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GENERIC 3-DIMENSIONAL VOLUME-PRESERVING DIFFEOMORPHISMS WITH SUPEREXPONENTIAL GROWTH OF NUMBER OF PERIODIC ORBITS

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Abstract. Let M be a compact manifold of dimension three with a non-degenerate volume form Ω and $\mathrm{Diff}^r_\Omega(M)$ be the space of C^r-smooth (Ω-) volume-preserving diffeomorphisms of M with $2 \le r < \infty$. In this paper we prove two results. One of them provides the existence of a Newhouse domain $\mathcal{N}$ in $\mathrm{Diff}^r_\Omega(M)$. The proof is based on the theory of normal forms [13], construction of certain renormalization limits, and results from [23, 26, 28, 32]. To formulate the second one, associate to each diffeomorphism a sequence $P_n(f)$ which gives for each n the number of isolated periodic points of f of period n. The main result of this paper states that for a Baire generic diffeomorphism f in $\mathcal{N}$, the number of periodic points $P_n(f)$ grows with n faster than any prescribed sequence of numbers $\{a_n\}_{n\in\mathbb{Z}_+}$ along a subsequence, i.e., $P_{n_i}(f) > a_{n_i}$ for some $n_i \to \infty$ with $i \to \infty$. The strategy of the proof is similar to the one of the corresponding 2-dimensional non volume-preserving result [16]. The latter one is, in its turn, based on the Gonchenko-Shilnikov-Turaev Theorem [8, 9].

To Vitusia and Matyusha who helped so much writing this paper

1. Introduction.

1.1. Newhouse domains.

Let (M, Ω) be a C^∞ smooth compact manifold of dimension 3 equipped with a C^∞ smooth non-degenerate volume form Ω. Denote by $\mathrm{Diff}^r_\Omega(M)$ the space of C^r smooth (Ω-) volume-preserving diffeomorphisms with the uniform C^r topology, and by $\mathrm{Diff}^r(M)$ the space of C^r smooth diffeomorphisms with the same topology. There are two main results in this paper. The first one establishes the existence of Newhouse domains in $\mathrm{Diff}^r_\Omega(M)$. The second result asserts genericity of diffeomorphisms with the arbitrarily fast growth of the number

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of periodic points in the Newhouse domains. We start with a brief overview of the known facts concerning homoclinic tangencies and Newhouse domains.

We say that a diffeomorphism f exhibits a *homoclinic tangency* if for a saddle periodic point $p = f^k(p)$ of some period $k \geq 1$ its stable and unstable manifolds, $W^s(p)$ and $W^u(p)$, respectively, have a point of tangency. Call an open set $\mathcal{N} \subset \text{Diff}^r(M)$ a *Newhouse domain* if it contains a C^r-dense set of diffeomorphisms exhibiting a homoclinic tangency. In the 70s Newhouse [23] proved existence of such domains in the space of C^r surface diffeomorphisms $\text{Diff}^r(M^2)$ for any $2 \leq r \leq \infty$. Moreover, he showed that in any neighborhood of a diffeomorphism exhibiting a homoclinic tangency there is a Newhouse domain. Later Palis-Viana [28] proved existence of multidimensional Newhouse domains in $\text{Diff}^r(M)$ arbitrarily close to a diffeomorphism with a homoclinic tangency, associated to a codimension 1 sectionally dissipative saddle[1]. Finally, Romero [32] removed the codimension one condition and sectional dissipativity. Using different ideas Bonatti-Diaz [5] proved that Newhouse phenomenon of coexistence of infinitely many sinks does occur for C^1-generic 3-dimensional diffeomorphisms inside of an open set in $\text{Diff}^1(M^3)$.

Newhouse domains are a source of many fascinating phenomena: cascade of period doubling bifurcations [36], generic coexistence of infinitely many sinks [23, 28], density of homoclinic tangencies of arbitrarily high orders [8, 9], superexponential growth of the number of periodic points [16], abundance of strange attractors [3, 6, 22, 37], prevalence of hyperbolicity [26, 29, 30], prevalence of absence of infinitely many coexisting sinks of finite complexity [11], etc (see [11, 26, 28, 32, 37] for further discussion). Newhouse domains of symplectic diffeomorphisms have also been extensively studied (see, e.g., [10, 24, 35] and references therein). In the present paper we investigate Newhouse domains for volume-preserving diffeomorphisms.

As the reader will see, there is an essential difference in proving existence of Newhouse domains in the space of non-conservative diffeomorphisms $\text{Diff}^r(M)$ and in the space of volume-preserving diffeomorphisms $\text{Diff}^r_\Omega(M)$. In this paper we restrict ourselves to 3-dimensional diffeomorphisms. It seems that the method might work in the higher dimensional case as well. The first result of this paper is the following

Theorem 1. *Let $2 \leq r < \infty$, $\dim M = 3$, and $f \in \text{Diff}^r_\Omega(M)$ be a diffeomorphism exhibiting a homoclinic tangency of a saddle periodic point with real eigenvalues. Then arbitrarily C^r-close to f there is an open set $\mathcal{N} \subset \text{Diff}^r_\Omega(M)$ with persistent homoclinic tangencies. In other words, there is a Newhouse domain $\mathcal{N}$ which is C^r-near f.*

Romero [32], relying on [28], proved the above theorem in the space of C^r smooth diffeomorphisms without restrictions on volume-preservation and the corresponding saddle to have real eigenvalues. Consider a diffeomorphism f exhibiting a homoclinic tangency, corresponding to a saddle periodic point p. By our assumption, eigenvalues are real. The key difference with the non-volume-preserving case is that if f is in $\text{Diff}^r(M)$ and has a saddle periodic point p, then generically one can apply Sternberg's theorem [1]. Namely, generically, the eigenvalues of the linearization $df^k(p)$ of p have no multiplicative resonances (see sect. 3 for the definition) and, therefore, in a neighborhood of p there are C^r-smooth normal coordinates, in which

[1]A saddle is called (codimension one) sectionally dissipative if it has just one expanding eigenvalue (positive Lyapunov exponent) and the product of any two eigenvalues has absolute value smaller than 1, i.e., any contracting eigenvalue is larger than the expanding one.

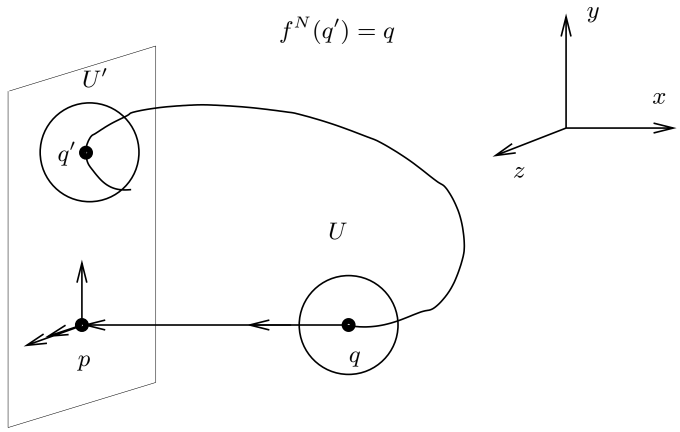

FIGURE 1. Homoclinic tangency.

the map f is linear. In the case when f is volume-preserving, i.e., in $\mathrm{Diff}^r_\Omega(M)$, the eigenvalues of $df(p)$ have the obvious resonance: the product of all of them equals 1. Absence of linearizing coordinates forces us to use another normal form: the one locally describing a saddle with essentially one resonance relation (multiplicatively single-resonant saddle, see sect. 3). The difference between the linear and the multiplicatively single-resonant normal form consists in the presence of a "small resonant" nonlinear term. We follow the strategy of the "standard" proof of existence of Newhouse domains (see, e.g., [26], ch. 6) and verify that the impact of the resonant nonlinear term is small enough, so that the proof goes through (see sect. 3 and 4 for details).

1.2. **Growth of the number of periodic points.** Now we turn to the problem of growth of the number of periodic points. For a diffeomorphism f consider the number of *isolated* periodic points of period n (i.e., the number of isolated fixed points of f^n)

$$P_n(f) = \#\{\text{isolated } p \in M : \ p = f^n(p)\}. \tag{1}$$

Call a diffeomorphism *Artin-Mazur or AM* if $P_n(f)$ grows at most exponentially fast. In 1965 Artin-Mazur [2] proved density of AM diffeomorphisms in $\mathrm{Diff}^r(M)$ for any $0 \le r \le \infty$. Notice that an AM mapping f could have a curve of periodic points γ, i.e., $\forall x \in \gamma,\ f^n(p) = p$ for some $n \in \mathbb{Z}_+$, because in this case γ consists of non-isolated periodic points. Artin-Mazur could not rule out such a seemingly rare phenomenon. In [15] there is a simple proof of density of AM diffeomorphisms with hyperbolic periodic points only, which solves the problem of existence of non isolated periodic points. In [16], using [9], it is shown that diffeomorphisms having a curve of non-isolated periodic points are dense in a Newhouse domain. To the best of our knowledge, density of AM diffeomorphisms in $\mathrm{Diff}^r_\Omega(M)$ and AM flows in the space of C^r smooth vector fields on a compact manifold M of dimension at least three are open problems.

The next natural question is whether AM diffeomorphisms form a Baire generic set. The answer is negative, see [16][2]. Moreover, in a Newhouse domain for surface diffeomorphisms, those exhibiting superexponential growth of the number of periodic points are Baire generic[3].

1.2.1. *Superexponential growth for surface diffeomorphisms.* According to a Palis' conjecture for surface diffeomorphisms [25], $\mathrm{Diff}^r(M)$ is a C^r-closure of hyperbolic ones and those exhibiting homoclinic tangency. Pujals-Sambarino [31] proved this conjecture for C^1-closure. It implies that diffeomorphisms with superexponential growth of the number of periodic points form a countable intersection of C^1-dense C^r-open sets outside of the open set of hyperbolic diffeomorphisms. If Palis' conjecture is true, then diffeomorphisms with superexponential growth of the number of periodic points form a C^r-Baire generic set outside of the open set of hyperbolic diffeomorphisms.

1.2.2. *Superexponential growth for multidimensional diffeomorphisms.* It looks attractive to prove C^r-Baire genericity of the set of diffeomorphisms with superexponential growth of the number of periodic points outside of the open set of hyperbolic diffeomorphisms in $\mathrm{Diff}^r(M^d)$ for $d > 2$. In this case Newhouse domains and hyperbolic diffeomorphisms do not form a dense set. There is another open set discovered by L. Diaz [7] (see also [4]). It is an open set $\mathcal{BD} \subset \mathrm{Diff}^r(M)$ of diffeomorphisms with a C^r dense subset of those having a heterodimensional heteroclinic cycle. In [20] we prove genericity of superexponential growth of the number of periodic points in the domain $\mathcal{BD}$ of 3-dimensional diffeomorphisms. The latter work is a step toward proving genericity of superexponential growth of the number of periodic points outside of hyperbolic diffeomorphisms. Indeed, one of Palis' conjectures [25] states that the set consisting of all hyperbolic diffeomorphisms, those exhibiting homoclinic tangency, and those having a heterodimensional cycle, is C^r-dense in the space of diffeomorphisms $\mathrm{Diff}^r(M)$ (see [17] for further discussion).

1.2.3. *Superexponential growth of the number of periodic points for 3-dimensional volume-preserving diffeomorphisms.* The second *main* result of this paper is the following.

Theorem 2. *Let $2 \le r < \infty$, and M be a compact 3-dimensional manifold. Let $\mathcal{N} \subset \mathrm{Diff}^r_\Omega(M)$ be a Newhouse domain. Then for an arbitrary sequence of positive integers $\alpha = \{a_n\}_{n=1}^\infty$ there exists a residual set $\mathcal{R}_\alpha \subset \mathcal{N}$, depending on this sequence, with the property that $f \in \mathcal{R}_\alpha$ implies that*

$$\limsup_{n\to\infty} \frac{P_n(f)}{a_n} = \infty. \tag{2}$$

This is an extension of the main result from [16] to the volume-preserving setting.

2. **Strategy of the proofs.**

[2] To the best of our knowledge, the first example of superexponential growth of the number of periodic points is constructed in [33].

[3] Growth of the number of periodic points for area-preserving diffeomorphisms of a surface is a superficial problem, as the following arguments show. Existence of elliptic periodic points is an open phenomenon. By the Birkhoff Normal Form Theorem (see, e.g., [21]), a diffeomorphism with an elliptic periodic point after a C^r-small perturbation can have an integrable elliptic point. A C^r-perturbation of an integrable elliptic point can produce the supergrowth of the number of periodic points.

2.1. **Plan of the proof of Theorem 1.** Let $f \in \mathrm{Diff}^r_\Omega(M),\ 2 \le r \le \infty$ be a diffeomorphism with a hyperbolic saddle point whose eigenvalues are real, and whose 1-dimensional stable manifold and 2-dimensional unstable manifold have a tangency at some point $q \in M$. Consider a generic 1-parameter unfolding $\{f_\varepsilon\}_{\varepsilon \in I},\ f_0 = f$ of the homoclinic tangency at q. It turns out that there is the following scaling limit. Fix a real number K and a sufficiently large positive n_0. Then for $n > n_0$ there are sequences of parameters $\varepsilon_n(K)$ near zero, and of open sets U_n near q, such that the return map $f^{n+N}_{\varepsilon_n} : U_n \to U$ is well-defined. Moreover, there is a sequence of renormalizations of the initial map, $\Phi^2_{\varepsilon_n,n} = R_n^{-1} \circ f^{n+N}_{\varepsilon_n} \circ R_n$, with $R_n : [-1,1]^3 \to U_n$ such that the following holds.
• Each $f^{n+N}_{\varepsilon_n} : U_n \to U$ (resp., $\Phi^2_{\varepsilon_n,n} : [-2,2]^3 \to [-2,2] \times \mathbb{R}^2$) has a sufficiently smooth locally invariant 2-dimensional surface $S_{\varepsilon_n,n}$ (resp., $R_n^{-1}(S_{\varepsilon_n,n})$);
• $R_n^{-1}(S_{\varepsilon_n,n})$ converges to a plane parallel to the XY-plane, and the restriction $\Phi^2_{\varepsilon_n,n}|_{R_n^{-1}(S_{\varepsilon_n,n})}$ converges to the Henon family $(x,y) \mapsto (y, y^2 + K)$.

Existence of a homoclinic tangency of the restriction $\Phi^2_{\varepsilon_n,n}|_{R_n^{-1}(S_{\varepsilon_n,n})}$ implies existence of a homoclinic tangency for the map f_{ε_n} itself. At this point we can reduce our 3-dimensional problem to the standard 2-dimensional problem. The 2-dimensional one is well-understood (see, e.g., [26] ch. 6). We work out this construction in Section 4.

2.2. **Plan of the proof of Theorem 2:** We start with a few definitions.

Definition 1. A periodic point $p = f^n(p)$ of some period k is called a *saddlenode* if the linearization $df^n(p)$ has all eigenvalues but one away from the unit circle.

In the 3-dimensional volume-preserving case it implies that the eigenvalues of $df^n(p)$, denoted by λ, μ, ν, are real and satisfy $|\lambda| < 1 = |\mu| < |\nu|$. Taking $f^{2n}(p)$ if necessary, we can assume that $\mu = 1$. Let f be C^r–smooth for some positive integer r and $p = f^n(p)$ be a saddlenode. Then the theorem on the invariant central manifold [34] Thm. III.2. implies that p has 1-dimensional stable, unstable, and central manifolds, denoted by $W^s(p),\ W^u(p)$, and $W^c(p)$, respectively, of smoothness $C^r,\ C^r$, and C^{r-1}, respectively. Introduce C^{r-1}-smooth coordinates on $W^c(p)$, denoted by y^c, so that $y^c = 0$ corresponds to the point p.

Definition 2. Let f be a C^r diffeomorphism, and $p = f^n(p)$ be its periodic point of saddlenode type. The point p is called m-saddlenode [4] for $m \le r-1$ if all the derivatives of the restriction $\phi(y^c) = f^n|_{W^c(p)(y^c)}$ to the central manifold of orders $2, \dots, m$ equal zero, i.e., $\phi^{(s)}(0) = 0$ for $2 \le s \le m$.

Fix any r, and suppose that $f \in C^\infty$ has an m-saddlenode $p = f^n(p)$ for some $m \ge r$. Then for any positive integer k by a C^r-perturbation of p one can create k distinct periodic points of the same period n, located in a small neighborhood of p.

Let p be a periodic saddle of a C^∞-smooth volume-preserving diffeomorphism f with the eigenvalues of the linearization satisfying $\lambda < 1 < \mu < \nu$, whose 2-dimensional unstable and 1-dimensional stable manifolds exhibit a homoclinic tangency. By volume-preservation $\lambda\mu\nu = 1$.

Definition 3. A saddle is called k-unstable if the eigenvalues of its linearization satisfy $\lambda < 1 < \mu < \mu^k < \nu$.

[4]Sometimes it is also called either a C^m-neutral periodic point or a saddlenode of order m.

Note that, due to volume-preservation, this condition is equivalent to $\lambda\mu^{k+1} < 1$. It resembles the corresponding definition of a k-shrinking saddle in [16], and serves two different purposes, one of which is similar to the one for k-shrinking saddle there.

The proof of Theorem 2, presented here, follows along the lines of the proof of the corresponding result in [16], which, in its turn, follows closely [8]. We reproduce the plan of the proof given there, and explain the modification at each step. Additional difficulties are caused by *the increase of dimension and absence of linear normal coordinates due to the preservation of volume* . Since the diffeomorphisms under consideration are volume-preserving, in general they have no linear normal form in a neighborhood of the saddle point, which is essential for the constructions in [16]. We assume instead that the only resonance relation is the one due to the preservation of volume, which can be achieved by a small perturbation of our system. This enables us to use a *multiplicatively single-resonant* normal form, see the definition in Sect. 3, and [13] for more details. Recall that $\mathrm{Diff}^r_\Omega(M)$ denotes the space of C^r diffeomorphisms preserving a smooth volume form Ω.

Definition 4. Let a diffeomorphism $f \in \mathrm{Diff}^r_\Omega(M)$ have a periodic saddle $p = f^k(p)$ with real eigenvalues $\lambda < 1 < \mu < \nu$, and let $W^{cs}(p)$ be the invariant manifold of p corresponding to the eigenvalues λ and μ. We say that f has a reduced homoclinic tangency if there is a C^r–smooth 2-dimensional f^k-locally invariant surface S inside $W^{cs}(p)$ such that the restriction $f^k|_S$ has a homoclinic tangency associated to p.

Notice that if f has a reduced homoclinic tangency, then it has a homoclinic tangency.

Now we are ready to outline the proof of Theorem 2. We start with f in a Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r_\Omega(M)$ from Theorem 1. Since C^∞ diffeomorphisms are C^r-dense in $\mathrm{Diff}^r_\Omega(M)$, we can always start with a C^∞ diffeomorphism $\tilde{f}$ which is C^r-near the starting C^r diffeomorphism f. In the proof we shall always make C^r-small C^∞-perturbations.

By a C^r-small C^∞-perturbation of $\tilde{f}$ we shall construct a diffeomorphism with an m-saddlenode of a large period. As we explained above, by a C^r-small C^∞-perturbation of the latter, one can create an arbitrary large number of periodic points of the same period.

Below we briefly describe the deformations leading to an m-saddlenode. Let $f \in \mathrm{Diff}^\infty_\Omega(M)$ be a diffeomorphism with a hyperbolic saddle point whose eigenvalues are real, and the corresponding 1-dimensional stable, and 2-dimensional unstable manifold have a tangency. Fix $2 \le r < \infty$. In the following plan, by a "perturbation", if not specified, we shall always mean "a C^r-small C^∞-perturbation in $\mathrm{Diff}^r_\Omega(M)$". Let $r \le m \le 2r$.

First Step. For an arbitrary r we construct a diffeomorphism with a reduced homoclinic tangency, associated to a $2r$-unstable saddle, C^r-close to f. The proof is close to the argument in sect. 2 [32] with the only difference that in our case f has no linear normal form in the neighborhood of p.

Second Step. By a small perturbation of a diffeomorphism with a reduced homoclinic tangency of a $2r$-unstable saddle, we create one with an m-floor tower (a heteroclinic contour). This is done in the same way as in [16]. Additional difficulties are caused by the 3-dimensionality and the preservation of volume.

Third Step. Using a normal form for a multiplicatively single-resonant saddle from [13] (see next section), we make a small perturbation of a diffeomorphism with an m-floor tower to obtain one with an m-th order homoclinic tangency.

Fourth Step. By a small perturbation of the diffeomorphism with an m-th order homoclinic tangency we construct one with a periodic m-saddlenode.

As we mentioned above, a periodic m-saddlenode can be “split” into any ahead given number of non-degenerate periodic points of the same period by a volume-preserving C^r-small C^∞-perturbation. By iterating the above procedure we achieve the result of Theorem 2 (see Sect. 2.7 [16]).

The paper is organized as follows. We start with a diffeomorphism $f \in \mathrm{Diff}^r_\Omega(M)$ having a generic saddle periodic point p with a homoclinic tangency. In section 3 we define the single-resonant normal form and analyze its iterates near p. In Section 4 we prove Theorem 1 and justify Step 1 of the proof of Theorem 2 on the way. This section also contains Theorem 4 which is an improvement of Theorem 1 and is of independent interest. In Section 5 we compute a return map near a homoclinic tangency. The rest of the paper is devoted to completing of the proof of Theorem 2: Step 2 is done in section 6, Step 3 — in Section 7, and finally, Step 4 — in Section 8. This suffices to complete the proof of Theorem 2.

3. **Normal form and estimates of nonlinear term.** We start with a C^∞ smooth volume-preserving diffeomorphism $f : M^3 \to M^3$ having a periodic point p of saddle type, as described in the Introduction. If p has period k, we replace f by f^k, and assume that p is a fixed point of f. Since f preserves the volume, the eigenvalues of the linearization at p have an inevitable resonance relation $\lambda\mu\nu = 1$. Making a small volume-preserving perturbation of f if necessary, we can assume that the only integer relations between the eigenvalues are $\lambda\mu\nu = 1$ and the ones that follow from it.

Recall that a set of n complex numbers $\lambda_1, \dots, \lambda_n$ has a *multiplicative resonance* if for a set of n integers $r_1, \dots, r_n$, $\sum_j |r_j| \geq 2$ we have $\lambda_1^{r_1} \dots \lambda_n^{r_n} = 1$. A set $\lambda_1, \dots, \lambda_n$ is called *multiplicatively single-resonant* if all multiplicative resonances follow from a single one. The following theorem from [13] (see also [14]) provides a finitely smooth normal form for our system in a neighborhood of the saddle point.

Theorem 3. *A deformation of a multiplicatively single-resonant hyperbolic germ of a C^∞ diffeomorphism is finitely smoothly equivalent to a local family*

$$x \mapsto Xg(u(x), \varepsilon), \quad X = \mathrm{diag}(x_1, \dots, x_n), \quad x \in (\mathbb{R}, 0),$$

where $g(u, \varepsilon)$ is a vector polynomial of a scalar variable u, whose coefficients depend smoothly on ε, and $u(x)$ is a resonant monomial. More precisely, given a positive integer m, there is a polynomial $g_m(u, \varepsilon)$ such that the initial family is C^m-smoothly equivalent to the normal form defined above with $g = g_m$ in some m-dependent neighborhood of zero in (x, ε)-space.

Application of this result to our setting. Denote $u = xyz$. Then for any m there is a C^m-smooth coordinate system such that f has the form:

$$f : (x, y, z) \to ((\lambda + g_1(u))x, (\mu + g_2(u))y, (\nu + g_3(u))z), \tag{3}$$

where $g_1(u), g_2(u), g_3(u)$ are polynomials vanishing at $u = 0$. In what follows we shall use these normal coordinates. Of course, the normalization does not preserve the Lebesgue volume form, but transforms it into another C^m-smooth non-degenerate volume form Ω'. The deformations of f constructed during the proof will preserve Ω'.

3.1. **Iterations of f in the normal coordinates.** The next calculation describes the n-th iterate f^n of the above normal form, and will be used later to prove that f^n is "close to" a linear transformation $(x, y, z) \mapsto (\lambda^n x, \mu^n y, \nu^n z)$ for certain (x, y, z). Let $\|g(u)\|_m$ denote C^m-norm of g, i.e., maximum taken over all partial derivatives of order up to m in the domain of its definition.

Lemma 1. *Fix $r \in \mathbb{Z}_+$. Let f be the normal form (3) and $u = xyz$. Then there exists n_0 such that for all $n \geq n_0$ and (x, y, z) such that $|u| < \lambda^n$ and all $\{f^j(x, y, z)\}_{j=0}^n$ belonging to the unit cube around zero we have:*

$$f^n(x, y, z) = (\lambda^n x, \mu^n y, \nu^n z) + \rho_n(x, y, z),$$

where

$$\rho_n(x, y, z) = (\rho_n^x(u)\, x, \rho_n^y(u)\, y, \rho_n^z(u)\, z),$$

$\{\rho_n^\xi(u)\}_{\xi \in \{x,y,z\}}$ are polynomials. Moreover, ρ_n^ξ's define polynomials by $\rho_n^\xi(u) = uP_n^\xi(u)$, $p_n^y(u) = \mu^{-n} P_n^y(u)$, and $p_n^z(u) = \nu^{-n} P_n^z(u)$, and these satisfy

$$\|P_n^x(u)\|_i \leq \lambda^n pol_i(n), \quad \|p_n^y(u)\|_i \leq pol_i(n), \quad \|p_n^z(u)\|_i \leq pol_i(n), \tag{4}$$

where $0 \leq i \leq m$ and for each i we have that $pol_i(n)$ is a polynomial of degree at most $2i + 3$.

Proof. Consider

$$K(u) := u(\lambda + g_1(u))(\mu + g_2(u))(\nu + g_3(u)) =: u + P(u). \tag{5}$$

By definition, $P(u)$ is a polynomial of some degree d in u having zero of order 2 at the origin. Denote $K^j(u) = K(K^{j-1}(u))$ for each $j \geq 1$. Thus, we have:

$$\rho_n^x(u) = (\lambda + g_1(u))(\lambda + g_1(K(u))) \dots (\lambda + g_1(K^{n-1}(u))) - \lambda^n,$$

and ρ_n^y and ρ_n^z satisfy the same formula with (λ, x, g_1) replaced by (μ, y, g_2) or (ν, z, g_3), respectively. These formulas imply that ρ_n^ξ is divisible by u for each $\xi \in \{x, y, z\}$. Therefore, polynomials P_n^ξ are well defined.

First let us estimate the n-th iterate of K for an arbitrary n:

$$K^n(u) = K^{n-1}(u) + P(K^{n-1}(u)).$$

In other words, K^n is a finite sum of the form

$$K^n(u) = u + P(u) + P(u + P(u)) + P\big(u + P(u) + P(u + P(u))\big) + \dots \ .$$

Notice that the highest degree of K^n, denoted by $s(n)$, is bounded by d^n. For any given $k \leq s(n)$ we want to estimate the coefficient $C_m(n)$ of the monomial u^m in the above sum. To do this, notice that u^m can only appear in the composition P^t for $t = t(m) \leq m/2$. Indeed, P is divisible by u^2. For a polynomial K, denote by $\bar{K}$ the same polynomial with all the coefficients replaced by their absolute values. Then $|C_m(n)|$ is less or equal to the coefficient of u^m in the following polynomial:

$$\begin{aligned} &\bar{K}^{t-1}(u) + \bar{P}(\bar{K}^{t-1}(u)) + \bar{P}(2\bar{K}^{t-1}(u)) + \dots + \bar{P}((n-1)\bar{K}^{t-1}(u)) \\ &\leq \bar{K}^{t-1}(u) + n\bar{P}(n\bar{K}^{t-1}(u)) \leq \bar{K}^t(n^2 u). \end{aligned}$$

Here $t = t(m) \leq m/2$, and the comparison of polynomials is component-wise.

It is left to estimate the coefficient of u^m in $\bar{K}^t(u)$. It is less than the largest coefficient in $\bar{K}^m$. The latter is less than c^m, where c is a constant depending only on the polynomial K.

In fact, we have proved that for all $j \le s(n)$

$$\bar{K}^j \le \bar{K}^n \le \sum_{m=1}^{s(n)} (cn^2 u)^m. \tag{6}$$

The same estimate, with different constants c and $s(n)$, holds for $g_i(\bar{K}^j)$. By assumption $|u| \le \lambda^n$. So for n sufficiently large, the product $(cn^2\lambda^n)$ gets as small as we like, and we can write the following estimate for $j \le s(n)$, $i = 1, 2, 3$:

$$g_i(\bar{K}^j) \le \sum_{k=1}^{s(n)} (cn^2 u)^k \le \sum_{k=1}^{\infty} (cn^2\lambda^n)^k \le 2cn^2\lambda^n =: b_n.$$

The latter constant goes to zero when n grows. By (5), we can estimate:

$$\|\rho_n^x(u)\|_0 \le \lambda^n (1 + b_n/\lambda)^n - \lambda^n \le 4cn^3\lambda^{2n-1};$$

0-norms of ρ_n^y and ρ_n^z are estimated in the same way. Formula (6) permits us to estimate the derivatives of ρ_n^ξ with respect to u. After we obtain estimates for ρ_n^ξ and its derivatives it is easy to derive similar estimates for P_n^ξ and p_n^ξ. This completes the proof. □

4. **Existence of Newhouse domains and density of r-unstable saddles with homoclinic tangency.** Here we prove Theorem 1 and realize Step 1 of the proof of Theorem 2. At some point we need to prove an analog of Lemma 2.1 from [32] for a multiplicatively single-resonance normal form instead of a linear one. Before we state the main result of this section we need an additional notion of a normally hyperbolic invariant manifold.

Recall that the *minimum norm* $m(L)$ of a linear transformation L is defined as

$$m(L) = \inf\{|Lx| : |x| = 1\}.$$

When L is invertible, $m(L) = \|L^{-1}\|^{-1}$. Let $f : M \to M$ be a C^r smooth diffeomorphism of a smooth manifold M and S be a C^r smooth invariant submanifold of M. Let T_SM be the tangent bundle of M over S. Suppose we have a df-invariant splitting into three subspaces

$$T_SM = W^u \oplus TS \oplus W^s,$$

i.e., for any $p \in M$ we have $df(p)W_p^s = W_{f(p)}^s$ and $df(p)W_p^u = W_{f(p)}^u$. Moreover, for some $C, \lambda > 1$ we have $|df^n(p)v| \ge C\lambda^n|v|$ for all $p \in S$, all $v \in W_p^u$ (resp., W_p^s), and all $n \in \mathbb{Z}_+$ (resp., $n \in \mathbb{Z}_-$). Denote

$$df^s(p) = df(p)|_{W^s}, \quad df^u(p) = df(p)|_{W^u}, \quad df^c(p) = df(p)|_{T_pS}.$$

Let k be positive. We say that f is *k-normally hyperbolic* at S if there is a Riemannian structure on TM such that for all $p \in S$ we have:

$$m(df_p^u) > \|df^c(p)\|^k, \quad m(df_p^c)^k > \|df^s(p)\|.$$

Let T_S be a tube neighborhood of S and $\pi_S : T_S \to S$ be the natural projection along directions normal to S. We say that f is *normally hyperbolic* on S if it is *k-normally hyperbolic* for some positive k.

The main result of this section is the proof of the following statement:

Theorem 4. *Let Ω be a C^∞ smooth volume form and $r \geq 1$. Let $\{f_\varepsilon\}_{\varepsilon \in I}$ be a generic 1-parameter family of C^∞-smooth Ω-preserving diffeomorphisms unfolding a quadratic homoclinic tangency associated to a saddle with real multiplicatively single-resonant eigenvalues. Then there exists a sequence of pairwise disjoint intervals I_n monotonically decreasing to zero in the parameter space such that for n sufficiently large the following holds:*
- *for any $\varepsilon \in I_n$ there is a C^{2r}-smooth 2-dimensional surface $S_{\varepsilon,n}$, which is locally invariant under f_ε^{n+N}, k_n-normally hyperbolic, and C^{2r}-smoothly depends on the parameter ε;*
- *the parameter $k_n \to \infty$ as $n \to \infty$;*
- *the family of restrictions to this surface $\{f_\varepsilon^{n+N}|_{S_{\varepsilon,n}}\}_{\varepsilon \in I_n}$ generically unfolds a quadratic homoclinic tangency [5];*
- *for any real number K there exists a sequence of parameters $\varepsilon_n(K) \in I_n$ tending to zero as n goes to infinity, and a sequence of changes of variables R_n such that the rescaled surface $R_n^{-1}(S_{\varepsilon,n})$ C^{2r}-tends to a plane parallel to the XY plane and the map*

$$\Phi^2_{\varepsilon_n(K),n} := R_n^{-1} \circ f^n \circ f^N_{\varepsilon_n(K)} \circ R_n$$

restricted to $R_n^{-1}(S_{\varepsilon_n(K),n})$ and written in (x,y)-coordinates, gets arbitrarily C^{2r}-close to

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ y^2 + K \end{pmatrix}$$

as n increases.

Remark 1. This theorem is a generalization of the corresponding 2-dimensional calculations near a homoclinic tangency, connecting planar homoclinic bifurcations to Henon-like families (see, e.g., [26]). If f has a linear normal form (which is the generic case for non–necessarily volume–preserving diffeomorphisms), then this theorem is an analog of Theorem C from [32]. Since our mappings are volume-preserving, they do not in general have a linear normal form. In this section recall the proof of Theorem C from [32] and describe the modifications we use to treat the non-linear case.

We speak about C^{2r} not C^r convergence above to fit to the notations of the rest of the paper. Below we derive from this Theorem Corollary 1, which in turn implies the first main result Theorem 1.

Proof. Definition of a homoclinic tangency from the introduction applies to C^r-smooth diffeomorphisms of a smooth 2-dimensional orientable manifold S. We distinguish two types of homoclinic tangencies. Let $\phi : S \to S$ be a C^r-smooth surface diffeomorphism with a saddle periodic point $\phi^k(p) = p$. Suppose the invariant manifolds $W^u(p)$ and $W^s(p)$ have a homoclinic tangency at some point q. Introduce local coordinates near p (not necessarily linear) so that invariant manifolds $W^s(p)$ and $W^u(p)$ coincide with the coordinate axis OX and OY, respectively. Scale the coordinates so that the tangency occurs at $q = (1, 0)$, and $W^s(p)$ has a tangency with the y-positive half of $W^u(p)$. Denote by $W^u_{q,loc}(p)$ the connected component of the intersection of $W^u(p)$ with a neighborhood of q that contains q. We call such a tangency *inner* (resp., *outer*) if the intersection of $W^u_{q,loc}(p)$ lies in $\{y \geq 0\}$ (resp., $\{y \leq 0\}$) (see Fig. 2 and, e.g., Fig. 1 in [16] or Fig. 3.1 in [27]). In what follows we

[5] Since this surface, $S_{\varepsilon,n}$, depends C^{2r}-smoothly on the parameter $\varepsilon \in I_n$, it does make sense to speak about the family of restrictions $\{f_\varepsilon|_{S_{\varepsilon,n}}\}_{\varepsilon \in I_n}$ unfolding a quadratic homoclinic tangency.

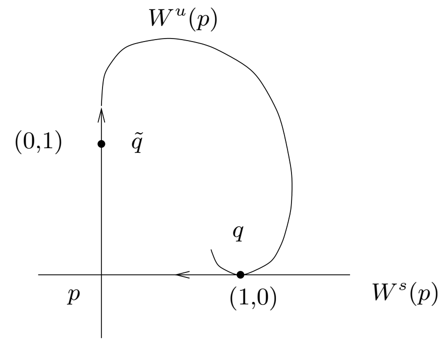

FIGURE 2. Inner tangency.

are mainly interested in inner tangencies. The reason is that for a 3-dimensional volume-preserving diffeomorphism f with a locally invariant 2-dimensional surface S such that the restriction $f|_S$ has an inner homoclinic tangency one can conclude the existence of a horseshoe (see Prop. 1). Recall definition 4 of a reduced homoclinic tangency.

Start with a C^∞ diffeomorphism f with a homoclinic tangency. Suppose that $q = (1, 0, 0)$ is a point of the orbit of tangency, and let $q' = (0, 1, \eta_z)$ be such that $f^N(q') = q$. Denote $\eta := (1, \eta_z)$. Let $\tilde{y} = (y - 1)$, $\tilde{z} = (z - \eta_z)$. Write f^N in the form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 + a_1 x + b_1 \tilde{y} + c_1 \tilde{z} + H_1(x, \tilde{y}, \tilde{z}) \\ a_2 x + b_2 \tilde{y} + c_2 \tilde{z} + d_2 \tilde{y}^2 + e_2 \tilde{y}\tilde{z} + f_2 \tilde{z}^2 + H_2(x, \tilde{y}, \tilde{z}) \\ a_3 x + b_3 \tilde{y} + c_3 \tilde{z} + d_3 \tilde{y}^2 + e_3 \tilde{y}\tilde{z} + f_3 \tilde{z}^2 + H_3(x, \tilde{y}, \tilde{z}) \end{pmatrix}, \tag{7}$$

where $a_i, \dots f_i$ are constants, and for $x = (y - 1) = (z - \eta_z) = 0$ we have:

$$H_i = \partial_j H_i = 0, \ i, j = 1, 2, 3.$$
$$\partial_i \partial_j H_2 = \partial_i \partial_j H_3 = 0, \quad i, j = 2, 3.$$

Perturbing if necessary assume $c_3 \neq 0$.

Remark 2. Let $V = (0, 1, v)$ be a vector such that $df^N(0, 1, \eta_z)V = (c, 0, 0)$ for some constant $c \neq 0$. Since we have a tangency with the OX-axis at q, such a vector V exists. Then in our notation we have $b_2 + vc_2 = b_3 + vc_3 = 0$.

We consider a 1-parameter unfolding f_ε^N of our system: add ε in the second line of (7). We shall denote the coordinate functions of f_ε^N by $(G^1, G^2_\varepsilon, G^3)$.

The main ingredient of the proof is the following lemma.

Lemma 2. *For any real numbers K_1, K_2, there exists a sequence of pairwise disjoint intervals I_n in the parameter space, monotonically tending to zero as n grows, and a sequence of changes of variables R_n such that the renormalized mapping*

$$\Phi^2_{\varepsilon,n} := R_n^{-1} \circ f^n \circ f_\varepsilon^N \circ R_n := (\bar{G}^1_n, \bar{G}^2_{\varepsilon,n}, \bar{G}^3_n)$$

has the following properties:

- *The first component, $\bar{G}^1_n(x, y, z)$, C^{2r}-converges to zero.*
- *The second and the third component are of the form*

$$\bar{G}^2_{\varepsilon_n,n}(x, y, z) = y^2 + K_1(y + k_0 z) + K_2 + k_1 yz + k_2 z^2 + h_1^n(x, y, z)$$
$$\bar{G}^3_n(x, y, z) = \nu^n(k_3 z + k_4 + \nu^{-n} h_2^n(x, y, z)),$$

where k_i are constants, only depending on the system (7), both h_1^n and $\nu^{-n}h_2^n$ converge to zero in C^{2r}-norm as $n \to \infty$ for any sequence $\{\varepsilon_n\}_{n\geq 1}$ such that $\varepsilon_n \in I_n$ for each n.

Proof. We split the proof of this lemma into two steps. First we consider the particular case when f is linear. In this case the statement coincides with the 3-dimensional version of Lemma 2.1 from [32]. We rewrite the proof presented there, since notations are quite different and we need to deal with the nonlinear case anyway. At the second step we make modifications in order to treat the general case, assuming that f has a multiplicatively single-resonant normal form.

4.1. **Linearizable case.** First we shall study a sequence of renormalizations of the system $f^n \circ f_\varepsilon^N$, where f^n coincides with its linear part $L^n(x,y,z) = (\lambda^n x, \mu^n y, \nu^n z)$. Consider

$$\Psi_n^u(x,y,z) = (\lambda^n x, y, z), \quad \Psi_n^s(x,y,z) = (x, \mu^{-n} y, \nu^{-n} z).$$

With this notation, $L^n = \Psi_n^u \circ (\Psi_n^s)^{-1}$. For each n we define

$$B_n = \begin{pmatrix} \beta_1\mu^{-n} & \beta_2\nu^{-n} \\ \beta_3\mu^{-n} & \beta_4\mu^{-n} \end{pmatrix}, \quad B_n^{-1} = \frac{1}{\det B_n}\begin{pmatrix} \beta_4\mu^{-n} & -\beta_2\nu^{-n} \\ -\beta_3\mu^{-n} & \beta_1\mu^{-n} \end{pmatrix}. \tag{8}$$

Here

$$\frac{1}{\det B_n} = \frac{\mu^{2n}(1+\kappa_n)}{\beta_1\beta_4},$$

where κ_n goes to zero when n goes to infinity.

We define Ψ_n to be of the form

$$\Psi_n(x,y,z) = (\delta_n x + 1, B_n(y,z) + \eta), \tag{9}$$

where constants β_i and δ_n are to be chosen later. Define the renormalization mapping as

$$R_n = \Psi_n^u \circ \Psi_n. \tag{10}$$

The renormalized mapping looks like:

$$\begin{aligned} \Phi_{\varepsilon,n}^{2,L} &= R_n^{-1} \circ L^n \circ f_\varepsilon^N \circ R_n = \Psi_n^{-1} \circ (\Psi_n^s)^{-1} \circ f_\varepsilon^N \circ \Psi_n^u \circ \Psi_n = \\ &= \Psi_n^{-1} \circ (\Psi_n^s)^{-1} \circ (G^1, G_\varepsilon^2, G^3) \circ \Psi_n^u \circ \Psi_n := (\bar G_n^{1,L}, \bar G_{\varepsilon,n}^{2,L}, \bar G_n^{3,L}). \end{aligned} \tag{11}$$

The corresponding sequence of changes of variables is illustrated by Fig. 3.

We compute:

$$\begin{aligned} &\Psi_n^{-1} \circ (\Psi_n^s)^{-1} = (\delta_n^{-1}(x-1), B_n^{-1}((y,z)-\eta)) \circ (x, \mu^n y, \nu^n z) \\ &= (\delta_n^{-1}(x-1),\ \frac{1}{\det B_n}(\beta_4 y - \beta_2 z - p_n),\ \frac{1}{\det B_n}(\beta_3 y + \beta_1 \frac{\nu^n}{\mu^n} z + q_n), \end{aligned} \tag{12}$$

where $p_n = -\beta_4\mu^{-n} + \beta_2\eta_z\nu^{-n}$, and $q_n = \beta_3\mu^{-n} - \beta_1\eta_z\nu^{-n}$.

$$\begin{aligned} \Psi_n^u \circ \Psi_n &= (\lambda^n(\delta_n x + 1), B_n(y,z) + \eta) = \\ &= \left(\lambda^n(\delta_n x+1),\ \frac{\beta_1 y}{\mu^n} + \frac{\beta_2 z}{\nu^n} + 1,\ \frac{\beta_3 y}{\mu^n} + \frac{\beta_4 z}{\mu^n} + \eta_z\right). \end{aligned} \tag{13}$$

Combining formulas (11–13) we get the following expressions for the components of $\Phi_{\varepsilon,n}^{2,L}$. For the first component, $\bar G_n^{1,L}$, we have:

$$\bar G_{\varepsilon,n}^{1,L} = \delta_n^{-1}(G^1 - 1) \circ \left(\lambda^n(\delta_n x+1),\ \frac{\beta_1 y}{\mu^n} + \frac{\beta_2 z}{\nu^n} + 1,\ \frac{\beta_3 y}{\mu^n} + \frac{\beta_4 z}{\mu^n} + \eta_z\right). \tag{14}$$

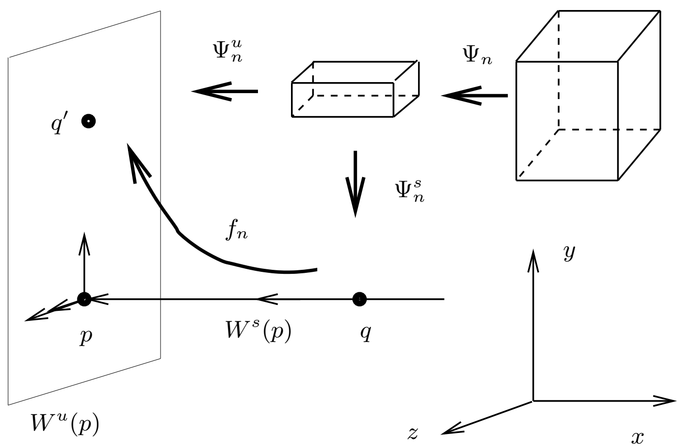

FIGURE 3. Renormalization of the return map.

We take δ_n such that $\delta_n^{-1}\mu^{-n} \to 0$ when n grows. Then $\bar{G}_n^{1,L}$ goes to zero with the growth of n.

For the second component, $\bar{G}_{\varepsilon,n}^{2,L}$, we have:

$$\begin{aligned}\bar{G}_{\varepsilon,n}^{2,L} =& \frac{\mu^{2n}(1+\kappa_n)}{\beta_1\beta_4}(\beta_4 G_\varepsilon^2 - \beta_2 G^3 + p_n)\circ \\ & \left(\lambda^n(\delta_n x+1),\ \frac{\beta_1 y}{\mu^n} + \frac{\beta_2 z}{\nu^n} + 1,\ \frac{\beta_3 y}{\mu^n} + \frac{\beta_4 z}{\mu^n} + \eta_z\right).\end{aligned} \tag{15}$$

- Let us first compute the linear part of $G_{\varepsilon,n}^{2,L}$, combining the above formula with (7). The coefficient of x goes to zero with the growth of n. The coefficient of y is

$$\frac{\mu^n(1+\kappa_n)}{\beta_1\beta_4}(\beta_1(\beta_4 b_2 - \beta_2 b_3) + \beta_3(\beta_4 c_2 - \beta_2 c_3)).$$

We choose β_i so that it equals K_1. The coefficient z is

$$\frac{\mu^{2n}(1+\kappa_n)}{\beta_1\beta_4}(\nu^{-n}\beta_2(\beta_4 b_2 - \beta_2 b_3) + \mu^{-n}\beta_4(\beta_4 c_2 - \beta_2 c_3))$$

Notice that, by Remark 2, $(\beta_4 b_2 - \beta_2 b_3) = -v(\beta_4 c_2 - \beta_2 c_3)$. Therefore, the coefficients of y and z vanish at the same time.

- Choosing ε properly we make the free term equal K_2: we take ε so that

$$\frac{\mu^{2n}(1+\kappa_n)}{\beta_1\beta_4}(\beta_4\varepsilon + p_n) = K_2.$$

In this case ε is of order μ^{-n}.

- The coefficients of y^2, yz and z^2 converge to constants as n goes to infinity. We choose β_j so that the coefficient of y^2 equals 1.

As for the third component, $\bar{G}_n^{3,L}$, we have:

$$\begin{aligned}\bar{G}_{\varepsilon,n}^{3,L} =& \frac{\mu^{2n}(1+\kappa_n)}{\beta_1\beta_4}(-\beta_3 G^2 + \frac{\nu^n}{\mu^n}\beta_4 G^3 + q^n)\circ\Psi_n = \\ & \frac{\mu^n\nu^n(1+\kappa_n)}{\beta_1\beta_4}(-\frac{\mu^n}{\nu^n}\beta_3 G^2 + \beta_4 G^3 + \frac{\mu^n}{\nu^n}q^n)\circ \\ & \left(\lambda^n(\delta_n x+1),\ \frac{\beta_1 y}{\mu^n} + \frac{\beta_2 z}{\nu^n} + 1,\ \frac{\beta_3 y}{\mu^n} + \frac{\beta_4 z}{\mu^n} + \eta_z\right).\end{aligned} \tag{16}$$

- The coefficient of y equals $\nu^n\beta_4(c_3\beta_4 + b_3\beta_1 + h_n^y)$, where h_n^y converges to zero with the growth of n. We choose β_4 and β_1 so that this expression equals zero.

- The coefficient of z equals $\nu^n(c_3\beta_3 + h_n^z)$, where h_n^z converges to zero. We take $\beta_3 \neq 0$.

- The free term converges to a constant times ν^n. Denote this constant by k_4.

- Coefficients of the higher order terms go to zero when n grows.

This completes the proof of Lemma 2 in the linearizable case.

To show that Lemma 2 implies Theorem 4 in the linearizable case we use arguments similar to [32]. Let $K_1 = 0$ in the Lemma above. In this case $\kappa_n = \kappa$ is a constant. Write the locally invariant surface as a graph $z(x, y)$ over the XY-plane. Notice that dependence of the main term in the third component $\bar{G}_n^{3,L}$ on x tends to zero as n goes to infinity. Therefore, the equation for an invariant surface is of the form:

$$\begin{aligned} x' &= \bar{G}_n^{1,L}(x,y,z(x,y)) = y + h_0(x,y,z(x,y)) \\ y' &= \bar{G}_{\varepsilon_n,n}^{2,L}(x,y,z(x,y)) = y^2 + K_1(y + k_0 z) + K_2 + (k_1 y + k_2 z)z + h_1^n(x,y,z(x,y)) \\ z'(x',y') &= \bar{G}_n^{3,L}(x,y,z(x,y)) = \nu^n(k_3 z + k_4 + \nu^{-n}h_2^n(x,y,z(x,y))),\end{aligned}$$

where all h_0^n, h_1^n, and $\nu^{-n}h_2^n$ tend to zero with any finite number of its derivatives. It implies that the graph $z(x, y)$ tends to the plane $\{k_4 + k_3 z = 0\}$, where k_3 is assumed to be nonzero. Thus, the locally invariant surface $S_{\varepsilon_n,n}$ has to be a C^{2r}-small perturbation of this plane and the C^{2r} error tends to zero as n goes to infinity. Substituting $z \approx -k_4/k_3$ we get that $y^2 + K_2 + k_1 yz + k_2 z^2 = y^2 + (K_2 - k_0 k_4/k_3) + (K_1 - k_1 k_4/k_3)y + k_2 k_4^2/k_3^2 +$ *error*, where *error* tends to zero as n tends to infinity. Choosing $K_1 = k_1 k_4/k_3$ we get that on the locally invariant surface $S_{\varepsilon_n,n}$ the restriction of $\Phi_{\varepsilon_n,n}^{2,L}$ converges to $y^2 + K$, $K = K_2 - k_0 k_4/k_3 + k_2 k_4^2/k_3^2$.

Now we justify the smoothness of the surface $S_{\varepsilon_n,n}$. Notice that the strong expansion in the OZ-direction, whose coefficient exponentially tends to infinity as $n \to \infty$, implies that $S_{\varepsilon_n,n}$ is k_n-normally hyperbolic. Fix any $r \geq 1$. Then for some $n_0 = n_0(r)$ and any $n > n_0$ normal hyperbolicity implies that a saddle fixed point p_{ε_n} is $2r$-unstable, i.e. the expanding eigenvalues satisfy $\mu^{2r} < \nu$. By the theorem on the invariant central manifold (see e.g. [34], Thm. III 2), we have that $S_{\varepsilon_n,n}$ is C^{2r} smooth. This, along with Hirsch-Pugh-Shub theory [12], implies that the surface $S_{\varepsilon_n,n}$ depends C^{2r}-smoothly on the parameter $\varepsilon_n \in I_n$. This proves Theorem 4 in the linearizable case.

4.2. **Non-linearizable case.** Without the linearizability assumption the idea is the following. Recall that the resonant term $\rho_n = f^n - L^n$ can be bounded by Lemma 1.

We define

$$\Psi_n^s(x,y,z) = (x, \mu^{-n}y, \nu^{-n}z), \quad \Psi_n^u(x,y,z) = f^n \circ \Psi_n^s.$$

With this notation, $f^n = \Psi_n^u \circ (\Psi_n^s)^{-1}$. Define Ψ_n and R_n by formulas (9) and (10). The renormalizations of our system look like:

$$\begin{aligned}\Phi_{\varepsilon,n}^2 =& R_n^{-1}\circ f^n \circ f_\varepsilon^N \circ R_n = \Psi_n^{-1}\circ(\Psi_n^s)^{-1}\circ f_\varepsilon^N \circ f^n \circ \Psi_n^s \circ \Psi_n = \\ & \Psi_n^{-1}\circ(\Psi_n^s)^{-1}\circ f_\varepsilon^N \circ (L^n + \rho_n)\circ\Psi_n^s\circ\Psi_n := (\bar{G}_n^1, \bar{G}_{\varepsilon,n}^2, \bar{G}_n^3).\end{aligned} \tag{17}$$

Notice that if $\rho_n = 0$, then this formula coincides with (11). Hence it is enough to estimate the C^{2r}-norm of $\rho_n \circ \Psi_n^s \circ \Psi_n$. We compute:

$$\Psi_n^s \circ \Psi_n = \left(\delta_n x + 1,\ \frac{\beta_1 y}{\mu^{2n}} + \frac{\beta_2 z}{\mu^n \nu^n} + \frac{1}{\mu^n},\ \frac{\beta_3 y}{\mu^{2n}} + \frac{\beta_4 z}{\mu^{2n}} + \frac{\eta_z}{\nu^n}\right). \tag{18}$$

equation

Note that each derivative of $\Psi_n^s \circ \Psi_n$ w.r.t. y brings a factor of μ^{-2n}, and a derivative w.r.t. z brings a factor of $\mu^{-n}\nu^{-n}$. Direct calculation shows that the estimates of the derivatives of ρ_n from Lemma 1 are sufficient to prove a C^{2r} vanishing of any term from $\xi\rho_n^\xi \circ \Psi_n^s \circ \Psi_n$. This completes the proof of the Lemma. □

A direct application of Lemma 1 shows that the corresponding nonlinear terms vanish. The derivation of Theorem 4 from Lemma 2 in the non-linearizable case is completely analogous to the one in the linearizable case. Thus, the proof of Theorem 4. □

Theorem 4 implies the following.

Corollary 1. *[Choice of the parameters providing $2r$-unstable saddles (Step 1)] With the notations of Theorem 4, for any positive r, there exists a sequence of subintervals $I_n' \subset I_n$ in the parameter space such that if n is sufficiently large and $\varepsilon \in I_n'$, then f_ε has an $2r$-unstable periodic saddle $p_\varepsilon = f_\varepsilon^{n+N}(p_\varepsilon)$. Moreover, for some $\varepsilon_n^* \in I_n'$, f has a reduced homoclinic tangency associated to an $2r$-unstable saddle $p_{\varepsilon_n^*}$. More precisely,*

- *$p_{\varepsilon_n^*}$ is multiplicatively single-resonant $2r$-unstable;*
- *for ε near ε_n^* in I_n' we have that $S_{\varepsilon,n}$ coincides with the local center-stable manifold of p_ε under f_ε^{n+N};*
- *for n sufficiently large the rescaled surface $R_n^{-1}(S_{\varepsilon,n})$ C^{2r}-tends to a plane parallel to the XY-plane, which permits us to write $\Phi^2_{\varepsilon_n,n}$ in xy-coordinates, where $\varepsilon_n \in I_n$;*
- *for some parameter ε_n^+ (resp. ε_n^-) in I_n' the surface map $f_\varepsilon^{n+N}|_{S_{\varepsilon,n}}$, i.e. the map f_ε^{n+N} restricted to the surface $S_{\varepsilon,n}$, has a homoclinic tangency associated with $p_{\varepsilon_n^\pm}$ of inner (resp., outer) type, and generically unfolds the tangency [6].*

Proof. Choose $K = -2$ in the above theorem. Then the restriction $\Phi^2_{\varepsilon_n(-2),n}$ is C^{2r}-close to the quadratic map $(x,y) \mapsto (y, y^2 - 2)$, $x,\ y \in [-2,2]$. This map has a fixed point $(2,2)$ with eigenvalues 4 and 0. Therefore, for a large enough n and $\varepsilon = \varepsilon_n(-2)$, $\Phi^2_{\varepsilon,n}$ has a saddle near $(2,2)$ on the surface $R_n^{-1}(S_{\varepsilon_n,n})$ with eigenvalues close to 4 and 0+. Using the standard technology varying K and the corresponding $\varepsilon = \varepsilon_n(K)$ we can get a saddle p_ε exhibiting a homoclinic tangency on the surface $S_{\varepsilon,n}$ of either inner or outer type (see [27] sect 6.3). Notice that a homoclinic tangency of the restriction to an invariant surface implies a homoclinic tangency of f_ε itself. The eigenvalues of f_ε^{n+N} at its fixed point p_ε inside $S_{\varepsilon,n}$ are bounded. The surface $S_{\varepsilon,n}$ is k_n-normally hyperbolic, where k_n can be arbitrary large provided n is large. Therefore, by volume-preservation, for any prescribed r and $n = n(r)$ large enough, p_ε is $2r$-unstable. □

Now we show that Corollary 1 implies Theorem 1.

Proof of Theorem 1. By Corollary 1, we have a sequence of disjoint intervals I_n' near $\varepsilon = 0$ and values $\varepsilon_n \in I_n'$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that f_{ε_n} has a $2r$-unstable periodic saddle p_{ε_n} of period $n + N$ with a quadratic homoclinic tangency.

[6]As we mentioned above, both $S_{\varepsilon,n}$ and the restriction $f_\varepsilon|_{S_{\varepsilon,n}}$ depend C^{2r}-smoothly on the parameter ε.

By Corollary 1, the local central-stable manifold $W^{cs}(p_{\varepsilon_n})$ is locally invariant and coincides with the surface $S_{\varepsilon_n,n}$. By Theorem 4, the surface $S_{\varepsilon_n,n}$ is C^{2r} smooth. Therefore, the family of restrictions $f^{n+N}_{\varepsilon_n}|_{S_{\varepsilon_n,n}}$ is C^{2r}-smooth.

Consider $f^n \circ f^N_{\varepsilon_n}$. Denoting $\tilde{y}_n = \mu^n y - 1$, we write this mapping in the form:

$$\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} 1 + a_1\lambda^n(1+x) + b_1\tilde{y}_n + c_1\nu^n\tilde{+}\tilde{H}_1(\lambda^n(1+x), \tilde{y}_n, \nu^n\tilde{z}) \\ a_2\lambda^n(1+x) + b_2\tilde{y}_n + c_2\nu^n\tilde{z} + \tilde{H}_2(\lambda^n(1+x), \tilde{y}_n, \nu^n\tilde{z}) \\ a_3\lambda^n(1+x) + b_3\tilde{y}_n + c_3\nu^n\tilde{z} + \tilde{H}_3(\lambda^n(1+x), \tilde{y}_n, \nu^n\tilde{z}) \end{pmatrix}. \tag{19}$$

Notice that there is exponentially large expansion in the OZ-direction. It implies that the local center-stable manifold $W^{cs}(p_{\varepsilon_n})$ is 2-dimensional. Since the restriction $f^{n+N}_{\varepsilon}|_{S_{\varepsilon_n,n}}$ generically unfolds a homoclinic tangency, there exists an open set $\tilde{I}_n \subseteq I'_n$ of parameters such that the restriction $f^{n+N}_{\varepsilon_n}|_{S_{\varepsilon_n,n}}$ has a persistent homoclinic tangency. Notice that a homoclinic tangency of the restriction $f^{n+N}_{\varepsilon}|_{S_{\varepsilon_n,n}}$ implies a homoclinic tangency of f itself. Therefore, we can apply the standard 2-dimensional arguments (see, e.g., [32] sect. 3.2, the proof of Theorem A or [27] sect. 6.3) to show that f_{ε_n} itself has a persistent homoclinic tangency. This completes the proof of Theorem 1.

□

5. **Return maps in a neighborhood of a homoclinic tangency.** Due to Corollary 1, after a C^r-small C^∞-perturbation we can start with a reduced homoclinic tangency (see definition 4). Let U and U' be sufficiently small neighborhoods of points of the homoclinic tangency q and q', respectively. Choose N so that $f^N(q') = q$ (see Fig. 1). Denote by $W^u_{\text{loc}}(p)$ the first connected component of the intersection $W^u(p) \cap U$. Below we shall use the coordinate systems in U and U', induced by the normal coordinates from the previous section.

A parallelepiped in U (resp., U') is called *right parallelepiped* if its sides are parallel to the coordinate planes.

Let f^k have a reduced homoclinic tangency associated to a $2r$-unstable periodic saddle $p = f^k(p)$ with a smooth 2-dimensional surface $S \subset W^{cs}(p)$ being locally invariant. Choose normal coordinates (3). Since the homoclinic tangency of f^k is reduced, f^k restricted to S, has a homoclinic tangency of inner type.

Proposition 1. *With the above notations, choose* c *and* c′ *sufficiently large. Put* $\delta_n = \mu^{-n}$. *For* n *sufficiently large consider the right parallelepiped* T_n *centered at* $(1, \delta_n, 0)$ *with edges* $2\delta_n^{1/2} \times c\delta_n^{3/2} \times c\nu^{-n}\delta_n^{1/2}$. *Then the image* $f^n(T_n)$ *is exponentially close to the right parallelepiped centered at* $(\lambda^n, 1, 0)$ *with edges* $2\delta_n^{1/2}\lambda^n \times c\delta_n^{1/2} \times c\delta_n^{1/2}$ *with corresponding directional errors bounded by* $(c'\lambda^{2n}(n+1)^3, c'\mu^n\lambda^n(n+1)^3, c'\nu^n\lambda^n(n+1)^3)$ *(see Fig. 4).*

Moreover, T_n *and* $f^{n+N}(T_n)$ *form a horseshoe with an* $(n+N)$*-periodic saddle* $\tilde{p}$, *and* $f^{n+N}(T_n)$ *is at most* $C\delta_n^{2r}$*-distant from* $W^s(p)$, *where* C *is a constant, independent of* n. *It is equivalent to the fact that* $f^{-N}(T_n)$ *and* $f^n(T^n) = T'_n$ *form a horseshoe (see Fig. 5).*

Proof. The first statement is proved by a computation with the normal form: by Lemma 1 we show that for the initial data in T_n, the first n iterates of f stay C^0-close to those of its linear part, see (4). Now recall that $pol_0(n)$ is of degree 3 and, therefore, bounded from above by $c'(n+1)^3$.

Notice that the restriction of (3) to the XY-plane gives a linear map. Thus, the second (horseshoe) part of Proposition follows from the corresponding planar

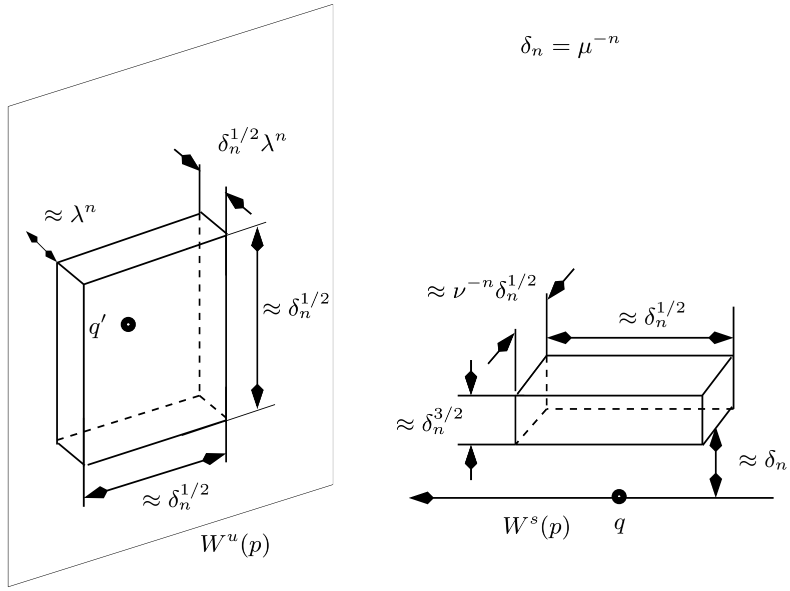

FIGURE 4. Parallelepipeds in neighborhoods of homoclinic tangencies.

statement about existence of a horseshoe (see, e.g., Prop.1 [16]). This proves the Proposition. □

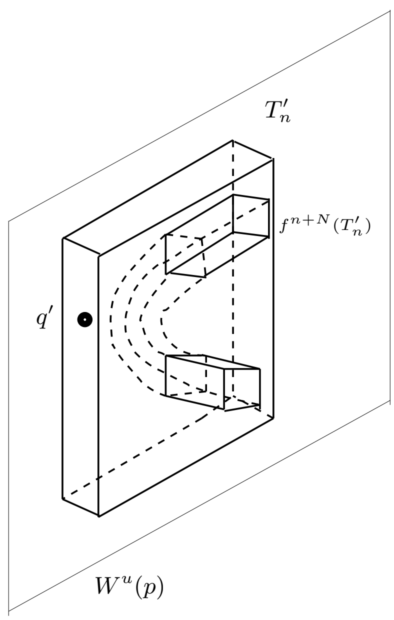

FIGURE 5. Horseshoe in U'.

6. **Construction of a tower of horseshoe saddles (Step 2).** In this section we adapt the idea of Gonchenko-Shilnikov-Turaev [8] of constructing a tower of heteroclinic connections in a neighborhood of a homoclinic tangency. The details of the construction are similar to those from [16] sect. 2.4. Additional complications are due to 3-dimensionality and volume-preservation. Construction of a tower is the second step in the proof of Theorem 2.

Definition 5. An m-floor tower is a collection of m saddle periodic points $p_1, \dots, p_m$ (possibly of different periods) with 1-dimensional stable manifolds $W^s(p_1), \dots, W^s(p_m)$ and 2-dimensional unstable manifolds $W^u(p_1), \dots, W^u(p_m)$, respectively, such that $W^s_{loc}(p_i)$ is tangent to $W^u_{loc}(p_{i+1})$ for $i = 1, \dots, m-1$ and $W^u_{loc}(p_m)$ intersects $W^s_{loc}(p_1)$, where $W^s_{loc}(p_i)$ (resp., $W^u_{loc}(p_i)$) denotes the connected component of the intersection of $W^s(p_i)$ (resp., $W^u(p_i)$) with a neighborhood of p_i which contains p_i.

Let $f \in C^\infty$ have a reduced homoclinic tangency of a $2r$-unstable saddle periodic point $p = f^k(p)$. Let f^k have a locally invariant 2-dimensional surface S, which is contained in the central-stable manifold $W^{cs}(p)$ of p as in definition 4. Suppose f^k, restricted to S, has a homoclinic tangency of inner type at some point q. Existence of such f near any C^r diffeomorphism with a homoclinic tangency associated to a saddle with real eigenvalues follows from Corollary 1.

In this section we prove the following

Lemma 3. *With the above notations, for any positive integer m there is a C^r-small C^∞-perturbation $\tilde f$ of f such that $\tilde f$ has an m-floor tower. If q is a point of homoclinic tangency of f, then the aforementioned tower of $\tilde f$ is located in a neighborhood U of q.*

Only for convenience of drawing pictures we shall construct a tower in a neighborhood U' of $q' = (0, 1, \eta_z)$ (see Fig. 1). Once it is constructed, it certainly implies the existence of a tower in a neighborhood U of $q = (1, 0, 0)$. For determines we shall construct a tower with saddles whose y-coordinate is smaller than 1, or saddles $p_1, p_2, \dots$ are located "below" the point of the homoclinic tangency q'.

Proof: We prove the above Lemma using the standard localized perturbation technique. As usually, consider the normal coordinates for a nonresonant saddle p. Induce the set of coordinates in a neighborhood U (resp., U') of q (resp., q') by normal coordinates for the point p and the diffeomorphism f (see Fig. 1). Application of Proposition 1 gives existence of a horseshoe (see Fig. 5). Now consider simultaneously several horseshoes in U', e.g., obtained by taking preimages f^{-N} of the rectangular parallelepipeds $\{T_{n_i}\}_i$ located in U and intersecting them with the corresponding images $\{T'_{n_i} = f^{n_i}(T_{n_i})\}_i$. We shall prove below that an application of Proposition 1 and an appropriate choice of n_i's guarantee the existence of the contour described on Fig. 5 in the case $m = 3$. Indeed, consider an increasing sequence of numbers $n_1, \dots, n_m$ such that for each $i = 1, \dots, m-1$ the following two properties hold:

1) Let T_{n_i} (resp., T'_{n_i}) denote the rectangular parallelepiped, defined in Proposition 1 (resp., the image of T_{n_i} under f^{n_i}) in U (resp., U'). By construction T_{n_i} intersects $f^{n_i+N}(T_{n_i})$ (resp., $f^N(T'_{n_i})$);

2) n_{i+1} is the largest number such that $T_{n_{i+1}}$ and $f^{n_i+N}(T_{n_i})$ intersects in an open set.

For each $i = 1, \dots, m$, by Proposition 1, the return map f^{n_i+N} has a horseshoe. It implies existence of at least one saddle point $p_i \in T_{n_i}$ (resp., $p'_i = f^{-N}(p_i) \in T'_{n_i}$)

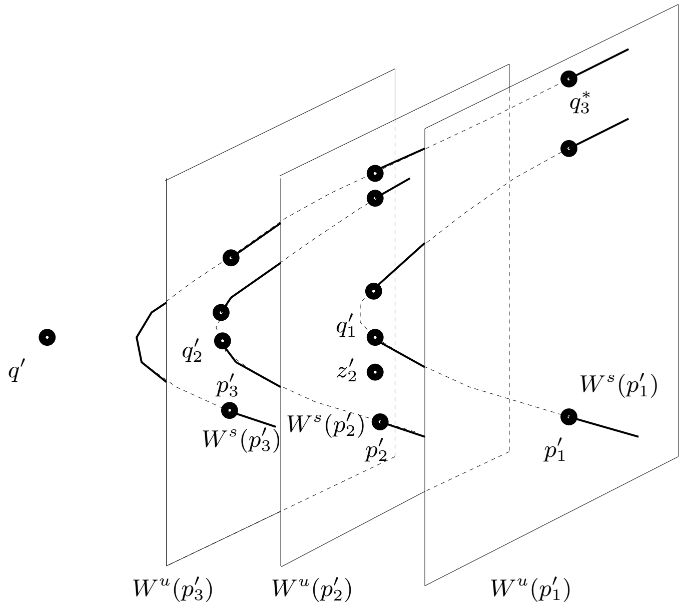

FIGURE 6. An uncompleted 3-floor tower.

of period $n_i + N$. However, we need additional information about the position of local stable and unstable manifolds of p_i (resp., p_i'). To obtain such a geometric information we study formulas for the renormalized return map (17) (see also Lemma 2) along with the diagram on Fig. 3 for $\varepsilon = 0$. Notice that the renormalization (11) of the return map f^{n+N}, by Theorem 4, last item, for $K > 1$ has a fixed point, denoted p_n, near the origin. Differentiating $\Phi^2_{\varepsilon_n,n}$ near the origin we see that local unstable manifold $W^u_{loc}(f^{-N}P_n')$ is almost parallel to the YZ-plane and local stable manifold $W^u_{loc}(P_n)$ is almost parallel to the OX-axis. We denote p_{n_i}' and p_{n_i} by p_i and p_i' respectively. For the construction below we need to keep in mind the relative position of both collections $\{T_{n_i}\}_{i=1}^m$ and $\{T_{n_i}'\}_{i=1}^m$ and the fact that f^N distorts distances by a bounded factor.

Recall that U and U' are equipped with normal coordinates. Define the maximal distance between $W^s_{loc}(p_i')$ and $W^u_{loc}(p_i')$, denoted by s_i', as the maximum of distance between any two points $x \in W^s_{loc}(p_i')$ and $y \in W^u_{loc}(p_i')$. Denote the distance between centers of T_{n_i} and $T_{n_{i+1}}$ (resp., T_{n_i}' and $T_{n_{i+1}}'$) by t_i (resp., t_i') (see Fig. 7). Since the horizontal width of T_n' is $\approx \mu^{-n/2}\lambda^n$, which is significantly smaller than the distance to the origin λ^n, $s_i' - t_i'$ is an approximate distance by which one should move $W^u_{loc}(p_i)$ to create heteroclinic tangency with $W^s_{loc}(p_{i+1})$. By Proposition 1 we get $t_i = \mu^{-n_i} - \mu^{-n_{i+1}}$. By the fact that f^N distorts distances only by a finite amount one can see that $\mu^{-n_{i+1}} \approx \lambda^{n_i}$. Now recall that the saddle p with a homoclinic tangency is $2r$-unstable and volume-preserving. It implies $\lambda < \mu^{-(2r+1)}$. Intuitively it means that for each $i = 1, \dots, m-1$ the rectangular parallelepipeds T_{n_i} and $T_{n_{i+1}}$ (resp., T_{n_i}' and $T_{n_{i+1}}'$) are well spaced one from the next one in the neighborhood U of q (resp., U' of q'. It allows us by a C^r-small C^∞-perturbation to create an m-floor tower as follows.

Proposition 2. *Consider a homoclinic tangency associated to a $2r$-unstable saddle p. Then the ratio $(s'_i - t'_i)(t'_i)^{-2r}$ is arbitrarily small for each $i = 1, \dots, m-1$.*

Proof. Let us use notations and quantitative estimate obtained in Proposition 1. Let p be $2r$-unstable. Recall that the rectangle parallelepiped T_n is centered at $(1, \delta_n = \mu^{-n}, 0)$ and has edges $3c\mu^{-n/2} \times 2c\mu^{-3n/2} \times \nu^{-n}\mu^{-n/2}$. Notice that y-edge is much smaller than μ^{-n}, and y-coordinate of the center is μ^{-n}. Another way to see this: μ^{-n} is the distance of a homoclinic tangency q to T_n. Also λ^n is the distance of a homoclinic tangency $q' = f^{-N}(q)$ to $T'_n = f^{-N}(T_n)$. By condition 2), $f^{n_i+N}(T_{n_i}) = f^N(T'_{n_i})$ intersects $T_{n_{i+1}}$. The fact that f^N distorts distances only by a bounded factor implies that $\mu^{-n_{i+1}} \approx \lambda^{n_i}$. Since p is $2r$-unstable we have $\lambda^n < \mu^{-(2r+1)n}$. It implies that $s'_i - t'_i < \mu^{-n_{i+1}} + \lambda^{n_i}\mu^{-n_i} < 2\mu^{-n_i} < C\mu^{-(2r+1)n_i} < \varepsilon\mu^{2rn_i} = \varepsilon(t'_{n_i})^r$, where ε can be chosen as small as we like by choosing n_1 sufficiently large (see Fig. 6 and Fig. 7). □

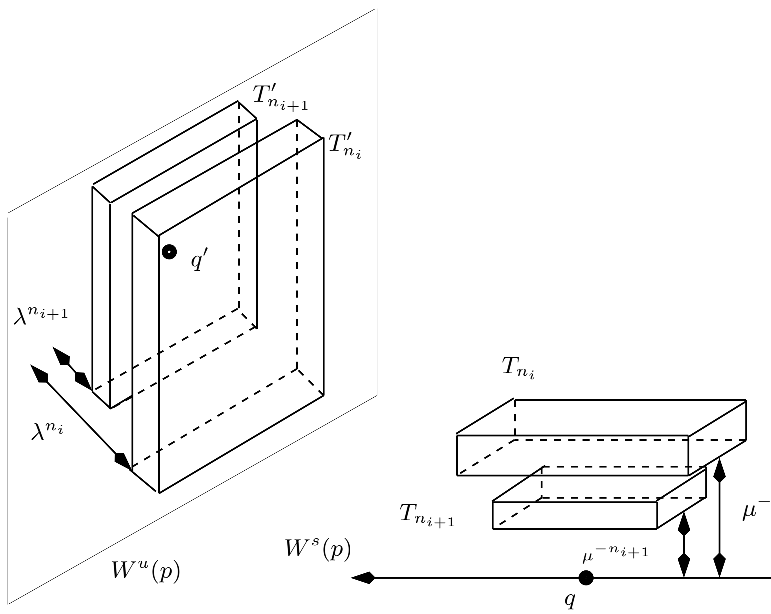

FIGURE 7. Layered rectangular parallelepipeds as building blocks for a tower.

Fix a point $\tilde{p}$ and a neighborhood $\tilde{U}$ of $\tilde{p}$ in 3-dimensional space $\mathbb{R}^3$. Fix a small positive number ρ and local coordinates (x, y, z) in $\tilde{U}$ so that $\tilde{p}$ is at zero and U' has diameter 1. We now construct a C^∞-smooth family $\{\phi_\varepsilon\}$ of volume-preserving self-maps in a neighborhood of $\tilde{p}$ such that this family moves the image of the horizontal plane $L_x = \{x = 0\}$ in the 3ρ-neighborhood of the origin. Consider cylindric coordinates $(x, y, z) = (\theta, r, z)$, where $x = r\cos\theta,\ y = r\sin\theta$. The family $\{\phi_\varepsilon\}_\varepsilon$, in local coordinates, is given as follows:

$$\phi_\varepsilon(\theta, r, z) = (\theta + \varepsilon\tau(r, z), r, z),$$

where $\tau : \mathbb{R}_+ \to \mathbb{T}$ is a C^∞ smooth function satisfying the following properties: it identically vanishes for $r \in (0, \rho] \cup [3\rho, +\infty)$ and any z, for $|z| > 3\rho$ and any r, and for $\tau(2\rho, 0) = \rho^{r+1}$. Under these conditions, the C^r-norm of ϕ_ε can be chosen

ρ-small. It is easy to see that ϕ_ε is volume-preserving for any ε, and the C^r-norm of ϕ_ε is of order of 1, independently of how small ρ is. Notice also that the image of $(0,2\rho,0)$ is $\phi_\varepsilon(0,2\rho,0)=(\varepsilon\rho^{r+1},2\rho,0)$. Call such a perturbation *axe symmetric twisting.*

Proposition 3. *Let z be a point on $W^u_{loc}(p'_i)$ which is equidistant from both $W^s_{loc}(p'_i)$ and $W^s_{loc}(p'_{i+1})$. If the ratio $(s'_i-t'_i)/(t'_i)^{2r}$ is sufficiently small, then there exists a C^r-small C^∞-perturbation inside of the ball B centered at z'_i of radius $t'_i/3$ (see Fig. 6 for location of z'_2) such that $W^s_{loc}(p'_{n_{i+1}})$ and $W^u_{loc}(p'_{n_i})$ have a point of a heteroclinic tangency.*

Proof. Let ε be a sufficiently small positive number. Using axe symmetric twisting inside of the ball B of radius $t'_i/3$ with an appropriately chosen axis one can find an ε C^r-small volume-preserving axe symmetric twisting that "lifts $W^u_{loc}(p'_{n_{i+1}})$ up" by $\approx\varepsilon(t'_i)^{2r}$ and creates a heteroclinic tangency with $W^s_{loc}(p'_{n_i})$. □

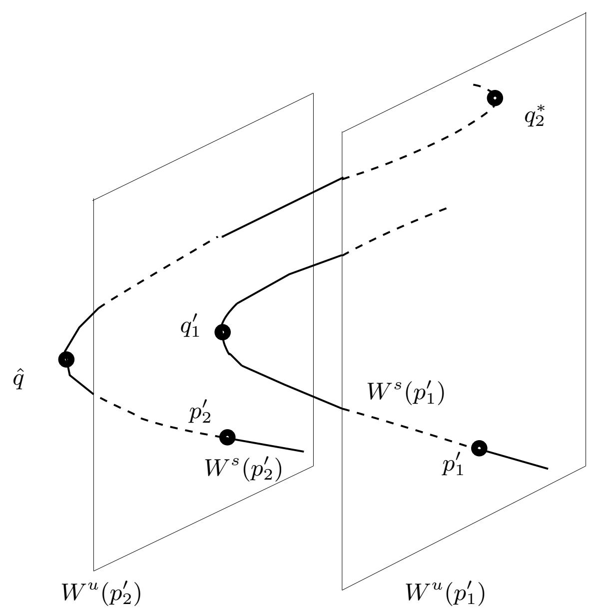

FIGURE 8. A localized perturbation for a floor of a tower.

7. **Construction of higher order tangencies using towers of horseshoe saddles (Step 3).** Before we start proving the existence of higher order tangencies we need a few definitions.

Definition 6. Let $r\le m\le 2r$. We say that a C^{2r}-smooth 1-dimensional curve $\Gamma\subset\mathbb{R}^3$ (resp., 3-dimensional manifold) has an m-th order tangency with a smooth 2-dimensional surface Π if there is a C^m curve lying in Π and having an m-th order tangency with Γ.

Fix $2r\ge m\ge r>1$. Consider a nonresonant fixed point $p=g(p)$ in the plane with real eigenvalues $1<\mu<\nu$ such that $\mu^{2r}<\nu$. It has a 1-dimensional strong (resp., weak) unstable manifold $W^{su}(p)$ (resp., $W^{wu}(p)$) tangent to the eigenvector with eigenvalue ν (resp., μ). Notice, however, that $W^{wu}(p)$ is not uniquely defined.

Lemma 4. *With the above notations, let $q \neq p$ and $q \notin W^{su}(p)$, and let γ be a C^∞ curve through q, transversal to the strong stable direction at q, $r \leq m$. Then after a C^r-small C^∞-perturbation away from a neighborhood* [7] *of p there exists a C^∞-smooth 1-dimensional invariant manifold $W^{wu}(p)$, which is m-tangent to γ at q.*

Proof. Since p is a nonresonant source, g has linearizing coordinates. Let OX (resp., OY) be the weak (resp., strong) stable direction. Locally we can write the curve γ as a graph $y_0(x)$ with $(x_0, y_0(x_0)) = q$. This is possible because γ is transversal to OY direction. Since γ is C^∞ smooth, we can write $y_0(x) = a_1x + a_2x^2 + \cdots + a_mx^m + r(x)$, where $r(x)$ vanishes at zero up to the order $m+1$. Consider the evolution of γ under backward iterations. The (pre)image of q is $g^k(q) = (\mu^k x_0, \nu^k y_0(x_0))$. We write locally the (pre)image of γ as a rescaled graph $g^m(\gamma)$ as follows: Start with $g^{-k}(x, y_0(x)) = (\mu^{-k}x, \nu^{-k}(a_1x + a_2x^2 + \cdots + a_mx^m + r(x))$ and rescale the first component: we have $x_k = \mu^{-k}x$

$$g^{-k}(\gamma) = \left(x_k, a_1 \left(\frac{\mu}{\nu}\right)^k x_k + \cdots + a_m \left(\frac{\mu^m}{\nu}\right)^k x_k^m + \nu^{-k} r(x_k) \right).$$

Distance of $g^{-k}(q)$ to the origin tends to zero as μ^{-k}. The first m-coefficients of γ_k tend to zero exponentially, too. The slowest one to tend to zero — the m-th derivative is of order $(\mu^m/\nu)^k$. To make a perturbation in a neighborhood of $g^{-k-1}(q)$ without affecting $g^{-k}(q)$ and $g^{-k-2}(q)$, one needs to choose a neighborhood of size $(\mu-1)\mu^{-m-2}$. If we make a C^r-small C^∞-perturbation of order small $\tau > 0$ inside of such a neighborhood, then its size should be of order $\tau\mu^{-rk}$. If $(\mu^m/\nu)^k < \tau\mu^{-rk}$ or $\mu^{m+r} < \nu$ and m is large, then such a perturbation is possible. Therefore, for k sufficiently large after a C^r-small C^∞-perturbation of the coordinate system one can create m-th order tangency of $g^{-k}(\gamma)$ and the perturbed OX-axis. This completes the proof of the Lemma. □

Remark 3. Suppose that we have a m-th order homoclinic tangency of the 1-dimensional stable and 2-dimensional unstable manifolds of a $2r$-unstable saddle p with $r \leq m\varepsilon 2r$. By a C^r-small C^∞-perturbation one can satisfy hypothesis of Lemma 4. Therefore, Lemma 4 implies that after a C^r-small C^∞-perturbation (if necessary), one can choose C^r-coordinates in a neighborhood of p so that the OX-axis coincides with the stable manifold $W^s_{loc}(p)$, the OZ-axis coincides with the strong unstable manifold $W^{su}_{loc}(p)$, the OY-axis locally coincides with a weak unstable manifold, which is m-tangent to the stable one at some q.

We assume without loss of generality that $q = (1, 0, 0) \in W^s_{loc}(p)$ is a point of the orbit of an m-th order homoclinic tangency, and $q' = (0, 1, 0)$ lies on $W^{wu}_{loc}(p)$. Moreover, $W^{wu}_{loc}(p)$ has an m-th order homoclinic tangency with $W^s(p)$. In this case we shall say that the tangency has a *geometric preliminary normal form.*

Following the same strategy as in sect. 2.5 of [16], we start with an m-floor tower. Denote by q'_i heteroclinic tangencies of $W^s_{loc}(p'_i)$ and $W^u_{loc}(p'_{i+1})$. Then we construct m consecutive perturbations as follows. We start with $W^s_{loc}(p_{m-1})$ tangent to $W^u_{loc}(p_m)$ at some q'_{m-1} and $W^s_{loc}(p_m)$ crossing $W^u_{loc}(p_1)$ at some q^*_m. After the first perturbation, localized in a neighborhood of q'_{m-1}, we obtain a quadratic heteroclinic tangency of $W^u(p_{m-1})$ and $W^s_{loc}(p_1)$ at some q^*_{m-1}, where q^*_{m-1} is close to q^*_m. After the second perturbation, localized in two neighborhoods of q'_{m-2}

[7] Size of such a neighborhood depends on γ, μ, and ν.

and q^*_{m-1}, resp., we obtain a third order heteroclinic tangency of $W^u(p_{m-2})$ and $W^s_{loc}(p_1)$ at some q^*_{m-2}, where q^*_{m-2} is close to q^*_{m-1}. After the third perturbation, localized in two neighborhoods of q'_{m-3} and q^*_{m-1}, resp., we obtain a 4-th order heteroclinic tangency of $W^u(p_{m-3})$ and $W^s_{loc}(p_1)$ at some q^*_{m-3}, and so on. Finally, after the $(m-1)$-st perturbation, we obtain an m-th order homoclinic tangency of $W^s(p_1)$ and $W^u(p_1)$ (see Fig. 8).

Consider saddles $p_i = f^{n_i+N}(p_i)$ of the return maps. By Corollary 1 and choice of the saddles p_i's, for n_1 sufficiently large, all of them are $2r$-unstable.

It suffices to prove that if $W^s(p_2)$ has a tangency of order $(m-1)$ with $W^u_{loc}(p_1)$ at some q^*_2, and $W^s_{loc}(p_1)$ has a quadratic tangency with $W^u_{loc}(p_2)$, then by a C^r-small C^∞-perturbation localized in neighborhoods of q^*_2 and q'_1 we can create a tangency of order m of $W^s(p_1)$ and $W^u_{loc}(p_1)$ near q^*_2 (see Fig. 7). As above we assume that the saddles have 1-dimensional stable and 2-dimensional unstable manifolds and $\mu^{m+r} < \nu$. Consider a neighborhood U_{p_2} of p_2 with coordinate system (x, y, z) centered at p_2, and assume that f is written in the normal form from Lemma 1 in these coordinates. Moreover, assume that the OX-axis coincides with $W^s_{loc}(p_2)$, the OZ-axis—with $W^{su}_{loc}(p_2)$, the OY-axis coincides with $W^{wu}_{loc}(p_2)$, and at the point q^*_2 this manifold is $(m-1)$-tangent to $W^s_{loc}(p_1)$. Lemma 4, this can be done by a C^r-small C^∞-perturbation, because the saddle is $m+r$-unstable. Recall that q'_1 denotes the point of a quadratic tangency between $W^u(p_1)$ and $W^s(p_2)$. Denote the preimage of q^*_2 that lies in U_{p_2} by $\hat{q} = f^{-N}(q^*)$ for some N. By Remark 3, there is a C^r-small C^∞-perturbation away from a neighborhood of p'_2 such that we can assume that $q_1 = (1, 0, 0)$, $\hat{q} = (0, 1, 0)$, and $W^s(p_1)$ has an m-th order tangency with the OY-axis. Let U, U^*, and $\hat{U}$ denote small neighborhoods of q_1, q^*_2 and $\hat{q}$, respectively, with coordinate systems, induced by the normal one.

Then $f^N : \hat{U} \mapsto U^*$, expressed in coordinates (x, y, z) has the form:

$$\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} a_1(y-1) + b_1 x + c_1 z + H_1(x, y-1, z) \\ a_2(y-1)^m + b_2 x + c_2 z + H_2(x, y-1, z) \\ a_3(y-1)^m + b_3 x + c_3 z + H_3(x, y-1, z) \end{pmatrix},$$

where for $x = (y-1) = z = 0$ we have

$$\begin{cases} H_i = \partial_1 H_i = \partial_2 H_i = \partial_3 H_i = 0, \quad i = 1, 2, 3, \quad j = 1, 2, \\ \partial_2^j H_i = 0, \quad i = 2, 3, \quad j = 1, \dots, m. \end{cases}$$

The idea of the proof is the the following: we choose a *curve* γ in $W^s_{loc}(p_1)$ that has a quadratic tangency with $W^u_{loc}(p_2)$ at q_1. Then we move $W^s(p_1)$ (and hence, γ) by an Ω-preserving transformation transversally to $W^u_{loc}(p_2)$ by ε. At the same time, we consider a *versal family* with $\varepsilon_0, \dots, \varepsilon_{m-1}$ — scalar and ε_v — functional parameters unfolding a heteroclinic tangency at q^*_2. A part of γ after a number of backward iterations will come to U^*. We are going to show that by changing the above parameters, we can obtain an m-th order tangency between the preimage of γ and $W^u_{loc}(p_1)$ near q^*_2. By construction this is the m-th order homoclinic tangency between the image of $W^s(p_1)$ and $W^u_{loc}(p_1)$.

We start with the linearizable case (supposing that the f^n has a linear diagonal normal form in some neighborhood of p). In the last subsection of this section we make the necessary modifications to treat the general case. This is done with the help of Lemma 1.

7.1. **The versal family.** Let $C > 0$ be a constant, which we shall make precise later, and consider the following deformation $f_\varepsilon^N : \hat{U} \to U^*$ of the above mapping:

$$\begin{pmatrix} x_n^* \\ y_n^* \\ z_n^* \end{pmatrix} = \begin{pmatrix} a_1(1+\varepsilon_v)(y-1) + b_1 x + c_1 z + H_1(x, y-1, z) \\ a_2(y-1)^m + \sum_{i=0}^{m-1} \varepsilon_i (y-1)^i + b_2 x + c_2 z + H_2(x, y-1, z) \\ a_3(y-1)^m + \sum_{i=0}^{m-1} C\varepsilon_i (y-1)^i + b_3 x + c_3 z + H_3(x, y-1, z) \end{pmatrix}.$$

Suppose $b_2 c_3 - b_3 c_2 \neq 0$, i.e., the corresponding 2×2 minor is non-degenerate. This can be achieved by a small perturbation. Recall that we are studying the linear case. Consider a mapping $f_\varepsilon^N \circ f^n : U_n \mapsto U^*$ for large enough n (U_n is an open subset of U, see Proposition 1 for the choice of $U_n = T_n$). We denote $\tilde{y}_n = \mu^n y - 1$, and write this mapping in the form:

$$\begin{pmatrix} x_n^* \\ y_n^* \\ z_n^* \end{pmatrix} = \begin{pmatrix} a_1(1+\varepsilon_v)\tilde{y}_n + b_1 \lambda^n (1+x) + c_1 \nu^n z + H_1(\lambda^n(1+x), \tilde{y}_n, \nu^n z) \\ a_2 \tilde{y}_n^m + \sum_{i=0}^{m-1} \varepsilon_i \tilde{y}_n^i + b_2 \lambda^n(1+x) + c_2 \nu^n z + H_2(\lambda^n(1+x), \tilde{y}_n, \nu^n z) \\ a_3 \tilde{y}_n^m + \sum_{i=0}^{m-1} C\varepsilon_i \tilde{y}_n^i + b_3 \lambda^n(1+x) + c_3 \nu^n z + H_3(\lambda^n(1+x), \tilde{y}_n, \nu^n z) \end{pmatrix}. \tag{20}$$

7.2. **The curve.** Without loss of generality, assume that

$$j := \frac{a_3 b_2 - a_2 b_3}{a_2 c_3 - a_3 c_2} \neq 0.$$

Locally, in the neighborhood U of q, we consider the curve $\gamma := W^u(p_1) \cap \{z = (\lambda/\nu)^n jx\}$. We can parameterize it near q as

$$x = t, \quad y = at^2 + g(t), \quad z = (\lambda/\nu)^n jt. \tag{21}$$

We shall construct a sequence of C^r-small C^∞-smooth Ω-preserving deformations of $W^u_{loc}(p_1)$ so that for each $n \in \mathbb{N}$ the deformed curve $\gamma(\varepsilon(n))$ looks like

$$x = t, \quad y = at^2 + \varepsilon(n) + g(t), \quad z = (\lambda/\nu)^n jt.$$

7.3. **Parameter choice.** The aim is to find parameters $\varepsilon_0(n), \dots, \varepsilon_{m-1}(n)$, $\varepsilon(n)$, the functional parameter $\varepsilon_v(n)$, and t_n in such a way that $\tilde{y}_n(t_n) = 0$,

$$y_n^*(t_n) = \left.\frac{\partial y_n^*}{\partial t}\right|_{t_n} = \dots = \left.\frac{\partial^m y_n^*}{\partial t^m}\right|_{t_n} = z_n^*(t_n) = \left.\frac{\partial z_n^*}{\partial t}\right|_{t_n} = \dots = \left.\frac{\partial^m z_n^*}{\partial t^m}\right|_{t_n} = 0, \tag{22}$$

and the deformed mapping is volume-preserving (this latter is done by the choice of $\varepsilon_v(n)$). Along the proof of it, we also show that $\left.\frac{\partial x_n^*}{\partial t}\right|_{t_n}$ is non-zero.

If we prove the above statement, then the curves $\Gamma(t) = (t, y_n^*(t), z_n^*(t))$ converge to a curve that is tangent at the point q^* to the x_n^*-axis as n tends to infinity.

Now we proceed similarly to [16], Prop. 5. On $\gamma(\varepsilon(n))$ we have:

$$\tilde{y}_n(t) = \mu^n(at^2 + g(t) - (\mu^{-1} - \varepsilon(n))).$$

To satisfy equation $\tilde{y}_n(t_n) = 0$, we need to choose t_n close to $\sqrt{a^{-1}(\mu^{-n} - \varepsilon(n))}$. Denote $\omega_n = (\mu^{-n} - \varepsilon(n))^{1/2}$, put $t_n = \omega_n \tau$ and $\tilde{g}_n(\tau) = g(\omega_n \tau)\omega_n^2$. Rewrite $\tilde{y}_n(t)$ in the form:

$$Y_n(\tau) = \tilde{y}_n(\omega_n \tau) = \mu^n \omega_n^2 (a\tau^2 - 1 + \tilde{g}_n(\tau)).$$

Denote

$$\pi_n = (b_2 + c_2 j)\lambda^n \omega_n, \quad \sigma_n = \mu^n \omega_n^2, \quad \varepsilon_0(n) = -(b_2 + c_2 j)\lambda^n - \pi_n/\sqrt{a}.$$

Now we rewrite formula

$$y_n^*(t) = a_2\tilde{y}_n^m(t) + \sum_{i=0}^{m-1} \varepsilon_i \tilde{y}_n^i(t) + b_2\lambda^n(1+t) + \\ c_2 j\lambda^n t + H_2(\lambda^n(1+t), \tilde{y}_n(t), c_2 j\lambda^n t)$$

in terms of $Y_n(\tau)$. Note that conditions (22) for y_n^* are equivalent to the equality to zero of Y_n^* and its derivatives with respect to τ. So we calculate the normalized function $Y_n^*(\tau) := y_n^*(\omega_n\tau)/\pi_n$ and get:

$$Y_n^*(\tau) = a_2\pi_n^{-1}\sigma_n^m(a\tau^2 - 1 + \tilde{g}_n(\tau))^m + \sum_{i=1}^{m-1} \varepsilon_i(n)\pi_n^{-1}\sigma_n^i(a\tau^2 - 1 + \tilde{g}_n(\tau))^i + \\ \tau - \frac{1}{\sqrt{a}} + \pi_n^{-1}H_2(\lambda^n(1+\omega_n\tau), \sigma_n(a\tau^2 - 1 + \tilde{g}_n(\tau)), j\lambda^n\omega_n\tau).$$

Consider a polynomial

$$Y^*(\tau) = \sum_{i=1}^{m} d_i(a\tau^2 - 1)^i + \tau - \frac{1}{\sqrt{a}}. \tag{23}$$

In Lemma 5 (see Lemma 3 [16]) we show that there exists a set of non-zero numbers $d_1, \dots, d_m$ such that

$$(Y^*)^{(s)}\left(\frac{1}{\sqrt{a}}\right) = 0, \quad s = 0, \dots, m. \tag{24}$$

Let us fix $\varepsilon_1(n), \dots, \varepsilon_{m-1}(n), \varepsilon(n)$ in such a way that

$$\lim_{n\to\infty} a_2\pi_n^{-1}\sigma_n^m = d_m, \quad \lim_{n\to\infty} \varepsilon_i\pi_n^{-1}\sigma_n^i = d_i. \tag{25}$$

Notice that $\varepsilon(n)$ appears implicitly in these formulas, because σ_n depends on it. We shall show that, with this choice of the parameters, in the limit we get:

$$\lim_n Y_n^*(\tau) = Y^*(\tau).$$

To do this, we shall estimate the remainder term $\pi_n^{-1}H_2(\lambda^n(1+\omega_n\tau), \sigma_n(a\tau^2 - 1 + \tilde{g}_n(\tau)), 0)$ and its derivatives of order $s \le m$, showing that they go to zero when n grows. By a standard calculation,

$$\left|\partial^s H_2|_{\tau=1/\sqrt{a}}\right| \le Const.\lambda^n\sigma_n.$$

Since $\pi_n = \lambda^n\omega_n$, $\sigma_n = \mu^n\omega_n^2$, $\pi_n^{-1}\sigma_n^m \approx const.$, we get:

$$\omega_n^2 \le \frac{\lambda^{n/m}}{\mu^n} \quad \text{and} \quad \sigma_n \le \lambda^{n/m}. \tag{26}$$

Then

$$\pi_n^{-1}\left|\partial^s H_2|_{\tau=1/\sqrt{a}}\right| \le \frac{Const.\lambda^n\sigma_n}{\lambda^n\omega_n} = \mu^n\omega_n \le (\mu^{1/2}\lambda^{1/2m})^n.$$

Since, by assumption, $\lambda < 1/\mu^{m+1}$, the latter goes to zero when n grows.

In order to make $z_n^*(t)$ and its derivatives vanish at t_n, set

$$C = \frac{b_3 + jc_3}{b_2 + jc_2}.$$

Note that constants j and C were chosen in such a way that $a_3 = Ca_2$. In the same way as above, we introduce a normalized function $Z_n^*(\tau) := z_n^*(\omega_n \tau)/\pi_n$:

$$Z_n^*(\tau) = Ca_2\pi_n^{-1}\sigma_n^m(a\tau^2 - 1 + \tilde{g}_n(\tau))^m + \sum_{i=0}^{m-1} C\varepsilon_i(n)\pi_n^{-1}\sigma_n^i(a\tau^2 - 1 + \tilde{g}_n(\tau))^i +$$

$$C\tau - C\frac{1}{\sqrt{a}} + \pi_n^{-1}H_3(\lambda^n(1+\omega_n\tau), \sigma_n(a\tau^2 - 1 + \tilde{g}_n(\tau)), j\lambda^n\omega_n\tau).$$

Exactly the same calculation as for H_2 works for H_3, providing that

$$\lim_n Z_n^*(\tau) = \sum_{i=1}^{m} Cd_i(a\tau^2 - 1)^i + C\tau - C\frac{1}{\sqrt{a}}.$$

We have proved that an arbitrarily small deformation of our system satisfies (22). Now consider the deformed system in the original coordinates and the matrix of its linear part. The minor which is compliment to the term with ε_v is nonzero, therefore we can compensate the change in $\det df_\varepsilon^N$ after we choose $\varepsilon_0, \ldots, \varepsilon_{m-1}$ and take ε_v as a functional parameter so that $\det df_\varepsilon^N \equiv \det df^N$. Then the resulting mapping is Ω-preserving.

As the last step, one verifies that, with such a choice of ε_v as above, $\pi_n^{-1}\frac{\partial x_n^*}{\partial \tau} \neq 0$ for τ close to $1/\sqrt{a}$. This proves the existence of an m-order homoclinic tangency between $W^u(p_1)$ and $W^s(p_1)$ after a C^r-small C^∞-perturbation of a m-floor tower and completes the justification of Step 3 in the linearizable case.

In the arguments above we used the following lemma, whose proof can be found in [16] (Lemma 3).

Lemma 5. *Let $Y^*(\tau)$ be as in (23). Then there exists a set of non-zero numbers $d_1, \ldots, d_m$ such that $Y^*(\tau)$ satisfies (24).*

7.4. **Modifications for the general case.** Now we treat the general case. Suppose that f^n is given by Lemma 1, and repeat the above argument with the following adjustment. Formula (20) transforms into the following:

$$x_n^* = a_1(1+\varepsilon_v)(\tilde{y}_n + y\rho_n^y) + b_1(\lambda^n(1+x) + x\rho_n^x) + c_1(\nu^n z + z\rho_n^z) + H_1,$$

$$y_n^* = a_2(\tilde{y}_n + y\rho_n^y)^m + \sum_{i=0}^{m-1}\varepsilon_i(\tilde{y}_n + y\rho_n^y)^i + b_2(\lambda^n(1+x) + x\rho_n^x) + c_2(\nu^n z + z\rho_n^z) + H_2,$$

$$z_n^* = a_3(\tilde{y}_n + y\rho_n^y)^m + \sum_{i=0}^{m-1}C\varepsilon_j(\tilde{y}_n + y\rho_n^y)^i + b_3(\lambda^n(1+x) + x\rho_n^x) + c_3(\nu^n z + z\rho_n^z) + H_3,$$

where $H_i = H_i((\lambda^n(1+x) + x\rho_n^x), (\tilde{y}_n + y\rho_n^y), (\nu^n z + z\rho_n^z))$.

It is left to show that on the curve γ, given by (21), the terms with ρ_n^ξ in the above system, together with their m derivatives with respect to τ, tend to zero as n tends to infinity. To see this, write:

$$u = xyz = t(at^2 + g(t) + \varepsilon(n))(\lambda/\nu)^n jt = \omega_n^2\sigma_n\lambda^n(\lambda^n\mu^n)j\tau^2(a\tau^2 - 1 + \tilde{g}_n(\tau)).$$

Application of the bounds gives that each term $\omega_n^2,\ \sigma_n,\ \lambda^n$, and $\lambda^n\mu^n$ is exponentially small in n. Application of the estimates from Lemma 1 shows that $\xi\rho_n^\xi$ and its partial derivatives derivatives with respect to τ are exponentially small in n.

8. **Creation of an m-saddlenode (Step 4).** In this section we prove the following theorem.

Theorem 5. *Let Ω be a smooth volume form, $r \leq m \leq 2r$. Let $\{f_\varepsilon\}_{\varepsilon\in I}$ be a generic m-parameter family of C^∞-smooth Ω-preserving diffeomorphisms unfolding a homoclinic tangency of order m corresponding to a $2r$-unstable saddle p. Then there exists a sequence of pairwise disjoint compact sets I_n in the parameter space tending to zero such that*
• *for any $\varepsilon \in I_n$ there is a C^{2r}-smooth 2-dimensional surface $S_{\varepsilon,n}$, which is locally invariant under f_ε^{n+N}, transversal to the OZ-axis, and C^{2r}-smoothly depends on the parameter ε;*
• *the family of restrictions to this surface $\{f_\varepsilon^{n+N}|_{S_{\varepsilon,n}}\}_{\varepsilon\in I_n}$ generically unfolds a homoclinic tangency of order m* [8].
• *for an arbitrary set of real numbers $\mathbf{K} = \{K_j\}_{j=0}^{k-1}$ there exists a sequence of parameters $\varepsilon_n(\mathbf{K})$ tending to zero when n grows, and a sequence of linear changes of variables R_n such that for any r the map*

$$\Phi^m_{\varepsilon_n(\mathbf{K}),n} := R_n \circ f^n \circ f^N_{\varepsilon_n(\mathbf{K})} \circ R_n^{-1},$$

restricted to $S_{\varepsilon,n}$, gets arbitrarily C^{2r}-close to

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ y^m + \sum_{i=0}^{m-1} K_i y^i \end{pmatrix}$$

as n increases.

The proof of this theorem follows from the lemma below.

Lemma 6. *Let the saddle p be $2r$-unstable, and $r \leq m \leq 2r$. For an arbitrary set of real numbers $\{K_j\}_{j=0}^{m-1}$ there exists a nested sequence of compact sets I_n in the parameter space, tending to zero when n grows, such that*
• *For an appropriate choice of β_n, the first component, $\bar{G}^{1,L}_n$, C^r-converges to zero (resp., to y).*
• *The second and the third component have the form*

$$\bar{G}^{2,L}_{\varepsilon,n} = y^m + \sum_{j=0}^{m-1} K_j y^j + h_1^n(x,y,z),$$

$$\bar{G}^{3,L}_n = c_3\nu^n z + h_2^n(x,y,z),$$

where c_3 is the constant in (28), and both h_1^n and $\nu^{-n}h_2^n$ converge to zero in C^r as $n \to \infty$ for any sequence $\{\varepsilon(n)\}_{n\geq 1}$ such that $\varepsilon(n) \in I_n$ for each n.

Proof. We start with discussing the linearizable case, and then describe the modification caused by the non-linearity of the single-resonant normal form (3).

8.1. **Linearizable case.** As before, consider $L(x,y,z) = (\lambda x, \mu y, \nu z)$. Since the saddle is $2r$-unstable and $m \leq 2r$, we can assume that the homoclinic tangency has the geometric preliminary normal form described after Remark 3. Then, in the

[8] Since this surface $S_{\varepsilon,n}$ depends C^{2r}-smoothly on the parameter $\varepsilon \in I_n$, it does make sense to speak about the family of restrictions $\{f_\varepsilon^{n+N}|_{S_{\varepsilon,n}}\}_{\varepsilon\in I_n}$ unfolding a homoclinic tangency of order $m \leq 2r$.

normal coordinates in the neighborhood of p, $f^N : \hat{U} \mapsto U$ can be written in the form:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 + a_1(y-1) + b_1 x + c_1 z + H_1(x, y-1, z) \\ a_2(y-1)^m + b_2 x + c_2 z + H_2(x, y-1, z) \\ a_3(y-1)^m + b_3 x + c_3 z + H_3(x, y-1, z) \end{pmatrix},$$

where for $x = (y-1) = z = 0$ we have:

$$\begin{cases} H_i = \partial_1 H_i = \partial_2 H_i = \partial_3 H_i = 0, \quad i = 1,2,3, \quad j = 1,2, \\ \partial_2^j H_i = 0, \quad i = 2,3, \quad j = 1, \dots, m. \end{cases} \tag{27}$$

Consider a generic m-parameter unfolding $f_\varepsilon^N(x,y,z)$ of the tangency with parameters ε_i, $i = 0, \dots, m-1$:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 + a_1(y-1) + b_1 x + c_1 z + H_1 \\ a_2(y-1)^m + \sum_{i=0}^{m-1} \varepsilon_i (y-1)^j + b_2 x + c_2 z + H_2 \\ a_3(y-1)^m + b_3 x + c_3 z + H_3 \end{pmatrix}. \tag{28}$$

In the linearizable case in order to study $L^n \circ f_\varepsilon^N := (G_n^{1,L}, G_{\varepsilon,n}^{2,L}, G_n^{3,L})$, (also denote $(G_{\varepsilon,n}^{2,L}, G_n^{3,L})$ by $G_{\varepsilon,n}^L$), we shall study a sequence of renormalizations of this system. To do this, consider

$$\Psi_n^u(x,y,z) = (\lambda^n x, y, z), \quad \Psi_n^s(x,y,z) = (x, \mu^{-n} y, \nu^{-n} z).$$

With this notation, $L^n = \Psi_n^u \circ (\Psi_n^s)^{-1}$. Let

$$\alpha = \frac{1}{m-1}, \quad 0 < \gamma < \alpha, \quad \delta = -c_2/c_3, \quad \eta_0 = (1,0),$$

β_n and κ be constants to be chosen later, and

$$B_n = \begin{pmatrix} \kappa \mu^{-\alpha n} & -\delta \mu^{-\gamma n} \nu^{-n} \\ 0 & \mu^{-(1+\gamma)n} \end{pmatrix}, \quad B_n^{-1} = \begin{pmatrix} \frac{1}{\kappa} \mu^{\alpha n} & \frac{\delta}{\kappa} \mu^{(\alpha+1)n} \nu^{-n} \\ 0 & \mu^{(1+\gamma)n} \end{pmatrix}, \tag{29}$$

We set

$$\Psi_n(x,y,z) = (\beta_n x + 1, B_n(y,z) + \eta_0),$$

cf. (9). Define the renormalization mapping by $R_n = \Psi_n^u \circ \Psi_n$, cf. (10). So the renormalization of our system has the same form as (11):

$$\Phi_{\varepsilon,n}^{m,L} = R_n^{-1} \circ L^n \circ f_\varepsilon^N \circ R_n := (\bar{G}_n^{1,L}, \bar{G}_{\varepsilon,n}^L) := (\bar{G}_n^{1,L}, \bar{G}_{\varepsilon,n}^{2,L}, \bar{G}_n^{3,L}). \tag{30}$$

First consider the second and the third components,

$$\bar{G}_{\varepsilon,n}^L = B_n^{-1}(U^n G_{\varepsilon,n}^L(\lambda^n(\beta_n x + 1), B_n(y,z)) - \eta_0),$$

where B_n^{-1} has the form (29), $U(y,z) = (\mu^n y, \nu^n z)$. The composition $B_n^{-1} U^n (G_{\varepsilon,n}^L)$ has the form $B_n^{-1} U^n (G_{\varepsilon,n}^L) = (\kappa^{-1} \mu^{(\alpha+1)n}(G_{\varepsilon,n}^{2,L} + \delta G_n^{3,L} - \mu^{-n}),\ \mu^{(1+\gamma)n} \nu^n G_n^{3,L})$. Therefore,

$$\begin{aligned} \bar{G}_n^{3,L} =& \mu^{(1+\gamma)n} \nu^n G_n^{3,L}(\lambda^n(\beta_n x + 1), B_n(y,z)) = \\ & \mu^{(1+\gamma)n} \nu^n \left[a_3 \left(\frac{\kappa y}{\mu^{\alpha n}} - \frac{\delta z}{\mu^{\gamma n} \nu^n} \right)^m + b_3 \lambda^n (\beta_n x + 1) + c_3 \frac{z}{\mu^{(\gamma+1)n}} + H_3 \right], \end{aligned}$$

where $H_3 = H_3 \left(\lambda^n(\beta_n x + 1),\ \frac{\kappa y}{\mu^{\alpha n}} - \frac{\delta z}{\nu^n} + 1,\ \frac{z}{\mu^n} \right)$. When we open the square brackets, we get the following: since $m\alpha = \alpha + 1$, and $\gamma < \alpha$, $\mu^{(1+\gamma)n} (\frac{y}{\mu^{\alpha n}} - \frac{\delta z}{\mu^{\gamma n} \nu^n})^m$ goes to zero when n grows; $\mu^{(1+\gamma)n} \lambda^n$ goes to zero when n grows. All the terms in H_3 do so too. The only term that persists is $c_3 \nu^n z$.

As for the second component, $\bar{G}^{2,L}_{\varepsilon,n}$, we have:

$$\bar{G}^{2,L}_{\varepsilon,n} = \kappa^{-1}\mu^{(\alpha+1)n}\Bigg[(a_2+\delta a_3)\left(\frac{\kappa y}{\mu^{\alpha n}} - \frac{\delta z}{\mu^{\gamma n}\nu^n}\right)^m + \\ + \sum_{i=0}^{m-1}\varepsilon_i\left(\frac{\kappa y}{\mu^{\alpha n}} - \frac{\delta z}{\mu^{\gamma n}\nu^n}\right)^i + (b_2+\delta b_3)\lambda^n(\beta_n x+1) + \\ + (c_2+\delta c_3)\frac{z}{\mu^{(\gamma+1)n}} - \frac{1}{\mu^n} + H_2 + \delta H_3\Bigg].$$

Opening the square brackets, we get: the coefficient of y^m is a constant (we can assume that it be non-zero, and take κ to make it equal 1), the coefficients of y^i can be made to equal our given constants by the choice of ε_i ($i = 0, ..., m-1$), the others (except for the one of z) go to zero when n grows, the same is true for $H_2 + \delta H_3$. The coefficient of z in the above sum equals zero since we have chosen δ so that $(c_2 + \delta c_3) = 0$.

For the first component we have:

$$\bar{G}^{1,L}_n = \beta_n^{-1}\left[a_1\left(\frac{\kappa y}{\mu^{\alpha n}} - \frac{\delta z}{\mu^{-\gamma n}\nu^n}\right) + b_1\lambda^n(\beta_n x+1) + c_1\frac{z}{\mu^n} + H_1\right].$$

If $\beta_n = 1$ (resp., $\beta_n = \kappa a_1\mu^{-\alpha n}$), then $\bar{G}^{1,L}_n$ tends to 0 (resp., y) and all the other terms vanish along with all its partial derivatives.

This completes the proof in the linearizable case. □

8.2. **Non-linearizable case.** Reduction of this case to the linearizable one is done exactly as in Section 4.2. This completes the proof of the Lemma.

Corollary 2. *Fix an arbitrary set of real numbers* $\{K_i\}_{i=0}^{m-1}$*. Let* $\Phi^m_{\varepsilon,n}$ *be as in the Lemma above. Then for* n *sufficiently large and* $\varepsilon \in I_n$ *the map* $\Phi^m_{\varepsilon,n}$ *has a* C^{2r}*-smooth invariant manifold* $S_{\varepsilon,n}$*. The restriction of* $\Phi^m_{\varepsilon,n}$ *to* $S_{\varepsilon,n}$ *is* C^{2r}*-close to the map*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ y^m + \sum_{i=0}^{m-1} K_i y^i \end{pmatrix}.$$

Corollary 3. *[Creation of an* m*-saddlenode (Step 4)] Let* Ω *be a smooth volume form. Suppose that a* C^{2r} *diffeomorphism* f *has a* $2r$*-unstable saddle with a homoclinic tangency of order* m*,* $r \le m \le 2r$*. Then there exists an arbitrarily* C^r*-small* C^∞*-smooth* Ω*-preserving deformation of* f *having a periodic* m*-saddlenode of arbitrarily high period (see definition 2 of an* m*-saddlenode).*

We have completed the proof of Step 4, which is the last part of the proof of Theorem 2 (see the end of Section 2.2). This completes the proof of the main result of the paper (Theorem 2).

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