ERGODIC THEOREMS FOR HOMOGENEOUS DILATIONS

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Abstract. In this paper we prove a general ergodic theorem for ergodic and measure–preserving actions of $\mathbb{R}^d$ on standard Borel spaces. In particular, we cover R.L. Jones' ergodic theorem on spheres. Our main theorem is concerned with ergodic averages with respect to homogeneous dilations of Rajchman measures on $\mathbb{R}^d$. We establish mean convergence in Hilbert spaces for general Rajchman measures, and give a criterion in terms of the Fourier dimension of the measure when almost everywhere pointwise convergence holds. Applications include averages over smooth submanifolds and polynomial curves.

1. Introduction

The first multidimensional pointwise ergodic theorem is due to N. Wiener [19], who proved that if $(X, \mathfrak{B}, \mu)$ is a standard Borel space and T is an ergodic measure–preserving action of $\mathbb{R}^d$ on X, then for all $f \in L^1(X)$, the limit

$$\lim_{\lambda \to \infty} \frac{1}{|B|} \int_B f(T_{\lambda t} x)\, dt = \int_X f \, d\mu,$$

exists almost everywhere on X, where B denotes the unit ball in $\mathbb{R}^d$ and $|B|$ the volume of B. It is not hard to extend this theorem to the more general setting where the normalized characteristic function of a ball is replaced by an absolutely continuous probability measure on $\mathbb{R}^d$.

In this paper we will be prove Wiener's ergodic theorem with respect to not necessarily absolutely continuous probability measures on $\mathbb{R}^d$; more precisely, we are interested in the class $\mathcal{C}_p$ of probability measures ν for which

$$\lim_{\lambda \to \infty} \int_X f(T_{\lambda t} x)\, d\nu(t) = \int_X f \, d\mu$$

exists almost everywhere on X for all f in $L^p(X)$, where the range of p is allowed to depend on ν. It is obviously necessary that ν is continuous, i.e. does not give positive mass to indivdual points, for this to true. We will say that Wiener's ergodic theorem holds for ν if the limit above exists almost everywhere on X. It was proved by R.L. Jones [9] that the induced Lebesgue measure on S^{d-1} in $\mathbb{R}^d$ for $d \geq 3$ belongs to the class $\mathcal{C}_p$ for $p > \frac{d}{d-1}$. This was later extended to $d = 2$ by M. Lacey [11].

Recall that the *Fourier dimension* of a probability measure ν on $\mathbb{R}^d$ is defined as the supremum over all $0 \leq a \leq d$ such that

$$|\hat{\nu}(\xi)| \leq C|\xi|^{-a/2} \quad \text{as } \xi \to \infty.$$

For instance, if S is a smooth hypersurface in $\mathbb{R}^d$ with non–vanishing Gaussian curvature and ν is the induced Lebesgue measure on S, then the Fourier dimension of ν is at least $n - 1$ [15], which motivates the terminology. Note that there are many non–smooth sets (e.g. random Cantor sets [2]) in $\mathbb{R}^d$ which support probability measures with high Fourier dimension. However, by Frostman's lemma (see e.g. [12]), the Fourier dimension is always bounded from above by the Hausdorff dimension of the support of ν.

In this paper we prove Wiener's ergodic theorem for probability measures ν on $\mathbb{R}^d$ with sufficiently large Fourier dimensions.

Theorem 1.1. *Let $(X, \mathfrak{B}, \mu)$ be a standard Borel probability measure space, and suppose T is a Borel measurable action of $\mathbb{R}^d$ on X which preserves μ. If ν is a compactly supported probabilty measure on $\mathbb{R}^d$ with Fourier dimension $a > 1$, then*

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}^d} f(T_{\lambda t}x)\, d\nu(t) = \int_X f\, d\mu$$

almost everywhere on X, for all f in $L^p(X)$ for $p > p_a$, where

$$p_a = \frac{1+a}{a}.$$

The question of mean convergence is much simpler. Recall that a probability measure ν on $\mathbb{R}^d$ is a *Rajchman measure* if the Fourier transform of ν decays to zero at infinity. Note that a Rajchman measure is always continuous, but not necessarily absolutely continuous. Indeed, there are Rajchman measures supported on the set of Liouville numbers in $\mathbb{R}$ [3], which have zero Hausdorff dimension. Therefore, the following theorem is somewhat surprising.

Theorem 1.2. *Let U be a unitary representation of $\mathbb{R}^d$ on a separable Hilbert space H. Let ρ be a Rajchman measure on $\mathbb{R}^d$, and define for $x \in H$, the operator*

$$A_\lambda x = \int_{\mathbb{R}^d} U_{\lambda t} x\, d\rho(t), \quad \lambda > 0.$$

Let P denote the projection onto the space of invariant vectors of U in H. Then

$$||A_\lambda x - Px||_H \to 0$$

for all x in H.

The special case of theorem 1.2 when ν is the push–forward of the Lebesgue measure on $[0,1]$ under a polynomial curve in $\mathbb{R}^d$ defined on $[0,1]$ was proved in [1]. Note that these measures in general have low Fourier dimensions, and theorem 1.1 does not apply in this situation. In order to extend parts of theorem 1.1 to this situation we have to allow more general dilations. Given positive real numbers $a_1, \ldots, a_d$, we define the associated *homogeneous dilation* by λ on $\mathbb{R}^d$:

$$\lambda.t = (\lambda^{a_1} t_1, \ldots, \lambda^{a_d} t_d), \quad t \in \mathbb{R}^d.$$

We will use the following two simple facts about homogeneous dilations; if $t, \xi \in R^d$, then

$$\lambda.(\xi + t) = \lambda.\xi + \lambda.t \quad \text{and} \quad \langle \xi, \lambda.t\rangle = \langle \lambda.\xi, t\rangle \quad \text{for all } \lambda > 0,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^d$. We use the dot–notation to separate between the standard dilations ($a_1 = \ldots = a_d = 1$) on $\mathbb{R}^d$ and homogeneous dilations.

For homogeneous dilations, we establish the following transfer principle.

Theorem 1.3. *Suppose ν is a Rajchman measure on $\mathbb{R}^n$ for which the maximal operator*

$$M\phi(x) = \sup_{\lambda>0} \Big| \int_{\mathbb{R}^d} \phi(\lambda.t)\, d\nu(t) \Big|, \quad \phi \in \mathcal{S}(\mathbb{R}^d)$$

is of weak type (p,p) for some $1 \le p < \infty$. For every standard Borel probability space $(X, \mathfrak{B}, \mu)$ and ergodic measure-preserving action T by $\mathbb{R}^d$ on X,

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}^d} f(T_{\lambda.t}x)\, d\nu(t) = \int_X f\, d\mu$$

almost everywhere on X for every f in $L^p(X)$.

By A.P. Calderon's transfer principle [7], the proof of this theorem is reduced to finding a dense subspace L_ν in $L^p(X)$ on which Wiener's ergodic theorem for ν holds almost everywhere. It turns out (see lemma 3.1) that we can choose this subspace, independently of the Rajchman measure ν, to consist of the constants and all functions of the form

$$f(x) = \int_{\mathbb{R}^d} f_0(T_s x)\phi(s)\, ds,$$

where $f_0 \in L^\infty(X)$ and $\phi \in L^1_0(\mathbb{R}^d)$.

Using the fundamental works by E.M. Stein on homgeneous dilations and maximal inequalities for dilated polynomial curves, we can establish the following theorem.

Theorem 1.4. *For every standard Borel probability space $(X, \mathfrak{B}, \mu)$ and ergodic measure-preserving action T by $\mathbb{R}^d$ on X,*

$$\lim_{\lambda\to\infty} \int_0^1 f(T_{\lambda.q(w)}x)\, dw = \int_X f\, d\mu$$

almost everywhere on X for every f in $L^p(X)$, where $p > 1$ and

$$q(w) = (q_1 w, \ldots, q_d w^d), \quad w \in [0, 1],$$

for some non-zero real numbers $q_1, \ldots, q_d$ and

$$\lambda.t = (\lambda t, \ldots, \lambda^d t), \quad t \in \mathbb{R}^d,\ \lambda > 0.$$

Extensions of this result to more general curves is possible; see e.g. [17].

2. Measure Theory and Fourier Analysis

2.1. **Basic Measure Theory.** Let G be a Hausdorff locally compact second countable group, and let $\mathfrak{B}_G$ denote the σ–algebra of Borel subsets of G. Let $(X, \mathfrak{B})$ be a measurable space such that there exists a Polish topology on X for which $\mathfrak{B}$ is the induced σ-algebra. If μ is a σ–finite measure on $\mathfrak{B}$, we refer to $(X, \mathfrak{B}, \mu)$ as a *standard Borel space*. We say that G admits a *Borel measurable action* on X if there is a map $\pi : G \times X \to X$ satisfying $\pi(gg', x) = \pi(g, g'x)$ for each g in G and x in X such that π is a measurable map from $(G \times X \times \mathfrak{B}_G \times \mathfrak{B})$ to $(X, \mathfrak{B})$. We will write $\pi(g, x) = gx$ for short. The action is measure–preserving if $\mu(gE) = \mu(E)$ for each g in G and E in $\mathfrak{B}$. All actions in this paper are assumed to be Borel measurable. For functions on X, measurability will refer to the completion of the σ–algebra $\mathfrak{B}$.

By a result due to V.S. Varadarajan [18], there is a compact metric G–space Y with a jointly continuous G–action and G–invariant Borel subset $Y_0 \subset Y$ together with a G–equivariant measurable and bijective map $\psi : X \to Y_0$. Let μ' denote the push–forward of the measure μ on X. This construction yields a Borel measurable measure–preserving action of G on the Borel subset Y_0.

Suppose $\lambda \mapsto \nu_t$ is a weakly continuous from $\mathbb{R}_+$ to the convex set of probability measures $M^1(G)$ on G. Define, for a continuous function f on Y and λ in $\mathbb{R}_+$, the linear operator

$$A_\lambda f(x) = \int_G f(g^{-1}x)\, d\nu_\lambda(g).$$

In this paper we will be concerned with the associated maximal function, i.e.

$$Mf(x) = \sup_{\lambda>0} \Big| A_\lambda f(x) \Big|.$$

Note that since $\lambda \mapsto \nu_\lambda$ is weakly continuous, Mf is a measurable function on Y_0, since we can realize it as

$$Mf(x) = \sup_{\substack{\lambda>0 \\ \lambda\in\mathbb{Q}}} \Big| \int_G f(g^{-1}x)\, d\nu_\lambda(g) \Big|.$$

It is easy to see that if a measurable function f is a pointwise monotone increasing limit of a sequence f_n of nonnegative functions for which the maximal functions Mf_n are measurable, then Mf is measurable. By rather technical, but standard, approximation arguments (see e.g. [13]), Mf is measurable for every measurable function f on X. Note however that this argument only ensures that Mf is measurable with respect to the *completion* of the σ–algebra $\mathfrak{B}$ with respect to the probability measure μ.

We now restrict our attention to the group $G = \mathbb{R}^d$ for $d \geq 1$. If μ is absolutely continuos with respect to the Haar measure on $\mathbb{R}^d$, weak–L^p bounds for the sublinear operator M are implied, via Calderón's transfer argument, by the corresponding weak L^p–bounds for the Hardy–Littlewood maximal function (see e.g. [7]). Analogous transfer arguments for maximal functions works equally well for singular measures and will be described in subsection 3.3.

2.2. **Fourier Analysis.** In this subsection we recall some basic notions from classical Fourier analysis. If ν is a complex measure on $\mathbb{R}^d$, we define the Fourier transform of ν by

$$\hat{\nu}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, t\rangle}\, dt, \quad \xi \in \mathbb{R}^d.$$

It is not hard to see that $\hat{\nu}$ is a uniformly continuous function on $\mathbb{R}^d$. We let $A^1(\mathbb{R}^d)$ denote the class of absolutely continuous complex measures ν on $\mathbb{R}^d$ for which $\hat{\nu}$ is integrable with respect to the Lebesgue measure, and by $A^1_0(\mathbb{R}^d)$, the subspace of $A^1(\mathbb{R}^d)$ with $\nu(\mathbb{R}^d) = 0$. If ν is in $A^1(\mathbb{R}^d)$, we can reproduce the density ρ of ν by

$$\rho(t) = \int_{\mathbb{R}^d} e^{i\langle \xi, t\rangle} \hat{\nu}(\xi)\, dm(\xi),$$

where m denotes the Plancherel measure on $\mathbb{R}^d$. The *Fourier dimension* of a probability measure ν is defined as the supremum over all real numbers $0 \leq a \leq d$ such that

$$|\xi|^a |\hat{\nu}(\xi)| \leq C, \quad \forall\, \xi \in \mathbb{R}^d,$$

for some finite constant C.

3. Ergodic Theorems

3.1. **Mean Ergodic Theorems.** Before we turn to the pointwise ergodic theorems, we will establish some results on mean convergence. Recall that a probability measure ν on $\mathbb{R}^d$ is Rajchman if the Fourier transform of ν decays to zero at infinity. If a homogeneous dilation has been chosen on $\mathbb{R}^d$, we let ν_λ denote the measure

$$\int_{\mathbb{R}^d} \phi(t)\, d\nu_\lambda(t) = \int_{\mathbb{R}^d} \phi(\lambda.t)\, d\nu(t), \quad \phi \in C_c(\mathbb{R}^d),$$

where $C_c(\mathbb{R}^d)$ is the space of compactly supported continuous functions on $\mathbb{R}^d$.

For homogeneous dilations, we have the following theorem.

Theorem 3.1. *Let U be a unitary representation of $\mathbb{R}^d$ on a separable Hilbert space H. Let ρ be a Rajchman measure on $\mathbb{R}^d$, and define for $x \in H$, the operator*

$$A_\lambda x = \int_{\mathbb{R}^d} U_{\lambda.t} x\, d\rho(t), \quad \lambda > 0.$$

Let P denote the projection onto the space of invariant vectors of U in H. Then

$$||A_\lambda x - Px||_H \to 0$$

for all x in H.

The proof consists of simple calculations with spectral measures. We give a proof for completeness.

Proof. We only have to prove that $A_\lambda x \to 0$ for all x for which $Px = 0$. Note that

$$\begin{aligned} ||A_\lambda x||_H^2 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \langle U_{\lambda.t}x, U_{\lambda.s}x\rangle \, d\rho(t)d\rho(s) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\langle \xi, \lambda.(t-s)\rangle}\, d\nu_x(\xi)d\rho(t)d\rho(s) \\ &= \int_{\mathbb{R}^d} |\hat{\rho}(\lambda.\xi)|^2 \, d\nu_x(\xi) \\ &\to \nu_x(\{0\}) \quad \text{as } \lambda \to \infty, \end{aligned}$$

where ν_x is the spectral measure of U located at x. Since x is in the complement of invariant vectors we have $\nu_x(\{0\}) = 0$, and the theorem is proved. □

Remark. The theorem also holds for isometric representations of more general uniformly convex Banach spaces from standard approximation arguments.

Note that the analogous theorem for representations of $\mathbb{Z}^d$ is completely false. This is essentially due to the discreteness of $\mathbb{Z}^d$. Let U be a unitary representation of $\mathbb{Z}^d$ on a separable Hilbert space H, and define for $x \in H$

$$A_n x = \sum_{k \in \mathbb{Z}^d} \rho_k U_{nk} x.$$

Theorem 3.2. *For all non-zero $x \in H$, and for any probability measure ρ on $\mathbb{Z}^d$,*

$$\liminf_{n\to\infty} \frac{||A_n x||_H}{||x||_H} \geq ||\rho||_{\ell^2(\mathbb{Z}^d)} > 0.$$

Proof. We define the sets

$$E_k = \{\theta \in \mathbb{T}^d \mid \langle k, \theta\rangle = 0\}, \quad k \in \mathbb{Z}^d.$$

Note that $E_0 = \mathbb{T}^d$, and by Parseval's relation,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} ||A_n x||_H^2 &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \rho_k\rho_l \langle U_{nk}x, U_{nl}x\rangle \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \rho_k\rho_l \int_{\mathbb{T}^d} e^{-2\pi i\langle\theta, n(k-l)\rangle}\, d\nu_x(\theta) \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_{\mathbb{T}^d} |\hat{\rho}(n\theta)|^2 \, d\nu_x(\theta) \\ &= \lim_{N\to\infty} \sum_{k\in\mathbb{Z}^d} (\rho * \check{\rho})_k \frac{1}{N}\sum_{n=0}^{N-1} \hat{\nu}_x(-nk) \\ &= \sum_{k\in\mathbb{Z}^d} (\rho*\check{\rho})_k \nu_x(E_k) \geq ||\rho||_{\ell^2(\mathbb{Z}^d)}^2 ||x||_H^2 > 0, \end{aligned}$$

for all $N \geq 1$. Thus, $||A_n x||_H$ does not converge to 0 for *any* non-zero x in H. □

3.2. **Pointwise Ergodic Theorems.** We follow the general approach to pointwise ergodic theorems: We first establish Wiener's ergodic theorem for a *dense* subspace of $L^p(X)$, and then prove a maximal inequality, which implies that the subspace of functions for which the Wiener ergodic theorems are true is closed in $L^p(X)$. Since the necessary maximal inequalities follow from A.P. Calderon's transfer lemma, which we present below, the key result in this paper is the following lemma.

Lemma 3.1. *The linear span L of all functions on the form $f(x) = \int_{\mathbb{R}^d} f_0(T_s x)\phi(s)\,ds$, where $f_0 \in L^\infty(X)$ and $\phi \in {}^1_0(\mathbb{R}^d)$, is dense in $L^1_0(X)$ and if ν is a Rajchman measure on $\mathbb{R}^d$, then*

$$A_\lambda f(x) = \int_{\mathbb{R}^d} f(T_{\lambda.t}x)\,d\nu(t) \to 0$$

almost everywhere on X for all $f \in L$.

Proof. Suppose there is a function $h \in L^\infty_0(X)$ such that

$$\begin{aligned}\int_X h(x)\left(\int_{\mathbb{R}^d} f_0(T_s x)\,\phi(s)\,ds\right)d\mu(x) &= \int_{\mathbb{R}^d}\phi(s)\left(\int_X f_0(T_s x)h(x)\,d\mu(x)\right)ds\\ &= \int_X f_0(x)\left(\int_{\mathbb{R}^d} h(T_{-s}x)\,\phi(s)\,ds\right)d\mu(x) = 0,\end{aligned}$$

for all $f_0 \in L^\infty(X)$ and $\phi \in S_0(\mathbb{R}^d)$. Thus,

$$\int_{\mathbb{R}^d} h(T_{-s}x)\,\phi(s)\,ds = 0, \quad \text{a.e. } [\mu].$$

Thus, h is invariant under the action of T and by ergodicity, h must be almost everywhere zero.

We now turn to almost everywhere convergence for f in the subspace L. We can without loss of generality assume that f is of the form

$$f(x) = \int_X f_0(T_s x)(\phi * \psi)(s)\,ds,$$

where $\phi \in L^1(\mathbb{R}^d)$ and $\psi \in A^1_0(\mathbb{R}^d)$, since every element in $L^1(\mathbb{R}^d)$ can be approximated arbitrarily well in $L^1_0(\mathbb{R}^d)$ by such functions. We note that

$$\begin{aligned}\int_{\mathbb{R}^d} f(T_{\lambda.t}x)\,d\nu(t) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(T_{\lambda.t+s}x)\psi(s)\phi(t-s)\,ds d\nu(t)\\ &= \int_{\mathbb{R}^d} f_0(T_s x)\left(\int_{\mathbb{R}^d}\psi(s-\lambda.t-r)\phi(r)\,dr d\nu(t)\right)ds\\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f_0(T_s x)\phi(r+s)\,ds\right)\left(\int_{\mathbb{R}^d}\psi(-\lambda.t-r)\,d\nu(t)\right)dr\end{aligned}$$

We define, for a fixed x in X, the functions

$$g(r) = \int_{\mathbb{R}^d} f_0(T_s x)\phi(r+s)\,ds \quad \text{and} \quad h_\lambda(r) = \int_{\mathbb{R}^d}\psi(-\lambda.t-r)\,d\nu(t).$$

We observe that for a conull subset of X, g is a bounded and uniformly continuous function on $\mathbb{R}^n$ and the set $\{h_\lambda\}_{\lambda>0}$ is contained in some ball of finite radius in $L^1_0(\mathbb{R}^d)$ for all $\lambda > 0$. We think of h_λ as Borel measures on $\mathbb{R}^n$ with uniformly bounded total variations. It now suffices to prove that the sequence h_λ tends to zero in the weak topology, i.e.

$$\int_{\mathbb{R}^d} g\,dh_\lambda \to 0,$$

for all *bounded and continuous* functions on $\mathbb{R}^n$. Since the variation norm of h_λ is uniformly bounded in λ, this is implied by the uniform convergence of the Fourier transforms of h_λ to 0 (See e.g. theorem 3.2.1. in [4]). Note that

$$\hat{h}_\lambda(\xi) = \hat{\nu}_\lambda(\xi)\psi(\xi),$$

and the uniform convergence of this sequence to 0 is immediate from the fact that ν is a Rajchman measure and $\hat{\psi}(0) = 0$. □

Remark. Note that lemma 3.1 also implies that $L \cap L_0^p(X)$ is dense in $L_0^p(X)$ for all $1 \leq p < \infty$.

3.3. **Maximal Inequalities.** Let $\lambda \mapsto \nu_\lambda$ be a weakly continuous map from $\mathbb{R}^+$ into $M^1(\mathbb{R}^d)$. Suppose $(X, \mathfrak{B}, \mu)$ is a standard Borel probability measure space, and T a Borel measurable action of $\mathbb{R}^d$ on X, which preserves μ. We define

$$A_\lambda f(x) = \int_X f(T_t x)\, d\nu_\lambda(t), \quad \lambda > 0.$$

and

$$Mf(x) = \sup_{\lambda > 0} |A_\lambda f(x)|.$$

By the arguments in section 2.1, Mf is measurable with respect to the completion of the σ–algebra $\mathfrak{B}$. We also define

$$D_\lambda \phi = \phi * \nu_\lambda, \quad \phi \in \mathcal{S},$$

and the corresponding maximal operator

$$S\phi(x) = \sup_{\lambda > 0} |D_\lambda \phi(x)|.$$

Both of these operators have been extensively studied in classical harmonic analysis; see e.g. [14] and [16]. We recall the following lemma by A.P. Calderon [7] which allows us to transfer bounds on S to bounds on M. We include a proof for completeness.

Lemma 3.2 (Calderon's Transfer Lemma)**.** *Suppose $\lambda \mapsto \nu_\lambda$ is a weakly continuous map from $\mathbb{R}^+$ into $M^1(\mathbb{R}^d)$. If $(X, \mathfrak{B}, \mu)$ is a Borel standard probability space and T a Borel measurable action of $\mathbb{R}^d$ on X, which preserves the measure μ, then if S is of strong or weak type (p,p) for some $1 \leq p \leq \infty$, then so is M.*

Proof. We only give the proof in the case when S if of strong type (p,p). Suppose f is in $L^p(X)$ and define $F(s,x) = f(T_s x)$ for $s \in \mathbb{R}^d$ and $x \in X$. Note that $F(s+t, x) = F(s, T_t x)$ for all s, t in $\mathbb{R}^d$ and for almost every $x \in X$. Now set

$$G(s,x) = \sup_{\lambda > 0} \left| \int_{\mathbb{R}^d} F(s, T_t x)\, d\nu_\lambda(t) \right|,$$

and observe that $G(s+r, x) = G(s, T_r x)$ for all s, r in $\mathbb{R}^d$. For $R > 0$, we define

$$F_R(s,x) = \begin{cases} F(s,x) & \text{if } |s| < R \\ 0 & \text{otherwise,} \end{cases}$$

and $G_R(s, \cdot) = SF_R(s, \cdot)$. Since S is a positive sublinear operator,

$$\begin{aligned} G(s,\cdot) &= SF(s,\cdot) = S(F_{R+\varepsilon}(s,\cdot) + F(s,\cdot) - F_{R+\varepsilon}(s,\cdot)) \\ &\leq SF_{R+\varepsilon}(s,\cdot) + S(F(s,\cdot) - F_{R+\varepsilon}(s,\cdot)), \end{aligned}$$

for all $\varepsilon > 0$. The last term is zero, since the function $F - F_{R+\varepsilon}$ vanishes in the region $|s| < R + \varepsilon$. Let B_R denote the euclidean ball in $\mathbb{R}^d$ of radius R. Note that if S is strong type (p,p), then

$$\begin{aligned}
||Mf||_p^p &= \int_X G(0,x)^p \, dm(x) \\
&= \frac{1}{m(B_R)} \int_{B_R} \int_X G(s,x)^p \, d\mu(x) dm(s) \\
&\leq \frac{1}{m(B_R)} \int_{B_R} \int_X G_{R+\varepsilon}(s,x)^p \, dm(s) d\mu(x) \\
&= \frac{1}{m(B_R)} \int_X \int_{B_R} SF_{R+\varepsilon}(s,x)^p \, dm(s) d\mu(x) \\
&\leq \frac{C^p}{m(B_R)} \int_X \int_{B_R} |F_{R+\varepsilon}(s,x)|^p \, dm(s) d\mu(x) \\
&= C^p \frac{m(B_{R+\varepsilon})}{m(B_R)} ||f||_p^p.
\end{aligned}$$

If we let $R \to \infty$ and use the polynomial volume growth of balls in $\mathbb{R}^d$, we conclude that

$$||Mf||_p \leq C||f||_p,$$

for all f in $L^p(X)$. □

The following theorem is now an easy and straightforward consequence of lemmata 3.1 and 3.2.

Theorem 3.3. *Suppose ν is a probability measure on $\mathbb{R}^n$ for which the maximal operator*

$$M\phi(x) = \sup_{\lambda > 0} \Big| \int_{\mathbb{R}^d} \phi(\lambda . t) \, d\nu(t) \Big|, \quad \phi \in \mathcal{S}(\mathbb{R}^d)$$

is of weak type (p,p) for some $1 \leq p < \infty$. For every standard Borel probability space $(X, \mathfrak{B}, \mu)$ and ergodic measure-preserving action T by $\mathbb{R}^d$ on X,

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^d} f(T_{\lambda . t} x) \, d\nu(t) = \int_X f \, d\mu$$

almost everywhere on X for every f in $L^p(X)$.

Proof. We can without loss of generality restrict our attention to f in the subspace $L_0^p(X)$. The almost sure convergence above holds for f in the dense subspace L of $L_0^p(X)$ defined in lemma 3.1. Thus, it suffices to prove that the subspace

$$\mathcal{C} = \{ f \in L^p(X) \mid \lim_{\lambda \to \infty} A_\lambda f(x) \quad \text{exists a.e.} \}$$

is closed in $L_0^p(X)$. Suppose f_k is a sequence in $\mathcal{C}$ which converge to f in $L_0^p(X)$, and note that if λ and η are positive, then for all k,

$$\begin{aligned}
|A_\lambda f(x) - A_\eta f(x)| &= |A_\lambda f_k(x) - A_\eta f_k(x)| + \\
&+ |A_\eta (f - f_k)(x)| + |A_\lambda (f - f_k)(x)|.
\end{aligned}$$

The first term clearly goes to zero almost everywhere, since f_k is in $\mathcal{C}$. Thus, since M is of weak type (p,p) we have

$$\begin{aligned}
\mu(\{x \in X \mid \limsup_{\lambda, \eta \to \infty} |A_\lambda f(x) - A_\eta f(x)| > \alpha\}) &\leq \mu(\{x \in X \mid 2M(f - f_k) > \alpha\}) \\
&\leq C \left(\frac{2}{\alpha} \right)^p ||f - f_k||_p^p,
\end{aligned}$$

which can be made arbitrarily small for k large. □

3.4. **Rubio de Francia's Maximal Inequality.** Let ν be a probability measure on $\mathbb{R}^d$ and define the maximal function

$$S\phi(x) = \sup_{\lambda>0} \Big| \int_{\mathbb{R}^d} \phi(\lambda t)\, d\nu(t) \Big|.$$

These functions were studied by J.L. Rubio de Francia in [14], where the following striking theorem was established.

Theorem 3.4. *Suppose ν is a compactly supported measure in $\mathbb{R}^d$ with Fourier exponent $a > 1$. Then S is strong type (p,p) for all $p > p_a$, where*

$$p_a = \frac{1+a}{a}.$$

By theorem 3.3, this result immediately implies the following theorem.

Theorem 3.5. *Let $(X, \mathfrak{B}, \mu)$ be a standard Borel probability measure space, and suppose T is a Borel measurable action of $\mathbb{R}^d$ on X which preserves μ. If ν is a compactly supported probabilty measure on $\mathbb{R}^d$ with Fourier exponent $a > 1$, then*

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}^d} f(T_{\lambda t}x)\, d\nu(t) = \int_X f\, d\mu$$

almost everywhere on X, for all f in $L^p(X)$ for $p > p_a$, where

$$p_a = \frac{1+a}{a}.$$

Weaker versions of Theorem 3.4 has been extended to a more general class of homogeneous dilations in [8].

4. Applications

We will now present some straightforward applications of theorems 3.5 and 3.3.

4.1. **Ergodic Theorems for Dilations of Submanifolds.** The following theorem can be found in chapter *VIII* in [15]

Theorem 4.1. *Suppose S is a smooth hypersurface in $\mathbb{R}^d$, whose Gaussian curvature is non-zero everywhere, and let $d\nu = \psi d\sigma$, where the support of $\psi \in C_0^\infty(\mathbb{R}^d)$ is assumed to intersect S in a compact set. Then the Fourier exponent of ν is at least $d-1$.*

By theorem 3.3 we have the following theorem.

Corollary 4.1. *Let S and ν be as in the theorem 4.1 above, and suppose $d \geq 3$. For every standard Borel probability space $(X, \mathfrak{B}, \mu)$ and ergodic measure-preserving action T by $\mathbb{R}^d$ on X,*

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}^d} f(T_{\lambda t}x)\, d\nu(t) = \int_X f\, d\mu$$

almost everywhere on X for every f in $L^p(X)$, where

$$p > \frac{d}{d-1}.$$

Remark. J. Bourgain [5] proved that if $d = 2$ and ν denotes the arclength measure on the boundary of a smooth centrally symmetric convex body in $\mathbb{R}^2$, then maximal operator in subsection 3.4 is of strong type (p,p) for $p > 2$. Hence, the analogue of corollary 4.1 holds in this case. The special case of circles was proved by M. Lacey in [11].

4.2. **Ergodic Theorems for Polynomial Curves.** Let $\lambda > 0$ and $q_1, \ldots, q_d$ non-zero real numbers. We define the polynomial curve $q : [0,1] \to \mathbb{R}^d$ by

$$q(w) = (q_1 w, \ldots, q_d w^d), \quad w \in [0,1]\,.$$

Let m denote the Lebesgue measure on $[0,1]$ and set $\nu = q_* m$, where q_* denotes the push–forward induced by q. Note that the Fourier dimension of ν is generally of order $2/d$ and thus theorem 1.1 does not apply. We introduce the homogeneous dilation on $\mathbb{R}^d$ defined by

$$\lambda.t = (\lambda t_1, \ldots, \lambda^d t_d), \quad t \in \mathbb{R}^d.$$

For $\phi \in \mathcal{S}$ we define the maximal operator

$$M\phi(t) = \sup_{\lambda > 0} \Big| \int_0^1 \phi(s - \lambda.q(w))\, dw \Big|.$$

The following theorem is due to E.M Stein and S. Wainger [17], who also studied maximal inequalities for more general curves.

Theorem 4.2. *The maximal operator M is of strong type (p,p) for $1 < p < \infty$.*

Corollary 4.2. *For every standard Borel probability space $(X, \mathfrak{B}, \mu)$ and ergodic measure-preserving action T by $\mathbb{R}^d$ on X,*

$$\lim_{\lambda \to \infty} \int_0^1 f(T_{\lambda.q(w)} x)\, dw = \int_X f\, d\mu$$

almost everywhere on X for every f in $L^p(X)$, where

$$p > \frac{d}{d-1}.$$

4.3. **Ergodic Theorems for Salem Sets.** The Hausdorff dimension of a subset E of $\mathbb{R}^d$ can be defined as the supremum of the a in $[0,n]$ such that, for some probability measure ν which is supported on E,

$$\int_{|\xi| \geq 1} |\hat{\nu}(\xi)|^2\, |\xi|^{a-n}\, d\xi < \infty.$$

This reformulation of the standard definition is justified by Frostman's lemma; see e.g. [12]. Note that this does not imply that ν is a Rajchman measure. However, if

$$|\hat{\nu}(\xi)| \leq \frac{C}{|\xi|^{a/2}},$$

then the integral above is finite, and we conclude that the Fourier dimension of ν is always majorized by the Hausdorff dimension. If E supports a probability measure with Fourier dimension equal to the Hausdorff dimension of E, we say that E is *Salem set*. With no known exception, these sets are constructed either from probabilistic arguments or number theoretic arguments. It is known (see e.g. [10]) that the image of any closed subset E' of $\mathbb{R}_+$ under a d–dimensional Brownian motion B is almost surely a Salem set with Fourier dimension equal to twice the Hausdorff dimension of E'. Thus, if E' is any compact subset of $\mathbb{R}_+$ of Hausdorff dimension $b > 1/2$ and $d \geq 2$ then, almost surely, there is a probability measure ν on the (almost surely totally disconnected) on $B(E')$ such that theorem 1.1 holds for all $p \geq \frac{2b}{2b-1}$. The measure ν depends of course on the Brownian motion. This is reminiscent to J. Bourgain's [6] theorem on random subsets of $\mathbb{Z}$ for which the pointwise ergodic theorem holds.

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