the derivative in ε of the minimal distance between $\Gamma^+(\theta, \varepsilon)$ and $\Gamma^-(\theta, \varepsilon)$. Finally, we shall derive the statement of the main theorem. We begin by constructing a piece $\Gamma_0^+(\theta, \varepsilon)$ of the unstable invariant curve for $\theta \in I_n + \omega$.

9.1. *Construction of the invariant curves* Γ^+ *and* Γ^-

For an arbitrary $n > 0$, fix an $\varepsilon \in [\varepsilon_{n-1}^-, \varepsilon_n^-)$, and let us omit the dependence on ε. From the inductive construction in the previous section we know that hypotheses $(\mathcal{C}_1)_n$–$(\mathcal{C}_3)_n$ hold true. By (8.13), for our fixed ε we have $\tilde{A}_n(\varepsilon) \cap \tilde{B}_n(\varepsilon) = \emptyset$.

The last part of lemma 8.2 implies condition $(\mathcal{C}_1)_{n+1}$. This condition will be used several times during the proof. Choose a sequence of positive integers $T_k > T_{k-1}$, $k \geqslant 0$, satisfying

$$I_n - T_k\omega \subset \Theta_n \qquad \text{for } k \geqslant 0.$$

The possibility of such a choice was proved in step 3 of the inductive procedure above. Denote

$$J_k = I_n - T_k\omega, \qquad C_k = J_k \times R_u, \qquad \tilde{C}_k = \Phi^{T_k+1}(C_k), \qquad k \geqslant 0.$$

Each curvilinear rectangle $\tilde{C}_k$ can be thought of as the kth approximation to Γ_0^+. Since $J_k,\ J_{k-1} \subset \Theta_n$, condition (8.23) implies

$$\Phi^{T_k - T_{k-1}}(C_k) \subset J_{k-1} \times R_u = C_{k-1}.$$

Therefore, the curvilinear rectangles $\tilde{C}_k$ form a nested sequence. Let

$$\Gamma_0^+ = \bigcap_{k=0}^{\infty} \tilde{C}_k.$$

Then Γ_0^+ is defined for all $\theta \in I_n + \omega$. Moreover, the horizontal widths of $\tilde{C}_k$ decay very fast with k. Indeed, estimate (8.22), together with the fact that $(I_n - T_k\omega) \subset \Theta_n \subset \Theta_{n-1}$, permits us to apply lemma 7.1 with $M = T_k$. Denote by $c_k^\pm(\theta)$ the upper and lower boundaries of $\tilde{C}_k$. Then (7.2) and (7.4) imply

$$\max_{\theta \in I_n+\omega} |c_k^+(\theta) - c_k^-(\theta)| \leqslant 5^{-T_k+1}.$$

Hence, Γ_0^+ is the graph of a continuous function, defined for all $\theta \in I_n + \omega$. Let

$$\Gamma^+ = \bigcup_{j=0}^{\infty} \Phi^j(\Gamma_0^+).$$

Note that every point of the curve $(\theta, \Gamma^+(\theta))$ is a point of an orbit that starts in $\Theta_n \times R_u$. Therefore, by (8.23),

$$\Gamma^+(\theta) \notin R_u \Rightarrow \theta \in \Sigma_n^F. \tag{9.1}$$

We have to verify that this curve is a graph of a function over $\mathbb{T}$. Suppose the contrary: there exist an integer p, points $\theta_0, \theta_1 \in I_n$ and $r \neq s$, such that

$$r = \Gamma_0^+(\theta_0), \qquad (\theta_0, s) = \Phi^p(\theta_1, \Gamma_0^+(\theta_1)).$$

Take k such that $\Phi^p(\tilde{C}_k)$ does not intersect $\tilde{C}_k$. By the definition of $\tilde{C}_k$,

$$\Phi^{-T_k-1}(\tilde{C}_k) = C_k = J_k \times R_u.$$

Therefore, the point

$$\Phi^{-T_k-1+p}(\theta_1, \Gamma_0^+(\theta_1)) = \Phi^{-T_k-1}(\theta_0, s)$$

lies in $J_k \times (\mathbb{T} \setminus R_u)$. But this point belongs to an orbit that starts in $\Theta_n \times R_u$. Therefore, by (8.23), it has to be in $J_k \times R_u$. This contradiction proves that Γ^+ is a continuous curve. It is forward invariant by definition and backward invariant since Φ is a diffeomorphism. It is a general fact that this curve is smooth in both θ and ε, see, e.g., [7, 8].

Recall that Γ_0^+ is the pointwise limit of c_k^+. By lemma 7.1 with $M = T_k$ and $\varphi^\pm = c_k^\pm$, and, in particular, (7.6) and (7.8), $c_k^\pm(\theta)$ satisfy the estimates $1/5 \leqslant \partial_\varepsilon c_k^\pm(\theta) \leqslant 2$ and $|\partial_{\varepsilon\varepsilon} c_k^\pm(\theta)| \leqslant 1$ for all $\theta \in I_n + \omega$. Since Γ_0^+ is smooth, this implies the same estimates for Γ_0^+:

$$1/5 \leqslant \partial_\varepsilon \Gamma_0^+(\theta) \leqslant 2, \qquad |\partial_{\varepsilon\varepsilon} \Gamma_0^+(\theta)| \leqslant 1, \quad \theta \in I_n + \omega. \tag{9.2}$$

The curve Γ^- can be constructed in the same way. By lemma 7.2 we get the estimates

$$|\partial_\varepsilon \Gamma_0^-(\theta)| \leqslant 1/20, \qquad |\partial_{\varepsilon\varepsilon} \Gamma_0^-(\theta)| \leqslant 1, \quad \theta \in I_n + \omega. \tag{9.3}$$

We want to stress the following property, arising from $(\mathcal{C}1)_{n+1}^B$:

$$\Gamma^-(\theta) \notin R_s \Rightarrow \theta \in \Sigma_n^B. \tag{9.4}$$

Recall that, by construction, $\Sigma_n^F \cap \Sigma_n^B = \emptyset$ (see the definition of Σ_n^F and Σ_n^F in section 3 and remember that the intervals $I_j - (M_j+1)\omega, \ldots, I_j + (M_j+1)\omega$ are disjoint due to the estimates on M_j and $|I_j|$, $j \in [0, n]$). This fact, together with (9.1) and (9.4), implies the following important property:

$$\Gamma^-(\theta) \notin R_s \Rightarrow \Gamma^+(\theta) \in R_u \qquad \text{and} \qquad \Gamma^+(\theta) \notin R_u \Rightarrow \Gamma^-(\theta) \in R_s. \tag{9.5}$$

In other words, one of the curves is always in its 'good' region.

9.2. The minimal distance, $\delta(\varepsilon) = \delta$, between Γ^+ and Γ^- is attained at a point $\theta_0 \in (I_n + \omega)$

Note that, by construction,

$$\Gamma^+(\theta) \subset \tilde{A}_n \subset \tilde{A}_{n-1}, \qquad \Gamma^-(\theta) \subset \tilde{B}_n \subset \tilde{B}_{n-1} \quad \text{for } \theta \in I_n + \omega,$$

so, by $(8.7)_{n-1}$, we have the following *a priori* estimate for δ:

$$\delta \leqslant \min_{\theta \in (I_0+\omega)} |\Gamma^+(\theta) - \Gamma^-(\theta)| \leqslant 2 \times 5^{-M_{n-1}+1} < 1/1000. \tag{9.6}$$

Indeed, since $\varepsilon \in [\varepsilon_{n-1}^-, \varepsilon_n^-) \subset \mathcal{E}_{n-1}$ we have from $(8.21)_{n-1}$ that $\tilde{A}_{n-1} \cap \tilde{B}_{n-1} \neq \emptyset$, and therefore δ is smaller than the sum of widths of $\tilde{A}_{n-1}$ and $\tilde{B}_{n-1}$, which is estimated as above.

Let the minimal distance between Γ^+ and Γ^- be attained at the point θ_0. Denote

$$u_0 = \Gamma^+(\theta_0), \qquad s_0 = \Gamma^-(\theta_0).$$

Then $\delta = |u_0 - s_0|$. Let us study different possibilities for the location of θ_0.

(a) Suppose $\theta_0 \notin \Sigma_n^F \cup \Sigma_n^B$. Then $u_0 \in R_u$ and $s_0 \in R_s$, see (9.1), (9.4). Thus, the distance between the two curves is larger than 5, and the minimal distance cannot be attained for this value of θ.

(b) Suppose that $\theta_0 \in (\Sigma_n^B \setminus I_0) \cup (\Sigma_n^F \setminus (I_0 + \omega))$. Consider first $\theta_0 \in \Sigma_n^F \setminus (I_0 + \omega)$. Then $s_0 \in R_s = [1/100, 1/5]$. Since, by assumption, $|s_0 - u_0| < 1/1000$, we have $u_0 \in [1/1000, 1/4]$. By lemma 4.2, both u_{-1} and $s_{-1} \in R_s$. Then, by (6.1)

$$|u_{-1} - s_{-1}| = |u_0 - s_0||u_{-1}s_{-1}| \leqslant \delta/25 < \delta,$$

which contradicts δ being the minimal distance. Hence, the minimal distance cannot be attained at $\theta_0 \in \Sigma_n^F \setminus (I_0 + \omega)$. Assuming that $\theta_0 \in \Sigma_n^B \setminus I_0$, we arrive at the contradiction in a similar way.

Combination of (a) and (b) shows that the only possible location of θ_0 is

$$\theta_0 \in (\Sigma_n^B \cap I_0) \cup (\Sigma_n^F \cap (I_0 + \omega)).$$

Now we shall restrict this possibility to

$$\theta_0 \in (\Sigma_n^B \cap I_n) \cup (\Sigma_n^F \cap (I_n + \omega)).$$

(c) Suppose that $\theta_0 \in (\Sigma_n^F \cap (I_k + \omega)) \setminus (I_{k+1} + \omega))$ for some $k = 0, \ldots, n-1$. We shall derive a contradiction with δ being the minimal distance between the invariant curves. Consider three cases. First, let

$$(\theta_0, u_0) \in \tilde{A}_k, \qquad (\theta_0, s_0) \in \tilde{B}_k.$$

Since $k \leqslant n-1$, hypotheses $(\mathcal{C}_1)_k$–$(\mathcal{C}_3)_k$ hold true. By (8.19), $|u_0 - s_0| \geqslant 4 \times 5^{-M_k+1}$. This contradicts the *a priori* estimate (9.6) for the minimal distance δ.

Second, let

$$(\theta_0, u_0) \notin \tilde{A}_k.$$

Consider the point $\theta_0 - M_k\omega \in \Theta_{k-1}$. Since $\Phi^{-M_k}(\tilde{A}_k) = A_k = I_k \times R_u$, we have that $u_{-M_k} = \Gamma^+(\theta_0 - M_k\omega) \notin R_u$. Then, by (9.5), $s_{-M_k} \in R_s$. By lemma 4.2, both $s_{-(M_k+2)} \in R_s$ and $u_{-(M_k+2)} \in R_s$. By (4.2) and (6.6), $|s_{-(M_k+2)} - u_{-(M_k+2)}| \leqslant 11^{2(M_k+2)}\delta$. Moreover, $\theta_0 - (M_k+2)\omega \in \Theta_{k-1}$. Let N be the minimal positive integer number such that $\theta_0 - (M_k + 2 + N) \in I_k$. Then $N > \text{const}|I_k|^{-1/\tau} - M_k \gg 10M_k$. By $(\mathcal{C}_1)_n$,

$$|s_{-(M_k+2+N)} - u_{-(M_k+2+N)}| \leqslant 11^{2(M_k+2)} \cdot 5^{-N}\delta \ll \delta.$$

This contradicts δ being the minimal distance.

Finally, assuming that $(\theta_0, s_0) \notin \tilde{B}_k$, we arrive at a contradiction by the same argument.

A similar analysis proves that θ_0 cannot lie in $(\Sigma_n^B \cap I_0)\backslash I_n$.

(d) To finish the proof we note the following. Suppose that $\theta_0 \in I_n$. Since $I_n \cap \Sigma_n^F = \emptyset$, it follows from (9.1) that $\Gamma^+(\theta) \in R_u$ for $\theta \in I_n$. Hence $u_0 \in R_u$, and s_0 must be very close to u_0. Therefore $|\Gamma^+(\theta_0 + \omega) - \Gamma^+(\theta_0 + \omega)| = |u_1 - s_1| = |u_0 - s_0|/|u_0 s_0| < |u_0 - s_0|$, which is a contradiction.

9.3. Estimation of the rate of change of $\delta(\varepsilon)$.

Let

$$\varepsilon_\infty = \lim_{n\to\infty} \varepsilon_n^-.$$

Note that the construction of the hyperbolic invariant curves of Φ_ε was done for an arbitrary n and an arbitrary fixed ε in the interval $[\varepsilon_n^-, \varepsilon_{n+1}^-)$, i.e. for any $\varepsilon < \varepsilon_\infty$. By (9.6), the minimal distance $\delta = \delta(\varepsilon)$ between the invariant curves goes to zero when $\varepsilon \to \varepsilon_\infty$. Define by continuity $\delta(\varepsilon_\infty) = 0$. Let us show that, for $\theta \in (I_n + \omega)$, there exists a constant $-3 \leqslant \alpha \leqslant -1/10$, such that

$$\delta(\varepsilon) = \delta(\varepsilon) - \delta(\varepsilon_\infty) = \alpha(\varepsilon - \varepsilon_\infty) + o(\varepsilon - \varepsilon_\infty) \qquad \text{as } \varepsilon \nearrow \varepsilon_\infty. \tag{9.7}$$

Since $\delta(\varepsilon) - \delta(\varepsilon_\infty) = \partial_\varepsilon \delta(\tilde{\varepsilon})(\varepsilon - \varepsilon_\infty)$ by the mean value theorem, it is enough to show that $\partial_\varepsilon \delta(\varepsilon)$ converges to a finite constant when ε goes to ε_∞. Let (ε_n), $n = 1, 2, \ldots$, be any increasing sequence such that $\varepsilon_n \in [\varepsilon_{n-1}^-, e_n^-)$, i.e. converging to ε_∞ from below. Since, by (9.2),

$$|\partial_{\varepsilon\varepsilon}\delta(\varepsilon_n)| \leqslant \max_{\theta \in I_n + \omega} |\partial_{\varepsilon\varepsilon}\Gamma^+(\theta, \varepsilon_n)| + \max_{\theta \in I_n + \omega} |\partial_{\varepsilon\varepsilon}\Gamma^-(\theta, \varepsilon_n)| \leqslant 2$$

for all n, the numbers $(\partial_\varepsilon \delta(\varepsilon_n))$ form a Cauchy sequence and, therefore, converge. It is evident that the limit does not depend on the choice of the sequence (ε_n). Since $-3 \leqslant \partial_\varepsilon \delta(\varepsilon) \leqslant -1/10$ by (9.2) and (9.3), the same is true for the limit.

9.4. Estimation of the rate of change of $\Delta(\varepsilon)$.

Recall that we have been working in the coordinates which are tangents of the angles. Let us return to the angular coordinates. Consider again $\varepsilon \in [\varepsilon_n^-, \varepsilon_{n+1}^-)$ for some $n > 0$. From section 9.1 we get the invariant curves $\Gamma^\pm$ (in tangent coordinates). Thus, in angular coordinates we have the two invariant curves,

$$\gamma^+(\theta) = \arctan(\Gamma^+(\theta)), \qquad \gamma^-(\theta) = \arctan(\Gamma^-(\theta)).$$

Define the distance between these curves in the following way:

$$d(\theta) = \min_{k \in \mathbb{Z}} |\tilde{\gamma}^+(\theta) - \tilde{\gamma}^-(\theta) + k\pi|.$$

From (9.5) it follows that there is no $\theta \in \mathbb{T}$ such that both $|\Gamma^+(\theta)| > 100$ and $|\Gamma^-(\theta)| > 100$. Thus the coordinate change can scale the minimum by at most a factor of $1/(1 + (200)^2)$ $(\mathrm{d}\arctan(x)/\mathrm{d}x = 1/(1 + x^2))$. By the same analysis as in parts (a)–(c) of section 9.2, we can prove (using the fact just mentioned) that the minimal distance has to be attained over $I_n \cup (I_n + \omega)$. Now we can proceed as in section 9.3 to get

$$\Delta(\varepsilon) = \alpha(\varepsilon - \varepsilon_\infty) + o(\varepsilon - \varepsilon_\infty) \qquad \text{as } \varepsilon \nearrow \varepsilon_\infty, \tag{9.8}$$

for some positive constant α.

Remark 2. If the function $V(\theta)$ is even (which it is in our model case), then the minimal distance $\Delta(\varepsilon)$ is attained twice: at some $\tilde{\theta} \in I_0 + \omega$ and at $(-\tilde{\theta} - \omega) \in I_0$.

Note that if $V(\theta)$ is even, then $\Gamma^-(\theta)$ can be given by a simple formula. One can verify that the function

$$\Gamma^-(\theta) = -\frac{1}{\Gamma^+(-\theta + \omega)} \tag{9.9}$$

is invariant under Φ_ε, i.e. $\Gamma^-(\theta + \omega) = \varepsilon V(\theta) - 10 + 1/\Gamma^-(\theta)$. This function is different from $\Gamma_u(\theta)$—this can be verified by comparing the integrals over $\mathbb{T}$ of the logarithm of the absolute value of these two functions. Since in this case we can have at most two invariant curves, $\Gamma^-(\theta)$, given by this formula, is indeed the stable invariant curve. Expression (9.9) implies that

$$\gamma^-(\theta) = \gamma^+(-\theta + \omega) - \pi/2 \qquad \text{for all } \theta \in \mathbb{T}.$$

Therefore,

$$d(\theta + \omega/2) = d(-\theta + \omega/2),$$

which implies the desired result.

9.5. Estimation of the Lyapunov exponent $\Lambda^+(\varepsilon)$.

Let

$$\Theta_\infty = \bigcup_{j=0}^{\infty} \bigcup_{m=-M_j-1}^{M_j+1} (I_j + m\omega).$$

Then $\Theta_\infty \subset \Theta_n$ for all n. Moreover, by the estimates on $|I_j|$ and M_j $(j \geqslant 0)$ it follows that $|\Theta_\infty| \to 1$ as $\lambda \to \infty$. In particular, Θ_∞ has a positive measure for large $\lambda > 0$.

Consider now $\varepsilon \in [0, \varepsilon_\infty]$. If $\varepsilon < \varepsilon_\infty$, then there is an $n \geqslant 0$ such that $\tilde{A}_n(\varepsilon) \cap \tilde{B}_n(\varepsilon) = \emptyset$. Take $\theta_0 \in \Theta_\infty$ and $r_0 \in R_u$, and let $0 < T_1 < T_2 < T_3 < \cdots$ be the times when $\theta_{T_i} \in I_n$. By the estimates in the 'Furthermore …' part of lemma 8.2 we get

$$\limsup_{k\to\infty} \frac{1}{k} \log |r_0 r_1 \cdots r_k| \geqslant \limsup_{i\to\infty} \frac{1}{T_i} \log |r_0 r_1 \cdots r_{T_i}| \geqslant 0.5 \log 5.$$

If $\varepsilon = \varepsilon_\infty$ we do as follows. Take $\theta_0 \in \Theta_\infty$ and $r_0 \in R_u$, and for each $n \geqslant 0$, let $T_n > 0$ be such that $\theta_{T_n} \in I_n$. Then $T_n > M_n \to \infty$ as $n \to \infty$. By condition $(\mathcal{C}1)_n$, which holds for each n, we again get

$$\limsup_{k\to\infty} \frac{1}{k} \log |r_0 r_1 \cdots r_k| \geqslant \limsup_{i\to\infty} \frac{1}{T_i} \log |r_0 r_1 \cdots r_{T_i}| \geqslant 0.5 \log 5.$$

Since Θ_∞ has a positive measure, this implies that $\Lambda^+(\varepsilon) \geqslant 0.5 \log 5$ for all $\varepsilon \in [0, \varepsilon_\infty]$.

Appendix

In this appendix we show how to use Herman's subharmonic trick [6] to establish a positive Lyapunov exponent for all ε and any irrational ω. Note that in the proof of the main theorem, we get the bound $\Lambda^+(\varepsilon) > \log(5)/2$ for all $\varepsilon \in [0, \varepsilon_c]$ (but we only needed C^2 assumptions on V). Here we will get a better lower estimate, which holds for all ε.

Fix ε and ω, and let

$$M(\theta) = \begin{pmatrix} \varepsilon V(\theta) - 10 & 1 \\ 1 & 0 \end{pmatrix},$$

where $V(\theta) = 1/(1 + 4\lambda \cos^2(\pi\theta))$. Since $4\cos^2(\pi\theta) = e^{2\pi i\theta} + e^{-2\pi i\theta} + 2$, we see that if

$$F(z) = \begin{pmatrix} \dfrac{\varepsilon}{1 + \lambda(z + z^{-1} + 2)} - 10 & 1 \\ 1 & 0 \end{pmatrix}$$

then we have $M(\theta) = F(e^{2\pi i\theta})$. The expression $\frac{\varepsilon}{1+\lambda(z+z^{-1}+2)}$ can be written as $\frac{z\varepsilon}{\lambda(z-z_0)(z-z_1)}$ with

$$z_0 = \frac{-2\lambda - 1 + \sqrt{4\lambda + 1}}{2\lambda}, \qquad z_1 = \frac{-2\lambda - 1 - \sqrt{4\lambda + 1}}{2\lambda}.$$

Note that $|z_0| < 1 < |z_1|$ and that $z_0, z_1 \to -1$ as $\lambda \to \infty$. We now let

$$G(z) = (z - z_0) F(z) = \begin{pmatrix} \dfrac{\varepsilon z}{\lambda(z - z_1)} - 10(z - z_0) & (z - z_0) \\ (z - z_0) & 0 \end{pmatrix}$$

Then $G(z)$ is analytic in the disc $|z| < |z_1|$, which contains the disc $|z| \leqslant 1$. Next we introduce the notation $e_n = \mathrm{e}^{\mathrm{i}2\pi n\omega}$. Using this, we now get

$$\begin{aligned}
\Lambda^+(\varepsilon) &= \inf_n \frac{1}{n} \int_0^1 \log \|M^n(\theta)\| \, \mathrm{d}\theta \\
&= \inf_n \frac{1}{n} \int_0^1 \log \|F(e_{n-1}\mathrm{e}^{2\pi\mathrm{i}\theta}) \cdots F(e_0\mathrm{e}^{2\pi\mathrm{i}\theta})\| \, \mathrm{d}\theta \\
&= \inf_n \frac{1}{n} \left(-\sum_{k=0}^{n-1} \int_0^1 \log |e_k \mathrm{e}^{\mathrm{i}2\pi\theta} - z_0| \, \mathrm{d}\theta \right. \\
&\qquad \left. + \int_0^1 \log \|G(e_{n-1}\mathrm{e}^{2\pi\mathrm{i}\theta}) \cdots G(e_0\mathrm{e}^{2\pi\mathrm{i}\theta})\| \, \mathrm{d}\theta \right).
\end{aligned}$$

By an application of Jensen's formula, for example, the first sum vanishes ($|z_0| < 1$). Moreover, the function $\log \|G(ze_{n-1}) \cdots G(ze_0)\|$ is subharmonic in $|z| < |z_1|$, so the second integral is greater or equal to $\log \|G(0)^n\|$. Thus, we have

$$\begin{aligned}
\Lambda^+(\varepsilon) &\geqslant \inf_n \frac{1}{n} \log \left\| \begin{pmatrix} 10z_0 & -z_0 \\ -z_0 & 0 \end{pmatrix}^n \right\| \\
&= \log |z_0| + \log \left(\text{spectral radius of} \begin{pmatrix} 10 & -1 \\ -1 & 0 \end{pmatrix} \right) \\
&= \log |z_0| + \log(5 + \sqrt{26}) > \log 10
\end{aligned}$$

if λ is sufficiently large. Recall that $|z_0| \to 1$ as $\lambda \to \infty$.

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