Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2004
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Homework assignment 5, Differential Equations

1. Let f be a C^1 vector field on a neighborhood of the annulus

$$A = \{x \in \mathbb{R}^2 : 1 \leq |x| \leq 2\}.$$

Assume that f has no zeros and that f is transversal to the boundary and points inwards.

(a) Prove that there is a closed orbit that solves $\dot{x} = f(x)$.

(b) Show that if there are exactly 7 closed orbits of the vector field then to one of these orbits there are associated orbits that spiral towards the closed orbits from both sides.

2. Prove that depending on whether $ad - bc > 0$ or $ad - bc < 0$, the index with respect to the origin of the linear vector field is

$$f_0(x, y) = (ax + by, cx + dy)$$

is equal to ± 1.

3. Assume that $f(x, y) = (f_1(x, y), f_2(x, y))$ is a C^1 vector field with an isolated critical point in $0 \in \mathbb{R}^2$ and that the derivative of f at $(0, 0)$ is the linear map in Exercise 1 above. Show that if $ad - bc > 0$ then f has index $+1$ at $(0, 0)$ and if $ad - bc < 0$ then f has index -1 in $(0, 0)$.

4. Let $f(z) = z^k$ where $z = x + iy$ and z^k is the k:th power of the complex number z. Consider f as a vector field on $\mathbb{R}^2$. Prove that the index of f at 0 is k.

5. Let $f(z) = \bar{z}^k$ where $\bar{z} = x - iy$. Consider f as a vector field on $\mathbb{R}^2$. Show that the index of f in 0 is $-k$.
6. Give an example of a C^∞ vector field f in the plane that has the unit circle as the only non-trivial periodic orbit.

7. ($*$) Assume that f is a C^1 planar vector field and that x is a point in $\mathbb{R}^2$, the orbit of which is defined and bounded for $t \geq 0$. Assume that $\omega(x)$ is not a periodic orbit or a critical point

   (a) Show that $\omega(x)$ can be written as a disjoint union $\omega(x) = C \cup S$ where C consist of critical points and S only consist of stable and unstable manifolds or limit cycles.

   (b) Show that the set S in the previous statement only consist of at most countably many orbits.

8. Assume that A is a 2×2 real matrix. Let $X(x) = Ax$ be the linear vector field on $\mathbb{R}^2$ defined by A. Show that every non-wandering point of X is a critical point or a periodic orbit.