CONVERGENCE RATES FOR AN OPTIMALLY CONTROLLED GINZBURG-LANDAU EQUATION

MATTIAS SANDBERG

Abstract. An optimal control problem related to the probability of transition between stable states for a thermally driven Ginzburg-Landau equation is considered. The value function for the optimal control problem with a spatial discretization is shown to converge quadratically to the value for the original problem. This is done by using that the value functions solve Hamilton-Jacobi equations, the equation for the original problem being defined on an infinite dimensional Hilbert space. Time discretization is performed using the Symplectic Euler method. Imposing a reasonable condition this method is shown to be convergent of order one in time, with a constant independent of the spatial discretization.

Contents

1. Introduction

We shall in this paper study the convergence of the Symplectic Euler scheme for approximating optimal control of the real Ginzburg-Landau equation. This follows the work developed in [19], where a convergence result for the value function to an optimally controlled ODE is obtained using the corresponding Hamilton-Jacobi equation. As there exists a rigorous theory also for infinite-dimensional Hamilton-Jacobi equations, developed by M. Crandall and P.-L. Lions [6–12], it is possible to perform a convergence analysis for a spatial discretization of an optimally controlled PDE, using that the value function is a viscosity solution to an infinite-dimensional Hamilton-Jacobi equation. In this paper the analysis is performed for the

2000 *Mathematics Subject Classification.* 49M29, 65M12.

Key words and phrases. Ginzburg-Landau Equation, Optimal Control, Hamilton-Jacobi, Error Estimates, Stochastic Partial Differential Equation, Symplectic Euler.

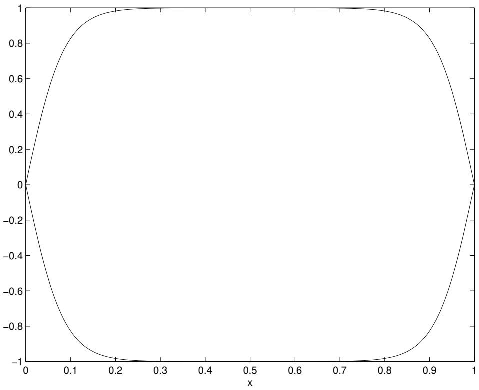

FIGURE 1.1. Upper curve: φ_+. Lower curve: φ_-.

specific problem at hand, but hopefully the analysis is clear enough to make adaptations to other circumstances (fairly) easy.

Consider the stochastic PDE

$$\varphi_t = \delta\varphi_{xx} - \delta^{-1}V'(\varphi) + \sqrt{\varepsilon}\eta, \quad \text{in } [0,T]\times[0,1], \tag{1.1}$$

where η is white noise in two dimensions; this means that η is a random Gaussian distribution with zero mean and covariance

$$E\big(\eta(x,t),\eta(x',t')\big) = \bar\delta(x-x')\bar\delta(t-t'),$$

where E denotes the expectation and $\bar\delta$ the Dirac delta distribution. The "state" variable φ satisfies the Dirichlet boundary conditions

$$\varphi(t,0) = \varphi(t,1) = 0,$$

and V is the "double-well" potential

$$V(\varphi) = \frac{1}{4}(\varphi^2-1)^2;$$

see Figure 1.2. In one space dimension the stochastic PDE (1.1) makes sense. Taking $\varepsilon = 0$, the solutions to (1.1) generically approach one of the two stable critical points, φ_+ or φ_-, (see Figure 1.1), which constitute minima to the energy

$$\int_0^1 \big(\frac{\delta}{2}\varphi_x^2 + \delta^{-1}V(\varphi)\big)dx.$$

With a small ε, the solutions to (1.1) spend most of the time in the vicinity of either φ_+ or φ_-, but as rare events make the transition from one to the other. The equation (1.1) may therefore be taken as a model for thermally driven phase transitions, nucleations, etc.

The probability of jumping from φ_+ to φ_- in the finite time T is related to the action functional

$$I(\varphi) = \frac{1}{2}\int_0^T\int_0^1 \big(\varphi_t - \delta\varphi_{xx} + \delta^{-1}V'(\varphi)\big)^2 dx\,dt. \tag{1.2}$$

Introduce the probability P_T that a solution φ to (1.1), with $\varphi(0) = \varphi_+$, satisfies $\varphi(T) \in S$, where S is an open subset of the set of continuous functions on the spatial interval $[0, 1]$. Theory of large deviations in [16] gives that

$$-I(S) \leq \liminf_{\varepsilon \to 0} \varepsilon \log P_T$$

and

$$\limsup_{\varepsilon \to 0} \varepsilon \log P_T \leq -I(\bar{S})$$

where

$$I(S) = \inf I(\varphi),$$

with the infimum in the last equality taken over all continuous functions φ in $[0, T] \times [0, 1]$ starting at φ_+ and ending in S, and where $\bar{S}$ is the closure of S. By taking S a small neighborhood of φ_- we can for small ε approximate the probability of transition from φ_+ to φ_- with

$$P_T \approx e^{-I(S)/\varepsilon}.$$

In [14] the minimization of (1.2) is performed for $\varphi(0) = \varphi_+$ and $\varphi(T) = \varphi_-$ using optimization of a finite difference approximation. In this paper $\varphi(T)$ will not be held fixed, but instead a penalty cost at the final time is added to the functional (1.2) in order to force the solution to end up close to φ_-. The optimal control problem which will be considered here is the following. Minimize, over all $\alpha \in L^2\big(0, T; L^2(0, 1)\big)$, the functional $v_{\varphi_+,0}(\alpha)$, defined by

$$v_{\varphi_0,t_0}(\alpha) = \int_{t_0}^{T} h\big(\alpha(t)\big)dt + g\big(\varphi(T)\big) \tag{1.3}$$

where φ is a mild solution to

$$\varphi_t = \delta\varphi_{xx} - \delta^{-1}V'(\varphi) + \alpha, \quad \varphi(t_0) = \varphi_0. \tag{1.4}$$

In order to define mild solutions we denote by $S(t)$ the contraction semigroup of bounded linear operators on $L^2(0, 1)$ generated by $\delta\, d^2/dx^2$ defined on $H_0^1(0, 1) \cap H^2(0, 1)$. A mild solution to (1.4) is a function $\varphi \in C(t_0, T; L^2)$ such that, for all $t_0 \leq t \leq T$,

$$\varphi(t) = S(t - t_0)\varphi_0 + \int_{t_0}^{T} S(t - s)\big(- \delta^{-1}V'(\varphi(s)) + \alpha(s)\big)ds. \tag{1.5}$$

In the appendix existence and uniqueness of weak solutions in $C(t_0, T; H_0^1)$ of (1.4) is proved. Such weak solutions are also mild solutions (see e.g. page 105 in [18]). Furthermore, with α bounded in $L^2(t_0, T; L^2)$, the weak solution is bounded in $C(t_0, T; H_0^1)$. Hence, the potential V may be changed outside an interval $[-s, s]$ without changing the result of the optimal control problem. For simplicity, we shall henceforth use the potential $\tilde{V}$ in Figure 1.2 and quickly change notation, so that we let $V \equiv \tilde{V}$, i.e. V is given by the

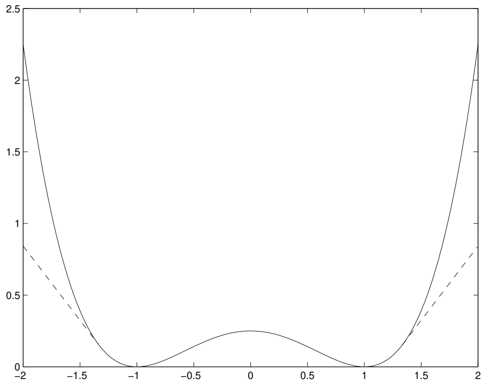

FIGURE 1.2. The original potential V is drawn with the solid line. The modified potential $\tilde{V}$ coincides with V in the interval $[-s,s]$ and is drawn with dashed lines outside that interval.

dashed line. Letting the transition from the interval $[-s,s]$ to the outside be a smooth one we can assume that arbitrarily many derivatives of V are bounded. The running cost, h, corresponds to the action functional (1.2) as

$$h(\alpha) = ||\alpha||^2_{L^2(0,1)}/2, \tag{1.6}$$

and the final cost is the squared L^2 distance from φ_-,

$$g(\varphi) = K||\varphi - \varphi_-||^2_{L^2(0,1)}, \tag{1.7}$$

where K is a constant large enough to force $\varphi(T) \approx \varphi_-$. We denote by u the *value function*, i.e. the best possible value of (1.3) for each starting position (φ_0, t_0):

$$u(\varphi_0, t_0) = \inf\left\{v_{\varphi_0,t_0}(\alpha) \ : \ \alpha \in L^2\big(t_0, T; L^2(0,1)\big)\right\} \tag{1.8}$$

Notation: We henceforth let $||\cdot||$ and $(\cdot,\cdot)$ be the L^2 norm and inner product on $(0,1)$, and $|\cdot|$ be the supremum norm on $\mathbb{R}$. The Dirac delta distribution will be denoted $\bar{\delta}$, as δ is used as the diffusivity constant.

Outline: Section 2 contains some facts regarding the value function, which are applied in Section 3 when the error from the spatial discretization is established. In Section 4 convergence of the time discretization using the Symplectic Euler method is examined. Under a reasonable condition, this method is shown to be convergent of order one, with a constant independent of the spatial discretization. Numerical results with examples of the convergence rate for discretization in both space and time is given in Section 5.

2. PRELIMINARIES

This section contains some results which will be needed when the spatial discretization error bound is established in Section 3. We start with a theorem about the boundedness of optimal controls.

Theorem 2.1. *For all $\varphi_0 \in L^2(0,1)$ and $0 \le t_0 \le T$ the value function u satisfies*

$$u(\varphi_0, t_0) = \inf\{v_{\varphi_0,t_0}(\alpha) \ : \ ||\alpha||_{L^\infty(t_0,T;L^2)} \le E||\varphi_0|| + F\}$$

where the constants E and F depend on δ, K, φ_-, T, $|V'|$ and $|V''|$, but not on φ_0 and t_0.

Proof. It is first shown that with $\alpha(t) = 0$, for all t, the state variable at the terminal time, $\varphi(T)$, is bounded in L^2 by a constant which depends on the starting point φ_0. This can be done by taking the inner product with φ in (1.4), noting that $(\varphi, \varphi_t) = \frac{d}{dt}||\varphi||^2/2$, and using $||\varphi_x||^2 \ge 0$. Hence

$$\frac{d}{dt}||\varphi||^2/2 \le -\delta^{-1}\big(V'(\varphi), \varphi\big) \le \delta^{-1}|V'| \cdot ||\varphi||,$$

and thereby

$$\frac{d}{dt}||\varphi|| \le \delta^{-1}|V'|,$$

so that $||\varphi(T)||$ is bounded by $||\varphi_0|| + \delta^{-1}|V'|T$. Hence the final cost, $g\big(\varphi(T)\big)$, is bounded in terms of the starting position:

$$\begin{aligned} g\big(\varphi(T)\big) = K||\varphi(T) - \varphi_-||^2 &\le 2K\big(||\varphi(T)||^2 + ||\varphi_-||^2\big) \\ &\le 4K||\varphi_0||^2 + 4K\delta^{-2}|V'|^2T^2 + 2K||\varphi_-||^2 =: M. \end{aligned}$$

It therefore holds that

$$u(\varphi_0, t_0) = \inf\{v_{\varphi_0,t_0}(\alpha) \ : \ ||\alpha||^2_{L^2(t_0,T;L^2)} \le 2M\}$$

For all α bounded by $\sqrt{2M}$ in $L^2(t_0, T; L^2)$ we have that $\varphi(T)$ is bounded, again by taking the inner product with φ in (1.4):

$$\frac{1}{2}\frac{d}{dt}||\varphi||^2 \le \delta^{-1}|V'| \cdot ||\varphi|| + ||\alpha|| \cdot ||\varphi||,$$

which implies $\frac{d}{dt}||\varphi|| \le \delta^{-1}|V'| + ||\alpha||$ and so

$$\begin{aligned} ||\varphi(T)|| &\le ||\varphi_0|| + \delta^{-1}|V'|T + \int_{t_0}^T ||\alpha||dt \\ &\le ||\varphi_0|| + \delta^{-1}|V'|T + \sqrt{T}||\alpha||_{L^2(t_0,T;L^2)} \\ &\le ||\varphi_0|| + \delta^{-1}|V'|T + \sqrt{2T}\sqrt{M} \le E||\varphi_0|| + F, \end{aligned}$$

for some constants E and F which do not depend on φ_0 and t_0.

It shall now be proved that changing the control α a small amount changes the state φ a small amount. We shall therefore compare two solutions, φ^1 and φ^2, both starting at (φ_0, t_0), such that φ^1 solves (1.4) with control α^1 and φ^2 with control α^2. Subtract the two evolution equations and take the inner product with $\varphi^1 - \varphi^2$ to obtain

$$\frac{1}{2}\frac{d}{dt}||\varphi^1 - \varphi^2||^2 + \delta||\varphi^1_x - \varphi^2_x||^2 = \\ \delta^{-1}(-V'(\varphi^1) + V'(\varphi^2), \varphi^1 - \varphi^2) + (\alpha^1 - \alpha^2, \varphi^1 - \varphi^2)$$

which, by the boundedness of V'', entails

$$\frac{d}{dt}||\varphi^1 - \varphi^2|| \leq \delta^{-1}|V''| \cdot ||\varphi^1 - \varphi^2|| + ||\alpha^1 - \alpha^2||.$$

By Grönwall's lemma we therefore have that

$$||\varphi^1(T) - \varphi^2(T)|| \leq \exp(\delta^{-1}|V''|T) \int_{t_0}^{T} ||\alpha^1 - \alpha^2|| dt,$$

so, provided α^1 and α^2 are both bounded by $\sqrt{2M}$ in $L^2(t_0, T; L^2)$, the difference in terminal cost has the following bound:

$$\begin{aligned}
|g\big(\varphi^1(T)\big) - g\big(\varphi^2(T)\big)| &= K\big|||\varphi^1(T) - \varphi_-||^2 - ||\varphi^2(T) - \varphi_-||^2\big| \\
&= K\big|(\varphi^1(T) + \varphi^2(T) - 2\varphi_-, \varphi^1(T) - \varphi^2(T))\big| \\
&\leq 2K(E||\varphi_0|| + F + ||\varphi_-||) \cdot ||\varphi^1(T) - \varphi^2(T)|| \\
&\leq R \int_{t_0}^{T} ||\alpha^1 - \alpha^2|| dt,
\end{aligned}$$

where

$$\begin{aligned}
R &= 2K \exp(\delta^{-1}|V''|T)(E||\varphi_0|| + F + ||\varphi_-||) \\
&=: E'||\varphi_0|| + F',
\end{aligned} \tag{2.1}$$

with the constants E' and F' independent of φ_0 and t_0. Let now α^1 be a control bounded by $\sqrt{2M}$ in $L^2(t_0, T; L^2)$ and let α^2 be a modification of α^1:

$$\alpha^2(t) = \begin{cases} \alpha^1(t), & \text{for all } t \text{ such that } ||\alpha^1(t)|| \leq 2R, \\ 0, & \text{otherwise.} \end{cases}$$

The difference in the terminal cost thus has the bound

$$|g(\varphi^1(T)) - g(\varphi^2(T))| \leq R \int_{\{t:||\alpha^1(t)||>2R\}} ||\alpha^1|| dt,$$

while the difference in running cost is

$$\int_{\{t:||\alpha^1(t)||>2R\}} \frac{||\alpha^1||^2}{2} dt \geq R \int_{\{t:||\alpha^1(t)||>2R\}} ||\alpha^1|| dt,$$

so

$$v_{\varphi_0, t_0}(\alpha^2) \leq v_{\varphi_0, t_0}(\alpha^1).$$

Hence for any control bounded in $L^2(t_0, T; L^2)$ there is another control, bounded by $2E'||\varphi_0|| + 2F'$ in $L^\infty(t_0, T; L^2)$, which gives a smaller or equal value functional. $\square$

With Theorem 2.1 the theory in [4] may be used, which establishes existence of optimal controls to u in (1.8). We state this in a corollary.

Corollary 2.2. *For each* $(\varphi_0, t_0) \in L^2(0,1) \times [0,T]$ *there exists a minimizer* α, *bounded in* $L^\infty(t_0, T; L^2)$, *in* (1.8).

Proof. Use Theorem 2.1 and [4]. $\square$

Theorem 2.1 is also used when proving Theorem 2.3 about semiconcavity. In [3] a theorem on semiconcavity on $L^2(0,1) \times [0,T)$ is established. This result could have been used in this paper, but as only the weaker result of semiconcavity on $H_0^1(0,1) \times [0,T]$ is needed for our purposes, an easier proof is given for this case.

Theorem 2.3. *The restriction of the value function to* $H_0^1 \times [0,T]$ *is semiconcave.*

Proof. It will be shown that for every constant B, every closed interval $I \subset [0,T)$, and all starting positions (φ_0^1, t^1) and (φ_0^2, t^2) with $||\varphi_0^1||_{H_0^1(0,1)} + ||\varphi_0^2||_{H_0^1(0,1)} \leq B$ and $t^1, t^2 \in I$, there is a constant C such that

$$u(\varphi_0^1, t^1) + u(\varphi_0^2, t^2) - 2u\Big(\frac{\varphi_0^1 + \varphi_0^2}{2}, \frac{t^1 + t^2}{2}\Big) \leq C(||\varphi_0^1 - \varphi_0^2||_{H_0^1}^2 + |t^1 - t^2|^2).$$

In order to keep constants simple we use that u may be defined in $H_0^1(0,1) \times (-\infty, T]$, so that we may set $t^1 = h$ and $t^2 = -h$, and realize that the result for other times follows analogously. In this proof we let C be any constant which may depend on B.

Let $\alpha : [0,T] \to L^2$ be an optimal control for the cost functional $v_{\frac{\varphi_0^1+\varphi_0^2}{2},0}$ defined in (1.3), and let $\varphi^3 : [0,T] \to H_0^1$ be the corresponding state. Define controls for solutions starting in (φ_0^1, h) and $(\varphi_0^2, -h)$ by dilations of α as

$$\alpha^1(t) = \alpha\Big(T\frac{t-h}{T-h}\Big),$$
$$\alpha^2(t) = \alpha\Big(T\frac{t+h}{T+h}\Big),$$

and let the corresponding states be denoted $\varphi^1 : [h,T] \to H_0^1$ with $\varphi^1(h) = \varphi_0^1$ and $\varphi^2 : [-h,T] \to H_0^1$ with $\varphi^2(-h) = \varphi_0^2$. The evolution equation (1.4) for φ^1 and φ^2 is now transformed to the interval $[0,T]$. The following equations are thereby obtained:

$$\varphi_t^1 = \frac{T-h}{T}(\delta\varphi_{xx}^1 - \delta^{-1}V'(\varphi^1) + \alpha), \quad \varphi^1(0) = \varphi_0^1,$$
$$\varphi_t^2 = \frac{T+h}{T}(\delta\varphi_{xx}^2 - \delta^{-1}V'(\varphi^2) + \alpha), \quad \varphi^2(0) = \varphi_0^2,$$
$$\varphi_t^3 = \delta\varphi_{xx}^3 - \delta^{-1}V'(\varphi^3) + \alpha, \quad \varphi^3(0) = \frac{\varphi_0^1 + \varphi_0^2}{2}.$$

The function

$$z(t) = \varphi^1(t) + \varphi^2(t) - 2\varphi^3(t)$$

is now introduced. We will obtain a bound for $||z(T)||$. The equation solved by z is

$$z_t = \delta z_{xx} - \delta^{-1}\big(V'(\varphi^1) + V'(\varphi^2) - 2V'(\varphi^3)\big) + \frac{\delta h}{T}(\varphi^2 - \varphi^1)_{xx} + \frac{\delta h}{T}\big(V'(\varphi^1) - V'(\varphi^2)\big). \quad (2.2)$$

It is therefore necessary to find a bound for $\varphi^1 - \varphi^2$. The evolution equation for $\varphi^1 - \varphi^2$ is

$$(\varphi^1 - \varphi^2)_t = \delta(\varphi^1 - \varphi^2)_{xx} - \delta^{-1}\big(V'(\varphi^1) - V'(\varphi^2)\big) - \frac{\delta h}{T}(\varphi^1 + \varphi^2)_{xx} + \frac{\delta h}{T}\big(V'(\varphi^1) + V'(\varphi^2)\big) - \frac{2h}{T}\alpha. \quad (2.3)$$

After the inner product is taken with $\varphi^1 - \varphi^2$ the following inequality is obtained:

$$\frac{1}{2}\frac{d}{dt}||\varphi^1 - \varphi^2||^2 \le \delta^{-1}|V''|\cdot||\varphi^1 - \varphi^2||^2 + \frac{\delta h}{T}||\varphi^1_{xx} + \varphi^2_{xx}||\cdot||\varphi^1 - \varphi^2|| + \frac{\delta^{-1}h}{T}||V'(\varphi^1) + V'(\varphi^2)||\cdot||\varphi^1 - \varphi^2|| + \frac{2h}{T}||\alpha||\cdot||\varphi^1 - \varphi^2||,$$

and hence

$$\frac{d}{dt}||\varphi^1 - \varphi^2|| \le \delta^{-1}|V''|\cdot||\varphi^1 - \varphi^2|| + \frac{\delta h}{T}||\varphi^1_{xx} + \varphi^2_{xx}|| + \frac{\delta^{-1}h}{T}||V'(\varphi^1) + V'(\varphi^2)|| + \frac{2h}{T}||\alpha||.$$

Thus, by Grönwall's Lemma,

$$||\varphi^1(t) - \varphi^2(t)|| \le e^{\delta^{-1}|V''|T}||\varphi^1(0) - \varphi^2(0)|| + e^{\delta^{-1}|V''|T}\frac{h}{T}\int_0^T \big(\delta||\varphi^1_{xx} + \varphi^2_{xx}|| + \delta^{-1}||V'(\varphi^1) + V'(\varphi^2)|| + 2||\alpha||\big)\,dt.$$

Since $||\varphi^1_0||_{H^1_0(0,1)} + ||\varphi^2_0||_{H^1_0(0,1)} \le B$ it follows that φ^1 and φ^2 are bounded by a constant C in $L^2(0,T;H^2)$; see [15]. Together with the fact that V' is bounded this implies that

$$||\varphi^1(t) - \varphi^2(t)|| \le C(||\varphi^1_0 - \varphi^2_0|| + h), \quad \text{for all } 0 \le t \le T. \quad (2.4)$$

Equation (2.3) is now used once again together with the fact that $|V'(\varphi^1) - V'(\varphi^2)| \le |V''|\cdot|\varphi^1 - \varphi^2|$ and Theorem 5 on page 360 in [15], to draw the conclusion that

$$\operatorname*{ess\,sup}_{0\le t\le T}||\varphi^1(t) - \varphi^2(t)||_{H^1_0} + ||\varphi^1_{xx} - \varphi^2_{xx}||_{L^2(0,T;L^2)} \le C(||\varphi^1_0 - \varphi^2_0||_{H^1_0} + h). \quad (2.5)$$

There is also the term $V'(\varphi^1) + V'(\varphi^2) - 2V'(\varphi^3)$ in (2.2). This can be handled as

$$\begin{aligned}
&|V'(\varphi^1) + V'(\varphi^2) - 2V'(\varphi^3)| \\
&\quad \le |V'(\varphi^1) + V'(\varphi^2) - 2V'(\frac{\varphi^1+\varphi^2}{2})| + 2|V'(\frac{\varphi^1+\varphi^2}{2}) - V'(\varphi^3)| \\
&\qquad \le \frac{|V'''|}{2}|\varphi^1 - \varphi^2|^2 + |V''| \cdot |z|. \qquad (2.6)
\end{aligned}$$

We are now ready to take the inner product with z in (2.2) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||z||^2 \le \frac{\delta^{-1}|V'''|}{2}\int_0^1 (\varphi^1-\varphi^2)^2|z|dx + \delta^{-1}|V''|\cdot||z||^2 \\
+ \frac{\delta h}{T}||\varphi^1_{xx} - \varphi^2_{xx}|| \cdot ||z|| + |V''|\frac{\delta^{-1}h}{T}||\varphi^1 - \varphi^2|| \cdot ||z||,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d}{dt}||z|| \le \delta^{-1}|V''|\cdot||z|| + \frac{\delta^{-1}|V'''|}{2}||\varphi^1-\varphi^2||^2_{L^4(0,1)} \\
+ \frac{\delta h}{T}||\varphi^1_{xx} - \varphi^2_{xx}|| + |V''|\frac{\delta^{-1}h}{T}||\varphi^1 - \varphi^2||.
\end{aligned}$$

By Grönwall's Lemma

$$\begin{aligned}
||z(T)|| \le e^{\delta^{-1}|V''|T}\int_0^T \Big(\frac{\delta^{-1}|V'''|}{2}||\varphi^1-\varphi^2||^2_{L^4(0,1)} \\
+ \frac{\delta h}{T}||\varphi^1_{xx} - \varphi^2_{xx}|| + |V''|\frac{\delta^{-1}h}{T}||\varphi^1 - \varphi^2||\Big)dt. \qquad (2.7)
\end{aligned}$$

Sobolev's inequality gives that $||\varphi^1-\varphi^2||_{L^4(0,1)} \le C||\varphi^1-\varphi^2||_{H^1_0(0,1)}$, so (2.5) together with (2.7) implies that

$$||z(T)|| \le C\big(||\varphi^1_0 - \varphi^2_0||^2_{H^1_0(0,1)} + h^2\big).$$

This fact is now used to show that

$$v_{\varphi^1_0,h}(\alpha^1) + v_{\varphi^2_0,-h}(\alpha^2) - 2v_{\frac{\varphi^1_0+\varphi^2_0}{2},0}(\alpha) \le C\big(||\varphi^1_0-\varphi^2_0||^2 + h^2\big), \qquad (2.8)$$

The terminal cost is treated first. We use the notation $\varphi_T \equiv \varphi(T)$ and perform a simple rearrangement:

$$\begin{aligned}
&|g(\varphi^1_T) + g(\varphi^2_T) - 2g(\varphi^3_T)| \\
&\quad = \big|\frac{K}{2}||\varphi^1_T - \varphi^2_T||^2 + K\big(\frac{\varphi^1_T+\varphi^2_T}{2} + \varphi^3_T - 2\varphi_-, \varphi^1_T + \varphi^2_T - 2\varphi^3_T\big)\big| \\
&\qquad \le C\big(||\varphi^1_0 - \varphi^2_0||^2_{H^1_0} + h^2\big), \qquad (2.9)
\end{aligned}$$

where (2.4), (2.7), and the fact that φ_T^1, φ_T^2 and φ_T^3, are bounded are used. The running costs must also be treated. A simple calculation shows that

$$\int_h^T ||\alpha^1||^2 dt + \int_{-h}^T ||\alpha^2||^2 dt - 2\int_0^T ||\alpha||^2 dt = 0. \tag{2.10}$$

The desired result (2.8) follows from (2.9) and (2.10). □

3. Discretization in space

We shall compare the value functions associated with our original problem and a finite element approximation. The value function we want to approximate is u defined in (1.8). The approximate value function is, similarly as in (1.8),

$$\bar{u}(\bar{\varphi}_0, t_0) = \inf_{\bar{\alpha}\in L^2(t_0,T;V)} \Big(g(\bar{\varphi}(T)) + \int_t^T h(\bar{\alpha}) ds \Big) \quad , \bar{\varphi}(t_0) = \bar{\varphi}_0, \tag{3.1}$$

where $\bar{\varphi} \in C(t_0, T; V)$ solves

$$(\bar{\varphi}_t, v) = -\delta(\bar{\varphi}_x, v_x) + (-\delta^{-1} V'(\bar{\varphi}) + \bar{\alpha}, v), \quad \text{for all } v \in V, \tag{3.2}$$

and V is the space of continuous piecewise linear functions on $[0,1]$ which are zero at 0 and 1 and linear on the intervals $(0, \Delta x)$, $(\Delta x, 2\Delta x)$, and so on. We note that the infima in (1.8) and (3.1) are attained, using Corollary 2.2 for the original problem (1.8) and the easier theory in [5] for the approximation problem (3.1). Therefore we can replace the infima with minima.

We now introduce some notation needed in Theorem 3.1. We denote by P the L^2 projection from $L^2(0,1)$ to V or from $L^2(0,1)\times\mathbb{R}$ to $V\times\mathbb{R}$. Let Ω be an open subset of a Hilbert space X, and $z : \Omega \to \mathbb{R}$. For any $x_0 \in \Omega$ the superdifferential $D^+z(x_0)$ is defined as follows:

$$D^+z(x_0) = \Big\{ p \in X \big| \limsup_{x\to x_0} \frac{z(x) - z(x_0) - (p, x - x_0)_X}{|x - x_0|_X} \le 0 \Big\}.$$

Semiconcave functions in finite dimensional spaces possess (one-sided) directional derivatives

$$\partial z(x, \theta) = \lim_{h\to 0^+} \frac{z(x + h\theta) - z(x)}{h}$$

at any point x and in any direction θ; see [5]. The Hamiltonian, H, for the optimal control problem (1.8) is given by

$$H(\lambda, \varphi) = -\delta(\lambda_x, \varphi_x) - \delta^{-1}\big(\lambda, V'(\varphi)\big) - ||\lambda||^2/2, \tag{3.3}$$

for all λ, $\varphi \in H_0^1(0,1)$. The restrictions of u to the subspaces $V \times [0,T]$ and $H_0^1 \times [0,T]$ are denoted u_V and u_H.

Theorem 3.1. *Let $\varphi_0 \in V$. Denote an optimal pair (control and state) for $u(\varphi_0, t_0)$ by $\alpha : [t_0, T] \to L^2$ and $\varphi : [t_0, T] \to L^2$ and an optimal pair for*

$\bar u(\varphi_0,t_0)$ by $\bar\alpha:[t_0,T]\to V$ and $\bar\varphi:[t_0,T]\to V$. Then

$$\begin{aligned}\int_{t_0}^{T}\Big(p_t^*(s)+H\big(Pp_\varphi^*(s),\bar\varphi(s)\big)\Big)ds&\\ &\le \bar u(\varphi_0,t_0)-u(\varphi_0,t_0)\\ \le g\big(P\varphi(T)\big)-g\big(\varphi(T)\big)+\int_{t_0}^{T}\Big(H\big(p_\varphi^{\#}(s),P\varphi(s)\big)-H\big(p_\varphi^{\#}(s),\varphi(s)\big)\Big)ds&\end{aligned}\tag{3.4}$$

where $p^*(s)=\big(p_\varphi^*(s),p_t^*(s)\big)\in L^2(0,1)\times\mathbb{R}$ is any measurable function with values in $D^+u_H(\bar\varphi(s),s)$, and $p^{\#}(s)=\big(p_\varphi^{\#}(s),p_t^{\#}(s)\big)\in V\times\mathbb{R}$ is any measurable function with values in $D^+\bar u(P\varphi(s),s)$.

Proof. We divide the proof into two steps: In *Step 1* we obtain a lower bound for $\bar u(\varphi_0,t_0)-u(\varphi_0,t_0)$, and in *Step 2* we do likewise for $u(\varphi_0,t_0)-\bar u(\varphi_0,t_0)$.

Step 1. Using the definitions (1.8) and (3.1) for u and $\bar u$, the fact that $\bar u\big(\bar\varphi(T),T\big)=g\big(\bar\varphi(T)\big)$, and that u_H is the restriction of u to $H_0^1\times[0,T]$, we can write

$$\begin{aligned}\bar u(\varphi_0,t_0)-u(\varphi_0,t_0)&=u_H\big(\bar\varphi(T),T\big)-u_H\big(\bar\varphi(t_0),t_0\big)+\int_{t_0}^{T}h(\bar\alpha)ds\\ &=\int_{t}^{T}\Big(\frac{d}{ds}u_H\big(\bar\varphi(s),s\big)+h\big(\bar\alpha(s)\big)\Big)ds,\end{aligned}\tag{3.5}$$

since $u_H(\bar\varphi(s),s)$ is absolutely continuous as u is locally Lipschitz continuous (see [3]) and $\bar\varphi$ is absolutely continuous as a function of s.

We now use that u_H is a semiconcave function (with linear modulus), so that for every $p\in D^+u_H(z_0)$ there exists a constant K such that

$$u_H(z)-u_H(z_0)-(p,z-z_0)\le K|z-z_0|^2\tag{3.6}$$

for all z in a neighborhood of $z_0\in H_0^1(0,1)\times\mathbb{R}$; see [2]. Let now $p^*(s)=\big(p_\varphi^*(s),p_t^*(s)\big)$ be any element in $D^+u_H\big(\bar\varphi(s),s\big)\cap\big(L^2(0,1)\times\mathbb{R}\big)$. Consider a point s where the derivative $\bar\varphi_t(s)$ exists. A lower bound for the backward derivative of $u_H\big(\bar\varphi(s),s\big)$ will now be obtained. We split the difference quotient approximating the backward derivative at s:

$$\begin{aligned}&\frac{u_H(\bar\varphi(s),s)-u_H(\bar\varphi(s-\Delta s),s-\Delta s)}{\Delta s}\\ &=-\ \big[u_H\big(\bar\varphi(s-\Delta s),s-\Delta s\big)-u_H\big(\bar\varphi(s),s\big)\\ &\quad-p_t^*(s)(-\Delta s)-\big(p_\varphi^*(s),\bar\varphi(s-\Delta s)-\bar\varphi(s)\big)\big]/\Delta s\\ &\quad+p_t^*(s)+\Big(p_\varphi^*(s),\frac{\bar\varphi(s)-\bar\varphi(s-\Delta s)}{\Delta s}\Big).\end{aligned}$$

If equation (3.6) is used together with the fact that $\bar\varphi$ is differentiable at s it can be deduced that the quotient involving the square bracket in the above equation is greater than or equal to $-K'\Delta s$, for some constant K'. Letting $\Delta s \to 0$ we see that

$$\frac{d}{ds} u_H\big(\bar\varphi(s), s\big) \geq p_t^*(s) + \big(p_\varphi^*(s), \bar\varphi_t(s)\big),$$

where (temporarily) d/ds denotes the backward derivative. In order to be able to apply (3.2) we note that $\bar\varphi_t \in V$ implies

$$(p_\varphi^*, \bar\varphi_t) = (Pp_\varphi^*, \bar\varphi_t).$$

Thus the integrand in (3.5), using the backward derivative, can be bounded from below as follows:

$$\begin{aligned}
\frac{d}{ds} u_H\big(\bar\varphi(s), s\big) + h\big(\bar\alpha(s)\big) &\geq p_t^*(s) + \big(\bar\varphi_t(s), Pp_\varphi^*(s)\big) + h\big(\bar\alpha(s)\big) \\
&= p_t^*(s) - \delta\big(\bar\varphi_x(s), (Pp_\varphi^*(s))_x\big) - \delta^{-1}\Big(V'\big(\bar\varphi(s)\big), Pp^*(s)\Big) \\
&\qquad + \big(\bar\alpha(s), Pp_\varphi^*(s)\big) + \frac{1}{2}||\bar\alpha(s)||^2 \\
&\qquad\qquad \geq p_t^*(s) + H\big(Pp_\varphi^*(s), \bar\varphi(s)\big),
\end{aligned}$$

since

$$H(\lambda,\varphi) = \min_{\alpha\in L^2(0,1)} \Big(- \delta(\varphi_x, \lambda_x) - \delta^{-1}\big(V'(\varphi), \lambda\big) + (\alpha, \lambda) + \frac{1}{2}||\alpha||^2\Big).$$

The double sided and the backward time derivatives of $u_H(\bar\varphi(s), s)$ differ on a set of measure zero, so there is no problem in using the backward derivative in (3.5).

Step 2. Lower bound for $u(\varphi_0, t_0) - \bar u(\varphi_0, t_0)$. It is now assumed that $\varphi_0 \in V$. Similarly as in *Step 1* we write, noting that $\bar u$ is only defined on $V \times [0, T]$,

$$\begin{aligned}
&u(\varphi_0, t_0) - \bar u(\varphi_0, t_0) \\
&= g\big(\varphi(T)\big) - g\big(P\varphi(T)\big) + \bar u\big(P\varphi(T), T\big) - \bar u\big(P\varphi(t_0), t_0\big) + \int_{t_0}^T h\big(\alpha(s)\big)ds \\
&= g\big(\varphi(T)\big) - g\big(P\varphi(T)\big) + \int_{t_0}^T \Big(\frac{d}{ds}\bar u\big(P\varphi(s), s\big) + h\big(\alpha(s)\big)\Big)ds =: I + II.
\end{aligned} \tag{3.7}$$

A lower bound for part II is obtained by splitting the difference quotient approximating the backward derivative at s:

$$\begin{aligned}
&\frac{\bar u\big(P\varphi(s), s\big) - \bar u\big(P\varphi(s-\Delta s), s-\Delta s\big)}{\Delta s} \\
&= - \Big[\bar u\big(P\varphi(s-\Delta s), s-\Delta s\big) - \bar u\big(P\varphi(s), s\big) \\
&\quad - p_t^\#(s)(-\Delta s) - \big(p_\varphi^\#(s), P\varphi(s-\Delta s) - P\varphi(s)\big)\Big]/\Delta s \\
&\quad + p_t^\#(s) + \Big(p_\varphi^\#(s), \frac{P\varphi(s) - P\varphi(s-\Delta s)}{\Delta s}\Big).
\end{aligned} \tag{3.8}$$

The derivative $\varphi_t(s)$ exists for $t_0 < s < T$ by the theory in e.g. Chapter 3 in [17], where we have used also that the control, $\alpha = -\lambda$, solves an adjoint backward parabolic PDE, and therefore is Hölder continuous. It is now used that $||Px|| \leq ||x||$, $\bar{u}$ is semiconcave (see e.g. [5]), and that

$$\Big(p_\varphi^\#(s), \frac{P\varphi(s) - P\varphi(s-\Delta s)}{\Delta s}\Big) = \Big(p_\varphi^\#(s), \frac{\varphi(s) - \varphi(s-\Delta s)}{\Delta s}\Big)$$

in equation (3.8), so that we have, similarly as in *Step 1*, that

$$\frac{d}{ds}\bar{u}(P\varphi(s), s) \geq p_t^\#(s) + (p_\varphi^\#(s), \bar{\varphi}_t(s)).$$

By further using Chapter 3 in [17] it is known that equation (1.4) is satisfied in the L^2 sense, with $\varphi(s) \in H^2(0,1) \cap H_0^1(0,1)$, for $t_0 < s < T$. Similarly as in *Step 1*, the integrand in (3.7), using the backward derivative can be bounded from below:

$$\frac{d}{ds}\bar{u}(P\varphi(s), s) + h(\alpha(s)) \geq p_t^\#(s) + H(p_\varphi^\#(s), \varphi(s)).$$

As $\bar{u}$ is a viscosity solution to the Hamilton-Jacobi equation for the discrete value function it holds that

$$p_t^\#(s) + H(p_\varphi^\#(s), P\varphi(s)) \geq 0,$$

which proves the second inequality in (3.4). □

Theorem 3.1 will be used when the error between the original and the approximate value functions is computed. For this to work some knowledge about the superdifferential D^+u_H is needed. The dual equation

$$-\lambda_t = \delta\lambda_{xx} - \delta^{-1}\lambda V''(\varphi), \tag{3.9a}$$

$$\lambda(T) = 2K\big(\varphi(T) - \varphi^-\big), \tag{3.9b}$$

is introduced. Let α and φ be optimal pairs as in Theorem 3.1. According to Theorem 2.1 it is possible to choose a bounded control. For the mild solution λ to (3.9) there exists, according to Theorem 3.1 in [4], a subset $\mathcal{L} \subset [t_0, T]$, of full measure, such that, for all $t \in \mathcal{L}$,

$$\varphi(t) \in H_0^1(0,1) \cap H^2(0,1) \implies \Big(\lambda(t), -H\big(\lambda(t), \varphi(t)\big)\Big) \in D^+u\big(\varphi(t), t\big). \tag{3.10}$$

By the same theorem, it holds that for almost every $t \in [t_0, T]$,

$$\big(\lambda(t), \alpha(t)\big) + \frac{||\alpha||^2}{2} = \min_{\substack{a \in L^2(0,1) \\ ||a|| \leq L}} \Big(\big(\lambda(t), a\big) + \frac{||a||^2}{2}\Big), \tag{3.11}$$

where L is the bound on the control from Theorem 2.1. This bound is included in order to be able to use the aforementioned Theorem 3.1 in [4], but since α and λ correspond to the original problem (1.8) with no bound we could also have used any constant greater than L in (3.11). Hence we see that (3.11) holds also without the requirement that a is bounded. From this we draw the conclusion that $\lambda(t) = -\alpha(t)$ a.e. The mild solutions φ and λ therefore satisfy the system

$$\varphi(t) = S(t-t_0)\varphi_0 + \int_{t_0}^{t} S(t-s)\big(-\delta^{-1}V'(\varphi(s)) - \lambda(s)\big)ds, \tag{3.12a}$$

$$\varphi(t_0) = \varphi_0, \tag{3.12b}$$

$$\lambda(t) = S(T-t)\lambda(T) - \delta^{-1}\int_{t}^{T} S(s-t)\big(\lambda(s)V''(\varphi(s))\big)ds, \tag{3.12c}$$

$$\lambda(T) = 2K(\varphi(T) - \varphi^{-}), \tag{3.12d}$$

where $S(t)$ is the contraction semigroup of linear operators generated by $\delta\frac{d^2}{dx^2}$, (3.12a) is equation (1.5) with $\lambda = -\alpha$, and (3.12c) is the equation for mild solutions to (3.9a); see e.g. Theorem 3.1 in [4]. Following the notation in [17] we introduce

$$A \equiv -\delta\frac{d^2}{dx^2}.$$

The operator A has eigenvalues $k_n = \delta\pi^2 n^2$, $(n = 1, 2, 3, \ldots)$ with corresponding eigenfunctions $\psi_n(x) = \sqrt{2}\sin(n\pi x)$. Fractional powers of A may be defined using this:

$$A^{\gamma}\varphi = \sum_{n=1}^{\infty} k_n^{\gamma}(\psi_n, \varphi)\psi_n, \tag{3.13}$$

for $\gamma \geq 0$. The domain for A^{γ} is given by

$$D(A^{\gamma}) = \Big\{\varphi \in L^2(0,1) : \sum_{n=1}^{\infty} k_n^{2\gamma}(\psi_n, \varphi)^2 < \infty\Big\}. \tag{3.14}$$

For $\gamma = 1$ and $\gamma = 1/2$ we have that $||A\varphi|| = \delta||\varphi_{xx}||$ and $||A^{1/2}\varphi|| = \sqrt{\delta}||\varphi_x||$. We state a few useful properties of the fractional powers of A, which may be found in e.g. [17]. For any $K > 0$ and all $0 < \gamma < K$ there exists a constant C such that

$$||A^{\gamma}S(t)|| \leq Ct^{-\gamma}, \quad \text{for } t > 0, \tag{3.15a}$$

and if $0 < \gamma \leq 1, \varphi \in D(A^{\gamma})$,

$$||(S(t) - I)\varphi|| \leq \frac{1}{\gamma}Ct^{\gamma}||A^{\gamma}\varphi||. \tag{3.15b}$$

It also holds that

$$A^{\gamma^1}A^{\gamma^2} = A^{\gamma^2}A^{\gamma^1} = A^{\gamma^1+\gamma^2} \text{ on } D(A^{\gamma^1+\gamma^2}) \text{ when } \gamma^1, \gamma^2 \geq 0, \tag{3.15c}$$

$$A^{\gamma}S(t) = S(t)A^{\gamma} \text{ on } D(A^{\gamma}),\, t > 0. \tag{3.15d}$$

In the following Theorem it is shown how an element in $D^+u_H(\bar{\varphi}(s), s)$ can be obtained, which is needed according to Theorem 3.1.

Theorem 3.2. *When $\varphi_0 \in V$ and $0 \leq s \leq T$ one element in $D^+u_H(\varphi_0, s)$ is given by*

$$\Big(\lambda(s), -H\big(\lambda(s), \varphi_0\big)\Big),$$

where λ is a mild solution to (3.9) *and φ is an optimal solution to* (1.8) *with $t_0 = s$.*

Proof. Using Chapter 3 in [17] we have that $\varphi(t) \in H_0^1(0,1) \cap H^2(0,1)$ for $t_0 < t < T$, so by (3.10)

$$\big(\lambda(t), -H(\lambda(t), \varphi(t))\big) \in D^+u(\varphi(t), t), \quad \text{a.e.}$$

It follows from the definition that then also

$$\big(\lambda(t), -H(\lambda(t), \varphi(t))\big) \in D^+u_H(\varphi(t), t), \quad \text{a.e.}$$

We shall verify that the semiconcavity of u_H implies that

$$z_n \to z_0,\ D^+u_H(z_n) \ni p_n \to p \implies p \in D^+u_H(z_0). \tag{3.16}$$

In order to prove this we use (3.6) at the points z_n. We thereby have that

$$\begin{aligned}
&u_H(z) - u_H(z_0) - (p, z - z_0) \\
=&u_H(z) - u_H(z_n) - (p_n, z - z_n) \\
&\quad + u_H(z_n) - u_H(z_0) + (p_n - p, z - z_0) + (p_n, z - z_n) \\
\leq& K|z - z_0|^2 + \varepsilon,
\end{aligned}$$

where ε can be made arbitrarily small by using the convergence $z_n \to z_0$, $p_n \to p$, and that u_H is continuous (it is even locally Lipschitz continuous, see [3]). Hence

$$u_H(z) - u_H(z_0) - (p, z - z_0) \leq K|z - z_0|^2,$$

which implies that $p \in D^+u_H(z_0)$, and (3.16) holds. (As can be seen in the above argument it suffices that $p_n \rightharpoonup p$ weakly, but we will not need this here.)

Since the Hamiltonian $H : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$ is locally Lipschitz continuous and (3.10) holds, what remains is to prove that φ and λ are continuous as functions of time with values in $H_0^1(0,1)$ at t_0. By equation (3.12a) we have that

$$\begin{aligned}
A^{1/2}\big(\varphi(t) - \varphi_0\big) = &\big(S(t - t_0) - I\big)A^{1/2}\varphi_0 \\
&+ \int_{t_0}^{t} A^{1/2}S(t - s)\big(-\delta^{-1}V'(\varphi(s)) - \lambda(s)\big)ds,
\end{aligned} \tag{3.17}$$

where passing $A^{1/2}$ under the integral sign is justified by the fact that $A^{1/2}$ is a closed operator. By (3.14) it is a straightforward calculation to confirm that $V \subset D(A^\gamma)$ for $\gamma < 3/4$. Since $\varphi_0 \in V$, (3.15b) may be used to get a bound for the first term in the right hand side of (3.17):

$$||\big(S(t - t_0) - I\big)A^{1/2}\varphi_0|| \leq 10Ct^{1/10}||A^{3/5}\varphi_0||.$$

The norm of the integral in (3.17) converges to zero as $t \to t_0$ by (3.15a) and the fact that V' and λ (since it equals $-\alpha$) are bounded. Hence

$$||A^{1/2}\big(\varphi(t) - \varphi_0\big)|| \to 0, \quad \text{as } t \searrow t_0.$$

The function λ is also continuous as a function with values in $H_0^1(0,1)$ when $t \searrow t_0$, as, by Theorem 3.5.2 in [17], $||A^\gamma \lambda_t||$ exists when $t < T$ and $\gamma < 1$, e.g. $\gamma = 1/2$. □

In order to be able to use Theorem 3.1 and 3.2 a few results about the regularity for the state and the dual is established. The original setting, without discretization in space, is considered first.

Theorem 3.3. *For every $C > 0$ and all starting positions (φ_0, t_0) satisfying $||\varphi_0||_{H_0^1(0,1)} \le C$, $0 \le t_0 \le T$, there exists a $D > 0$ and an optimal state φ to problem* (1.8)*, with corresponding dual λ, solving* (3.9)*, such that for all $t_0 \le t \le T$,*

$$||\varphi(t)||_{H_0^1(0,1)} \le D, \tag{3.18a}$$
$$||\varphi(t)||_{H^2(0,1)} \le D(t-t_0)^{-1/2}, \tag{3.18b}$$
$$||\lambda(t)||_{H_0^1(0,1)} \le D, \tag{3.18c}$$
$$||\lambda(t)||_{H^2(0,1)} \le D(T-t_0)^{-1/2}. \tag{3.18d}$$

If φ_0 satisfies the higher regularity $||A^{5/7}\varphi_0|| \le C$ it further holds that

$$||\varphi(t)||_{H^2(0,1)} \le D(t-t_0)^{-2/7}, \tag{3.18e}$$
$$||\lambda(t)||_{H^2(0,1)} \le D(T-t_0)^{-2/7}. \tag{3.18f}$$

Proof. In the proof, we will write D for any constant which may depend on C, but not on t_0.

Step 1. By theorem 2.1 it is possible to choose an optimal control α such that $||\alpha(t)|| \le L$, for some constant L which only depends on C. Since $\lambda = -\alpha$ the same holds for λ.

Step 2. Since A^γ and $S(t)$ commute (see (3.15d)) we can operate with $A^{1/2}$ on equation (3.12a) to obtain

$$A^{1/2}\varphi(t) = S(t-t_0)A^{1/2}\varphi_0 + \int_{t_0}^t A^{1/2}S(t-s)\big(-\delta^{-1}V'(\varphi(s)) - \lambda(s)\big)ds.$$

As $S(t)$ is a contraction semigroup and by the boundedness of V' and $||\lambda(s)||$ together with (3.15a) it therefore holds that

$$||A^{1/2}\varphi(t)|| \le ||A^{1/2}\varphi_0|| + D\int_{t_0}^t (t-s)^{-1/2}ds,$$

and hence $||\varphi_x(t)|| \le D$. By a Poincaré inequality (e.g. Proposition 5.3.5 in [1]) (3.18a) holds.

Step 3. Since $\lambda(T) = 2K(\varphi(T) - \varphi^-)$, boundedness of $\lambda(T)$ in $H_0^1(0,1)$ follows. The same analysis for (3.12c) as was performed in *Step 2* may therefore be used. Using that $||\lambda(s)||$ is bounded for all s gives (3.18c).

Step 4. Operate with A on (3.12a) to obtain

$$A\varphi(t) = A^{1/2}S(t-t_0)A^{1/2}\varphi_0 + \int_{t_0}^{t} A^{1/2}S(t-s)A^{1/2}\big(-\delta^{-1}V'(\varphi(s)) - \lambda(s)\big)ds. \tag{3.19}$$

Since $\varphi(s)$ and $\lambda(s)$ are bounded in $H_0^1(0,1)$ for all s it holds that

$$||A^{1/2}\big(- \delta^{-1}V'(\varphi(s)) - \lambda(s)\big)|| < D.$$

Therefore

$$||A\varphi(t)|| \leq D(t-t_0)^{-1/2} + D\int_{t_0}^{t}(t-s)^{-1/2}ds \leq D(t-t_0)^{-1/2},$$

as T is finite. So (3.18b) holds.

Step 5. In this last step we use the operator A in equation (3.12c) in the following way:

$$A\lambda(t) = S(T-t)A\lambda(T) - \delta^{-1}\int_{t}^{T} A^{1/2}S(s-t)A^{1/2}\Big(\lambda(s)V''\big(\varphi(s)\big)\Big)ds. \tag{3.20}$$

The bound for $\varphi(T)$ in $H^2(0,1)$ may be transferred to $\lambda(T)$ by (3.12d), which yields $||\lambda(T)|| \leq D(T-t_0)^{-1/2}$. As both $\varphi(t)$ and $\lambda(t)$ are bounded in $H_0^1(0,1)$ for all $t_0 < t < T$ the integral in (3.20) is bounded by

$$D\int_{t}^{T}(s-t)^{-1/2}ds.$$

Since T is finite (3.18d) holds.

Step 6. Equations (3.18e) and (3.18f) can be proved similarly as in *Step 4* and *Step 5* by changing the first term in the right hand side of (3.19) to $A^{2/7}S(t-t_0)A^{5/7}\varphi_0$. By this we see that $||A\varphi(t)|| \leq D(t-t_0)^{-2/7}$, which implies (3.18f), just as in *Step 5*. □

A regularity result for the spatially discretized case is now to be established. According to theory in e.g. [5] the optimal control problem (3.1), (3.2) has a minimizing control $\bar{\alpha}$. The corresponding state is denoted $\bar{\varphi}$. The value function is differentiable along optimal paths, i.e. the derivative exists at $\bar{u}(\bar{\varphi}(s), s)$ for all $s \in (t_0, T]$. The spatial Gâteaux derivative of $\bar{u}$ at $(\bar{\varphi}(s), s)$ will be denoted $\bar{\lambda}(s)$. The optimal state, $\bar{\varphi}$, and the Gâteaux derivative $\bar{\lambda}$ satisfy the following system:

$$(\bar{\varphi}_t, v) = -\delta(\bar{\varphi}_x, v_x) - (\delta^{-1}V'(\bar{\varphi}) + \bar{\lambda}, v), \quad \text{for all } v \in V, \tag{3.21a}$$

$$\bar{\varphi}(t_0) = \bar{\varphi}_0, \tag{3.21b}$$

$$-(\bar{\lambda}_t, v) = -\delta(\bar{\lambda}_x, v_x) - \delta^{-1}(\bar{\lambda}V''(\bar{\varphi}), v), \quad \text{for all } v \in V, \tag{3.21c}$$

$$\bar{\lambda}(T) = 2K(\bar{\varphi}(T) - \mathrm{Proj}_V\varphi^-). \tag{3.21d}$$

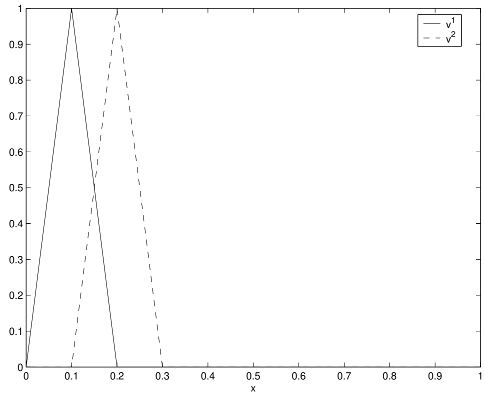

FIGURE 3.1. Two of the nine basis functions when $\Delta x = 1/10$.

Furthermore, the theory in [5] reveals that the optimal control, $\bar{\alpha}$, satisfies $\bar{\alpha} = -\bar{\lambda}$. Some new notation is now introduced. We let the interval $[0,1]$ be divided into M subintervals with $\Delta x = 1/M$, and let $\{v^i\}_{i=1}^{M-1}$ be the standard nodal basis in V; see Figure 3.1. The *interpolant*, I, is the element in V which coincides with the original function at $i\Delta x$, $i = 1, \dots, M-1$, so that, for instance, $I\lambda(i\Delta x) = \lambda(i\Delta x)$. The second difference quotient matrix D^2 and the mass matrix B are introduced as

$$D^2 = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}, \tag{3.22}$$

$$B = \begin{pmatrix} 2/3 & 1/6 & 0 & \cdots & 0 \\ 1/6 & 2/3 & 1/6 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1/6 & 2/3 & 1/6 \\ 0 & \cdots & 0 & 1/6 & 2/3 \end{pmatrix}. \tag{3.23}$$

If $\bar{\varphi}(t)$ and $\bar{\lambda}(t)$ are written in the basis $\{v^i\}_{i=1}^{M-1}$ as

$$\bar{\varphi}(t) =: \sum_{i=1}^{M-1} \zeta^i(t) v^i, \quad \bar{\lambda}(t) =: \sum_{i=1}^{M-1} \theta^i(t) v^i, \tag{3.24}$$

then equations (3.21a) and (3.21c) may be rewritten as

$$B\zeta' = \delta D^2\zeta - \frac{\delta^{-1}}{\Delta x}p - B\theta, \tag{3.25a}$$

$$-B\theta' = \delta D^2\theta - \frac{\delta^{-1}}{\Delta x}r, \tag{3.25b}$$

where

$$\begin{gathered}\zeta = \begin{pmatrix}\zeta^1\\ \vdots\\ \zeta^{M-1}\end{pmatrix}, \quad \theta = \begin{pmatrix}\theta^1\\ \vdots\\ \theta^{M-1}\end{pmatrix},\\ p = \begin{pmatrix}(V'(\bar\varphi), v^1)\\ \vdots\\ (V'(\bar\varphi), v^{M-1})\end{pmatrix}, \text{ and } r = \begin{pmatrix}(\bar\lambda V''(\bar\varphi), v^1)\\ \vdots\\ (\bar\lambda V''(\bar\varphi), v^{M-1})\end{pmatrix}.\end{gathered} \tag{3.26}$$

We now state a Lemma which will be used in the proofs of Theorem 3.5 and 3.6.

Lemma 3.4. *For any element $\psi \in H^2(0,1) \cap H^1_0(0,1)$ the projection $P\psi$, written in the nodal basis $\{v^i\}$ as*

$$P\psi =: \sum_{i=1}^{M-1} \xi^i v^i,$$

satisfies

$$\big(\Delta x \sum_{i=1}^{M-1} (D^2\xi)_i^2\big)^{1/2} \le C||\psi_{xx}||, \tag{3.27}$$

with a constant C independent of Δx.

Proof. The vector ξ is defined as

$$\xi = \begin{pmatrix}\xi^1\\ \vdots\\ \xi^{M-1}\end{pmatrix},$$

and is given by $\xi = \frac{1}{\Delta x}B^{-1}q$, where

$$q = \begin{pmatrix}(\psi, v^1)\\ \vdots\\ (\psi, v^{M-1})\end{pmatrix},$$

which follows from the fact that $(\psi, v^i) = (P\psi, v^i)$ for all v^i. The matrices D^2 and B^{-1} commute, since the eigenvectors of D^2 and B (and B^{-1}) are equal, so $D^2\xi = \frac{1}{\Delta x}B^{-1}D^2 q$. Every element of the vector D^2q, except the first and the last, is a $L^2(0,1)$ scalar product between ψ and a translate of the function v in Figure 3.2. It is easy to check that there exists a primitive function $\bar v$ to v such that $\bar v(0) = \bar v(4\Delta x) = 0$, and that there exists

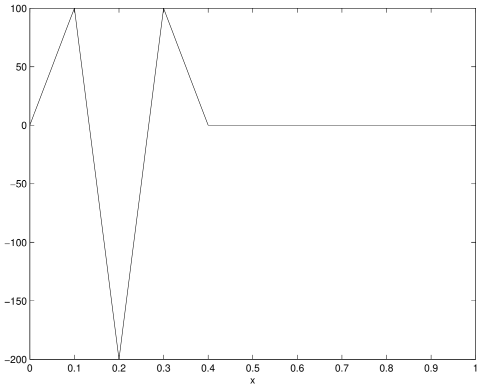

FIGURE 3.2. The function v admits values in the interval $[-2/\Delta x^2, 1/\Delta x^2]$ (here $\Delta x = 1/10$).

a primitive function $\bar{\bar{v}}$ to $\bar{v}$ such that $\bar{\bar{v}}(0) = \bar{\bar{v}}(4\Delta x) = 0$. Furthermore $\max \bar{\bar{v}} \le E$, for a constant E which does not depend on Δx. Hence,

$$|(\psi, v)| = |(\psi_{xx}, \bar{\bar{v}})| \le E \int_0^{4\Delta x} |\psi_{xx}| dx \le 2E\sqrt{\Delta x}\Big(\int_0^{4\Delta x} \psi_{xx}^2 dx\Big)^{1/2}.$$

The same sort of bound may be obtained also for the first and the last elements of $D^2 q$ by using a 2-periodic, odd extension of ψ outside $[0,1]$. We therefore have that

$$\begin{aligned} ||D^2 q||_2^2 \le 4E^2 \Delta x\Big(& \underbrace{\int_0^{\Delta x} \psi_{xx}^2 dx + \int_0^{3\Delta x} \psi_{xx}^2 dx}_{\text{From } (D^2 q)_1} + \underbrace{\int_0^{4\Delta x} \psi_{xx}^2 dx}_{\text{From } (D^2 q)_2} + \dots \\ & + \underbrace{\int_{(M-3)\Delta x}^{1} \psi_{xx}^2 dx + \int_{(M-1)\Delta x}^{1} \psi_{xx}^2 dx}_{\text{From } (D^2 q)_{M-1}}\Big) \\ & \le 16E^2 \Delta x ||\psi_{xx}||^2, \end{aligned}$$

where $||\cdot||_2$ denotes the standard Euclidean vector norm. The eigenvalues of B^{-1} lie in the interval $[1,3]$, and hence it holds that

$$||D^2 \xi||_2 \le \frac{12E}{\sqrt{\Delta x}} ||\psi_{xx}||,$$

which is equivalent to (3.27). $\square$

Theorem 3.5. *There are constants E and F, depending on the parameters of the optimal control problem, as in Theorem 2.1, but not on φ_0, $t_0 \in [0,T]$*

and the size of the spatial discretization, such that for all $t_0 \le t \le T$,

$$||\bar{\varphi}_x(t)|| + ||\bar{\lambda}_x(t)|| \le E||(\varphi_0)_x|| + F,$$
$$||\bar{\varphi}_t(t)|| \le E\big(\Delta x \sum_{i=1}^{M-1} (D^2\zeta(t_0))_i^2\big)^{1/2} + F,$$

and

$$||\bar{\lambda}_t(t)|| \le \Big(E\big(\Delta x \sum_{i=1}^{M-1} (D^2\zeta(t_0))_i^2\big)^{1/2} + F\Big)^2,$$

where

$$\varphi_0 = \sum_{i=1}^{M-1} \zeta^i(t_0) v^i.$$

Proof. The proof uses the same kind of techniques as in the proof of Theorem 5 on page 360 in [15]. Here, however, the regularity of $\bar{\varphi}$ must also be conveyed to $\bar{\lambda}$. In the proof, we will write E and F for any constants that may depend on the parameters of the problem, but not on Δx, t_0, and φ_0.

Step 1. As in the infinite dimensional case, treated in Theorem 2.1, it holds that the infimum in (3.1) can be changed to $\inf_{||\bar{\alpha}||_{L^\infty(t_0,T;V)} \le E||\varphi_0||+F}$. The proof goes just as the proof for Theorem 2.1. By a Poincaré inequality (see e.g. Theorem 5.3.5 in [1]) it holds that $||\varphi_0|| \le C||(\varphi_0)_x||$ and therefore the infimum can be written $\inf_{||\bar{\alpha}||_{L^\infty(t_0,T;V)} \le E||(\varphi_0)_x||+F}$. It is therefore possible to let $v = \bar{\varphi}$ in (3.2) and use this boundedness of $\bar{\alpha}$ to see that $||\bar{\varphi}(t)|| \le E||(\varphi_0)_x|| + F$, for all $t_0 \le t \le T$.

Step 2. As already noted $\bar{\lambda} = -\bar{\alpha}$, and so by *Step 1* the same bound on $||\bar{\lambda}(t)||$ also holds.

Step 3. With $v = \bar{\varphi}_t$ in (3.21a) we have

$$\begin{aligned}
&||\bar{\varphi}_t||^2 + \frac{\delta}{2}\frac{d}{dt}||\bar{\varphi}_x||^2 = -\delta^{-1}(V'(\bar{\varphi}), \bar{\varphi}_t) - (\bar{\lambda}, \bar{\varphi}_t) \\
&\quad \le \delta^{-1}|V'| \cdot ||\bar{\varphi}_t|| + ||\bar{\lambda}|| \cdot ||\bar{\varphi}_t|| \le \delta^{-2}|V'|^2 + \frac{||\bar{\varphi}_t||^2}{4} + \frac{||\bar{\lambda}||^2}{2} + \frac{||\bar{\varphi}_t||^2}{2}.
\end{aligned}$$

The boundedness of $(\bar{\varphi}_0)_x$ and $||\bar{\lambda}(t)||$ thus implies that

$$||\bar{\varphi}_x(t)|| \le E||(\varphi_0)_x|| + F,$$
$$||\bar{\varphi}_t||_{L^2(t_0,T;V)} \le E||(\varphi_0)_x|| + F.$$

By this the same sort of bound holds for $\bar{\lambda}_x(T)$. Letting $v = \bar{\lambda}_t$ in (3.21c) gives that

$$||\bar{\lambda}_x(t)|| \le E||(\varphi_0)_x|| + F,$$
$$||\bar{\lambda}_t||_{L^2(t_0,T;V)} \le E||(\varphi_0)_x|| + F,$$

similarly as for $\bar{\varphi}$.

Step 4. All eigenvalues of B lie in the interval $[1/3,1]$, and so $||B^{-1}||_2 \le 3$ (independently of Δx), where $||\cdot||_2$ denotes the operator 2-norm. From this and (3.25) it follows that

$$||\bar\varphi_t(t_0)|| \le E\big(\Delta x \sum_{i=1}^{M-1} (D^2\zeta(t_0))_i^2\big)^{1/2} + F.$$

We now introduce the notation $\hat\varphi \equiv \bar\varphi_t$ and $\hat\lambda \equiv \bar\lambda_t$ and differentiate equation (3.21a) with respect to time:

$$(\hat\varphi_t, v) = -\delta(\hat\varphi_x, v_x) - \delta^{-1}(V''(\bar\varphi)\hat\varphi, v) - (\hat\lambda, v), \quad \text{for all } v \in V.$$

Let $v = \hat\varphi$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\hat\varphi||^2 &= -\delta||\hat\varphi_x||^2 - \delta^{-1}(V''(\bar\varphi)\hat\varphi,\hat\varphi) - (\hat\lambda,\hat\varphi)\\ &\qquad\qquad \le \delta^{-1}|V''|\cdot||\hat\varphi||^2 + \frac{||\hat\lambda||^2}{2} + \frac{||\hat\varphi||^2}{2}.\end{aligned}$$

The fact that $||\bar\varphi_t(t_0)|| \le D$ together with the result on boundedness of $||\hat\lambda||_{L^2(t_0,T;V)}$ and $||\hat\varphi||_{L^2(t_0,T;V)}$ in *Step 3* implies that

$$||\hat\varphi(t)|| \le E\big(\Delta x \sum_{i=1}^{M-1} (D^2\zeta(t_0))_i^2\big)^{1/2} + F, \quad \text{for } t_0 \le t \le T.$$

Step 5. In this step it will be shown that $|\big((P\varphi_-)_x, w_x\big)| \le D||w||$ for all $w \in V$. Some new notation is introduced:

$$P\varphi_- \equiv \sum_{i=1}^{M-1} \xi^i v^i, \quad w \equiv \sum_{i=1}^{M-1} \eta^i v^i,$$

with corresponding vectors ξ and η. By means of a partial integration

$$|\big((P\varphi_-)_x, w_x\big)| \le \Delta x|\eta^T D^2\xi| \le \Delta x||\eta||_2 \cdot ||D^2\xi||_2, \tag{3.28}$$

with $||\cdot||_2$ denoting the Euclidean vector norm. Since the eigenvalues of B lie in $[1/3,1]$ it holds that

$$||w||^2 = \Delta x \eta^T B\eta \ge \frac{\Delta x}{3}||\eta||_2^2,$$

and so by Lemma 3.4, (3.28) gives:

$$|\big((P\varphi_-)_x, w_x\big)| \le \Delta x||D^2\xi||_2\cdot||\eta||_2 \le \Delta x\frac{D||(\varphi_-)_{xx}||}{\sqrt{\Delta x}}\cdot\frac{\sqrt{3}}{\sqrt{\Delta x}}||w|| = D||w||.$$

Step 6. It holds by equation (3.21d) that $\bar\lambda_x(T) = 2K(\bar\varphi_x(T) - (P\varphi_-)_x)$. Using this in (3.21c) as well as (3.21a) gives

$$\begin{aligned}-(\hat\lambda(T), v) &= -2K\delta(\bar\varphi_x(T), v_x) + 2K\delta((P\varphi_-)_x, v_x) - \delta^{-1}(\bar\lambda(T)V''(\bar\varphi(T)), v)\\ &= 2K(\hat\varphi(T), v) + 2K(\delta^{-1}V'(\bar\varphi(T)) + \bar\lambda(T), v)\\ &\quad + 2K\delta((P\varphi_-)_x, v_x) - \delta^{-1}(\bar\lambda(T)V''(\bar\varphi(T)), v).\end{aligned}$$

With $v = \hat{\lambda}(T)$ it follows that

$$||\hat{\lambda}(T)|| \leq E\big(\Delta x \sum_{i=1}^{M-1} (D^2\zeta(t_0))_i^2\big)^{1/2} + F,$$

by the results in *Step 4* and *Step 5*. In order to bound $\hat{\lambda}$ at all times, equation (3.21c) is differentiated with respect to time:

$$-(\hat{\lambda}_t, v) = -\delta(\hat{\lambda}_x, v_x) - \delta^{-1}(\hat{\lambda}V''(\bar{\varphi}), v) - \delta^{-1}(\bar{\lambda}V'''(\bar{\varphi})\hat{\varphi}, v).$$

With $v = \hat{\lambda}$ in the previous equation the following bound is obtained:

$$-\frac{1}{2}\frac{d}{dt}||\hat{\lambda}||^2 \leq \delta^{-1}|V''| \cdot ||\hat{\lambda}||^2 + \delta^{-1}|V'''| \int_0^1 |\bar{\lambda}\hat{\varphi}\hat{\lambda}| dx.$$

Since $\bar{\lambda}$ is bounded by $E\big(\Delta x \sum_{i=1}^{M-1}(D^2\zeta(t_0))_i^2\big)^{1/2} + F$ in H_0^1 for all times it is similarly bounded in L^∞. The last integral in the previous inequality may therefore be estimated as follows:

$$\int_0^1 |\bar{\lambda}\hat{\varphi}\hat{\lambda}| dx \leq |\bar{\lambda}| \cdot ||\hat{\varphi}|| \cdot ||\hat{\lambda}|| \leq \Big(E\big(\Delta x \sum_{i=1}^{M-1} (D^2\zeta(t_0))_i^2\big)^{1/2} + F\Big)^2 ||\hat{\lambda}||.$$

Using $\frac{d}{dt}||\hat{\lambda}||^2 = 2||\hat{\lambda}||\frac{d}{dt}||\hat{\lambda}||$, Grönwall's Lemma, and the boundedness of $||\hat{\lambda}(T)||$ we see that $||\hat{\lambda}(t)||$ is bounded by $\Big(E\big(\Delta x \sum_{i=1}^{M-1}(D^2\zeta(t_0))_i^2\big)^{1/2} + F\Big)^2$ for all times. $\square$

With the error representation in Theorem 3.1, Theorem 3.2 about D^+u_H, and the regularity results of Theorem 3.3 and 3.5, it is possible to prove Theorem 3.6 about spatial convergence. We first give the idea of the proof. When the first integral in (3.4) is estimated the optimal path $\bar{\varphi}(s)$ is used. In order to obtain an element in $D^+u_H\big(\bar{\varphi}(s), s\big)$ the system of equations (3.12) is considered with $\big(\bar{\varphi}(s), s\big)$ playing the role of (φ_0, t_0). By Theorem 3.2 the computed $\lambda(s)$ is the spatial part of an element in $D^+u_H\big(\bar{\varphi}(s), s\big)$. Therefore, equation (3.12) must be used for every starting position $\big(\bar{\varphi}(s), s\big)$, $0 \leq s \leq T$, as depicted by Figure 3.3. Similarly, when the second integral in (3.4) is estimated, the optimal path $\varphi(s)$ is used, and for each $\big(P\varphi(s), s\big)$ as starting positions the solution $\bar{\lambda}(s)$ to (3.21) is computed.

Theorem 3.6. *For every constant $C > 0$ and all starting points φ_0 with*

$$\varphi_0 = \sum_{i=1}^{M-1} \xi^i v^i \in V \tag{3.29}$$

such that

$$\big(\Delta x \sum_{i=1}^{M-1} (D^2\xi)_i^2\big)^{1/2} \leq C, \tag{3.30}$$

there is a constant $D > 0$ such that

$$|u(\varphi_0, 0) - \bar{u}(\varphi_0, 0)| \leq D\Delta x^2. \tag{3.31}$$

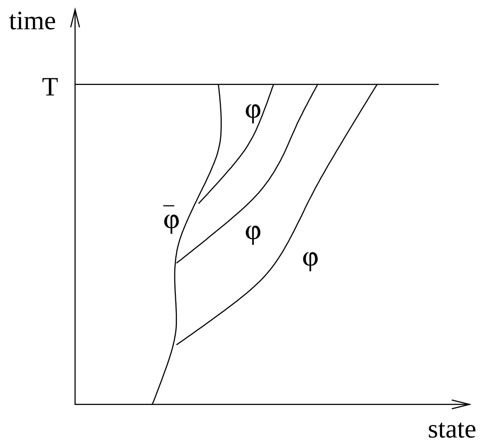

FIGURE 3.3. Sketch showing the idea in Theorem 3.6.

Remark 3.7. *Every reasonable approximation in V of φ_+, for any Δx, satisfies* (3.30). *The interpolant and the projection are possible choices.*

Proof. As in the proof of Theorem 3.5, whenever D is written in this proof it means a constant independent of Δx, but (possibly) dependent on C.

Step 1. It will be shown that condition (3.30) implies

$$||A^{5/7}\varphi_0|| \leq D. \tag{3.32}$$

Using (3.13) it follows that

$$||A^{\gamma}\varphi|| = \big(\sum_{n=1}^{\infty} k_n^{2\gamma}(\psi_n,\varphi)^2\big)^{1/2}, \tag{3.33}$$

where k_n and ψ_n are the eigenvalues and eigenfunctions of A. Two partial integrations imply that

$$(\psi_n,\varphi_0) = -\frac{1}{n^2\pi^2}(\psi_n,(\varphi_0)_{xx}),$$

where $(\varphi_0)_{xx}$ is the distributional second derivative of φ_0, i.e.

$$(\varphi_0)_{xx} = \Delta x \sum_{i=1}^{M-1}(D^2\xi)_i\bar{\delta}_{i\Delta x},$$

with $\bar{\delta}_{i\Delta x}$ the Dirac delta distribution in $x = i\Delta x$. Since $|\psi_n(x)| \leq \sqrt{2}$ for all $n \in \mathbb{N}$ and all $x \in [0,1]$ it holds that

$$\begin{aligned}|(\psi_n,\varphi_0)| &\leq \frac{\sqrt{2}}{n^2\pi^2}\Delta x\sum_{i=1}^{M-1}|(D^2\xi)_i| \leq \frac{\sqrt{2}}{n^2\pi^2}\Delta x\sqrt{M-1}\big(\sum_{i=1}^{M-1}(D^2\xi)_i^2\big)^{1/2}\\ &\leq \frac{\sqrt{2}}{n^2\pi^2}\big(\Delta x\sum_{i=1}^{M-1}(D^2\xi)_i^2\big)^{1/2} \leq \frac{\sqrt{2}C}{n^2\pi^2}.\end{aligned}$$

It thereby follows that the sum in (3.33) is finite (and proportional to C) when $\gamma < 3/4$, e.g. $\gamma = 5/7$.

Step 2. In this step it is shown that there exists a solution λ to (3.12) with $\bar{\varphi}(s)$ playing the role of φ_0, such that

$$|H\big(P\lambda(s), \bar{\varphi}(s)\big) - H\big(\lambda(s), \bar{\varphi}(s)\big)| \leq D\Delta x^2 (T-s)^{-1/2}. \tag{3.34}$$

We start by showing that the starting position φ_0 in (3.29) is bounded in $H_0^1(0,1)$:

$$\begin{aligned} ||(\varphi_0)_x||^2 &= -\big((\varphi_0)_{xx}, \varphi_0\big) \\ &\leq D\big(\Delta x \sum_{i=1}^{M-1} (D^2\xi)_i^2\big)^{1/2} ||\varphi_0||_{L^\infty(0,1)} \leq D||(\varphi_0)_x||, \end{aligned}$$

where the last inequality follows by a Sobolev inequality. Hence $||(\varphi_0)_x|| \leq D$. By Theorem 3.5 it follows that $||\bar{\varphi}_x(s)|| \leq D$, for all $t_0 \leq s \leq T$. By Theorem 3.3 it then follows that $||\lambda(s)||_{H^2(0,1)} \leq D(T-s)^{-1/2}$. The Hamiltonian, H, consists of three parts; see (3.3). The first of these is the most difficult when (3.34) is to be proved, so we will focus on this one and let the other two parts be treated by the reader. The difference between the first parts of the Hamiltonians in (3.34) is given by

$$-\Big(\bar{\varphi}_x(s), \big(\lambda(s) - P\lambda(s)\big)_x\Big) = \big(\bar{\varphi}_{xx}(s), \lambda(s) - P\lambda(s)\big),$$

where the factor δ is left out for convenience. We reuse notation and let

$$\bar{\varphi}(s) =: \sum_{i=1}^{M-1} \xi^i v^i,$$

so that

$$\bar{\varphi}_{xx}(s) = \Delta x \sum_{i=1}^{M-1} (D^2\xi)_i \bar{\delta}_{i\Delta x}.$$

As $||\bar{\varphi}_t(s)|| \leq D$ by Theorem 3.5 it, again using (3.25), holds that

$$\big(\Delta x \sum_{i=1}^{M-1} (D^2\xi)_i^2\big)^{1/2} \leq D.$$

Hence it holds that

$$\begin{aligned} |\big(\bar{\varphi}_{xx}(s), \lambda(s) - P\lambda(s)\big)| &\leq \Delta x \sum_{i=1}^{M-1} |(D^2\xi)_i \big(\lambda(i\Delta x) - P\lambda(i\Delta x)\big)| \\ &\leq \big(\Delta x \sum_{i=1}^{M-1} (D^2\xi)_i^2\big)^{1/2} \Big(\Delta x \sum_{i=1}^{M-1} \big(\lambda(i\Delta x) - P\lambda(i\Delta x)\big)^2\Big)^{1/2} \\ &\leq D\Big(\Delta x \sum_{i=1}^{M-1} \big(\lambda(i\Delta x) - P\lambda(i\Delta x)\big)^2\Big)^{1/2}. \end{aligned} \tag{3.35}$$

By the use of the interpolant the last parenthesis in (3.35) may be written

$$\Big(\Delta x \sum_{i=1}^{M-1} \big(I\lambda(i\Delta x) - P\lambda(i\Delta x)\big)^2\Big)^{1/2}.$$

This minor difference simplifies the situation as $I\lambda(s) - P\lambda(s) \in V$, which makes comparison with the L^2 norm possible. For an element

$$\kappa = \sum_{i=1}^{M-1} \eta^i v^i \in V$$

it holds that

$$||\kappa||^2 = \Delta x(\eta, B\eta)_2,$$

where $(\cdot,\cdot)_2$ is the Euclidean scalar product on $\mathbb{R}^d$. All eigenvalues of B lie in the interval $[1/3, 1]$, and hence

$$||I\lambda - P\lambda||^2 \geq \frac{1}{3}\Delta x \sum_{i=1}^{M-1} (\lambda(i\Delta x) - P\lambda(i\Delta x))^2.$$

It also holds that

$$||I\lambda - P\lambda|| = ||P(I\lambda - \lambda)|| \leq ||I\lambda - \lambda|| \leq D\Delta x^2 ||\lambda||_{H^2(0,1)},$$

where the last inequality may be found in e.g. [1]. By equation (3.18d) we have that

$$||\lambda(s)||_{H^2(0,1)} \leq D(T-s)^{-1/2},$$

and so (3.34) holds. By Theorem 3.2, an element in $D^+u_H\big(\bar\varphi(s), s\big)$ is given by $\Big(\lambda(s), -H\big(\lambda(s), \bar\varphi(s)\big)\Big)$, and thereby it is clear that the first integral in (3.4) may be bounded by

$$D\Delta x^2 \int_{t_0}^{T} (T-s)^{-1/2} ds \leq D\Delta x^2.$$

Step 3. In this step a bound for the second integral in (3.4) is derived. In *Step 1* it was proved that $||A^{5/7}\varphi_0|| \leq D$. Theorem 3.3 then implies that $||\varphi(s)||_{H^2(0,1)} \leq D(s-t_0)^{-2/7}$. Therefore, by Lemma 3.4 and Theorem 3.5 there exists a solution $\bar\lambda$ to (3.21), with $\big(P\varphi(s), s\big)$ in the role of (φ_0, t_0), such that

$$||\bar\lambda_t(s)|| \leq \big(D(s-t_0)^{-2/7} + F\big)^2 \leq D(s-t_0)^{-4/7}.$$

By (3.25b) it holds that

$$\Big(\Delta x \sum_{i=1}^{M-1} \big(D^2\theta(s)\big)_i^2\Big)^{1/2} \leq D(s-t_0)^{-4/7},$$

where θ is given by (3.24) and (3.26). In order to be able to use the above information to get a bound of the second integral in (3.4), we need that $\bar\lambda(s)$ is the spatial part of an element in $D^+\bar u(P\varphi(s), s)$. This follows from Lemma 3.3.16 and Theorem 7.4.17 in [5]. As in *Step 2* we are satisfied with considering only the first parts of the Hamiltonians. The only difference is that now the partial integration is performed so that $\bar\lambda(s)$ is distributionally differentiated twice:

$$\begin{aligned}\Big|\Big(\bar\lambda_x(s), \big(\varphi(s) - P\varphi(s)\big)_x\Big)\Big| &= \big|\big(\bar\lambda_{xx}(s), \varphi(s) - P\varphi(s)\big)\big| \\ &\le \Big(\Delta x \sum_{i=1}^{M-1} \big(D^2\theta(s)\big)_i^2\Big)^{1/2} \Big(\Delta x \sum_{i=1}^{M-1} \big(\varphi(i\Delta x, s) - P\varphi(i\Delta x, s)\big)^2\Big)^{1/2} \\ &\le D(s-t_0)^{-4/7}\Delta x^2 ||\varphi_{xx}(s)|| \le D\Delta x^2 (s-t_0)^{6/7},\end{aligned}$$

similarly as in *Step 2*. The second integral in (3.4) is therefore bounded by a term

$$D\Delta x^2 \int_{t_0}^T (s-t_0)^{-6/7} ds = D\Delta x^2.$$

Step 4. It remains to show that the difference $g(P\varphi(T)) - g(\varphi(T))$ is of the order Δx^2. Since $||\varphi(T)||$ is uniformly bounded for all starting positions in a bounded set in $L^2(0,1)$ we see by (2.1) that the difference in final costs is less than $D||P\varphi(T) - \varphi(T)||$. Since $PI\varphi = I\varphi$, where I is the interpolant, introduced in *Step2*, we have that

$$\begin{aligned}||P\varphi(T) - \varphi(T)|| &\le ||P\big(\varphi(t) - I\varphi(T)\big)|| + ||I\varphi(T) - \varphi(T)|| \\ &\le 2||I\varphi(T) - \varphi(T)|| \le D\Delta x^2 ||\varphi_{xx}(T)|| \le D\Delta x^2,\end{aligned}$$

where the last inequality follows by Theorem 3.3. □

The next theorem provides an error estimate which makes comparison with the situation where φ_+ is used as initial position possible.

Theorem 3.8. *There exists a constant $D > 0$ such that*

$$|u(\varphi_+, 0) - u(P\varphi_+, 0)| + |u(\varphi_+, 0) - u(I\varphi_+, 0)| \le D\Delta x^2. \tag{3.36}$$

Remark 3.9. *The theorem shows that both the projection and the interpolant can be chosen when approximating φ_+ in V.*

Proof. The semiconcavity of u_H implies that for every bounded set $X \subset H_0^1(0,1)$ there exists a constant D, such that

$$\varphi^1, \varphi^2 \in X \implies u_H(\varphi^1, 0) - u_H(\varphi^2, 0) - (p, \varphi^1 - \varphi^2) \le D||\varphi^1 - \varphi^2||^2_{H_0^1(0,1)}, \tag{3.37}$$

where p is the spatial part of any element in $D^+u_H(\varphi^2, 0)$ (compare (3.6)). In Theorem 3.2 it was proved that $\Big(\lambda(0), -H\big(\lambda(0), \varphi^1\big)\Big)$ is one such element, where λ is a solution to (3.12) with $\varphi_0 = \varphi^2$. We may therefore take $p = \lambda(0)$ in (3.37). By Theorem 3.3, $||p|| \le D$ for some constant D (it is even bounded in $H_0^1(0,1)$, but this is not needed here). Plugging this boundedness into (3.37) results in the inequality

$$u_H(\varphi^1, 0) - u_H(\varphi^2, 0) \le D\big(||\varphi^1 - \varphi^2|| + ||\varphi^1 - \varphi^2||^2_{H_0^1(0,1)}\big).$$

We may change places for φ^1 and φ^2 everywhere above, and thereby obtain

$$|u_H(\varphi^1,0) - u_H(\varphi^2,0)| \le D\big(||\varphi^1-\varphi^2|| + ||\varphi^1-\varphi^2||^2_{H^1_0(0,1)}\big). \tag{3.38}$$

Consider now $\varphi^1 = \varphi_+$ and $\varphi^2 = I\varphi_+$ or $\varphi^2 = P\varphi_+$. For the interpolant, I, the following bounds hold:

$$\begin{aligned} ||\varphi_+ - I\varphi_+|| &\le D||(\varphi_+)_{xx}||_{L^\infty(0,1)}\Delta x^2, \\ ||\varphi_+ - I\varphi_+||_{H^1_0(0,1)} &\le D||(\varphi_+)_{xx}||\Delta x, \end{aligned} \tag{3.39}$$

with a constant D independent of Δx and φ_+. Since $(\varphi_+)_{xx}$ is bounded in $L^\infty(0,1)$ this together with (3.38) directly shows that the interpolant part of (3.36) is correct. The projection part is proved by using the result from [13], that the L^2 projection is stable in $H^1_0(0,1)$, i.e.

$$||P\varphi||_{H^1_0(0,1)} \le D||\varphi||_{H^1_0(0,1)}.$$

It therefore holds that

$$\begin{aligned} &||\varphi_+ - P\varphi_+||_{H^1_0(0,1)} \le ||\varphi_+ - I\varphi_+||_{H^1_0(0,1)} + ||I\varphi_+ - P\varphi_+||_{H^1_0(0,1)} \\ &= ||\varphi_+ - I\varphi_+||_{H^1_0(0,1)} + ||P(I\varphi_+ - \varphi_+)||_{H^1_0(0,1)} \le (1+D)||\varphi_+ - I\varphi_+||_{H^1_0(0,1)}. \end{aligned} \tag{3.40}$$

The same technique as in (3.40) can also be used for the L^2 norm, now using the obvious bound $||P\varphi|| \le ||\varphi||$, which implies that

$$||\varphi_+ - P\varphi_+|| \le 2||\varphi_+ - I\varphi_+||. \tag{3.41}$$

The equations (3.38), (3.39), (3.40) and (3.41) imply that also the projection part of (3.36) is correct. □

Theorems 3.6 and 3.8 directly imply the following corollary.

Corollary 3.10. *There exists a constant D, such that*

$$|u(\varphi_+,0) - \bar u(P\varphi_+,0)| + |u(\varphi_+,0) - \bar u(I\varphi_+,0)| \le D\Delta x^2.$$

4. Discretization in time

In [19] the method *Symplectic Pontryagin* for approximation of optimally controlled ODE:s is constructed and analyzed. It is a Symplectic Euler discretization for a Hamiltonian system, involving the state and dual variables associated with the control problem, with a regularized Hamiltonian. In the present situation, when the Hamiltonian is smooth, the Symplectic Pontryagin method reduces to ordinary Symplectic Euler, since no need for regularization exists. The theory in [19] can be used to show that the difference between the value function for a system with only spatial discretization and the value function for a system with discretization in space *and* time, is of the order Δt, where Δt is the size of the temporal discretization. It is, however, desirable to achieve more than this. In order for the estimate on the temporal discretization to be useful the constant in front of Δt in the error estimate needs to be really constant, i.e. independent of Δx.

The theorems in [19] do not directly provide the desired result. This has to do with the fact that the second order difference quotient operator D^2, defined in (3.22), has norm proportional to $1/\Delta x^2$. Furthermore, the proof in [19] requires a bound on the derivatives $\partial\tilde{\lambda}^{n+1}/\partial\tilde{\varphi}^n$, where $\tilde{\varphi}$ and $\tilde{\lambda}$ are obtained with the Symplectic Pontryagin method. The problem of large norm of D^2 can be handled using that it is a negative operator. But in addition to this we also need to bound $\partial\tilde{\lambda}^{n+1}/\partial\tilde{\varphi}^n$ independently of Δx in some proper sense.

The proof of convergence of the Symplectic Euler method given here is based on another technique. It uses that the present problem admits optimal controls which are regular by Theorem 3.5. It also involves an assumption about the derivative $\tilde{\varphi}_x$, and another similar assumption. Under these assumptions it is shown in Theorem 4.1 that a minimum of a forward Euler approximation of control problem (3.1), (3.2) has an error $C\Delta t$ in the objective, where C does not depend on Δx. In Theorem 4.5 it is shown that the solution to this minimization problem is equivalent to the solution of a Symplectic Euler scheme, and hence the desired property for the Symplectic Euler scheme is achieved. The main difference in the result when the present method is used compared to a result using the theory in [19] is that the present result needs an assumption on the derivative $\tilde{\varphi}_x$ whereas [19] needs control over $\partial\tilde{\lambda}^{n+1}/\partial\tilde{\varphi}^n$. The assumptions on $\tilde{\varphi}_x$ seem easier to verify. The numerical tests performed in Section 5 support that it is true.

We now present the setting of the aforementioned discretized optimization problem. Consider the time-discrete state $\{\tilde{\varphi}^n\}_{n=0}^N$, which is a forward Euler approximation of the state $\bar{\varphi}$ in (3.2) and is given by

$$(\tilde{\varphi}^{n+1}, v) = (\tilde{\varphi}^n, v) + \Delta t\big(-\delta(\tilde{\varphi}^n_x, v_x) + (-\delta^{-1}V'(\tilde{\varphi}^n) + \tilde{\alpha}^n, v)\big), \quad \text{for all } v \in V, \tag{4.1}$$

where $\{\tilde{\alpha}^n\}_{n=0}^{N-1}$ is a time-discrete control. The discrete state $\tilde{\varphi}^n$ therefore corresponds to $\bar{\varphi}(t_n)$, where $t_n = n\frac{T}{N} \equiv n\Delta t$. By (4.1) it is possible to define a discrete value function for all times t_m:

$$\tilde{u}(\tilde{\varphi}_0, t_m) = \min_{\{\tilde{\alpha}^n\}_{n=m}^{N-1}} \Big(g(\tilde{\varphi}_N) + \Delta t \sum_{n=m}^{N-1} h(\tilde{\alpha}^n)\Big), \tag{4.2}$$

where $\{\tilde{\varphi}^n\}$ solves (4.1) and $\tilde{\varphi}^m = \tilde{\varphi}_0$. For the proof of Theorem 4.1 we also introduce the discrete state $\{\mathring{\varphi}^n\}_{n=0}^N$. It is also given by a forward Euler time stepping scheme, but its evolution is determined by an optimal control $\bar{\alpha}$ to the time-continuous problem (3.2):

$$(\mathring{\varphi}^{n+1}, v) = (\mathring{\varphi}^n, v) + \Delta t\big(-\delta(\mathring{\varphi}^n_x, v_x) + (-\delta^{-1}V'(\mathring{\varphi}^n) + \bar{\alpha}(t_n), v)\big), \text{ for all } v \in V. \tag{4.3}$$

We will consider starting positions $\bar{\varphi}_0$ in finite element spaces V satisfying

$$\delta\big((\bar{\varphi}_0)_x, v_x\big) + \delta^{-1}\big(V'(\bar{\varphi}_0), v\big) = 0, \quad \text{for all } v \in V. \tag{4.4}$$

We are now ready for the theorem on time discretization convergence.

Theorem 4.1. *Assume there exists a function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\Delta t \leq r(\Delta x)$ there are solutions $\{\tilde{\varphi}^n\}_{n=0}^N$ and $\{\mathring{\varphi}^n\}_{n=0}^N$ with $\tilde{\varphi}^0 = \mathring{\varphi}^0 = \bar{\varphi}_0$, where $\bar{\varphi}_0$ satisfies (4.4), and*

$$||\tilde{\varphi}_x^{n+1} - \tilde{\varphi}_x^n|| + ||\mathring{\varphi}_x^{n+1} - \mathring{\varphi}_x^n|| \leq C\Delta t \tag{4.5}$$

for all $0 \leq n < N$, where C does not depend on Δx. Then

$$|\tilde{u}(\bar{\varphi}_0, 0) - \bar{u}(\bar{\varphi}_0, 0)| \leq D\Delta t$$

for $\Delta t \leq r(\Delta x)$, where D does not depend on Δx.

Remark 4.2. *By the numerical computations performed in Section 5 it seems plausible that (4.5) holds.*

Remark 4.3. *The proof would be valid without inclusion of the function r. However, since the forward Euler method is used it seems reasonable to believe that (4.5) would not be valid for all Δt.*

Proof. As for Theorem 3.1 the proof is divided into two steps. We obtain in the first step a lower bound for $\tilde{u}(\bar{\varphi}_0, 0) - \bar{u}(\bar{\varphi}_0, 0)$, and in the second step a corresponding upper bound. The first step in this proof is similar to the first step in the proof of Theorem 3.1, while the corresponding second steps differ. We denote an optimal pair (control and state) for $\bar{u}$ by $\bar{\alpha}$ and $\bar{\varphi}$, and an optimal pair for $\tilde{u}$ by $\{\tilde{\alpha}^n\}$ and $\{\tilde{\varphi}^n\}$.

Step 1. This part of the proof starts by an extension of the initially time-discrete state $\{\tilde{\varphi}^n\}$ to a piecewise linear time-continuous function $\tilde{\varphi} : [0, T] \to V$ as follows:

$$\tilde{\varphi}(t) \equiv \frac{t_{n+1} - t}{\Delta t}\tilde{\varphi}^n + \frac{t - t_n}{\Delta t}\tilde{\varphi}^{n+1}, \quad \text{for } t_n \leq t \leq t_{n+1}.$$

As in the proof of Theorem 3.1 we have

$$\tilde{u}(\bar{\varphi}_0, 0) - \bar{u}(\bar{\varphi}_0, 0) = \int_0^T \frac{d}{ds}\bar{u}(\tilde{\varphi}(s), s)ds + \Delta t \sum_{i=0}^{N-1} h(\tilde{\alpha}^i). \tag{4.6}$$

In order to be able to use that $\bar{u}$ solves a Hamilton-Jacobi equation we note that the right hand side in (4.6) may be written

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \big(\frac{d}{ds}\bar{u}(\tilde{\varphi}(s), s) + h(\tilde{\alpha}^i)\big)ds,$$

and thus we may focus our attention on one time interval $[t_n, t_{n+1}]$. We also note that equation (3.2) defines a flow $\bar{f} : V \times V \to V$ which is defined by

$$(\bar{f}(\bar{\varphi}, \bar{\alpha}), v) = -\delta(\bar{\varphi}_x, v_x) + (\delta^{-1}V'(\bar{\varphi}) + \bar{\alpha}, v), \quad \text{for all } v \in V. \tag{4.7}$$

Let now $p(s) = \big(p_\varphi(s), p_t(s)\big)$ be any element in $D^+\bar{u}\big(\tilde{\varphi}(s), s\big)$. Similarly as in the proof of Theorem 3.1 we have for almost every $s \in [t_n, t_{n+1}]$

$$\begin{aligned}
&\frac{d}{ds}\bar{u}(\tilde{\varphi}(s), s) + h(\tilde{\alpha}^i) \geq p_t(s) + \big(p_\varphi(s), \bar{f}(\tilde{\varphi}^n, \tilde{\alpha}^n)\big) \\
&= p_t(s) + \underbrace{\big(p_\varphi(s), \bar{f}(\tilde{\varphi}(s), \tilde{\alpha}^n)\big) + h(\tilde{\alpha}^n)}_{\geq H(p_\varphi(s), \tilde{\varphi}(s))} + \big(p_\varphi(s), \bar{f}(\tilde{\varphi}^n, \tilde{\alpha}^n) - \bar{f}(\tilde{\varphi}(s), \tilde{\alpha}^n)\big) \\
&\geq \big(p_\varphi(s), \bar{f}(\tilde{\varphi}^n, \tilde{\alpha}^n) - \bar{f}(\tilde{\varphi}(s), \tilde{\alpha}^n)\big),
\end{aligned} \tag{4.8}$$

since $p_t(s) + H(p_\varphi(s), \tilde{\varphi}(s)) \geq 0$ as $\bar{u}$ is a Hamilton-Jacobi viscosity solution. By assumption (4.5) it follows that $\tilde{\varphi}_x(s)$ is bounded for $0 \leq s \leq T$ independently of Δx. We are therefore free to use $\tilde{\varphi}(s)$ as $\bar{\varphi}_0$ in Theorem 3.5 so that the $\bar{\lambda}(s)$ (corresponding to $\bar{u}(\tilde{\varphi}(s), s)$) is bounded in H_0^1 independently of Δx. For such a $\bar{\lambda}(s)$ we have

$$\begin{aligned}
&\big|\big(\bar{\lambda}(s), \bar{f}(\tilde{\varphi}^n, \tilde{\alpha}^n) - \bar{f}(\tilde{\varphi}(s), \tilde{\alpha}^n)\big)\big| \\
&\leq \delta\big|\big(\bar{\lambda}_x(s), \tilde{\varphi}_x^n - \tilde{\varphi}_x(s)\big)\big| + \delta^{-1}\big|\big(V'(\tilde{\varphi}^n) - V'(\tilde{\varphi}(s)), \bar{\lambda}(s)\big)\big| \leq C\Delta t,
\end{aligned}$$

with C independent of Δx by (4.5). It is now used that $\bar{\lambda}(s)$ is the spatial part of an element in $D^+\bar{u}(\tilde{\varphi}(s), s)$. It thereby holds that the right hand side in (4.8) is less than $C\Delta t$ in magnitude.

Step 2. We start by noting that

$$\bar{u}(\bar{\varphi}_0, 0) - \tilde{u}(\bar{\varphi}_0, 0) \geq g(\bar{\varphi}(T)) + \int_0^T h(\bar{\alpha})dt - \big(g(\mathring{\varphi}^N) + \Delta t \sum_{i=0}^{N-1} h(\bar{\alpha}(t_i))\big). \tag{4.9}$$

The difference between the running costs in (4.9) is

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \big(h(\bar{\alpha}(t)) - h(\bar{\alpha}(t_n))\big)dt.$$

Using that $h(\alpha) = ||\alpha||^2/2$ we have that

$$\begin{aligned}
|h(\bar{\alpha}(t)) - h(\bar{\alpha}(t_n))| &= \frac{1}{2}|(\bar{\alpha}(t) + \bar{\alpha}(t_n), \bar{\alpha}(t) - \bar{\alpha}(t_n))| \\
&\leq \frac{1}{2}||\bar{\alpha}(t) + \bar{\alpha}(t_n)|| \cdot ||\bar{\alpha}(t) - \bar{\alpha}(t_n)|| \leq C\Delta t,
\end{aligned}$$

where we have used the result in Theorem 3.5 on the boundedness of the control and its derivative (remember that $\bar{\alpha} = -\bar{\lambda}$). It remains to show that the difference between the terminal costs in (4.9) behaves similarly. As in *Step 1* we now extend the discrete state $\{\mathring{\varphi}^n\}$ to a continuous function:

$$\mathring{\varphi}(t) \equiv \frac{t_{n+1} - t}{\Delta t}\mathring{\varphi}^n + \frac{t - t_n}{\Delta t}\mathring{\varphi}^{n+1}, \quad \text{for } t_n \leq t \leq t_{n+1}.$$

For $t_n < t < t_{n+1}$ the evolution equations for $\bar{\varphi}$ and $\mathring{\varphi}$ look as follows:

$$
\begin{aligned}
(\bar{\varphi}_t, v) &= -\delta(\bar{\varphi}_x, v_x) + (-\delta^{-1}V'(\bar{\varphi}) + \bar{\alpha}, v),\\
(\mathring{\varphi}_t, v) &= -\delta(\mathring{\varphi}^n_x, v_x) + (-\delta^{-1}V'(\mathring{\varphi}^n) + \bar{\alpha}(t_n), v),
\end{aligned}
$$

for all $v \in V$. Subtract these two equations and let $v = \bar{\varphi} - \mathring{\varphi}$ to get:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\bar{\varphi} - \mathring{\varphi}||^2\\
&= -\delta(\bar{\varphi}_x - \mathring{\varphi}^n_x, \bar{\varphi}_x - \mathring{\varphi}_x) + \delta^{-1}(V'(\mathring{\varphi}^n) - V'(\bar{\varphi}), \bar{\varphi} - \mathring{\varphi})\\
&\qquad + (\bar{\alpha} - \bar{\alpha}(t_n), \bar{\varphi} - \mathring{\varphi})\\
&= -\delta||\bar{\varphi}_x - \mathring{\varphi}_x||^2 + \delta(\mathring{\varphi}^n_x - \mathring{\varphi}_x, \bar{\varphi}_x - \mathring{\varphi}_x)\\
&\qquad + \delta^{-1}(V'(\mathring{\varphi}^n) - V'(\bar{\varphi}), \bar{\varphi} - \mathring{\varphi}) + (\bar{\alpha} - \bar{\alpha}(t_n), \bar{\varphi} - \mathring{\varphi})\\
&\le -\delta||\bar{\varphi}_x - \mathring{\varphi}_x||^2 + \delta\frac{||\mathring{\varphi}^n_x - \mathring{\varphi}_x||^2}{2} + \delta\frac{||\bar{\varphi}_x - \mathring{\varphi}_x||^2}{2}\\
&\quad + \delta^{-1}|V''| \cdot ||\mathring{\varphi}^n - \mathring{\varphi} + \mathring{\varphi} - \bar{\varphi}|| \cdot ||\bar{\varphi} - \mathring{\varphi}|| + \frac{||\bar{\alpha} - \bar{\alpha}(t_n)||^2}{2} + \frac{||\bar{\varphi} - \mathring{\varphi}||^2}{2}\\
&\le \delta\frac{||\mathring{\varphi}^n_x - \mathring{\varphi}_x||^2}{2} + \frac{\delta^{-1}|V''|}{2}||\mathring{\varphi}^n - \mathring{\varphi}||^2\\
&\qquad + \frac{\delta^{-1}|V''| + 1}{2}||\bar{\varphi} - \mathring{\varphi}||^2 + \frac{||\bar{\alpha} - \bar{\alpha}(t_n)||^2}{2}.
\end{aligned}
$$

According to a Poincaré inequality (see e.g. Theorem 5.3.5 in [1]), using the Dirichlet conditions, we have that $||\mathring{\varphi}^n - \mathring{\varphi}|| \le C||\mathring{\varphi}^n_x - \mathring{\varphi}_x||$. If now Grönwall's Lemma is used together with the fact that $||\bar{\alpha} - \bar{\alpha}^n|| + ||\mathring{\varphi}^n_x - \mathring{\varphi}_x|| \le C\Delta t$, we have that $||\bar{\varphi}(T) - \mathring{\varphi}(T)|| \le \Delta t$. Since $g(\bar{\varphi}(T))$ is bounded independently of Δx we have, similarly as in the proof of Theorem 2.1, that $|g(\bar{\varphi}(T)) - g(\mathring{\varphi}(T))| \le C\Delta t$. □

As convergence of the forward Euler method has now been proved, the *Symplectic Euler method*, which can be used to find the forward Euler solution, is now presented. It is given by the system

$$
\begin{aligned}
(\tilde{\varphi}^{n+1}, v) &= (\tilde{\varphi}^n, v) + \Delta t H_\lambda(\tilde{\lambda}^{n+1}, \tilde{\varphi}^n; v)\\
&= (\tilde{\varphi}^n, v) + \Delta t\big(-\delta(\tilde{\varphi}^n_x, v_x) - \delta^{-1}(V'(\tilde{\varphi}^n), v) - (\tilde{\lambda}^{n+1}, v)\big),\\
\tilde{\varphi}^0 &= \bar{\varphi}_0,\\
(\tilde{\lambda}^n, v) &= (\tilde{\lambda}^{n+1}, v) + \Delta t H_\varphi(\tilde{\lambda}^{n+1}, \tilde{\varphi}^n; v)\\
&= (\tilde{\lambda}^{n+1}, v) + \Delta t\big(-\delta(\tilde{\lambda}^{n+1}_x, v_x) - \delta^{-1}(\tilde{\lambda}^{n+1}V''(\tilde{\varphi}^n), v)\big),\\
\tilde{\lambda}^N &= g'(\tilde{\varphi}^N) = 2K(\tilde{\varphi}^N - P\varphi_-),
\end{aligned}
\tag{4.10}
$$

where g' is a Gâteaux derivative and $H_\lambda(\cdot; v)$, $H_\varphi(\cdot; v)$ are Gâteaux derivatives in the direction v. For every minimizer $\{\tilde{\alpha}^n\}$ in (4.2) there exists a solution to (4.10) with $\tilde{\lambda}^{n+1} = -\tilde{\alpha}^n$ for all n. In order to prove this we first state a lemma.

Lemma 4.4. *The value function $\tilde{u}(\cdot, t_n)$ is semiconcave for every n.*

Proof. Consider the starting positions $\tilde{\varphi}_1^0$, $\tilde{\varphi}_2^0$ and $\frac{\tilde{\varphi}_1^0+\tilde{\varphi}_2^0}{2}$ at time 0. The time-discrete cost functional $\tilde{v}$ is introduced:

$$\tilde{v}_{\tilde{\varphi}_0,t_m}(\{\tilde{\alpha}^n\}) = \big(g(\tilde{\varphi}_N) + \Delta t \sum_{n=m}^{N-1} h(\tilde{\alpha}^n)\big),$$

where $\{\tilde{\varphi}^n\}$ solves (4.1) and $\tilde{\varphi}_m = \tilde{\varphi}_0$. Let $\{\tilde{\alpha}^n\}$ be an optimal control for the starting position $(\frac{\tilde{\varphi}_1^0+\tilde{\varphi}_2^0}{2}, 0)$. We can thus write

$$\begin{aligned}\tilde{u}(\tilde{\varphi}_1^0, 0) + \tilde{u}(\tilde{\varphi}_2^0, 0) - 2\tilde{u}(\frac{\tilde{\varphi}_1^0 + \tilde{\varphi}_2^0}{2}, 0)&\\ \le \tilde{v}_{\tilde{\varphi}_1^0,0}(\{\tilde{\alpha}^n\}) + \tilde{v}_{\tilde{\varphi}_2^0,0}(\{\tilde{\alpha}^n\}) &- 2\tilde{v}_{\frac{\tilde{\varphi}_1^0+\tilde{\varphi}_2^0}{2},0}(\{\tilde{\alpha}^n\}).\end{aligned}$$

The states starting in $\tilde{\varphi}_1^0$, $\tilde{\varphi}_2^0$ and $\frac{\tilde{\varphi}_1^0+\tilde{\varphi}_2^0}{2}$, all using the control $\{\tilde{\alpha}^n\}$, are called

$$\tilde{\varphi}_1^n \equiv \sum_{i=1}^{M-1} \xi_{1,i}^n v^i, \ \tilde{\varphi}_2^n \equiv \sum_{i=1}^{M-1} \xi_{2,i}^n v^i, \text{ and } \tilde{\varphi}_3^n \equiv \sum_{i=1}^{M-1} \xi_{3,i}^n v^i.$$

Introducing the notation

$$\xi_m^n = \begin{pmatrix} \xi_{m,1}^n \\ \vdots \\ \xi_{m,M-1}^n \end{pmatrix}, \quad p_m^n = \begin{pmatrix} \big(V'(\tilde{\varphi}_m^n), v^1\big) \\ \vdots \\ \big(V'(\tilde{\varphi}_m^n), v^{M-1}\big) \end{pmatrix}, \quad a^n = \begin{pmatrix} a_1^n \\ \vdots \\ a_{M-1}^n \end{pmatrix},$$

where m can be 1, 2, or 3 and

$$\tilde{\alpha}^n \equiv \sum_{i=1}^{M-1} a_i^n v^i,$$

we can, using the mass matrix B in (3.23) and the second difference operator D^2 in (3.22), write the equation for $\tilde{\varphi}_m^n$, $m = 1, 2, 3$, as follows:

$$B\xi_m^{n+1} = B\xi_m^n + \Delta t\big(\delta D^2 \xi_m^n - \frac{\delta^{-1}}{\Delta x} p_m^n + Ba^n\big). \tag{4.11}$$

Introducing the state $z^n = \xi_1^n + \xi_2^n - 2\xi_3^n$ and using (4.11) gives

$$Bz^{n+1} = Bz^n + \Delta t\big(\delta D^2 z^n - \frac{\delta^{-1}}{\Delta x}(p_1^n + p_2^n - 2p_3^n)\big). \tag{4.12}$$

Every element in the vector

$$p_1^n + p_2^n - 2p_3^n = \begin{pmatrix} \big(V'(\tilde{\varphi}_1^n) + V'(\tilde{\varphi}_2^n) - 2V'(\tilde{\varphi}_3^n), v^1\big) \\ \vdots \\ \big(V'(\tilde{\varphi}_1^n) + V'(\tilde{\varphi}_2^n) - 2V'(\tilde{\varphi}_3^n), v^{M-1}\big) \end{pmatrix}$$

can be bounded in magnitude by

$$
\begin{aligned}
&||V'(\tilde\varphi_1^n)+V'(\tilde\varphi_2^n)-2V'(\tilde\varphi_3^n)||\cdot||v^i||\\
&\qquad\qquad =\sqrt{\frac{2}{3}\Delta x}||V'(\tilde\varphi_1^n)+V'(\tilde\varphi_2^n)-2V'(\tilde\varphi_3^n)||\\
&\qquad\qquad =\sqrt{\frac{2}{3}\Delta x}||V'(\tilde\varphi_1^n)+V'(\tilde\varphi_2^n)-2V'(\frac{\tilde\varphi_1^n+\tilde\varphi_2^n}{2})||\\
&\qquad\qquad\qquad + 2\sqrt{\frac{2}{3}\Delta x}||V'(\frac{\tilde\varphi_1^n+\tilde\varphi_2^n}{2})-V'(\tilde\varphi_3^n)|| =: I+II,
\end{aligned}
$$

using the triangle inequality as in (2.6). We first treat term I above:

$$
\begin{aligned}
&||V'(\tilde\varphi_1^n)+V'(\tilde\varphi_2^n)-2V'(\frac{\tilde\varphi_1^n+\tilde\varphi_2^n}{2})||^2\le\frac{|V'''|^2}{2^2}\int_0^1|\tilde\varphi_1^n-\tilde\varphi_2^n|^4dx\\
&\qquad\le\frac{|V'''|^2}{2^2}\big(\max|\tilde\varphi_1^n-\tilde\varphi_2^n|\big)^4=\frac{|V'''|^2}{2^2}\big(\max|\xi_1^n-\xi_2^n|\big)^4\\
&\qquad\qquad\le\frac{|V'''|^2}{2^2}\big(\sum_{i=1}^{M-1}|\xi_{1,i}^n-\xi_{2,i}^n|^2\big)^2=\frac{|V'''|^2}{2^2}||\xi_1^n-\xi_2^n||_2^4,
\end{aligned}
$$

where $||\cdot||_2$ denotes the Euclidean vector norm. Part II may be bounded as follows:

$$
2||V'(\frac{\tilde\varphi_1^n+\tilde\varphi_2^n}{2})-V'(\tilde\varphi_3^n)||\le A||\tilde\varphi_1^n+\tilde\varphi_2^n-2\tilde\varphi_3^n||\le B||z^n||_2.
$$

These facts in (4.12) give that

$$
||z^{n+1}||_2\le C||z^n||_2+D||\xi_1^n-\xi_2^n||_2^2. \tag{4.13}
$$

By subtracting the equations for $m=1$ and $m=2$ in (4.11) we see that

$$
||\xi_1^{n+1}-\xi_2^{n+1}||_2\le E||\xi_1^n-\xi_2^n||_2\le\ldots\le E^{n+1}||\xi_1^0-\xi_2^0||_2,
$$

so that in (4.13) we could really write $D||\xi_1^0-\xi_2^0||_2^2$ instead of $D||\xi_1^n-\xi_2^n||_2^2$. Thereby, since $z^0=0$, it holds that $||z^N||_2\le F||\xi_1^0-\xi_2^0||_2^2$. Note that the constants A – F are allowed to depend on Δx. Similarly as in the proof of Theorem 2.3, semiconcavity of $\tilde u$ is a consequence of this. □

We are now ready for the promised theorem about the Symplectic Euler method.

Theorem 4.5. *For every minimizer* $\{\tilde\alpha^n\}$ *in* (4.2) *there exists a solution to* (4.10) *with* $\tilde\lambda^{n+1}=-\tilde\alpha^n$ *for* $0\le n\le N-1$.

Proof. The proof is divided into three steps. In the first step it is shown that the value function $\tilde u$ is differentiable along the optimal path $\tilde\varphi^n$. In the second step it is proved that the dual variable $\tilde\lambda^n$ equals the Gâteaux derivative of $\tilde u$, and in the last step it is shown that $\tilde\lambda^{n+1}=\tilde\alpha^n$.

Step 1. In order to show that the discrete value function $\tilde{u}$ is differentiable at $(\tilde{\varphi}^n, t_n)$ for $0 < n \leq N$ the function

$$r(\tilde{\alpha}) \equiv \tilde{u}(\tilde{\varphi}, t_{n+1}) + \Delta t h(\tilde{\alpha}) \tag{4.14}$$

is introduced, where $\tilde{\varphi} = \tilde{\varphi}^n + \Delta t \bar{f}(\tilde{\varphi}^n, \tilde{\alpha})$ and $\bar{f}$ is given by (4.7). Assume that $\tilde{u}$ is not differentiable at $(\tilde{\varphi}^{n+1}, t_{n+1})$. Because $\tilde{u}$ is semiconcave it then follows that the superdifferential $D^+\tilde{u}(\tilde{\varphi}^{n+1}, t_{n+1})$ (which we let designate the superdifferentials in the Gâteaux sense) contains more than one point. For all $\tilde{\alpha}$ in a neighborhood of $\tilde{\alpha}^n$ it holds that

$$\begin{aligned}
&r(\tilde{\alpha}) - r(\tilde{\alpha}^n)\\
=&\tilde{u}\big(\tilde{\varphi}^n + \Delta t \bar{f}(\tilde{\varphi}^n, \tilde{\alpha})\big) - \tilde{u}\big(\tilde{\varphi}^n + \Delta t \bar{f}(\tilde{\varphi}^n, \tilde{\alpha}^n)\big) + \Delta t\big(h(\tilde{\alpha}) - h(\tilde{\alpha}^n)\big)\\
\leq&\Delta t\big(p, \bar{f}(\tilde{\varphi}^n, \tilde{\alpha}) - \bar{f}(\tilde{\varphi}^n, \tilde{\alpha}^n)\big) + \Delta t\big(h'(\tilde{\alpha}^n), \tilde{\alpha} - \tilde{\alpha}^n\big) + K||\tilde{\alpha} - \tilde{\alpha}^n||^2\\
=&\Delta t(\tilde{p}, \tilde{\alpha} - \tilde{\alpha}^n) + \Delta t\big(h'(\tilde{\alpha}^n), \tilde{\alpha} - \tilde{\alpha}^n\big) + K||\tilde{\alpha} - \tilde{\alpha}^n||^2,
\end{aligned} \tag{4.15}$$

where p is an element in $D^+\tilde{u}(\tilde{\varphi}^{n+1}, t_{n+1})$ and $\tilde{p}$ is given by a linear bijection of s, since $\bar{f}$ is linear in the α variable. Since there are more than one element $p \in D^+\tilde{u}(\tilde{\varphi}^{n+1}, t_{n+1})$, there are also more than one possible $\tilde{p}$ in equation (4.15). It is therefore possible to choose the element $\tilde{p}$ such that the linear term in (4.15) is non-vanishing. It follows that there exists $\tilde{\alpha}$ such that $r(\tilde{\alpha}) < r(\tilde{\alpha}^n)$, which is the sought contradiction. By this reasoning we see that $\tilde{u}$ is differentiable at $(\tilde{\varphi}^n, t_n)$ for $0 < n \leq N$.

Step 2. It follows directly that $\tilde{\lambda}^N = g'(\tilde{\varphi}^N)$, i.e. the Gâteaux derivative of $\tilde{u}(\cdot, t_N)$. Assume that $\tilde{\lambda}^{n+1} = \tilde{u}_\varphi(\tilde{\varphi}^{n+1}, t_{n+1})$. It will follow from this that $\tilde{\lambda}^n = \tilde{u}_\varphi(\tilde{\varphi}^n, t_n)$. Since it is known that $\tilde{u}$ is differentiable at both $(\tilde{\varphi}^n, t_n)$ and $(\tilde{\varphi}^{n+1}, t_{n+1})$ the Gâteaux derivative of $\tilde{u}$ at $(\tilde{\varphi}^{n+1}, t_{n+1})$ equals the Gâteaux derivative at $\tilde{\varphi}^n$ of the function

$$s(\tilde{\varphi}) \equiv \tilde{u}(\tilde{\varphi} + \Delta t \bar{f}(\tilde{\varphi}, \tilde{\alpha}^n), t_n) + \Delta t h(\tilde{\alpha}^n),$$

where $\tilde{\alpha}^n$ is fixed. The Gâteaux derivative of s at $\tilde{\varphi}^n$ is given by

$$s'(\tilde{\varphi}^n) = \tilde{u}_\varphi(\tilde{\varphi}^{n+1}, t_{n+1}) \circ (I + \Delta t \bar{f}'(\tilde{\varphi}^n)),$$

where $\tilde{u}_\varphi(\tilde{\varphi}^{n+1}, t_{n+1}) = \tilde{\lambda}^{n+1}$ is a function from V to $\mathbb{R}$ and $\bar{f}'(\tilde{\varphi}^n)$ is a function from V to V. This equation coincides with the $\tilde{\lambda}$ equation in (4.10), which gives that $\tilde{\lambda}^n = \tilde{u}_\varphi(\tilde{\varphi}^n, t_n)$. By induction in n it follows that $\tilde{\lambda}^n = \tilde{u}_\varphi(\tilde{\varphi}^n, t_n)$ for $0 < n \leq N$.

Step 3. Knowing that $\tilde{u}$ is differentiable at $(\tilde{\varphi}^n, t_n)$ for $0 < n \leq N$ the function (4.14) can be differentiated. Since $\tilde{\alpha}^n$ is a minimizer of r the derivative at this argument must be zero:

$$r'(\tilde{\alpha}^n) = \tilde{\lambda}^{n+1} \Delta t \circ I + \Delta t \tilde{\alpha}^n = 0,$$

where it is used that $\tilde{u}_\varphi(\tilde{\varphi}^{n+1}, t_{n+1}) = \tilde{\lambda}^{n+1}$, $\bar{f}_\alpha = I$ and $h'(\tilde{\alpha}^n) = \tilde{\alpha}^n$. It follows that $\tilde{\lambda}^{n+1} = -\tilde{\alpha}^n$ for $0 \leq n \leq N - 1$. $\square$

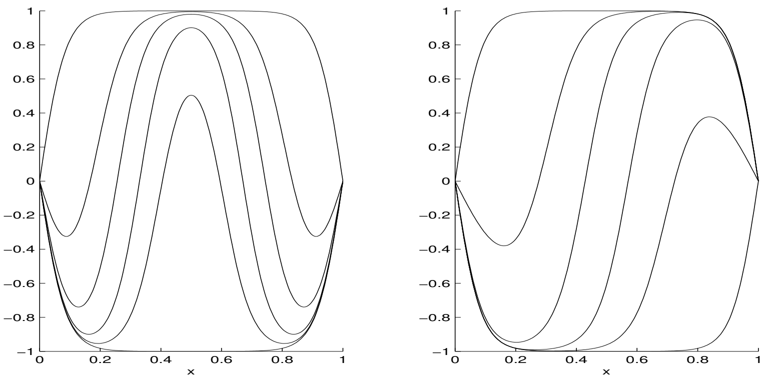

FIGURE 5.1. Snapshots of transitions between the two stable configurations where φ is shown at times 0, 0.2, 0.4, 0.6, 0.8 and 1 ($= T$). To the right propagation of one wall and to the left propagation of two walls. In these examples $K = 10^9$ and $\delta = 0.06$ was used.

5. Numerical Results

We here present some numerical results for the Symplectic Euler scheme for a finite difference discretization of (1.4), (1.8). The numerics is performed in this setting, partly because it is slightly simpler than using finite elements, partly because a finite difference discretization is used in [14]. The system we will consider is therefore

$$\begin{aligned} \xi^{n+1} &= \xi^n + \Delta t\big(\delta D^2 \xi^n - \delta^{-1} V'(\xi^n) - \eta^{n+1}\big), \\ \eta^n &= \eta^{n+1} + \Delta t\big(\delta D^2 \eta^{n+1} - \delta^{-1}\eta^{n+1} V'(\xi^n)\big), \\ \eta^N &= 2K(\xi^N - \xi^-), \end{aligned} \tag{5.1}$$

where ξ^- is a finite difference approximation to φ_-, D^2 is defined in (3.22) and ξ^n and η^n correspond to the nodal values of $\bar\varphi^n$ and $\bar\lambda^n$, respectively. The approximate value used together with this scheme is

$$K\Delta x ||\xi^N - \xi^-||_2^2 + \Delta t \Delta x \sum_{n=1}^{n=N} ||\eta^n||_2^2/2, \tag{5.2}$$

where $||\cdot||_2$ denotes the ordinary Euclidean vector norm. As noted in [14] there are several local minima to (5.2), corresponding to different "strategies" to overcome the potential barrier V. The switching between the two stable points proceeds by "nucleation", which involves a large control α, followed by propagation of domain walls. In Figure 5.1 the transition is shown for the cases propagation of one and two domain walls. The λ variable, which equals the negative control, is shown in Figure 5.2 for the case of propagation of two walls.

Apart from the Symplectic Forward Euler method previously mentioned, the Symplectic Backward Euler method can also be used. This method is

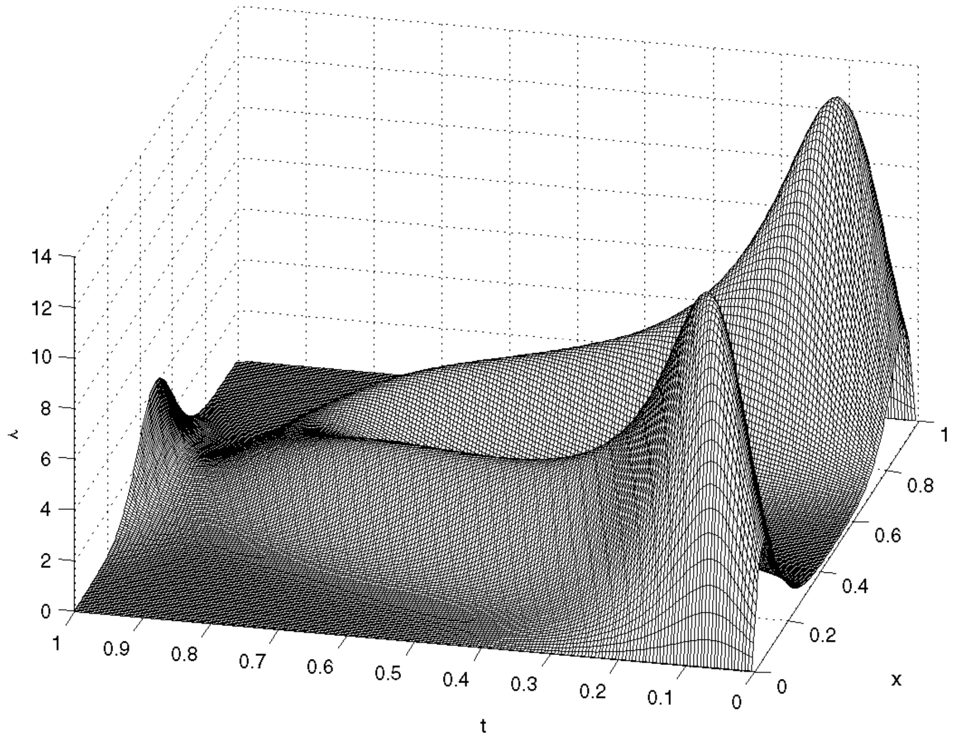

FIGURE 5.2. The dual variable λ for the case of two propagating walls corresponding to the left part of Figure 5.1.

given by

$$\begin{aligned}\xi^{n+1} &= \xi^n + \Delta t(\delta D^2 \xi^{n+1} - \delta^{-1} V'(\xi^{n+1}) - \eta^n),\\ \eta^n &= \eta^{n+1} + \Delta t(\delta D^2 \eta^n - \delta^{-1}\eta^n V'(\xi^{n+1})),\\ \eta^N &= 2K(\xi^N - \xi^-).\end{aligned}$$

The approximate value for the Symplectic Backward Euler method is given by (see Chapter 4.4 in [19])

$$K\Delta x||\xi^N - \xi^-||_2^2 + \Delta t \Delta x \sum_{n=0}^{n=N-1} ||\eta^n||_2^2/2. \tag{5.3}$$

An advantage with the Backward Euler method is that it enables using a small Δx even when Δt is not small. This feature is however not as profound for the present case of control of a parabolic equation as for the uncontrolled case, as the control compensates for the instability, which makes it possible to use smaller Δx. Another good thing about the Backward Euler method is that it seems to underestimate the optimal value while it seems to be overestimated by the Forward Euler method. Figure 5.3 shows the dependence on $\Delta x = \Delta t$ of the values (5.2) and (5.3). By extrapolating these fairly

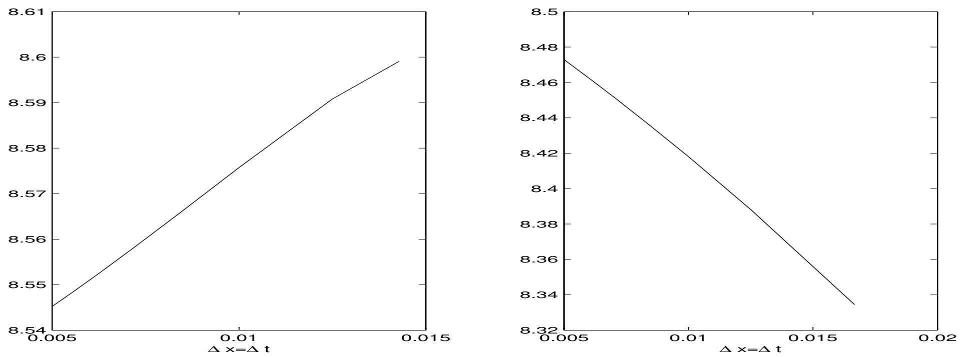

FIGURE 5.3. Convergence of the optimal values (5.2) and (5.3) for the case of equal spacing in space and time, i.e. $\Delta x = \Delta t$. The left figure shows the values obtained by the Forward Euler method and the right shows the values of the Backward Euler method.

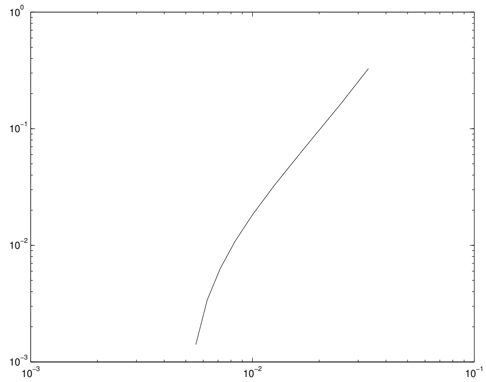

FIGURE 5.4. Convergence of the optimal value (y-axis) with respect to Δx (x-axis). The case with two propagating domain walls and $\Delta t = 1/200$, $\delta = 0.03$ and $K = 10^9$.

straight curves to $\Delta x = \Delta t = 0$ an approximate value of the optimal control problem is obtained. The extrapolated value from the Forward Euler curve is 8.517, and the approximate value from the Backward Euler curve is 8.526.

We now indicate the dependence of the spatial discretization error on Δx. This is done by changing the spatial discretization Δx while keeping the time discretization Δt constant. We let the value obtained for the smallest spatial discretization Δx be the reference value which takes the role of an "exact" solution. A convergence plot can be found in Figure 5.4. The slope of the upper part of this curve corresponds to a convergence rate of approximately $(\Delta x)^{2.37}$.

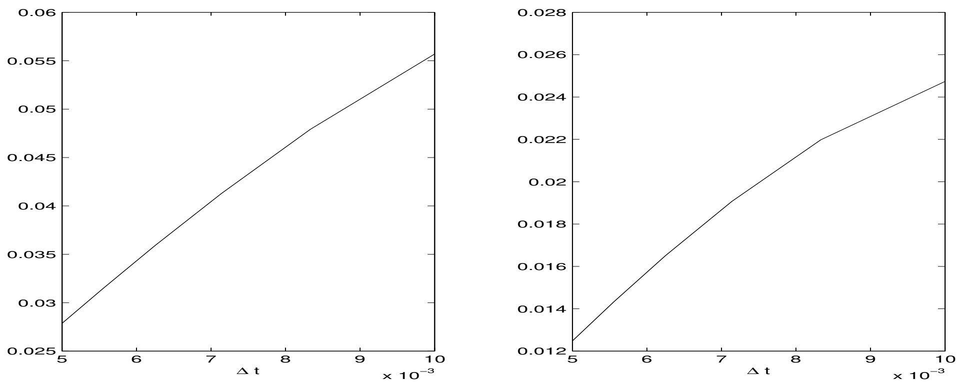

FIGURE 5.5. Time discretization convergence for two different spatial discretizations, $\Delta x = 1/30$ (left), and $\Delta x = 1/100$ (right). The case with two propagating domain walls and $\delta = 0.03$ and $K = 10^9$.

For the time discretization error we want to show that it is less than a linear function of Δt with a constant which does not depend on Δx. Time discretization convergence is therefore considered for two spatial discretizations, one having $\Delta x = 1/30$ and the other $\Delta x = 1/100$. Since the Forward and Backward Euler methods in the limit $\Delta t \to 0$ shall have the same value we may extrapolate the values from diagrams similar to the ones in Figure 5.3, but with the exception that Δx is held fixed. The following values are obtained from these extrapolations in the case of two propagating domain walls, using $\delta = 0.03$ and $K = 10^9$:

- Forward Euler, $\Delta x = 1/30$: 8.841
- Forward Euler, $\Delta x = 1/100$: 8.547
- Backward Euler, $\Delta x = 1/30$: 8.849
- Backward Euler, $\Delta x = 1/100$: 8.555

The mean of the above values for Forward and Backward Euler can be taken as an "exact" reference value when convergence is studied. Hence for $\Delta x = 1/30$ the reference value is taken to be 8.845 and for $\Delta x = 1/100$ it is taken to be 8.551. The two convergence plots can be found in Figure 5.5. Note that the inclination in the left curve, the values using $\Delta x = 1/30$, is larger than the inclination in the right curve ($\Delta x = 1/100$). This is in harmony with Theorem 4.1 since it is allowed that (and good if) we have faster convergence for smaller Δx.

The system (5.1) can be (and has been) solved in two steps. The first step gives a starting position for the second step, and may be performed on a coarse grid, i.e. using large Δx and Δt. The method is to choose an initial guess ξ_0 (a vector containing all time steps) and with it compute the dual, η_0, using (5.1). This computed η_0 is used in (5.1) to compute ξ_{upd}, an updated ξ. Using a damping ν, a new state $\xi_1 = \nu\xi_0 + (1-\nu)\xi_{upd}$ is computed which is used to obtain the dual η_1, which in turn is used to compute a new ξ_{upd}, and so on. When the difference $\xi_{upd} - \xi_n$ is sufficiently small the iterations are terminated, and a starting point (ξ, η) is obtained for the second step.

Step two consists of Newton iterations of (5.1). Since the sparse Jacobian can be computed explicitly this second step converges at a quadratic rate, making it computationally cheap to reach an accurate solution. In the examples presented in this chapter the Newton iterations continued until the difference between two consecutive ξ:s and η:s was less than 10^{-13} in each space-time component. After convergence has been reached for some discretization, a space-time interpolation of ξ and η can be used as a starting position for a Newton iteration on a new grid. It is also possible to gradually change the parameters δ and K in the Newton iterations in order to be able to treat a favorite case. When the starting point is sufficiently good the Newton method terminates after 5-7 iteration steps, making it fast. As comparison, when in [14] a limited memory BFGS method is used, about 550 iterations is needed to decrease the L_2-norm of the objective gradient to 10^{-10}, even when a clever approximation of the initial Hessian was used.

6. Appendix

In order to show existence and uniqueness of solutions to (1.4) we introduce the notion of *weak* solutions (see [15]). We will let $\langle \cdot, \cdot \rangle$ denote the pairing between H^{-1} and H_0^1.

Definition 6.1. *We say a function*

$$\varphi \in L^2(0,T;H_0^1(0,1)), \ \textit{with } \varphi_t \in L^2(0,T;H^{-1}(0,1)),$$

is a weak solution *of (1.4) with $\varphi_0 \in L^2(0,1)$ provided*

$$\langle \varphi_t, v \rangle + (\varphi_x, v_x) = (-V'(\varphi) + \alpha, v)$$

for each $v \in H_0^1(0,1)$ and a.e. time $t_0 \leq t \leq T$, and

$$\varphi(t_0) = \varphi_0.$$

Weak solutions are in fact more regular than is required in the definition when the initial state $\varphi_0 \in H_0^1(0,1)$, which is used when proving the following theorem.

Theorem 6.2. *There is a unique weak solution φ to (1.4) in $C([t_0,T];H_0^1)$ when $\varphi_0 \in H_0^1(0,1)$. This solution satisfies*

$$||\varphi_x(t)||_{L^2}^2 \leq ||(\varphi_0)_x||_{L^2}^2 + \frac{\delta^{-2}}{2}||\varphi_0||_{L^4}^4 - \delta^{-2}||\varphi_0||_{L^2}^2 + \delta^{-1}||\alpha||_{L^2(0,T;L^2)}^2 + \frac{\delta^{-1}}{2}, \tag{6.1}$$

for all $t \in [t_0, T]$.

Proof. We start by proving existence and uniqueness of solutions to the equation (1.4) when the potential $\tilde{V}$ is used; see figure 1.2. Let $\breve{\varphi} \in L^\infty(t_0, T; H_0^1)$ and $\breve{\varphi}(t_0) = \varphi_0$ and define $\tilde{\varphi}$ by

$$\tilde{\varphi}_t = \tilde{\varphi}_{xx} - \tilde{V}'(\breve{\varphi}) + \alpha, \quad \tilde{\varphi}(t_0) = \varphi_0.$$

The solution then satisfies $\tilde{\varphi} \in L^\infty(t_0, T; H_0^1)$, so we can define a map

$$\begin{aligned} A : &L^\infty(t_0, T; H_0^1) \to L^\infty(t_0, T; H_0^1) \\ &\breve{\varphi} \mapsto \tilde{\varphi} \end{aligned}$$

which is single valued (see [15]). It is now shown that A is a contraction on $L^\infty(t_0, T; H_0^1)$ if T is small enough. Let $\tilde{\varphi} = A(\breve{\varphi})$ and $\tilde{\psi} = A(\breve{\psi})$. Subtracting the equations for $\tilde{\varphi}$ and $\tilde{\psi}$ gives

$$(\tilde{\varphi} - \tilde{\psi})_t = (\tilde{\varphi} - \tilde{\psi})_{xx} - (\tilde{V}'(\breve{\varphi}) - \tilde{V}'(\breve{\psi})),$$

which entails

$$||\tilde{\varphi} - \tilde{\psi}||_{L^\infty(t_0,T;H_0^1)} \leq K||\tilde{V}'(\breve{\varphi}) - \tilde{V}'(\breve{\psi})||_{L^2(t_0,T;L^2)},$$

where the constant K decreases when T decreases (see [15]). The right hand side in the previous inequality may be estimated as

$$\begin{aligned} &||\tilde{V}'(\breve{\varphi}) - \tilde{V}'(\breve{\psi})||_{L^2(t_0,T;L^2)} \leq ||\tilde{V}''||_{L^\infty}||\breve{\varphi} - \breve{\psi}||_{L^2(t_0,T;L^2)} \\ &\leq ||\tilde{V}''||_{L^\infty}||\breve{\varphi} - \breve{\psi}||_{L^2(t_0,T;H_0^1)} \leq ||\tilde{V}''||_{L^\infty}\sqrt{T - t_0}||\breve{\varphi} - \breve{\psi}||_{L^\infty(t_0,T;H_0^1)}, \end{aligned}$$

so that A is a contraction when T is small enough. By splitting the interval $[t_0, T]$ into smaller subintervals and using the contraction property on each such interval we obtain the existence and uniqueness of solutions to (1.4) when the potential $\tilde{V}$ is used. There exists a continuous representative of solutions to (1.4) in the equivalence class in $L^\infty(t_0, T; H_0^1)$ (see [15] again) which we call φ. Since the solution lives in one space dimension it is continuous as a function of both space and time. So for each $M > ||\varphi_+||$ there is a time T^* such that $||\varphi(t)||_{C(0,1)} < M$ for all $t \leq T^*$. Thus, in a certain time interval the solution φ is only affected by the unchanged potential V (it never touches the level where V changes into $\tilde{V}$). Consider a time in this interval, and take the inner product with φ_t in (1.4) to get (using $V'(\varphi) = \varphi^3 - \varphi$):

$$||\varphi_t||_{L^2}^2 + \frac{\delta}{2}\frac{d}{dt}||\varphi_x||_{L^2}^2 + \frac{\delta^{-1}}{4}\frac{d}{dt}||\varphi||_{L^4}^4 - \frac{\delta^{-1}}{2}\frac{d}{dt}||\varphi||_{L^2}^2 \leq \frac{1}{2}||\alpha||_{L^2}^2 + \frac{1}{2}||\varphi_t||_{L^2}^2.$$

The $||\varphi_t||_{L^2}^2$ terms are dropped and the resulting inequality is integrated from t_0 to T^*:

$$\begin{aligned} &\delta||\varphi_x(T^*)||_{L^2}^2 + \frac{\delta^{-1}}{2}||\varphi(T^*)||_{L^4}^4 - \delta^{-1}||\varphi(T^*)||_{L^2}^2 \\ &\qquad \leq \delta||(\varphi_0)_x||_{L^2}^2 + \frac{\delta^{-1}}{2}||\varphi_0||_{L^4}^4 - \delta^{-1}||\varphi_0||_{L^2}^2 + \int_{t_0}^{T^*} ||\alpha||_{L^2}^2 dt. \end{aligned}$$

It is now used that

$$\frac{\delta^{-1}}{2}||\varphi(T^*)||_{L^4}^4 - \delta^{-1}||\varphi(T^*)||_{L^2}^2 = \delta^{-1}\int_0^1 \big(\frac{\varphi(x,T^*)^4}{2} - \varphi(x,T^*)^2\big)dx \geq -\frac{1}{2}$$

so that the previous inequality implies

$$\delta||\varphi_x(T^*)||_{L^2}^2 \leq \delta||(\varphi_0)_x||_{L^2}^2 + \frac{\delta^{-1}}{2}||\varphi_0||_{L^4}^4 - \delta^{-1}||\varphi_0||_{L^2}^2 + \int_{t_0}^{T^*} ||\alpha||_{L^2}^2 dt + \frac{1}{2}. \tag{6.2}$$

By Sobolev's inequality we thereby obtain a bound on the continuous function $\varphi(T^*)$ in the supremum norm. Consequently, for all controls $\alpha \in L^2(t_0, T; L^2)$ it is possible to choose the border point s in Figure 1.2 between V and $\tilde{V}$ large enough so that the solution φ is affected only by the unchanged potential V. Note also that it is possible to choose $T^* = T$ in (6.2). Such a solution is a weak solution to (1.4) with the original potential V. It is unique in $C(t_0, T; H_0^1)$, for non-uniqueness would otherwise also hold for some modified potential $\tilde{V}$. The error bound (6.1) follows from (6.2). □

References

[1] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1994.

[2] P. Cannarsa. Regularity properties of solutions to Hamilton-Jacobi equations in infinite dimensions and nonlinear optimal control. *Differential Integral Equations*, 2(4):479–493, 1989.

[3] P. Cannarsa and H. Frankowska. Value function and optimality condition for semilinear control problems. II. Parabolic case. *Appl. Math. Optim.*, 33(1):1–33, 1996.

[4] Piermarco Cannarsa and Halina Frankowska. Value function and optimality conditions for semilinear control problems. *Appl. Math. Optim.*, 26(2):139–169, 1992.

[5] Piermarco Cannarsa and Carlo Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control.* Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[6] M. G. Crandall and P.-L. Lions. Hamilton-Jacobi equations in infinite dimensions. VI. Nonlinear A and Tataru's method refined. In *Evolution equations, control theory, and biomathematics (Han sur Lesse, 1991)*, volume 155 of *Lecture Notes in Pure and Appl. Math.*, pages 51–89. Dekker, New York, 1994.

[7] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. VII. The HJB equation is not always satisfied. *J. Funct. Anal.*, 125(1):111–148, 1994.

[8] Michael G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. IV. Hamiltonians with unbounded linear terms. *J. Funct. Anal.*, 90(2):237–283, 1990.

[9] Michael G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. V. Unbounded linear terms and B-continuous solutions. *J. Funct. Anal.*, 97(2):417–465, 1991.

[10] Michael G. Crandall and Pierre-Louis Lions. Hamilton-Jacobi equations in infinite dimensions. I. Uniqueness of viscosity solutions. *J. Funct. Anal.*, 62(3):379–396, 1985.

[11] Michael G. Crandall and Pierre-Louis Lions. Hamilton-Jacobi equations in infinite dimensions. II. Existence of viscosity solutions. *J. Funct. Anal.*, 65(3):368–405, 1986.

[12] Michael G. Crandall and Pierre-Louis Lions. Hamilton-Jacobi equations in infinite dimensions. III. *J. Funct. Anal.*, 68(2):214–247, 1986.

[13] M. Crouzeix and V. Thomée. The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. *Math. Comp.*, 48(178):521–532, 1987.

[14] Weinan E, Weiqing Ren, and Eric Vanden-Eijnden. Minimum action method for the study of rare events. *Comm. Pure Appl. Math.*, 57(5):637–656, 2004.

[15] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[16] William G. Faris and Giovanni Jona-Lasinio. Large fluctuations for a nonlinear heat equation with noise. *J. Phys. A*, 15(10):3025–3055, 1982.

[17] Daniel Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[18] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[19] Mattias Sandberg and Anders Szepessy. Convergence rates of symplectic Pontryagin approximations in optimal control theory. *M2AN Math. Model. Numer. Anal.*, 40(1):149–173, 2006.

Institutionen för Matematik, Kungl. Tekniska Högskolan, S-100 44 Stockholm

E-mail address: `mattiass@math.kth.se`