Combinatorics and zeros of multivariate polynomials

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Abstract

This thesis consists of five papers in algebraic and enumerative combinatorics. The objects at the heart of the thesis are combinatorial polynomials in one or more variables. We study their zeros, coefficients and special evaluations.

Hyperbolic polynomials may be viewed as multivariate generalizations of real-rooted polynomials in one variable. To each hyperbolic polynomial one may associate a convex cone from which a matroid can be derived - a so called hyperbolic matroid. In Paper A we prove the existence of an infinite family of non-representable hyperbolic matroids parametrized by hypergraphs. We further use special members of our family to investigate consequences to a central conjecture around hyperbolic polynomials, namely the generalized Lax conjecture. Along the way we strengthen and generalize several symmetric function inequalities in the literature, such as the Laguerre-Turán inequality and an inequality due to Jensen. In Paper B we affirm the generalized Lax conjecture for two related classes of combinatorial polynomials: multivariate matching polynomials over arbitrary graphs and multivariate independence polynomials over simplicial graphs. In Paper C we prove that the multivariate d-matching polynomial is hyperbolic for arbitrary multigraphs, in particular answering a question by Hall, Puder and Sawin. We also provide a hypergraphic generalization of a classical theorem by Heilmann and Lieb regarding the real-rootedness of the matching polynomial of a graph.

In Paper D we establish a number of equidistributions between Mahonian statistics which are given by conic combinations of vincular pattern functions of length at most three, over permutations avoiding a single classical pattern of length three.

In Paper E we find necessary and sufficient conditions for a candidate polynomial to be complemented to a cyclic sieving phenomenon (without regards to combinatorial context). We further take a geometric perspective on the phenomenon by associating a convex rational polyhedral cone which has integer lattice points in correspondence with cyclic sieving phenomena. We find the half-space description of this cone and investigate its properties.

Sammanfattning

Denna avhandling består av fem artiklar i algebraisk och enumerativ kombinatorik. Objekten som ligger till hjärtat av avhandlingen är kombinatoriska polynom i en eller flera variabler. Vi studerar deras nollställen, koefficienter och speciella evalueringar.

Hyperboliska polynom kan ses som multivariata generaliseringar av reell-rootade polynom i en variabel. Till varje hyperboliskt polynom kan en konvex kon associeras från vilket en matroid kan härledas - en så kallad hyperbolisk matroid. I Artikel A bevisar vi existensen av en oändlg familj av icke-representerbara hyperboliska matroider som parametriseras av hypergrafer. Vidare använder vi speciella medlemmar av vår familj för att undersöka konsekvenser till en central förmodan kring hyperboliska polynom, nämligen den generaliserade Lax förmodan. Längst vägen stärker och generaliserar vi ett flertal symmetriska olikheter i literaturen så som Laguerre-Túran olikheten och en olikhet av Jensen. I Artikel B bekräftar vi den generaliserade Lax förmodan för två relaterade klasser av kombinatoriska polynom: multivariata matchningspolynom över godtyckliga grafer, samt multivariata oberoende-polynom över simpliciala grafer. I Artikel C bevisar vi att det multivariata d-matchningspolynomet är hyperboliskt för godtyckliga multigrafer vilket i synnerhet besvarar en fråga av Hall, Puder och Sawin. Vi tillhandhåller även en hypergrafisk generalisering av en klassisk sats av Heilmann och Lieb angående reell-rotenheten hos matchningspolynomet för en graf.

I Artikel D fastställer vi en rad olika ekvidistributioner mellan Mahoniska statistiker som ges av koniska kombinationer av generaliserade mönsterfunktioner av längd som mest tre, över permutationer som undviker ett enstaka klassiskt mönster av längd tre.

I Artikel E hittar vi nödvändiga och tillräckliga villkor för att ett kandidatpolynom ska kunna komplementeras till ett cykliskt sållfenomen (utan hänsyn till kombinatoriskt kontext). Vi tar dessutom ett geometrisk perspektiv på fenomenet genom att associera en konvex rationell polyhedral kon vars gitterpunkter är i korrespondens med cykliska sållfenomen. Vi finner halvrymdsbeskrivningen av denna kon och undersöker dess egenskaper.

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Contents

Part I

Introduction and summary

1 Overview

Polynomials have a long history in mathematics and remain relevant to almost all branches of mathematical science. In combinatorics, polynomials are an indispensable tool for studying quantitative properties associated with discrete structures. In this thesis this manifests itself in at least three different ways:

- The geometry of zeros of combinatorial polynomials
- Generating polynomials of combinatorial statistics
- Counting via evaluation of polynomials

The geometry of zeros of combinatorial polynomials

The problem of locating zeros of polynomials is almost as old as mathematics itself and includes fundamental theoretical contributions by mathematicians such as Cauchy, Fourier, Gauss, Hermite, Laguerre, Newton, Pólya, Schur and Szegö.

In combinatorics there are numerous examples of polynomials which are known to have zero sets confined to a prescribed region in the complex plane. Many of them are polynomials associated with combinatorial objects such as graphs, matroids, posets and lattice polytopes etc. For a combinatorialist the zero set of a univariate polynomial is mainly interesting due its relationship with the polynomial coefficients. This relationship is especially pronounced when the polynomial vanishes only at real points, a property which is known to imply both unimodality and log-concavity of the coefficients. Unimodality and log-concavity are properties exhibited by many important combinatorial sequences and have been the subject of much research. More recently, with breakthroughs by Borcea, Brändén and others, analogues of real-rootedness in multivariate polynomials have attracted a lot of attention. These ideas are captured in the notion of hyperbolic/stable polynomials which is fundamentally the subject of papers A, B and C in this thesis. Although hyperbolic polynomials originated in PDE-theory with the works of Gårding, Hörmander and others, they have recently found applications in diverse areas such as optimization, real algebraic geometry, computer science, probability theory and combinatorics. They were notably used by Marcus, Spielman and Srivastava in 2013 to give an affirmative answer to the longstanding Kadison-Singer problem from 1959 - a problem originally formulated in the area of operator theory but with far-reaching consequences for other areas of mathematics. Linear transformations preserving stability were fully characterized in seminal work of Borcea and Brändén, completing a century old classification program going back to Pólya and Schur. Their characterization have since been applied to a multitude of combinatorial settings as a tool for establishing stability through primarily linear differential operators.

Generating polynomials of combinatorial statistics

A combinatorial statistic may be loosely defined as a function which associates to each object in a combinatorial set a non-negative integer which is derived in some concrete way from the object. Generating polynomials are standard tools in enumerative combinatorics for reasoning about multi-dimensional arrays of combinatorial data. In essence, the coefficients of a generating polynomial represent the number of objects in the combinatorial set grouped by the statistics under consideration. Two tuples of statistics (on possibly different combinatorial objects) are said to be equidistributed if their generating polynomials have the same coefficients. Many interesting and sometimes unexpected equidistributions have been identified in combinatorics through a variety of different techniques, ranging from generating function manipulations to concrete bijective proofs. Perhaps the most well-known equidistribution is that between the inversion statistic and the major index statistic on permutations.

Pattern avoidance is an area of combinatorics which has seen considerable expansion in the last couple of decades, now even boasting a dedicated annual conference. The study of pattern avoidance in permutations was pioneered by Donald Knuth. He showed in his book *The art of computer programming Vol 1*, that a permutation is sortable by a stack if and only if it avoids the pattern 231, and moreover that these permutations are enumerated by the Catalan numbers. Since then, a main objective in the community have been to enumerate pattern classes and finding similar pattern restrictions in sorting procedures with other data structures. However the study has now expanded well beyond this endeavour.

More recently people including Claesson-Kitaev and Sagan-Savage have combined the study of combinatorial statistics with pattern avoidance in order to refine patterns classes and study statistic-preserving bijections between them. This is the context for paper D in this thesis.

Counting via evaluation of polynomials

The chromatic polynomial of a graph and the Ehrhart polynomial of a lattice polytope are examples of combinatorial polynomials which when evaluated at a natural number n count the number of n-colourings of a graph and the number of lattice points inside the nth dilation of a lattice polytope respectively. The evaluation of combinatorial polynomials at non-natural numbers may sometimes count interesting quantities too, despite there being no a priori reason for it to do so. A prime example of this so called *combinatorial reciprocity* is due to Stanley and occurs when the chromatic polynomial is evaluated at -1. By a combinatorial miracle this evaluation amounts to the number of acyclic orientations of G, a quantity which is seemingly unrelated to counting colourings. Other examples of this phenomenon occurs when counting fixed points under a cyclic action. The phenomenon is exhibited when the evaluations of a combinatorial polynomial at roots of unity coincides with the number of fixed points under a cyclic action on a combinatorial

set. This so called *cyclic sieving phenomenon* was introduced by Reiner, Stanton and White, and there are plenty of examples of it in the literature. Again there is no a priori reason why evaluating a combinatorial polynomial at roots of unity should mean anything at all. In paper E we look closer at the nature of the cyclic sieving phenomenon.

2 Background

Stable polynomials

For a subset $\Omega \subseteq \mathbb{C}^n$, a polynomial $P(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ is called Ω-*stable* if $P(z) \neq 0$ for all $z \in \Omega$. Let $H := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, denote the open upper complex half-plane. Conventionally H^n-stable polynomials are simply referred to as *stable*. If P is a stable polynomial with only real coefficients, then P is referred to as a *real stable polynomial*. It is worth noting that real stable polynomials in one variable are precisely the real-rooted polynomials. Indeed if a real univariate polynomial is non-vanishing on H, then it must also be non-vanishing on $-H$ since its complex roots come in conjugate pairs. Therefore all roots must lie on the real line. In this sense real stability is a multivariate generalization of the notion of real-rootedness. Examples of stable polynomials occurring in combinatorics include:

- *Elementary symmetric polynomials*:
$$e_d(\mathbf{z}) := \sum_{\substack{S \subseteq [n] \\ |S|=d}} \prod_{i \in S} z_i.$$
- *Spanning tree polynomials*:
$$P_G(\mathbf{z}) := \sum_{T} \prod_{e \in T} z_e,$$
where the sum runs over all spanning trees T of a graph G.
- *Matching polynomials*:
$$\mu_G(\mathbf{z}) := \sum_{M} (-1)^{|M|} \prod_{ij \in M} z_i z_j,$$
where the sum runs over all matchings M of a graph G.
- *Eulerian polynomials*:
$$A(\mathbf{y}, \mathbf{z}) := \sum_{\sigma} \prod_{i \in \mathrm{DB}(\sigma)} y_i \prod_{j \in \mathrm{AB}(\sigma)} z_j,$$
where the sum runs over all permutations σ in $\mathfrak{S}_n$ and $\mathrm{DB}(\sigma)$ (resp. $\mathrm{AB}(\sigma)$) denote the set of descent (resp. ascent) bottoms of σ.

Linear transformations preserving stability

A common technique for proving that a polynomial is stable is to realize the polynomial as the image of a known stable polynomial under a stability preserving linear transformation.

Stable polynomials satisfy a number of basic closure properties:

(i) *Permutation*: for any permutation $\sigma \in \mathfrak{S}_n$, $P(\mathbf{z}) \mapsto P(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$.

(ii) *Scaling*: for $\lambda \in \mathbb{C}$ and $\mathbf{a} \in \mathbb{R}_+^n$, $P(\mathbf{z}) \mapsto \lambda P(a_1 x_1, \ldots, a_n z_n)$.

(iii) *Diagonalization*: for $1 \leq i < j \leq n$, $f(\mathbf{z}) \mapsto f(\mathbf{z})|_{z_i = z_j}$.

(iv) *Specialization*: for $1 \leq i \leq n$ and $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) \geq 0$, $f(\mathbf{z}) \mapsto f(\mathbf{z})|_{z_i = \zeta}$.

(v) *Translation*: $f(\mathbf{z}) \mapsto f(\mathbf{z} + \mathbf{t}) \in \mathbb{C}[\mathbf{z}, \mathbf{t}]$.

(vi) *Inversion*: if $\deg_{z_i}(f) = d$, $f(\mathbf{z}) \mapsto z_i^d f(z_1, \ldots, z_{i-1}, -z_i^{-1}, z_{i+1}, \ldots, z_n)$.

(vii) *Differentiation*: for $1 \leq i \leq n$, $f(\mathbf{z}) \mapsto (\partial/\partial z_i) f(\mathbf{z})$.

Despite the elementary nature of the above facts they accomplish a fair amount. For instance, both the Newton inequalities and the Gauss-Lucas theorem are straightforward consequences of the last two facts.

It is natural to ask more generally, which linear transformations preserve stability? For real univariate polynomials this question was already considered by Pólya and Schur in [57] where they characterized diagonal operators preserving real-rootedness. However it was not until nearly a century later that Borcea and Brändén gave a complete answer to this question. They later generalized their results to the multivariate setting [11, 12], in the most general case characterizing stability preservers on Cartesian products of open circular domains (i.e. images of H under Möbius transformations). We state one version of the characterization below. The key to the characterization is an associated $2n$-variate polynomial which characterizes the stability-preserving properties of the linear transformation.

Let $\boldsymbol{\kappa} \in \mathbb{N}^n$ and let $\mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n]$ be the space of polynomials $P \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\deg_{z_i}(P) \leq \kappa_i$ for each $1 \leq i \leq n$. Given a linear transformation $T : \mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$, define its *algebraic symbol* G_T by

$$G_T(\mathbf{z}, \mathbf{w}) := T\left(\prod_{j \in [n]} (x_j + w_j)^{\kappa_j}\right) \in \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n].$$

Theorem 2.1 (Borcea-Brändén [11]). *A linear transformation $T : \mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ preserves stability if and only if either*

(i) T has range of dimension at most one and is of the form

$$T(f) = \alpha(f) P,$$

where α is a linear functional on $\mathbb{C}_{\boldsymbol{\kappa}}[z_1, \ldots, z_n]$ and P is a stable polynomial, or

(ii) $G_T(\mathbf{z}, \mathbf{w})$ is stable.

Stable multiaffine polynomials

A polynomial $P(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be *multiaffine* if each variable occurs to at most the first power in P, that is, $\deg_{z_i}(P) \leq 1$ for all $i = 1, \ldots, n$. Stable multiaffine polynomials play a special role in the theory and applications of stable polynomials, primarily due to important results by Grace-Walsh-Szegö, Borcea-Brändén-Liggett and Choe-Oxley-Sokal-Wagner.

The Grace-Walsh-Szegö theorem is a cornerstone which is often relied upon when proving results on stability. The theorem is in essence a polarization procedure which proclaims the equivalence between stability and multiaffine stability.

Theorem 2.2 (Grace-Walsh-Szegö [31, 68, 66])**.** *Suppose $P(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ is a polynomial of degree at most d in the variable z_n. Write*

$$P(\mathbf{z}) = \sum_{k=0}^{d} P_k(z_1, \ldots, z_{n-1}) z_n^k.$$

Let Q be the polynomial in variables $z_1, \ldots, z_{n-1}, w_1, \ldots, w_{n-1}$ given by

$$Q = \sum_{k=0}^{d} P_k(z_1, \ldots, z_{n-1}) \frac{e_k(w_1, \ldots, w_d)}{\binom{d}{k}}.$$

Then P is stable if and only if Q is stable.

The following corollary is nearly a restatement of Theorem 2.2, often quoted in practise to depolarize symmetries in a multiaffine polynomial for achieving a reduction in the number of variables.

Corollary 2.3. *If $P(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$ is a multiaffine and symmetric polynomial, then $P(z_1, \ldots, z_n)$ is stable if and only if $P(z, \ldots, z) \in \mathbb{C}[z]$ is stable.*

Example 2.4. The elementary symmetric polynomial $e_d(z_1, \ldots, z_n)$ is a multiaffine and symmetric polynomial of degree d. By Corollary 2.3 we have that $e_d(z_1, \ldots, z_n)$ is stable if and only if $e_d(z, \ldots, z) = \binom{n}{d} z^d$ is stable, the latter of which is clear since $\binom{n}{d} z^d$ is trivially a real-rooted univariate polynomial.

Brändén [14] proved that real stability in multiaffine polynomials is equivalent to certain polynomial inequalities being satisfied.

Theorem 2.5. *Let $P(\mathbf{z}) \in \mathbb{R}[z_1, \ldots, z_n]$ be a multiaffine polynomial. Then P is stable if and only if*

$$\frac{\partial P}{\partial z_i}(\mathbf{z})\frac{\partial P}{\partial z_j}(\mathbf{z}) \geq \frac{\partial^2 P}{\partial z_i \partial z_j}(\mathbf{z})P(\mathbf{z})$$

for any $\mathbf{z} \in \mathbb{R}^n$ and $i, j \in [n]$.

The inequalities in Theorem 2.5 are similar, but stronger than those satisfied by the partition function of a Rayleigh measure, leading to an interesting connection between stable polynomials and probability theory. This topic was investigated closer in a paper by Borcea, Brändén and Liggett [13].

The significance of stable multivariate polynomials in combinatorics first became apparent in a long paper by Choe, Oxley, Sokal and Wagner [22]. The authors discovered a highly fascinating connection between matroids and stable homogeneous multiaffine polynomials. Matroids are structures which try to capture the combinatorial essence of independence. They admit several cryptomorphic axiomatizations which is an important reason why they serve as useful abstractions. The definition we give here is the most relevant for our current purposes. We refer to [54] for further background on matroid theory.

A *matroid* is a pair $(\mathcal{M}, E)$, where $\mathcal{M}$ is a collection of subsets of a finite ground set E satisfying,

(1) If $B \in \mathcal{M}$ and $A \subseteq B$, then $A \in \mathcal{M}$,

(2) The collection $\mathcal{B}(\mathcal{M})$ of maximal (with respect to inclusion) elements of $\mathcal{M}$ satisfies the *basis exchange axiom*:

$$A, B \in \mathcal{B}(\mathcal{M}) \text{ and } x \in A \backslash B \text{ implies } y \in B \backslash A \text{ such that } A \backslash \{x\} \cup \{y\} \in \mathcal{B}(\mathcal{M}).$$

The elements of $\mathcal{M}$ are called *independent sets* and the elements of $\mathcal{B}(\mathcal{M})$ are called *bases* of $\mathcal{M}$. The *support*, $\mathrm{supp}(P)$, of a polynomial $P(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a(\boldsymbol{\alpha}) \prod_{i=1}^n z_i^{\alpha_i}$ is defined by

$$\mathrm{supp}(P) := \{\boldsymbol{\alpha} \in \mathbb{N}^n : a(\boldsymbol{\alpha}) \neq 0\}.$$

Theorem 2.6 (Choe-Oxley-Sokal-Wagner)**.** *The support of a stable homogeneous multiaffine polynomial is the set of bases of a matroid.*

In fact Brändén later proved that the support of an arbitrary stable polynomial posesses the structure of a so called *jump system*, see [14] for further details. The converse to Theorem 2.6 is false however, the *weighted bases generating polynomial*

$$P_{\mathcal{M}}(\mathbf{z}) := \sum_{B \in \mathcal{B}(\mathcal{M})} a(B) \prod_{i \in B} z_i$$

of every matroid is not necessarily a stable polynomial for some weighting $a(B) \in \mathbb{R}$, $B \in \mathcal{B}(\mathcal{M})$. One such example is given by the Fano matroid. A matroid is said

to have the *weak half-plane property* (WHPP) if $P_{\mathcal{M}}$ is a stable polynomial and is said to have the *half-plane property* (HPP) if $P_{\mathcal{M}}$ is stable with $a(B) = 1$ for all $B \in \mathcal{B}(\mathcal{M})$. Despite the Fano matroid there are many important matroid classes which have HPP and WHPP, e.g., the class of uniform matroids and the class of $\mathbb{C}$-representable matroids respectively. There are also matroids, e.g. the Pappus matroid, which have WHPP but not HPP. A natural question is thus to properly characterize these two matroid classes, but the problem remains elusive.

Hyperbolic polynomials

A polynomial $h(\mathbf{z}) \in \mathbb{R}[z_1, \ldots, z_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if

(1) $h(\mathbf{z})$ is a homogeneous polynomial (i.e., $h(t\mathbf{z}) = t^d h(\mathbf{z})$),

(2) $h(\mathbf{e}) \neq 0$,

(3) for all $\mathbf{x} \in \mathbb{R}^n$, the univariate polynomial

$$t \mapsto h(t\mathbf{e} - \mathbf{x})$$

has real zeros only.

Geometrically speaking hyperbolicity means that any line parallel to the direction $\mathbf{e}$ of hyperbolicity must intersect the real algebraic variety cut out by $h(\mathbf{z})$ in exactly d points (counting multiplicity), where d is the degree of $h(\mathbf{z})$. Thus the notion of hyperbolicity may, in addition to the notion of stability, be viewed as a multivariate generalization of real-rootedness. As we will point out in the next section, hyperbolicity is essentially a more general notion than real stability.

It is worth giving a brief explanation regarding the origins of this definition. Hyperbolic polynomials first appeared in the theory of partial differential equations with the works of Petrowsky, Gårding, Hörmander, Atiyah and Bott [6, 38, 42, 56]. Let $h(z_1, \ldots, z_n)$ be a polynomial and consider the Cauchy problem,

$$h(\partial/\partial z_1, \ldots, \partial/\partial z_n)u(\mathbf{z}) = f(\mathbf{z}),$$

where $f \in C_0^\infty(H)$ and $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{e} \geq 0\}$. The analytical significance of hyperbolicity is that the PDE above has a unique solution $u(\mathbf{z})$ supported on H for every $f \in C_0^\infty(H)$ if and only if h is a hyperbolic polynomial with respect to $\mathbf{e}$. Whenever $h(\mathbf{z})$ is a hyperbolic polynomial with respect to $\mathbf{e} \in \mathbb{R}^n$, such equations are therefore naturally referred to as *hyperbolic partial differential equations*. A classical example is the second order wave equation $(\partial^2/\partial z_1^2 - c^2\partial^2/\partial z_2^2)f = 0$ in two variables.

Example 2.7. Below we list a few examples of hyperbolic polynomials:

- Any product $h(\mathbf{z}) = \prod_{i=1}^{d} \ell_i(\mathbf{z})$ of linear forms $\ell_i(\mathbf{z})$ is a hyperbolic polynomial with respect to any direction $\mathbf{e} \in \mathbb{R}^n$ without a zero coordinate.
- The determinant polynomial $\det(Z)$, where $Z = (z_{ij})$ is a symmetric matrix with $\binom{n+1}{2}$ indeterminate entries, may be regarded as a quintessential example of a hyperbolic polynomial due to its prominent role in the theory. If X is a real symmetric $n \times n$ matrix and I is the identity matrix, then $t \mapsto \det(tI - X)$ is the characteristic polynomial of a symmetric matrix and is thus real-rooted. Hence $\det(Z)$ is a hyperbolic polynomial with respect to I.
- Let $h(\mathbf{z}) = z_1^2 - z_2^2 - \cdots - z_n^2$. Then $h(\mathbf{z})$ is hyperbolic with respect to $\mathbf{e} = (1, 0, \ldots, 0)^T$.

Hyperbolicity cones

Let h be a hyperbolic polynomial with respect to $\mathbf{e}$ of degree d. We may write

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x})),$$

where

$$\lambda_{\max}(\mathbf{x}) := \lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x}) =: \lambda_{\min}(\mathbf{x})$$

are called the *eigenvalues* of $\mathbf{x}$ (with respect to $\mathbf{e}$). By homogeneity of h one sees that

$$\lambda_j(s\mathbf{x}) = s\lambda_j(\mathbf{x}) \text{ and } \lambda_j(\mathbf{x} + s\mathbf{e}) = \lambda_j(\mathbf{x}) + s,$$

for all $j = 1, \ldots, d$, $\mathbf{x} \in \mathbb{R}^n$ and $s \in \mathbb{C}$. The *hyperbolicity cone* of h with respect to $\mathbf{e}$ is the set

$$\Lambda_+(h, \mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n : \lambda_{\min}(\mathbf{x}) \geq 0\}.$$

The interior of $\Lambda_+(h, \mathbf{e})$ is denoted $\Lambda_{++}(h, \mathbf{e})$. Note that $\mathbf{e} \in \Lambda_{++}(h, \mathbf{e})$ since $h(t\mathbf{e} - \mathbf{e}) = h(\mathbf{e})(t-1)^d$. We usually abbreviate and write $\Lambda_+(\mathbf{e})$, or even Λ_+, if there is no risk for confusion.

Example 2.8. Below we list the hyperbolicity cones associated with the hyperbolic polynomials in Example 2.7.

- $\Lambda_+(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \ell_i(\mathbf{x})e_i \geq 0 \text{ for all } i\}$.
- $\Lambda_+(I)$ is the cone of positive semidefinite matrices.
- $\lambda_+(1, 0, \ldots, 0) = \left\{\mathbf{x} \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}\right\}$ is the *Lorentz light cone.*

The following facts are due to Gårding.

Theorem 2.9 (Gårding)**.** *Let h be a hyperbolic polynomial with respect to $\mathbf{e}$. Then*

(i) $\Lambda_+(h, \mathbf{e})$ *is a convex cone.*

(ii) $\Lambda_+(h, \mathbf{e})$ *is the connected component of*

$$\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$$

which contains $\mathbf{e}$.

(iii) If $\mathbf{v} \in \Lambda_{++}(h, \mathbf{e})$*, then h is hyperbolic with respect to* $\mathbf{v}$*, and* $\Lambda_{++}(h, \mathbf{v}) = \Lambda_{++}(h, \mathbf{e})$.

(iv) $\lambda_{min} : \mathbb{R}^n \to \mathbb{R}$ *is a concave function.*

Another natural property of the hyperbolicity cone is its facial exposure, that is, the property that all its faces are intersections between the cone itself and one of its supporting hyperplanes (see [59]). The following elementary lemma is a consequence of Rolle's theorem from real analysis and states that taking directional derivatives of a hyperbolic polynomial relaxes the hyperbolicity cone.

Lemma 2.10. *If h is a hyperbolic polynomial and* $\mathbf{v} \in \Lambda_+$ *such that* $D_{\mathbf{v}}h \not\equiv 0$*, then* $D_{\mathbf{v}}h$ *is hyperbolic with respect to* $\mathbf{v}$ *and* $\Lambda_+(h, \mathbf{v}) \subseteq \Lambda_+(D_{\mathbf{v}}h, \mathbf{v})$.

Finally we remark on the connection between hyperbolic polynomials and homogeneous real stable polynomials.

Proposition 2.11. *Let* $P \in \mathbb{R}[z_1, \ldots, z_n]$ *be a homogeneous polynomial. Then P is stable if and only if P is hyperbolic with* $\mathbb{R}^n_+ \subseteq \Lambda_+(P)$.

It is also worth noting that the homogenization of a real stable polynomial is a polynomial hyperbolic with respect to any vector with non-negative coordinates. Therefore the real stable polynomials essentially form a subclass of hyperbolic polynomials with hyperbolicity cone containing the positive orthant.

Hyperbolic polymatroids

Let E be a finite set. A *polymatroid* is a function $r : 2^E \to \mathbb{N}$ satisfying

1. $r(\emptyset) = 0$,
2. $r(S) \leq r(T)$ whenever $S \subseteq T \subseteq E$,
3. r is *semimodular*, i.e.,

$$r(S) + r(T) \geq r(S \cap T) + r(S \cup T),$$

for all $S, T \subseteq E$.

Rank functions of matroids on E coincide with polymatroids r on E with $r(\{i\}) \leq 1$ for all $i \in E$. The connection between hyperbolic polynomials and polymatroids was noted by Gurvits in [35].

In analogy with the rank of a matrix, the *hyperbolic rank*, $\mathrm{rk}(\mathbf{x})$, of $\mathbf{x} \in \mathbb{R}^n$ is defined as the number of non-zero eigenvalues of $\mathbf{x}$, i.e., $\mathrm{rk}(\mathbf{x}) := \deg h(\mathbf{e} + t\mathbf{x})$. Note that the rank is independent of the direction $\mathbf{e}$ of hyperbolicity.

Theorem 2.12 (Gurvits)**.** *Let $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a tuple of vectors in $\Lambda_+(h, \mathbf{e})$. Define a function $r_{\mathcal{V}} : 2^{[m]} \to \mathbb{N}$, where $[m] := \{1, 2, \ldots, m\}$, by*

$$r_{\mathcal{V}}(S) = \mathrm{rk}\left(\sum_{i \in S} \mathbf{v}_i\right).$$

Then r is the rank function of a polymatroid.

The polymatroid constructed in Theorem 2.12 is called a *hyperbolic polymatroid*. If the vectors in $\mathcal{V}$ have rank at most one, then we obtain the hyperbolic rank function of a *hyperbolic matroid*.

Example 2.13. Let $A_1, \ldots, A_n$ be positive semidefinite matrices over $\mathbb{C}$. Define $r : 2^{[n]} \to \mathbb{N}$ by $r(S) = \dim\left(\sum_{i \in S} A_i\right)$ for all $S \subseteq [n]$. Then $r : 2^{[n]} \to \mathbb{N}$ is a hyperbolic polymatroid on $[n]$. In particular, if $A_1, \ldots, A_n$ are positive semidefinite matrices of rank at most one, then we obtain the rank function of a hyperbolic matroid on $[n]$. These are the matroids representable over $\mathbb{C}$.

Hyperbolic matroids are in fact equivalent to WHPP matroids, see [5].

The generalized Lax conjecture

The generalized Lax conjecture is one of the major outstanding problems in the theory of hyperbolic polynomials. Interest in it is largely driven by the connection between hyperbolic polynomials and convex optimization. The field of hyperbolic programming was introduced by Güler [36] for studying efficient optimization of linear functionals over hyperbolicity cones. A *hyperbolic program* is an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \text{ and} \\ & \mathbf{x} \in \Lambda_+, \end{aligned}$$

where $\mathbf{c} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ is a system of linear equations and Λ_+ is a hyperbolicity cone. Notable subfields of hyperbolic programming are linear programming (LP) and semidefinite programming (SDP). Linear programming arises by taking Λ_+ to be the positive orthant in $\mathbb{R}^n$ and semidefinite programming arises by taking Λ_+ to be the cone of positive semidefinite matrices. Recall that these cones are associated with the hyperbolic polynomials $h(\mathbf{z}) = z_1 \cdots z_n$ and $h(Z) = \det(Z)$ respectively.

The generalized Lax conjecture roughly asserts that hyperbolic programming is in fact not a generalization of semidefinite programming at all, but that the two fields are equivalent.

A convex cone in $\mathbb{R}^n$ is said to be *spectrahedral* if it is of the form

$$\left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \text{ is positive semidefinite} \right\}$$

where $A_1, \ldots, A_n$ are symmetric matrices such that there exists a vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n y_i A_i$ positive definite.

Remark 2.14. It is not difficult to see that spectrahedral cones are the hyperbolicity cones associated with the hyperbolic polynomials

$$h(\mathbf{z}) = \det \left(\sum_{i=1}^n z_i A_i \right).$$

The generalized Lax conjecture asserts more precisely that every hyperbolicity cone is conversely an affine section of the cone of positive semidefinite matrices.

Conjecture 2.15 (Generalized Lax conjecture (geometric version))**.** *All hyperbolicity cones are spectrahedral.*

Remark 2.16. Note that h_1 and h_2 are hyperbolic polynomials with respect to $\mathbf{e}$ if and only if $h_1 h_2$ is hyperbolic with respect to $\mathbf{e}$. In that case we also have

$$\Lambda_+(h_1 h_2, \mathbf{e}) = \Lambda_+(h_1, \mathbf{e}) \cap \Lambda_+(h_2, \mathbf{e}).$$

Moreover if C_1 and C_2 are two spectrahedral cones with respect to symmetric matrices $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ respectively, then their intersection

$$C_1 \cap C_2 = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix} \text{ is positive semidefinite} \right\},$$

is again spectrahedral. Hence it suffices to prove the generalized Lax conjecture for hyperbolicity cones associated with irreducible hyperbolic polynomials.

The generalized Lax conjecture can also be formulated algebraically as follows, see [41].

Conjecture 2.17 (Generalized Lax conjecture (algebraic version))**.** *If $h(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$ is hyperbolic with respect to $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$, then there exists a polynomial $q(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$, hyperbolic with respect to $\mathbf{e}$, such that $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$ and*

$$q(\mathbf{x}) h(\mathbf{z}) = \det \left(\sum_{i=1}^n z_i A_i \right) \tag{2.1}$$

for some real symmetric matrices $A_1, \ldots, A_n$ of the same size such that $\sum_{i=1}^n e_i A_i$ is positive definite.

Indeed if the conditions in Conjecture 2.17 are satisfied, then $\Lambda_+(qh, \mathbf{e})$ is a spectrahedral cone by Remark 2.14, and by Remark 2.16 we have that

$$\Lambda_+(qh, \mathbf{e}) = \Lambda_+(q, \mathbf{e}) \cap \Lambda_+(h, \mathbf{e}) = \Lambda_+(h, \mathbf{e}).$$

Conversely if $\Lambda_+(h, \mathbf{e})$ is a spectrahedral cone, then by Remark 2.14 there exists symmetric matrices $A_1, \ldots, A_n$ such that $\Lambda_+(h, \mathbf{e}) = \Lambda_+(f, \mathbf{e})$ where $f(\mathbf{z}) := \det(z_1 A_1 + \cdots + z_n A_n)$. By Remark 2.16 we may assume that h is irreducible. Furthermore h and f both vanish on the boundary $\partial\Lambda_+(h, \mathbf{e})$ of $\Lambda_+(h, \mathbf{e})$. Therefore h must divide f i.e. $f(\mathbf{z}) = q(\mathbf{z})h(\mathbf{z})$ for some hyperbolic polynomial $q(\mathbf{z})$ with respect to $\mathbf{e}$. Hence

$$\Lambda_+(q, \mathbf{e}) \cap \Lambda_+(h, \mathbf{e}) = \Lambda_+(f, \mathbf{e}) = \Lambda_+(h, \mathbf{e}),$$

implying that $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$. This establishes the equivalence between Conjecture 2.15 and Conjecture 2.17.

For hyperbolic polynomials $h(z_1, z_2, z_3)$ in three variables more is true, namely there exists symmetric matrices A_1, A_2, A_3 satisfying Conjecture 2.17 with $q(\mathbf{z}) \equiv 1$, i.e., h has a *definite determinantal representation*. This property was initially conjectured by Peter Lax [46] (originally known as the *Lax conjecture*), and was proved by Helton and Vinnikov [41] as pointed out in [48]. However the former conjecture cannot extend to more than three variable. This may be seen by comparing dimensions. The set of polynomials on $\mathbb{R}^n$ of the form $\det(x_1 A_1 + \cdots x_n A_n)$ with A_i a $d \times d$ symmetric matrix for $1 \leq i \leq n$, has dimension at most $n\binom{d+1}{2}$ (as an algebraic image $(A_1, \ldots, A_n) \mapsto \det(x_1 A_1 + \cdots x_n A_n)$ of a vector space of the same dimension) whereas the set of hyperbolic polynomials of degree d on $\mathbb{R}^n$ has non-empty interior in the space of homogeneous polynomials of degree d in n variables (see [53]) and therefore has the same dimension $\binom{n+d-1}{d}$.

Apart from the theorem by Helton and Vinnikov for $n = 3$, the generalized Lax conjecture, as it currently stands (Conjecture 2.17), is known to be true only in a few special cases, see [5] for an up to date summary at the time of writing.

Permutation patterns

There are many different notions of "patterns" in combinatorics involving objects such as graphs, matrices, partitions, words and permutations etc. In this section we shall give a brief (and by no means comprehensive) background on permutation patterns. For a more extensive introduction we refer to books by Kitaev [44] and Bona [10].

Let $\mathcal{S}_n$ denote the set of permutations on $[n]$. A permutation $\sigma \in \mathcal{S}_n$ is said contain an *occurrence* of the *classical pattern* $\pi \in \mathcal{S}_m$, $m \leq n$ if there exists a subsequence in σ whose letters are in the same relative order as those in π i.e. there exists $i_{\pi(1)} < i_{\pi(2)} < \cdots < i_{\pi(m)}$ such that $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_m)$.

Example 2.18. The permutation $\sigma = 241563 \in \mathcal{S}_6$ has four occurrences of the pattern $\pi = 231 \in \mathcal{S}_3$ given by the subsequences 241, 453, 463 and 563 in σ. On the other hand σ avoids the pattern 321.

Remark 2.19. It is also possible to visualize the definition using permutation matrices. Let M_σ denote the permutation matrix of $\sigma \in \mathcal{S}_n$. Then a permutation $\sigma \in \mathcal{S}_n$ contains an occurrence of the pattern $\pi \in \mathcal{S}_m$ if and only if M_π is a submatrix of M_σ.

For a set Π of patterns, let $\mathcal{S}_n(\Pi)$ denote the set of permutations in $\mathcal{S}_n$ avoiding all of the patterns in Π simultaneously. Two pattern classes Π_1 and Π_2 are called *Wilf-equivalent* if $|\mathcal{S}_n(\Pi_1)| = |\mathcal{S}_n(\Pi_2)|$. Unfortunately the problem of enumerating $\mathcal{S}_n(\Pi)$ is very difficult in general, even for small patterns. However one of the earliest results in the area relates to the enumeration of permutations avoiding patterns of length three, a result that goes back to MacMahon [49] and Knuth [45].

Theorem 2.20 (MacMahon, Knuth)**.** *If $\pi \in \mathcal{S}_3$, then $|\mathcal{S}_n(\pi)| = C_n$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the nth Catalan number.*

In other words the theorem says that all classical patterns of length three are Wilf-equivalent. This no longer remains true for classical patterns of length greater than three. Already for patterns of length four we have three different Wilf-equivalence classes, one of which has not yet been enumerated.

Another early result (famous from Ramsey theory) is due to Erdős and Szekeres [28] which in the language of permutation patterns states the following.

Theorem 2.21 (Erdős-Szekeres [28])**.** *Let a, b be positive integers and $n = (a-1)(b-1)+1$. Then any permutation $\sigma \in \mathcal{S}_n$ contains an occurrence of the pattern $123\cdots a$ or an occurrence of the pattern $b\cdots 321$.*

A milestone was reached when Marcus and Tardos [52] proved the Stanley-Wilf conjecture which asserts that for each pattern $\pi \in \mathcal{S}_m$ there exists a constant C such that $|\mathcal{S}_n(\pi)| \leq C^n$. The conjecture is equivalent to the following statement.

Theorem 2.22 (Marcus-Tardos [52])**.** *For any pattern $\pi \in \mathcal{S}_m$, the limit $\lim\limits_{n\to\infty} \sqrt[n]{|\mathcal{S}_n(\pi)|}$ exists and is finite.*

There are several different generalizations of classical patterns. One such generalization is the notion of a vincular pattern introduced by Babson and Steingrímsson. A *vincular pattern* is a permutation $\pi \in \mathcal{S}_m$ some of whose consecutive letters are underlined. If π contains $\underline{\pi(i)\pi(i+1)\cdots\pi(j)}$, then the letters corresponding to $\pi(i), \pi(i+1), \ldots, \pi(j)$ in an occurrence of π in $\sigma \in \mathcal{S}_n$ must be adjacent, whereas there is no adjacency condition for non-underlined consecutive letters. Moreover if π begins with $[\pi(1)$, then any occurrence of π in σ must begin with the leftmost letter of σ. Similarly if π ends with $\pi(m)]$, then any occurrence of π in σ must end with the rightmost letter of σ.

Example 2.23. Let $\sigma = 241563$.

Pattern π	Occurrences in σ
231	$241, 453, 463, 563$
$\underline{23}1$	$241, 563$
$2\underline{31}$	$241, 463, 563$
$[\underline{23}1$	241
$231]$	$453, 463, 563$

More recently vincular patterns have been generalized a step further to so called *mesh patterns* introduced by Brändén and Claesson in [17].

Permutation patterns and statistics

A *statistic* on a combinatorial set S is a function $\mathrm{stat} : S \to \mathbb{N}$ that keeps track of a particular quantity associated with S. A plethora of statistics have been studied on a number of different combinatorial objects in the literature. Many of them are currently being collected in the findstat database [61]. The *generating polynomial* of a statistic $\mathrm{stat} : S \to \mathbb{N}$ is given by

$$f^{\mathrm{stat}}(q) := \sum_{\sigma \in S} q^{\mathrm{stat}(\sigma)}$$

The polynomials $f^{\mathrm{stat}}(q)$ provide natural q-analogues to the enumeration sequence of the combinatorial family. Furthermore $f^{\mathrm{stat}}(q)$ may have other natural properties of interest such as real-rootedness and coefficient unimodality etc. Generating polynomials of statistics defined on two different combinatorial objects may occasionally coincide leading to new and sometimes unexpected connections in combinatorics and beyond.

Example 2.24. The inversion statistic is a particularly well-studied statistic on permutations. The *inversion set* of $\sigma \in \mathcal{S}_n$ is defined by $\mathrm{Inv}(\sigma) := \{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$. The *inversion statistic* $\mathrm{inv} : \mathcal{S}_n \to \mathbb{N}$ is given by $\mathrm{inv}(\sigma) := |\mathrm{Inv}(\sigma)|$. Rodrigues [60] showed in 1839 that

$$\sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!,$$

where $[n]_q! := [1]_q[2]_q \cdots [n]_q$ and $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$. It is not difficult to show that $[n]_q!$ is a polynomial with unimodal coefficients.

Example 2.25. The *descent set* of σ is defined by $\mathrm{Des}(\sigma) := \{i : \sigma(i) > \sigma(i+1)\}$ and the *descent statistic* by $\mathrm{des}(\sigma) := |\mathrm{Des}(\sigma)|$. The coefficients of the polynomial $f^{\mathrm{des}}(q)$ are given by the Eulerian numbers and the Eulerian polynomial $f^{\mathrm{des}}(q)$ is well-known to be real-rooted (see e.g. [55]).

Example 2.26. The *major index statistic* is defined by $\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i$. MacMahon [49] showed that the maj and inv statistics are equidistributed, i.e., $f^{\mathrm{maj}}(q) = f^{\mathrm{inv}}(q)$. Permutation statistics which are equidistributed with inv are called *Mahonian*.

Patterns give rise to statistics as well. A *pattern function* $(\pi) : \mathcal{S}_n \to \mathbb{N}$ is a statistic that is induced by a permutation pattern π, counting the number of occurrences of π in a permutation $\sigma \in \mathcal{S}_n$. The *length* of a pattern function is the length of its underlying pattern. Babson and Steingrímsson[7] classified (up to trivial bijections) all Mahonian statistics that are conic combinations of pattern functions of length at most 3. Among them are inv and maj.

Sagan and Savage [63] introduced a q-analogue of Wilf-equivalence in order to refine Wilf-classes by statistic equidistribution. Formally two sets of patterns Π_1 and Π_2 are said to be *st-Wilf equivalent* with respect to the statistic $\mathrm{st} : \mathcal{S}_n \to \mathbb{N}$ if

$$\sum_{\sigma \in \mathcal{S}_n(\Pi_1)} q^{\mathrm{st}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\Pi_2)} q^{\mathrm{st}(\sigma)}.$$

Clearly st-Wilf equivalence implies Wilf-equivalence but not conversely. Dokos et.al. [25] completed the inv-Wilf and maj-Wilf classifications over $\mathcal{S}_n(\pi)$ where π is a classical pattern of length three. The st-Wilf classification of other permutation statistics such as fixed points, exceedances, peak and valley have also been investigated in detail, see [9, 26].

The cyclic sieving phenomenon

Let C_n be a cyclic group of order n generated by σ_n, X a finite set on which C_n acts and $f(q) \in \mathbb{N}[q]$. Let $X^g := \{x \in X : g \cdot x = x\}$ denote the fixed point set of X under $g \in C_n$. A triple $(X, C_n, f(q))$ is said to exhibit the *cyclic sieving phenomenon (CSP)* if

$$f(\omega_n^k) = |X^{\sigma_n^k}|, \text{ for all } k \in \mathbb{Z}, \tag{2.2}$$

where ω_n is any fixed primitive n^{th} root of unity. The cyclic sieving phenomenon was introduced by Reiner, Stanton and White in [58]. Although it is always possible to find a (generally uninteresting) polynomial satisfying the equations in (2.2) when provided with a cyclic action, namely,

$$f(q) = \sum_{\mathcal{O} \in \mathrm{Orb}_{C_n}(X)} \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1}, \tag{2.3}$$

it sometimes happens that a polynomial $f(q) \in \mathbb{N}[q]$ can be found which satisfies (2.2) and is intrinsically related to the set X on which C_n acts. Generally we would consider a CSP "interesting" if for example

- $f(q) = \sum_{x \in X} q^{\mathrm{stat}(x)}$ where $\mathrm{stat} : X \to \mathbb{N}$ is a natural statistic on X.

- $f(q)$ is the formal character of some representation $\rho : C_n \to GL(V)$.
- $f(q)$ is the Hilbert series $\mathrm{Hilb}(R, q) := \sum_i \dim(R_i) q^i$ of some graded ring $R = \bigoplus_i R_i$.
- $f(q)$ at $q = p^d$ counts the number of points of a variety over a finite field $\mathbb{F}_q$.

There is no a priori reason why one would expect the existence of polynomials with any of the above properties. Nevertheless such situations occur quite ubiquitously in combinatorics, as witnessed by the growing literature on the phenomenon. See [62] for an extensive survey on CSP.

Example 2.27. The prototypical example of CSP is given by $X = \binom{[n]}{k}$ and

$$f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[m]_q! := [m]_q[m-1]_q \cdots [2]_q[1]_q$ and $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$. Here the generator σ_n of C_n acts on $S = \{i_1, \ldots, i_k\} \in X$ via

$$\sigma_n \cdot S := \{i_1 \pmod n + 1, \ldots, i_k \pmod n + 1\}.$$

By [58] the triple $(X, C_n, f(q))$ exhibits CSP. The following facts are also proved in [58]:

- If $\mathrm{sum} : X \to \mathbb{N}$ is the statistic defined by $\mathrm{sum}(S) := \sum_{i \in S} i$, then
$$f(q) = q^{-\binom{k+1}{2}} \sum_{S \in X} q^{\mathrm{sum}(S)}.$$
- Let $V = \bigwedge^k(\mathbb{C}^n)$ denote the kth exterior power of the vector space $\mathbb{C}^n$. The action of C_n on X induces an action of C_n on V, giving rise to a representation $\rho : C_n \to GL(V)$. Denote the character of ρ by $\chi_\rho(x_1, \ldots, x_n) : C_n \to \mathbb{C}[x_1, \ldots, x_n]$, defined for $\sigma \in C_n$ as the trace of the matrix $\rho(\sigma)$ with eigenvalues $x_1, \ldots, x_n$. Then
$$f(q) = q^{-\binom{k}{2}} \chi_\rho(1, q, q^2, \ldots, q^{n-1}).$$
- Let $\mathbb{Z}[\mathbf{x}]^G$ denote the ring of polynomials in variables $\mathbf{x} = (x_1, \ldots, x_n)$ invariant under the action of the group G. Then
$$f(q) = \mathrm{Hilb}(\mathbb{Z}[\mathbf{x}]^{\mathcal{S}_k \times \mathcal{S}_{n-k}} / \mathbb{Z}[\mathbf{x}]_+^{\mathcal{S}_n}, q),$$
where $\mathcal{S}_k \times \mathcal{S}_{n-k}$ and $\mathcal{S}_n$ act as usual on $\mathbb{Z}[\mathbf{x}]$, and $\mathbb{Z}[\mathbf{x}]_+$ denotes the ring of polynomials with positive degree.
- $f(q)$ counts the number of k-subspaces of a vector space of dimension n over a finite field $\mathbb{F}_q$ with q elements i.e. the number points in the Grassmanian variety $\mathrm{Gr}_{\mathbb{F}_q}(k, n)$.

3 Summary of results

Paper A [5]

In the wake of Helton and Vinnikov's celebrated proof of the Lax conjecture [41] the follow up question was how the theorem should be generalized to more than three variables. Stronger versions of Conjecture 2.17 were initially believed to be true. For instance it was conjectured in [41] that if $h(\mathbf{z})$ is a hyperbolic polynomial, then $h(\mathbf{z})^N$ has a definite determinantal representation for some positive integer N. This belief is not totally unreasonable given that for $p(\mathbf{z})$ homogeneous and irreducible, it is well-known that $p(\mathbf{z})^N$ has a (not necessarily definite) determinantal representation for some N, see [8]. The claim was however disproved by Brändén in [15] via the bases generating polynomial of a certain non-representable hyperbolic matroid.

The *Vámos matroid* V_8 is the matroid with ground set $E = \{1, \dots, 8\}$ and bases

$$\mathcal{B}(V_8) = \binom{[8]}{4} \setminus \{\{1,2,3,4\}, \{3,4,5,6\}, \{1,2,5,6\}, \{1,2,7,8\}, \{5,6,7,8\}\}.$$

Theorem 3.1 (Wagner-Wei [67])**.** *V_8 is a HPP matroid (and therefore hyperbolic).*

In 1969 Ingleton [43] proved a necessary condition for a matroid to be representable.

Theorem 3.2 (Ingleton)**.** *Suppose $r : 2^E \to \mathbb{N}$ is the rank function of a representable matroid and $A, B, C, D \subseteq E$. Then*

$$\begin{aligned} r(A \cup B) + r(A \cup C \cup D) + r(C) + r(D) + r(B \cup C \cup D) \leq \\ r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D) + r(C \cup D) \end{aligned} \tag{3.1}$$

Considering V_8 and setting

$$A = \{1,2\}, \quad B = \{3,4\}, \quad C = \{5,6\}, \quad D = \{7,8\},$$

the Ingleton inequality (3.1) reads, $4+4+2+2+4 \leq 3+3+3+3+3$ which is a contradiction. Hence V_8 cannot be representable.

Theorem 3.3 (Brändén)**.** *There exists no positive integer N such that $P_{V_8}(\mathbf{z})^N$ has a definite determinantal representation where $P_{V_8}(\mathbf{z})$ denotes the bases generating polynomial of V_8.*

Proof sketch. Suppose

$$P_{V_8}(\mathbf{z}) = \det(\sum_{i=1}^{n} z_i A_i),$$

for some positive integer N and symmetric matrices $A_1, \dots, A_n$. The bases generating polynomial $P_{V_8}(\mathbf{z})$ is stable by Theorem 3.1, so it is hyperbolic with respect to $\mathbf{1}$. The rank function of the hyperbolic matroid associated with the hyperbolic

polynomial $P_{V_8}(\mathbf{z})^N$ with respect to $\mathcal{V} = \{\delta_1, \ldots, \delta_8\} \subseteq \mathbb{R}^8_+ \subseteq \Lambda_+$ can be expressed as

$$r_{\mathcal{V}}(S) = \deg\left(P_{V_8}\left(\mathbf{1} + t\sum_{i\in S}\delta_i\right)^N\right) = N r_{V_8}(S)$$

where $\delta_1, \ldots, \delta_8$ denote the standard basis vectors of $\mathbb{R}^8$ and r_{V_8} denotes the rank function of the matroid V_8. Now consider the representable matroid given by

$$r(S) := \mathrm{rk}\left(\sum_{i\in S} A_i\right).$$

By initial assumption we have

$$r(S) = r_{\mathcal{V}}(S) = N r_{V_8}(S).$$

However we know that r_{V_8} violates the Ingleton inequalities (3.1) which contradicts the fact that r is the rank function of a representable matroid. □

Remark 3.4. Brändén [15] in fact proved a slightly stronger statement: There exists no positive integers M, N and no linear form $\ell(\mathbf{z})$ such that $\ell(\mathbf{z})^M P_{V_8}(\mathbf{z})^N$ has a definite determinantal representation.

It is not known whether $P_{V_8}(\mathbf{z})$ satisfies the generalized Lax conjecture (Conjecture 2.17). In order to find potential obstructions to the generalized Lax conjecture it is worthwhile understanding the role of non-representable hyperbolic matroids in the context of the conjecture and finding additional instances of them. Prior to Paper A, only the Vámos matroid V_8 and a certain generalization of it were known to be both non-representable and hyperbolic.

A *paving matroid* of rank r is a matroid such that all its circuits (minimal dependent sets) have size at least r. A paving matroid of rank r is called *sparse* if all its hyperplanes (flats of rank $r-1$) have size $r-1$ or r.

Further instances of non-representable hyperbolic matroids come from finite projective geometry. Sparse paving matroids of rank three can be obtained from finite point-line configurations in which every line contains three points. Such matroids are obtained by letting a subset of three points define a circuit hyperplane if and only if there is a line containing them. The Pappus and Desargues configurations are geometrical configurations with 9 and 10 points respectively such that every line contains three points and every point is incident to three lines (note that such configurations need not be unique). The *Non-Pappus* and *Non-Desargues* matroids are obtained from the Pappus and Desargues configurations by deleting one line. Both of these matroids are not representable over any field. However the Non-Pappus matroid can be shown to be representable over every skew-field e.g. the quaternions $\mathbb{H}$, see [43]. The Non-Desargues matroid on the other hand is not even representable over any skew-field [43], but is known to be representable over the octonions $\mathbb{O}$, see [37]. The algebras $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$ of Hermitian 3×3 matrices

over $\mathbb{H}$ and $\mathbb{O}$ respectively, are examples of real Euclidean Jordan algebras. All real Euclidean Jordan algebras A come equipped with a hyperbolic determinant polynomial $\det : A \to \mathbb{R}$, in particular realizing the cone of positive semidefinite matrices in $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$ as hyperbolicity cones. Hence we obtain:

Theorem 3.5. *The Non-Pappus and Non-Desargues matroids are hyperbolic matroids not representable over any field.*

Burton et.al. [19] defined a class of matroids V_{2n} for $n \geq 4$ with base set $\mathcal{B}(V_{2n}) := \binom{[2n]}{4} \setminus \mathcal{H}_{2n}$ where

$$\mathcal{H}_{2n} := \{1, 2, 2k-1, 2k\} \cup \{2k-1, 2k, 2k+1, 2k+1\} \text{ for } 2 \leq k \leq n,$$

extending the Vámos matroid. They made the following conjecture regarding the family V_{2n} for $n \geq 4$.

Conjecture 3.6 (Burton-Vinzant-Youm)**.** *For each $n \geq 4$, V_{2n} is a HPP matroid.*

Burton et.al. confirmed Conjecture 3.6 for $n = 5$. In Paper A we prove a sweeping generalization of Conjecture 3.6, in particular proving Conjecture 3.6 in the affirmative for all $n \geq 4$.

Theorem 3.7. *Let H be a d-uniform hypergraph on $[n]$, and let $E = \{1, 1', \ldots, n, n'\}$. Let*

$$\mathcal{B}(V_H) = \binom{E}{2d} \setminus \{e \cup e' : e \in E(H)\},$$

in which $e' := \{i' : i \in e\}$ for each $e \in E(H)$. Then $\mathcal{B}(V_H)$ is the set of bases of a sparse paving matroid V_H of rank $2d$.

Theorem 3.8. *If G is a simple graph, then V_G is a HPP matroid.*

Theorem 3.8 unfortunately does not admit a full generalization to matroids V_H parametrized by hypergraphs H. An obstruction is e.g. given by the complete 3-uniform hypergraph on [6]. Nevertheless we can prove the following.

Theorem 3.9. *If H is a d-uniform hypergraph, then V_H is a WHPP matroid.*

Remark 3.10. Since the class of hyperbolic matroids is equivalent to the class of WHPP matroids [5], all matroids V_H are hyperbolic by Theorem 3.9.

Remark 3.11. The family $\{V_{2n}\}_{n \geq 4}$ studied by Burton et al. [19] corresponds to V_{G_n} where G_n is an n-cycle with edges $\{1, i\}$, $i = 2, \ldots, n$, adjoined. Thus Theorem 3.8 implies Conjecture 3.6.

Remark 3.12. Since representability is closed under taking minors, any matroid V_H containing the Vámos V_8 as a minor is necessarily non-representable (and fails to satisfy Ingleton's inequality (3.1)).

The proof of Theorem 3.9 depends on certain symmetric function inequalities. These inequalities are also of independent interest.

Recall that a *partition* of a natural number d is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of natural numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = d$. We write $\lambda \vdash d$ to denote that λ is a partition of d. The *length*, $\ell(\lambda)$, of λ is the number of nonzero entries of λ. If λ is a partition and $\ell(\lambda) \leq n$, then the *monomial symmetric polynomial*, m_α, is defined as

$$m_\lambda(z_1, \ldots, z_n) := \sum z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n},$$

where the sum is over all distinct permutations $(\beta_1, \beta_2, \ldots, \beta_n)$ of $(\lambda_1, \ldots, \lambda_n)$. If $\ell(\lambda) > n$, we set $m_\lambda(\mathbf{z}) = 0$. The dth *elementary symmetric polynomial* is $e_d(\mathbf{z}) := m_{1^d}(\mathbf{z})$. Lemma 3.13 below is a refinement of the Laguerre-Turán inequalities

$$0 \leq r e_r(\mathbf{z})^2 - (r+1) e_{r-1}(\mathbf{z}) e_{r+1}(\mathbf{z}),$$

and is used in the proof of Theorem 3.14.

Lemma 3.13. *If $r \geq 1$, then*

$$m_{2^r}(\mathbf{z}) \leq r e_r(\mathbf{z})^2 - (r+1) e_{r-1}(\mathbf{z}) e_{r+1}(\mathbf{z}).$$

The theorem below is a central ingredient to the proof of Theorem 3.9.

Theorem 3.14. *Let $r \geq 2$ be an integer, and let*

$$M(\mathbf{z}) = \sum_{|S|=r} a(S) \prod_{i \in S} z_i^2 \in \mathbb{R}[z_1, \ldots, z_n],$$

where $0 \leq a(S) \leq 1$ for all $S \subseteq [n]$, where $|S| = r$. Then the polynomial

$$4 e_{r+1}(\mathbf{z}) e_{r-1}(\mathbf{z}) + \frac{3}{r+1} M(\mathbf{z})$$

is stable.

In light of Remark 3.4 it is natural to question whether it is possible to put any kind of restrictions on the factor $q(\mathbf{z})$ in Conjecture 2.17 when it comes to a prescribed bound on its degree and its number of irreducible factors. The answer turns out to be *no*. We construct a family of hyperbolic polynomials obtained from the bases generating polynomials of specific members of the family V_H, such that for sufficiently many variables $\mathbf{z} = (z_1, \ldots, z_n)$, the factor $q(\mathbf{z})$ in Conjecture 2.17 must either have an irreducible factor of large degree or have a large number of irreducible factors of low degree.

Given positive integers n and k, consider the k-uniform hypergraph $H_{n,k}$ on $[n+2]$ containing all hyperedges $e \in \binom{[n+2]}{k}$ except those for which $\{n+1, n+2\} \subseteq e$. By Theorem 3.9 the matroid $V_{H_{n,k}}$ is hyperbolic and therefore has a hyperbolic

bases generating polynomial $h_{V_{H_{n,k}}}(\mathbf{z})$ with respect to $\mathbf{1}$. The polynomial $h_{n,k}(\mathbf{z}) \in \mathbb{R}[z_1, \ldots, z_{n+2}]$, obtained from the multiaffine polynomial $h_{V_{H_{n,k}}}(\mathbf{z})$ by identifying the variables z_i and $z_{i'}$ pairwise for all $i \in [n+2]$ is therefore hyperbolic with respect to $\mathbf{1}$.

Theorem 3.15. *Let n and k be a positive integers. Suppose there exists a positive integer N and a hyperbolic polynomial $q(\mathbf{z})$ such that*

$$q(\mathbf{z})h_{n,k}(\mathbf{z})^N = \det\left(\sum_{i=1}^{n+2} z_i A_i\right) \tag{3.2}$$

with $\Lambda_+(h_{n,k}) \subseteq \Lambda_+(q)$ for some symmetric matrices $A_1, \ldots, A_{n+2}$ such that $A_1 + \cdots + A_{n+2}$ is positive definite and

$$q(\mathbf{z}) = \prod_{i=1}^{s} p_j(\mathbf{z})^{\alpha_i}$$

for some irreducible hyperbolic polynomials $p_1, \ldots, p_s \in \mathbb{R}[z_1, \ldots, z_{n+2}]$ of degree at most $k-1$ where $\alpha_1, \ldots, \alpha_s$ are positive integers. Then

$$n < (2s+1)k - 1.$$

Paper B [2]

Although there is not an extensive amount of evidence for the generalized Lax conjecture (Conjecture 2.17), the conjecture is known to hold for some specific classes of hyperbolic polynomials (see [5]). In particular Brändén [16] confirmed the conjecture for elementary symmetric polynomials, extending work of Zinchenko [69] and Sanyal [64]. Brändén applied the matrix-tree theorem, which implies that every spanning tree polynomial has a definite determinantal representation, and realized the spanning tree polynomial of a certain series-parallel graph as a product of elementary symmetric polynomials. A consequence of Brändén's result is that hyperbolic polynomials which are iterated derivatives of products of linear forms have spectrahedral hyperbolicity cones. Moreover the hyperbolicity cone of the spanning tree polynomial of a complete graph is linearly isomorphic to the cone of positive semidefinite matrices. Hence the generalized Lax conjecture is equivalent to the assertion that each hyperbolicity cone is an affine slice of the hyperbolicity cone of a spanning tree polynomial.

In Paper B we consider hyperbolicity cones of multivariate matching polynomials in context of the generalized Lax conjecture. Two main reasons for considering matching polynomials are the well-known facts that the univariate matching polynomial of a tree coincide with its characteristic polynomial and that every univariate matching polynomial divides the matching polynomial of a tree. Multivariate versions of the above two facts are important inputs for proving the generalized Lax

conjecture for the class of multivariate matching polynomials. As an application we reprove Brändén's result by realizing the elementary symmetric polynomials of degree k as a factor in the matching polynomial of the length-k truncated path tree of the complete graph.

Recall that a *k-matching* in a graph $G = (E, V)$ is a subset $M \subseteq E(G)$ of k edges, no two of which have a vertex in common. Let $\mathcal{M}(G)$ denote the set of all matchings in G and for $M \in \mathcal{M}(G)$, let $V(M)$ denote the set of vertices contained in M. Let $\mathbf{z} = (z_v)_{v \in V}$ and $\mathbf{w} = (w_e)_{e \in E}$ be indeterminates. Define the *homogeneous multivariate matching polynomial* $\mu(G, \mathbf{z} \oplus \mathbf{w}) \in \mathbb{R}[\mathbf{z}, \mathbf{w}]$ by

$$\mu(G, \mathbf{z} \oplus \mathbf{w}) := \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \prod_{v \notin V(M)} z_v \prod_{e \in M} w_e^2.$$

As a direct consequence of a theorem by Heilmann and Lieb [40], the polynomial $\mu(G, \mathbf{z} \oplus \mathbf{w})$ is hyperbolic with respect to $\mathbf{e} = \mathbf{1} \oplus \mathbf{0}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^V$ and $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^E$. Note that $\mu(G, \mathbf{z} \oplus \mathbf{w})$ specializes to the conventional univariate matching polynomial $\mu(G, t)$ by putting $\mathbf{z} \oplus \mathbf{w} = t\mathbf{1} \oplus \mathbf{1}$. The following recursion is immediate from the definition,

$$\mu(G, \mathbf{x} \oplus \mathbf{w}) = z_u \mu(G \setminus u, \mathbf{z} \oplus \mathbf{w}) - \sum_{v \in N(u)} w_{uv}^2 \mu((G \setminus u) \setminus v, \mathbf{z} \oplus \mathbf{w}).$$

Let G be a graph and $u \in V(G)$. The *path tree* $T(G, u)$ is the tree with vertices labelled by simple paths in G (i.e. paths with no repeated vertices) starting at u and where two vertices are joined by an edge if one vertex is labelled by a maximal subpath of the other. Godsil [32] proved the following divisibility relation for the univariate matching polynomial,

$$\frac{\mu(G \setminus u, t)}{\mu(G, t)} = \frac{\mu(T(G, u) \setminus u, t)}{\mu(T(G, u), t)}.$$

The above identity implies that $\mu(G, t)$ divides $\mu(T(G, u), t)$. To establish a multivariate version of the above relationship we must consider a natural change of variables. The technique used to prove the multivariate divisibility relation is very similar to its univariate counterpart. Let $\phi : \mathbb{R}^{T(G,u)} \to \mathbb{R}^G$ denote the linear change of variables defined by

$$\begin{aligned} z_p &\mapsto z_{i_k}, \\ w_{pp'} &\mapsto w_{i_k i_{k+1}}, \end{aligned}$$

where $p = i_1 \cdots i_k$ and $p' = i_1 \cdots i_k i_{k+1}$ are adjacent vertices in $T(G, u)$. For every subforest $T \subseteq T(G, u)$, define the polynomial

$$\eta(T, \mathbf{z} \oplus \mathbf{w}) := \mu(T, \phi(\mathbf{z}' \oplus \mathbf{w}'))$$

where $\mathbf{z}' = (z_p)_{p \in V(T)}$ and $\mathbf{w}' = (w_e)_{e \in E(T)}$.

Lemma 3.16. *Let $u \in V(G)$. Then*

$$\frac{\mu(G \setminus u, \mathbf{z} \oplus \mathbf{w})}{\mu(G, \mathbf{z} \oplus \mathbf{w})} = \frac{\eta(T(G,u) \setminus u, \mathbf{z} \oplus \mathbf{w})}{\eta(T(G,u), \mathbf{z} \oplus \mathbf{w})}.$$

In particular $\mu(G, \mathbf{z} \oplus \mathbf{w})$ divides $\eta(T(G,u), \mathbf{z} \oplus \mathbf{w})$.

The next lemma arises as a multivariate analogue to the fact that the matching polynomial of a tree T is equal to the characteristic polynomial of the adjacency matrix of T.

Lemma 3.17. *Let $T = (V, E)$ be a tree. Then $\mu(T, \mathbf{z} \oplus \mathbf{w})$ has a definite determinantal representation.*

Note that

$$\frac{\partial}{\partial z_u} \mu(G, \mathbf{z} \oplus \mathbf{w}) = \mu(G \setminus u, \mathbf{z} \oplus \mathbf{w}),$$

and therefore

$$\Lambda_+(\mu(G, \mathbf{z} \oplus \mathbf{w})) \subseteq \Lambda(\mu(G \setminus u, \mathbf{z} \oplus \mathbf{w})).$$

Using the above fact, Lemma 3.16 and Lemma 3.17 it follows, using an inductive argument, that multivariate matching polynomials $\mu(G, \mathbf{z} \oplus \mathbf{w})$ satisfy the generalized Lax conjecture for any graph G.

Theorem 3.18. *The hyperbolicity cone of $\mu(G, \mathbf{z} \oplus \mathbf{w})$ is spectrahedral.*

By considering the matching polynomial of the partial path tree of the complete graph K_n up to paths of length at most k, along with a suitable linear change of variables, we recover Brändén's result regarding the spectrahedrality of hyperbolicity cones of elementary symmetric polynomials. Hence Theorem 3.18 can be viewed as a generalization of this fact.

A subset $I \subseteq V(G)$ is called *independent* if no two vertices of I are adjacent in G. Let $\mathcal{I}(G)$ denote the set of all independent sets in G. Define the *homogeneous multivariate independence polynomial* $I(G, \mathbf{z} \oplus t) \in \mathbb{R}[\mathbf{z}, t]$ by

$$I(G, \mathbf{z} \oplus t) = \sum_{I \in \mathcal{I}(G)} (-1)^{|I|} \left(\prod_{v \in I} z_v^2 \right) t^{2|V(G)| - 2|I|}.$$

A graph is said to be *claw-free* if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. If G is a claw-free graph, then $I(G, \mathbf{z} \oplus t)$ is hyperbolic with respect to $\mathbf{e} = (0, \ldots, 0, 1)$. This fact is a simple consequence of the real-rootedness of the weighted univariate independence polynomial of a claw-free graph, due to Engström [27]. We prove that when G satisfies an additional technical condition (stronger than claw-freeness), then $I(G, \mathbf{z} \oplus t)$ satisfies Conjecture 2.17.

Matching polynomials and independence polynomials are intimately related. The *line graph* $L(G)$ of G is the graph having vertex set $E(G)$ and where two

vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident. The univariate matching polynomial of a graph G can be realized as the univariate independence polynomial of its line graph $L(G)$. With that said, the multivariate polynomial $I(G, \mathbf{z} \oplus t)$ does not strictly generalize $\mu(G, \mathbf{z} \oplus \mathbf{w})$ due to the dummy homogenization in the variable t. Unfortunately we were unsuccessful in constructing a hyperbolic refinement of $I(G, \mathbf{z} \oplus t)$ with respect to the variable t which reduces to $\mu(G, \mathbf{z} \oplus \mathbf{w})$ (after relabelling) when G is a line graph.

The key to proving the generalized Lax conjecture for $I(G, \mathbf{z} \oplus t)$ is to find a tree that plays a role similar to that of the path tree for the matching polynomial. Such a tree was constructed by Leake and Ryder in [47]. We outline its construction below.

An induced clique K in G is called a *simplicial clique* if for all $u \in K$ the induced subgraph $N[u] \cap (G \setminus K)$ of $G \setminus K$ is a clique. In other words the neighbourhood of each $u \in K$ is a disjoint union of two induced cliques in G. Furthermore, a graph G is said to be *simplicial* if G is claw-free and contains a simplicial clique. A connected graph G is a *block graph* if each 2-connected component is a clique.

Given a simplicial graph G with a simplicial clique K we recursively define a block graph $T^{\boxtimes}(G, K)$ called the *clique tree* associated to G and rooted at K.

We begin by adding K to $T^{\boxtimes}(G, K)$. Let $K_u = N[u] \setminus K$ for each $u \in K$. Attach the disjoint union $\bigsqcup_{u \in K} K_u$ of cliques to $T^{\boxtimes}(G, K)$ by connecting $u \in K$ to every $v \in K_u$. Finally recursively attach $T^{\boxtimes}(G \setminus K, K_u)$ to the clique K_u in $T^{\boxtimes}(G, K)$ for every $u \in K$.

Theorem 3.19 (Leake-Ryder)**.** *Let K be a simplicial clique of a simplicial graph G. Then*

$$\frac{I(G, \mathbf{z} \oplus t)}{I(G \setminus K, \mathbf{z} \oplus t)} = \frac{I(T^{\boxtimes}(G, K), \mathbf{z} \oplus t)}{I(T^{\boxtimes}(G, K) \setminus K, \mathbf{z} \oplus t)},$$

where $T^{\boxtimes}(G, K)$ is relabelled according to the natural graph homomorphism $\phi_K : T^{\boxtimes}(G, K) \to G$. Moreover $I(G, \mathbf{z} \oplus t)$ divides $I(T^{\boxtimes}(G, K), \mathbf{z} \oplus t)$.

The following lemma asserts that vertex deletion relaxes the hyperbolicity cone, providing the necessary setup for an inductive argument of spectrahedrality.

Lemma 3.20. *Let $v \in V(G)$. Then $\Lambda_+(I(G, \mathbf{z} \oplus t)) \subseteq \Lambda_+(I(G \setminus v, \mathbf{z} \oplus t))$.*

Using Theorem 3.19, Lemma 3.20 and the fact that the clique tree $T^{\boxtimes}(G, K)$ can be realized as the line graph of an actual tree, one proves the theorem below using an inductive argument which unfolds in an analogous manner to the proof of Theorem 3.18.

Theorem 3.21. *If G is a simplicial graph, then the hyperbolicity cone of $I(G, \mathbf{z} \oplus t)$ is spectrahedral.*

Paper C [3]

A graph G is called *Ramanujan* if the absolute value of its largest non-trivial eigenvalue is bounded above by the spectral radius $\rho(G)$ of its universal covering tree. We refer to [33] for undefined terminology. *Expanders* are graphs which can be informally characterized by being sparse and yet well-connected. Expanders are of importance in e.g. computer science where they serve as basic building blocks for robust network designs (among other things). Due to their spectral properties, Ramanujan graphs are considered optimal expanders in the sense that a random walk on a Ramanujan graph converges to the uniform distribution in the fastest possible way. The existence of Ramanujan graphs is a highly non-trivial issue. A longstanding open question asks about the existence of infinitely many k-regular Ramanujan graphs for every $k \geq 3$. Marcus, Spielman and Srivastava proved that every finite graph G has a 2-sheeted covering (or 2-covering for short) with maximum non-trivial eigenvalue (not induced by G) bounded above by $\rho(G)$, a so called *one-sided Ramanujan covering*. Since coverings of bipartite graphs are bipartite, and the spectrum of a bipartite graph is symmetric around zero, they were able to point to the existence of infinitely many k-regular bipartite Ramanujan graphs.

Subsequently Hall, Puder and Sawin [39] generalized the techniques in [50, 51] and proved that every loopless connected graph has a one-sided Ramanujan d-covering for every $d \geq 1$. An essential polynomial to the proof is the average matching polynomial of all d-coverings of G. For $d \geq 1$, the *d-matching polynomial* of G is defined by

$$\mu_{d,G}(z) := \frac{1}{|\mathcal{C}_{d,G}|} \sum_{H \in \mathcal{C}_{d,G}} \mu_H(z),$$

where $\mathcal{C}_{d,G}$ denotes the set of all d-coverings of G and

$$\mu_G(z) := \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i m_i z^{n-2i} \in \mathbb{Z}[z]$$

denotes the univariate matching polynomial of G. In particular if $d = 1$, then $\mu_{d,G}(z) = \mu_G(z)$.

Using the celebrated technique of interlacing families, developed by Marcus, Spielman and Srivastava, the authors prove that the maximum root of the expected characteristic polynomial over all d-coverings of G is bounded above by their uniform average, which in turn is proved to equal $\mu_{d,G}(z)$. The real roots of $\mu_{d,G}(z)$ on the other hand can easily be deduced to lie in the interval $[-\rho(G), \rho(G)]$ using a well-known theorem of Heilmann and Lieb [40]. Hence there is at least one covering in the family which has its maximal non-trivial eigenvalue less than the maximum root of the average $\mu_{d,G}(z)$, that is, less than $\rho(G)$ as desired.

As implied by the paragraph above we have in particular the following theorem.

Theorem 3.22 (Hall-Puder-Sawin)**.** *If G is a finite loopless graph, then $\mu_{d,G}(z)$ is real-rooted.*

The authors gave a rather long and indirect proof of Theorem 3.22. They further asked for a direct proof that includes graphs with loops. In Paper C we answer their question by proving that a multivariate version of the d-matching polynomial is stable, a statement which is more general than their original question. Define the *multivariate d-matching polynomial* of G by

$$\mu_{d,G}(\mathbf{z}) := \mathbb{E}_{H\in\mathcal{C}_{d,G}}\mu_H(\mathbf{z}),$$

where

$$\mu_G(\mathbf{z}) := \sum_M (-1)^{|M|} \prod_{v\in[n]\setminus V(M)} z_v,$$

and the sum runs over all matchings in G. By analysing the algebraic symbol it follows that the *multi-affine part* operator

$$\begin{aligned} \mathrm{MAP} : \mathbb{C}[z_1,\dots,z_n] &\to \mathbb{C}[z_1,\dots,z_n] \\ \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a(\boldsymbol{\alpha})\mathbf{z}^{\boldsymbol{\alpha}} &\mapsto \sum_{\boldsymbol{\alpha}:\alpha_i\leq 1, i\in[n]} a(\boldsymbol{\alpha})\mathbf{z}^{\boldsymbol{\alpha}} \end{aligned}$$

is a stability-preserving linear operator. Moreover one sees that

$$\mathrm{MAP}\left(\prod_{uv\in E(G)}(1-z_uz_v)\right) = \mu_G(\mathbf{z}),$$

proving that $\mu_G(\mathbf{z})$ is stable. By using MAP and the Grace-Walsh-Szegö theorem we prove:

Theorem 3.23. *Let G be a finite graph and $d\geq 1$. Then $\mu_{d,G}(\mathbf{z})$ is stable.*

Corollary 3.24. *Let G be a finite graph and $d\geq 1$. Then $\mu_{d,G}(z)$ is real-rooted.*

Proof. Follows by putting $\mathbf{z}=(z,\dots,z)$ in Theorem 3.23 □

In [40] Heilmann and Lieb proved that the matching polynomial $\mu_G(z)$ of any graph G is real-rooted. In analogy with graph matchings, a matching in a hypergraph consists of a subset of (hyper)edges with empty pairwise intersection. However the analogous matching polynomial for hypergraphs is not real-rooted in general, see e.g. [34]. A natural question is thus how to generalize the Heilmann-Lieb theorem to hypergraphs. We consider a relaxation of matchings in general hypergraphs that leads to an associated real-rooted polynomial which reduces to the conventional matching polynomial for graphs.

Consider the problem of assigning a subset of n people with prescribed competencies into teams of no less than two people, working on a subset of m different projects in such a way that no person is assigned to more than one project and each person has the competency to work on the project they are assigned to. We shall call such team assignments "relaxed matchings". More formally define a *relaxed*

matching in a hypergraph $H = (V(H), E(H))$ to be a collection $M = (S_e)_{e \in E}$ of edge subsets such that $E \subseteq E(H)$, $S_e \subseteq e$, $|S_e| > 1$ and $S_e \cap S_{e'} = \emptyset$ for all pairwise distinct $e, e' \in E$.

Remark 3.25. If H is a graph then the concept of relaxed matching coincides with the conventional notion of graph matching. Note also that a conventional hypergraph matching is a relaxed matching $M = (S_e)_{e \in E}$ for which $S_e = e$ for all $e \in E$.

Remark 3.26. The subsets S_e in the relaxed matching are labeled by the edge they are chosen from in order to avoid ambiguity. However if H is a *linear hypergraph*, that is, the edges pairwise intersect in at most one vertex, then the subsets uniquely determine the edges they belong to and therefore no labeling is necessary. Graphs and finite projective geometries (viewed as hypergraphs) are examples of linear hypergraphs.

Let $V(M) := \bigcup_{S_e \in M} S_e$ denote the set of vertices in the relaxed matching. Moreover let $m_k(M) := |\{S_e \in M : |S_e| = k\}|$ denote the number of subsets in the relaxed matching of size k. Define the *multivariate relaxed matching polynomial* of H by

$$\eta_H(\mathbf{z}) := \sum_M (-1)^{|M|} W(M) \prod_{i \in [n] \setminus V(M)} z_i,$$

where the sum runs over all relaxed matchings of H and

$$W(M) := \prod_{k=1}^{n-1} k^{m_{k+1}(M)}.$$

Let $\eta_H(z) := \eta_H(z\mathbf{1})$ denote the *univariate relaxed matching polynomial.*

Remark 3.27. If H is a graph, then $\eta_H(z) = \mu_H(z)$.

Theorem 3.28. *The polynomial $\eta_H(\mathbf{z})$ is stable. In particular*

$$\eta_H(z) = \sum_M (-1)^{|M|} W(M) z^{n - |V(M)|},$$

is a real-rooted polynomial for any hypergraph H.

Paper D [4]

Combining the study of pattern avoidance with combinatorial statistics is a paradigm which has been advocated in papers by Claesson-Kitaev [23] and Sagan-Savage [63] among others. Typically one is interested in the generating polynomial

$$f(q) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\mathrm{stat}(\sigma)},$$

for some pattern set Π and combinatorial statistic $\mathrm{stat} : \mathcal{S}_n(\Pi) \to \mathbb{N}$. Examples of questions one may ask about $f(q)$ have to do with equidistribution, recursion and unimodality/log-concavity/real-rootedness etc. In Paper D we focus on equidistributions of the form

$$\sum_{\sigma \in \mathcal{S}_n(\Pi_1)} q^{\mathrm{stat}_1(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\Pi_2)} q^{\mathrm{stat}_2(\sigma)},$$

where Π_1, Π_2 consist of a single classical pattern of length three and stat_1, stat_2 are Mahonian permutation statistics.

Let Π denote the set of vincular patterns of length at most d. A *d-function* is a statistic of the form

$$\mathrm{stat} = \sum_{\pi \in \Pi} \alpha_\pi \cdot (\pi),$$

where $\alpha_\pi \in \mathbb{N}$ and (π) is the statistic counting the number of occurrences of the pattern π. Babson and Steingrímsson classified all Mahonian 3-functions up to trivial symmetries. Several previously studied Mahonian statistics fall under the classification, including maj and inv. The complete table of Mahonian 3-functions may be found below along with their original references.

Name	Vincular pattern statistic	Reference
maj	$(1\underline{32}) + (2\underline{31}) + (3\underline{21}) + (\underline{21})$	MacMahon [49]
inv	$(\underline{23}1) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	MacMahon [49]
mak	$(1\underline{32}) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Foata-Zeilberger [30]
makl	$(1\underline{32}) + (2\underline{31}) + (\underline{32}1) + (\underline{21})$	Clarke-Steingrímsson-Zeng [24]
mad	$(2\underline{31}) + (2\underline{31}) + (\underline{31}2) + (\underline{21})$	Clarke-Steingrímsson-Zeng [24]
bast	$(\underline{13}2) + (\underline{21}3) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[7]
bast′	$(\underline{13}2) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[7]
bast″	$(1\underline{32}) + (3\underline{12}) + (3\underline{21}) + (\underline{21})$	Babson-Steingrímsson[7]
foze	$(\underline{21}3) + (3\underline{21}) + (\underline{13}2) + (\underline{21})$	Foata-Zeilberger [29]
foze′	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Foata-Zeilberger [29]
foze″	$(\underline{23}1) + (\underline{31}2) + (\underline{31}2) + (\underline{21})$	Foata-Zeilberger [29]
sist	$(\underline{13}2) + (\underline{13}2) + (2\underline{13}) + (\underline{21})$	Simion-Stanton [65]
sist′	$(\underline{13}2) + (\underline{13}2) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [65]
sist″	$(\underline{13}2) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [65]

Since all statistics in the table above are Mahonian, they are by definition equidistributed over $\mathcal{S}_n$. In Paper D we ask what equidistributions hold between the statistics if we restrict ourselves to permutations avoiding a classical pattern of

length three. Existing bijections $\phi : \mathcal{S}_n \to \mathcal{S}_n$ in the literature for proving the Mahonian nature of these statistics do not restrict to bijections over pattern classes. Therefore there is no a priori reason to expect that such equidistributions should continue to hold over $\mathcal{S}_n(\pi)$. Another motivation for studying equidistributions over $\mathcal{S}_n(\pi)$ where $\pi \in \mathcal{S}_3$, is that these pattern classes are enumerated by the Catalan numbers. Thus under appropriate bijections we may get induced equidistributions between combinatorial statistics on other Catalan structures (and vice versa). Below we give an example of such an induced equidistributions from Paper D.

Theorem 3.29. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)},$$

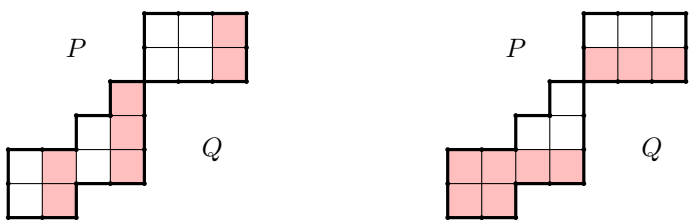
where $\mathrm{DB}(\sigma) := \{\sigma(i+1) : \sigma(i) > \sigma(i+1)\}$ *and* $\mathrm{DT}(\sigma) := \{\sigma(i) : \sigma(i) > \sigma(i+1)\}$.

The equidistribution in Theorem 3.29 is proved via an explicit involution $\phi : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ mapping maj to mak and preserving descent bottoms and descent tops in the process. The involution ϕ induces an equidistribution on shortened polyominoes (another Catalan structure) as we shall now describe.

A *shortened polyomino* is a pair (P, Q) of N (north), E (east) lattice paths $P = (P_i)_{i=1}^n$ and $Q = (Q_i)_{i=1}^n$ satisfying

1. P and Q begin at the same vertex and end at the same vertex.
2. P stays weakly above Q and the two paths can share E-steps but not N-steps.

Denote the set of shortened polyominoes with $|P| = |Q| = n$ by $\mathcal{H}_n$. Let $\mathrm{Valley}(Q) = \{i : Q_i Q_{i+1} = EN\}$ denote the set of indices of the *valleys* in Q and let $\mathrm{nval}(Q) = |\mathrm{Valley}(Q)|$. Define the statistics *valley-column area*, $\mathrm{vcarea}(P, Q)$, and *valley-row area*, $\mathrm{vrarea}(P, Q)$, as illustrated below.



(a) $\mathrm{vcarea}(P, Q) = 2 + 3 + 2 = 7$ (b) $\mathrm{vrarea}(P, Q) = 2 + 4 + 3 = 9$

Cheng, Eu and Fu [21] gave a creative bijection $\Psi : \mathcal{H}_n \to \mathcal{S}_n(321)$. In Paper D we show that

- $\mathrm{vcarea}(P, Q) = [(\underline{21}) + (\underline{31}2)]\Psi(P, Q)$.

- $\mathrm{vrarea}(P,Q) = [(\underline{21}) + (2\underline{31})]\Psi(P,Q)$.

From the involution ϕ in Theorem 3.29 one gets

$$[(\underline{21}) + (\underline{31}2)]\phi(\sigma) = [(\underline{21}) + (2\underline{31})]\sigma.$$

Hence by considering the composition $\Psi^{-1} \circ \phi \circ \Psi$ we get the following induced equidistribution.

Theorem 3.30. *For any $n \geq 1$,*

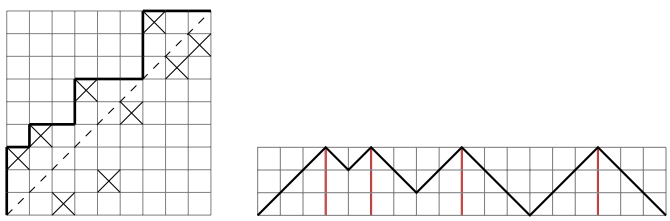
$$\sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vcarea}(P,Q)} t^{\mathrm{nval}(Q)} = \sum_{(P,Q)\in\mathcal{H}_n} q^{\mathrm{vrarea}(P,Q)} t^{\mathrm{nval}(Q)}.$$

Conversely we may prove equidistributions between Mahonian 3-functions via equidistributions over an intermediate Catalan structure. Below we give an example of this technique from Paper D.

Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never go below the x-axis. For convenience we denote the up-steps by U and the down-steps by D. Let $\mathcal{D}_n$ denote the set of Dyck paths of semi-length n. Under Krattenthaler's well-known bijection $\Gamma : \mathcal{S}_n(321) \to \mathcal{D}_n$, the statistic inv is mapped to the statistic *sumpeaks*, defined for Dyck paths $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$ by

$$\mathrm{spea}(P) := \sum_{p\in\mathrm{Peak}(P)} (\mathrm{ht}_P(p) - 1),$$

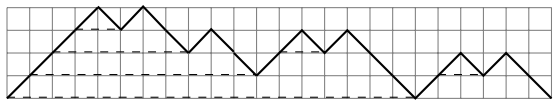
where $\mathrm{Peak}(P) := \{p : s_p s_{p+1} = UD\}$ and $\mathrm{ht}_P(p)$ is the y-coordinate of the pth step in P. The figure below illustrates the Dyck path corresponding to $\sigma = 341625978 \in \mathcal{S}_9(321)$ under Krattenthaler's bijection, mapping inv to spea.



Let $\mathrm{Valley}(P) := \{v : s_v s_{v+1} = DU\}$ denote the set of indices of the valleys in P. For each $v \in \mathrm{Valley}(P)$, there is a corresponding *tunnel* which is the subword $s_i \cdots s_v$ of P where i is the step after the first intersection of P with the line $y = \mathrm{ht}_P(v)$ to the left of step v (see figure below). The length, $v - i$, of a tunnel is always an even number. Let $\mathrm{Tunnel}(P) := \{(i,j) : s_i \cdots s_j \text{ tunnel in } P\}$ denote the set of pairs of beginning and end indices of the tunnels in P. Define the statistic *sumtunnels* by

$$\mathrm{stun}(P) := \sum_{(i,j)\in\mathrm{Tunnel}(P)} (j-i)/2.$$

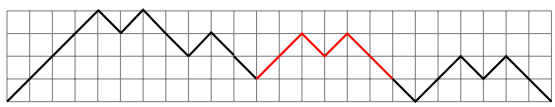
The tunnel lengths of the Dyck path below are highlighted by dashes.



Cheng, Elizalde, Kasraoui and Sagan [20] gave a bijection $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ mapping spea to stun. The *mass* corresponding to two consecutive U-steps, is half the number of steps between their matching D-steps (i.e. if $P = UUP'DP''D$, then the mass of the pair UU is $|P''|/2$). Define the statistics

$$\begin{aligned} \mathrm{mass}(P) &:= \text{ sum of masses over all occurrences of } UU \\ \mathrm{dr}(P) &:= \text{ number of double rises } UU \text{ in } P. \end{aligned}$$

The part of the Dyck path below contributing to the mass associated with the first double rise is highlighted in red.



In Paper D we give a bijection $\Phi : \mathcal{D}_n \to \mathcal{D}_n$, mapping stun to mass + dr. Finally via Knuth's standard bijection $\Delta : \mathcal{S}_n(231) \to \mathcal{D}_n$ defined recursively by $k\sigma_1\sigma_2 \mapsto U\Delta(\sigma_1)D\Delta(\sigma_2)$ where $\sigma_1 < k < \sigma_2$, we map the 3-Mahonian statistic mad to mass + dr. Combining all mentioned bijections we obtain the following theorem.

Theorem 3.31. *For any $n \geq 1$,*

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}(321)} q^{\mathrm{inv}(\sigma)} &= \sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{mass}(P)+\mathrm{dr}(P)} \\ &= \sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} \end{aligned}$$

As an aside we find several other related equidistributions with inv and mad over $\mathcal{S}_n(321)$ and $\mathcal{S}_n(231)$ respectively.

Consider the statistic

$$\mathrm{inc} := \iota_1 + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \iota_k$$

where $\iota_{k-1} = (12 \ldots k)$ is the statistic that counts the number of increasing subsequences of length k in a permutation. Using the Catalan continued fraction framework of Brändén, Claesson and Steingrímsson[18] we prove the following equidistribution.

Theorem 3.32. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(132)} q^{\mathrm{inc}(\sigma)}.$$

Let $\mathrm{Up}(P) := \{i : s_i = U\}$ denote the indices of the up-steps in $P = s_1 \cdots s_{2n}$. Define

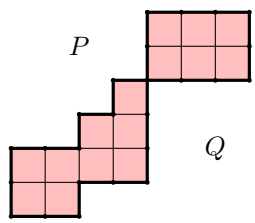
$$\mathrm{sups}(P) := \sum_{i \in \mathrm{Up}(P)} \lceil \mathrm{ht}_P(i)/2 \rceil.$$

By constructing a bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$, mapping sups to mass + dr, we deduce via Theorem 3.31 the following equidistribution.

Proposition 3.33. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{sups}(P)}.$$

If $(P, Q) \in \mathcal{H}_n$ is a shortened polyomino, then the *area statistic*, $\mathrm{area}(P, Q)$ is defined as the number of boxes enclosed by (P, Q).



It is finally worth mentioning the following equidistribution.

Theorem 3.34 (Cheng-Eu-Fu)**.** *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)} = \sum_{(P,Q) \in \mathcal{H}_n} q^{\mathrm{area}(P,Q)}.$$

See Paper D for the full table of established and conjectured Mahonian 3-function equidistributions.

Paper E [1]

Given a cyclic action of C_n on the set X, Reiner, Stanton and White [58] showed that the polynomial $f(q)$ in (2.3) always makes $(X, C_n, f(q))$ into a CSP triple. Many natural CSP triples occurring in the literature have the additional property that $f(q) = \sum_{x \in X} q^{\mathrm{stat}(x)}$ for some combinatorial statistic $\mathrm{stat} : X \to \mathbb{N}$. Conversely it is natural to ask under what circumstances a combinatorial polynomial

$f(q) = \sum_{x \in X} q^{\mathrm{stat}(x)}$ can be complemented with a cyclic action to a CSP? In Paper E we give a necessary and sufficient criterion for this to be the case. In particular the converse is not trivial in the sense that if $f(q) \in \mathbb{N}[q]$ is a polynomial such that $f(\omega_n^j) \in \mathbb{N}$ for all $1 \leq j \leq n$, then one cannot always find a cyclic action complementing $f(q)$ to a CSP. Our main theorem is the following.

Theorem 3.35. *Let $f(q) \in \mathbb{N}[q]$ and suppose $f(\omega_n^j) \in \mathbb{N}$ for each $j = 1, \ldots, n$. Let X be any set of size $f(1)$. Then there exists an action of C_n on X such that $(X, C_n, f(q))$ exhibits CSP if and only if for each $k|n$,*

$$\sum_{j|k} \mu(k/j) f(\omega_n^j) \geq 0. \tag{3.3}$$

The action complementing $f(q)$ to a CSP in Theorem 3.35 is given by the following generic construction.

Construction 3.36. *Let $X = \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \cdots \sqcup \mathcal{O}_m$ be a partition of a finite set X into m parts such that $|\mathcal{O}_i|$ divides n for $i = 1, \ldots, m$. Fix a total ordering on the elements of $\mathcal{O}_i$ for $i = 1, \ldots, m$. Let C_n act on X by permuting each element $x \in \mathcal{O}_i$ cyclically with respect to the total ordering on $\mathcal{O}_i$ for $i = 1, \ldots, m$.*

We call the action in Construction 3.36 an *ad-hoc* cyclic action. The action lacks combinatorial context and merely depends on the choice of partition and total order. By ordinary Möbius inversion, the sums $S_k = \sum_{j|k} \mu(k/j) f(\omega_n^j)$ represent the number of elements of order k under the action of C_n. Thus the only non-trivial issue in the proof of Theorem 3.35 is whether k divides S_k for all k. This is required for the elements to be evenly partitioned into orbits. Rather surprisingly it turns out that the divisibility property always hold as long as $f(\omega_n^j) \in \mathbb{Z}$ for all $1 \leq j \leq n$.

Although we would generally not consider a CSP "interesting" unless both the action and the polynomial are combinatorially meaningful, we think that our criteria serves a useful purpose in the way that a candidate polynomial can be quickly tested for CSP without having a combinatorial cyclic action at hand. A combinatorial polynomial passing the test may be a likely indication that a combinatorially meaningful cyclic action is present explaining the CSP.

Example 3.37. Let $f(q) = q^5 + 3q^3 + q + 9$. Then $f(\omega_6^j)$ takes values $7, 11, 4, 11, 7, 14$ for $j = 1, \ldots, 6$. On the other hand $S_k = \sum_{j|k} \mu(k/j) f(\omega_6^j)$ takes values $7, 4, -3, 0, 0, 6$ for $k = 1, \ldots, 6$. Since we cannot have a negative number of elements of order 3, there is no action of C_6 on a set X of size $f(1) = 14$ such that $(X, C_6, f(q))$ is a CSP-triple.

Thus even if $f(q) \in \mathbb{N}[q]$ satisfies $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \ldots, n$, we may not have an associated cyclic action complementing $f(q)$ to a CSP.

In the second part of Paper E we consider CSP from a more geometric perspective. Let $\mathrm{stat} : X \to \mathbb{N}$ be a statistic and denote $\mathrm{stat}_n(x) = \mathrm{stat}(x) \pmod{n}$. Consider

the joint distribution

$$\sum_{x\in X} q^{\mathrm{stat}_n(x)} t^{o(x)} = \sum_{i=0}^{n-1}\sum_{j=1}^{n} a_{ij} q^i t^j,$$

where $o(x)$ denotes the order of $x \in X$ under C_n. We can now restate CSP as follows.

Proposition 3.38. *Suppose X is a finite set on which C_n acts and let $f(q) = \sum_{x\in X} q^{stat(x)}$ where* stat $: X \to \mathbb{N}$ *is a statistic. Then the triple $(X, C_n, f(q))$ exhibits CSP if and only if $A_{(X,C_n,\mathrm{stat})} = (a_{ij})$ satisfies the condition that for each $1 \le k \le n$,*

$$\sum_{\substack{0\le i<n\\ 1\le j\le n}} a_{ij}\omega_n^{ki} = \sum_{0\le i<n}\sum_{j|k} a_{ij}. \tag{3.4}$$

where ω_n is a primitive nth root of unity.

We call a matrix $A = (a_{ij}) \in \mathbb{R}_{\ge 0}^{n\times n}$ a CSP matrix if it satisfies the linear equations in (3.4). Let CSP(n) denote the set of $n \times n$ CSP matrices.

Example 3.39. Consider all binary words of length 6, with group action being cyclic right-shift by one position and stat being the the major index statistic (sum of all descent indices). Then

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 11\\ 0 & 0 & 2 & 0 & 0 & 7\\ 0 & 0 & 0 & 0 & 0 & 11\\ 0 & 1 & 2 & 0 & 0 & 7\\ 0 & 0 & 0 & 0 & 0 & 11\\ 0 & 0 & 2 & 0 & 0 & 7 \end{pmatrix}$$

is the corresponding CSP matrix. The above matrix can be checked to satisfy (3.4) with $n = 6$. The entry in the upper left hand corner correspond to the two binary words 000000 and 111111. These have major index 0 and are fixed under a single shift, so they have order one. The words corresponding to the second column are 010101 and 101010. These have major index $6 \equiv 0 \pmod 6$ and $9 \equiv 3 \pmod 6$ respectively and are fixed under a minimum of two consecutive shifts, so they have order two etc.

Define the hyperplanes

$$H_k(\mathbf{x}) := \sum_{i=0}^{n-1}\sum_{\substack{j|n\\ j>1}} \alpha_{ijk} x_{ij} \in \mathbb{Z}[\mathbf{x}],$$

where

$$\alpha_{ijk} := \begin{cases} -n + \frac{n}{j}, & \text{if } i = k \text{ and } k \equiv 0 \pmod{\frac{n}{j}}, \\ -n, & \text{if } i = k \text{ and } k \not\equiv 0 \pmod{\frac{n}{j}}, \\ \frac{n}{j}, & \text{if } i \neq k \text{ and } k \equiv 0 \pmod{\frac{n}{j}}, \\ 0, & \text{if } i \neq k \text{ and } k \not\equiv 0 \pmod{\frac{n}{j}}. \end{cases}$$

Theorem 3.40. *We have*

$$CSP(n) \cong \{\mathbf{x} \in \mathbb{R}_{\geq 0}^{n(d-1)+1} : H_k(\mathbf{x}) \geq 0\},$$

where d denotes the number of divisors of n.

Thus we see that CSP(n) forms a convex rational polyhedral cone of dimension $n(d-1)+1$. The cone CSP(n) has several notable properties as summarized below.

- The integer lattice points $\mathrm{CSP}(n) \cap \mathbb{Z}^{n \times n}$ correspond to distributions that are realizable by a CSP triple $(X, C_n, f(q))$.
- Suppose that i and i' are indices such that $\gcd(n, i) = \gcd(n, i')$. Then the operation of swapping rows i and i' preserves the property of being a CSP matrix.
- Adding a matrix B with zero row and column-sum to a CSP matrix A preserves the property of being a CSP matrix provided $A + B \in \mathbb{R}_{\geq 0}^{n \times n}$.

About the joint paper contributions of the author

Papers A and E in this thesis are a result of joint collaboration with two different coauthors. The contribution of the author in each of these papers is described below.

Paper A was written together with the author's advisor Petter Brändén. While the author participated in all aspects of the project, many of the key breakthroughs regarding the symmetric function inequalities were made by the advisor. Initially the hyperbolicity of the matroids in our family was proved only for graphs. The main contribution of the author pertains to the generalization of the inequalities in the graphical case to strengthen the main result to matroids derived from hypergraphs. This later turned out to have consequences for the generalized Lax conjecture and produce instances of non-representable hyperbolic matroids without a Vámos minor. Some smaller results regarding the minor closure of the matroid family and facts regarding representability of matroids derived from tree-like hypergraphs was also contributed by the author. Paper E was written jointly with Per Alexandersson where both authors contributed approximately equal amounts to all aspects of the work.

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Part II

Scientific papers

Paper A

NON–REPRESENTABLE HYPERBOLIC MATROIDS

NIMA AMINI AND PETTER BRÄNDÉN

Abstract. The generalized Lax conjecture asserts that each hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices. Hyperbolic polynomials give rise to a class of (hyperbolic) matroids which properly contains the class of matroids representable over the complex numbers. This connection was used by the second author to construct counterexamples to algebraic (stronger) versions of the generalized Lax conjecture by considering a non–representable hyperbolic matroid. The Vámos matroid and a generalization of it are, prior to this work, the only known instances of non–representable hyperbolic matroids.

We prove that the Non–Pappus and Non–Desargues matroids are non-representable hyperbolic matroids by exploiting a connection between Euclidean Jordan algebras and projective geometries. We further identify a large class of hyperbolic matroids which contains the Vámos matroid and the generalized Vámos matroids recently studied by Burton, Vinzant and Youm. This proves a conjecture of Burton *et al.* We also prove that many of the matroids considered here are non–representable. The proof of hyperbolicity for the matroids in the class depends on proving nonnegativity of certain symmetric polynomials. In particular we generalize and strengthen several inequalities in the literature, such as the Laguerre–Turán inequality and an inequality due to Jensen. Finally we explore consequences to algebraic versions of the generalized Lax conjecture.

Contents

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1. Introduction

Although hyperbolic polynomials have their origin in PDE theory, they have during recent years been studied in diverse areas such as control theory, optimization, real algebraic geometry, probability theory, computer science and combinatorics, see [35, 36, 39, 40] and the references therein. To each hyperbolic polynomial is associated a closed convex (hyperbolicity) cone. Over the past 20 years methods have been developed to do optimization over hyperbolicity cones, which generalize semidefinite programming. A problem that has received considerable interest is the generalized Lax conjecture which asserts that each hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices (of some size). Hence if the generalized Lax conjecture is true then hyperbolic programming is the same as semidefinite programming.

Choe *et al.* [11] and Gurvits [20] proved that hyperbolic polynomials give rise to a class of matroids, called hyperbolic matroids or matroids with the weak half–plane property. The class of hyperbolic matroids properly contains the class of matroids which are representable over the complex numbers, see [11, 41]. This fact was used by the second author [7] to construct counterexamples to algebraic (stronger) versions of the generalized Lax conjecture. To better understand, and to identify potential counterexamples to the generalized Lax conjecture, it is therefore of interest to study hyperbolic matroids which are not representable over $\mathbb{C}$, or even better not representable over any (skew) field. However previous to this work essentially just two such matroids were known: The Vámos matroid V_8 [41] and a generalization V_{10} [10]. In this paper we first show that the Non-Pappus and Non-Desargues matroids are hyperbolic (but not representable over any field) by utilizing a known connection between hyperbolic polynomials and Euclidean Jordan algebras. Then, in Theorem 6.5, we construct a family of hyperbolic matroids which are parametrized by uniform hypergraphs, and prove that many of these matroids fail to be representable over any field, and more generally over any modular lattice. The proof of the main result is involved and uses several ingredients. In order to prove that the polynomials coming from our family of matroids are hyperbolic we need to prove that certain symmetric polynomials are nonnegative. The results obtained generalize and strengthen several inequalities in the literature, such as the Laguerre–Turán inequality and an inequality due to Jensen. Finally we explore some consequences to algebraic versions of the generalized Lax conjecture.

2. Hyperbolic and stable polynomials

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$, and if for all $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(t\mathbf{e} - \mathbf{x})$ has only real zeros. Note that if h is a hyperbolic polynomial of degree d, then we may write

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x})),$$

where

$$\lambda_{\max}(\mathbf{x}) = \lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x}) = \lambda_{\min}(\mathbf{x})$$

are called the *eigenvalues* of $\mathbf{x}$ (with respect to $\mathbf{e}$). The *hyperbolicity cone* of h with respect to $\mathbf{e}$ is the set $\Lambda_+(h, \mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \lambda_{\min}(\mathbf{x}) \geq 0\}$. We usually abbreviate and write Λ_+ if there is no risk for confusion. We denote by Λ_{++} the interior Λ_+.

Example 2.1. An important example of a hyperbolic polynomial is $\det(X)$, where $X = (x_{ij})_{i,j=1}^n$ is a symmetric matrix with $\binom{n+1}{2}$ indeterminate entries. If X is a real symmetric $n \times n$ matrix and I_n is the identity matrix of size $n \times n$, then $t \mapsto \det(tI_n - X)$ is the characteristic polynomial of a real symmetric matrix, so it has only real zeros. Hence $\det(X)$ is a hyperbolic polynomial with respect to I_n, and its hyperbolicity cone is the cone of positive semidefinite matrices.

The real linear space of complex hermitian matrices of size n is parametrized by matrices X in n^2 variables, and as above it follows that $\det(X)$ is a hyperbolic polynomial.

The next theorem follows from a theorem of Helton and Vinnikov [22], see [29]. It proved the Lax conjecture, after Peter Lax [28].

Theorem 2.2. *Suppose that $h(x, y, z)$ is of degree d and hyperbolic with respect to $\mathbf{e} = (e_1, e_2, e_3)^T$. Suppose further that h is normalized such that $h(\mathbf{e}) = 1$. Then there are symmetric $d \times d$ matrices A, B, C such that $e_1A + e_2B + e_3C = I_d$ and*

$$h(x, y, z) = \det(xA + yB + zC).$$

Remark 2.3. The exact analogue of the Helton-Vinnikov theorem fails for $n > 3$ variables. This may be seen by comparing dimensions. The space of degree d polynomials on $\mathbb{R}^n$ of the form $\det(x_1A_1 + \cdots x_nA_n)$ with A_i symmetric for $1 \le i \le n$, has dimension at most $n\binom{d+1}{2}$ whereas the space of hyperbolic polynomials on $\mathbb{R}^n$ has dimension $\binom{n+d-1}{d}$.

A convex cone in $\mathbb{R}^n$ is *spectrahedral* if it is of the form

$$\left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_iA_i \text{ is positive semidefinite}\right\}$$

where A_i, $i = 1, \ldots, n$ are symmetric matrices such that there exists a vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n y_iA_i$ positive definite. It is easy to see that spectrahedral cones are hyperbolicity cones. Indeed if $A_1, \ldots, A_n$ are real symmetric $d \times d$ matrices and $\mathbf{e} \in \mathbb{R}^n$ is a vector such that $\sum_{i=1}^n e_iA_i$ is positive definite, then $h(\mathbf{x}) = \det\left(\sum_{i=1}^n x_iA_i\right) \in \mathbb{R}[x_1, \ldots, x_n]$ is a hyperbolic polynomial with respect to $\mathbf{e}$, since for all $\mathbf{x} \in \mathbb{R}^n$ we have that $\det\left(tI_d - \sum_{i=1}^n x_iA_i\right) \in \mathbb{R}[t]$ is the characteristic polynomial of a real symmetric matrix and hence real-rooted. Therefore the hyperbolicity cone of $h(\mathbf{x})$ is precisely the spectrahedral cone $\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_iA_i$ is positive definite$\}$. A major open question asks if the converse is true.

Conjecture 2.4 (Generalized Lax conjecture (geometric version) [22, 39])**.** *All hyperbolicity cones are spectrahedral.*

We may reformulate Conjecture 2.4 as follows, see [22, 39].

Conjecture 2.5 (Generalized Lax conjecture (algebraic version) [22, 39])**.** *If $h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is hyperbolic with respect to $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$, then there exists a polynomial $q(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, hyperbolic with respect to $\mathbf{e}$, such that $\Lambda_{++}(h, \mathbf{e}) \subseteq \Lambda_{++}(q, \mathbf{e})$ and*

$$q(\mathbf{x})h(\mathbf{x}) = \det\left(\sum_{i=1}^n x_iA_i\right) \tag{2.1}$$

for some real symmetric matrices $A_1, \ldots, A_n$ of the same size such that $\sum_{i=1}^n e_iA_i$ is positive definite.

Here is an overview of known facts regarding Conjecture 2.4.

- Conjecture 2.4 is true for $n = 3$ by Theorem 2.2,
- Conjecture 2.4 is true for homogeneous cones [13], i.e., cones for which the automorphism group acts transitively on its interior,
- Conjecture 2.4 is true for quadratic polynomials, see e.g. [32],
- Conjecture 2.4 is true for elementary symmetric polynomials, see [8],
- Conjecture 2.4 is true for certain multivariate generalizations of matching and independence polynomials, see [1],
- Conjecture 2.4 is true for the first derivative relaxation of the positive semidefinite cone, see [38],
- Weaker versions of Conjecture 2.4 are true for smooth hyperbolic polynomials, see [27, 31].
- Stronger algebraic versions of Conjecture 2.4 are false, see [7].

A class of polynomials which is intimately connected to hyperbolic polynomials is the class of stable polynomials. Below we will collect a few facts about stable polynomials that will be needed in forthcoming sections. A polynomial $P(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ is *stable* if $P(z_1, \ldots, z_n) \neq 0$ whenever $\mathrm{Im}(z_j) > 0$ for all $1 \leq j \leq n$. Stable polynomials satisfy the following basic closure properties, see e.g. [40].

Lemma 2.6. *Let $P(x_1, \ldots, x_n)$ be a stable polynomial of degree d_i in x_i for $i = 1, \ldots, n$. Then for all $i = 1, \ldots, n$ we have*

(i) *Specialization: $P(x_1, \ldots, x_{i-1}, \zeta, x_{i+1}, \ldots, x_n)$ is stable or identically zero for each $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) \geq 0$.*
(ii) *Scaling: $P(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n)$ is stable for all $\lambda > 0$.*
(iii) *Inversion: $x_i^{d_i} P(x_1, \ldots, x_{i-1}, -x_i^{-1}, x_{i+1}, \ldots, x_n)$ is stable.*
(iv) *Permutation: $P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is stable for all $\sigma \in \mathfrak{S}_n$.*
(v) *Differentiation: $(\partial/\partial x_i) P(x_1, \ldots, x_n)$ is stable.*

Hyperbolic and stable polynomials are related as follows, see [5, Prop. 1.1] and [11, Thm. 6.1].

Lemma 2.7. *Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogenous polynomial. Then P is stable if and only if P is hyperbolic with $\mathbb{R}^n_+ \subseteq \Lambda_+$.*

Moreover all non-zero Taylor coefficients of a homogeneous and stable polynomial have the same phase, i.e., the quotient of any two non-zero coefficients is a positive real number.

Lemma 2.8 (Lemma 4.3 in [7])**.** *If $h \in \mathbb{R}[y_1, \ldots, y_n]$ is a hyperbolic polynomial, $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \Lambda_+$ and $\mathbf{v}_0 \in \mathbb{R}^n$, then the polynomial*

$$P(\mathbf{x}) = h(\mathbf{v}_0 + x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m)$$

is either identically zero or stable.

3. Hyperbolic polymatroids

We refer to [34] for undefined matroid terminology. The connection between hyperbolic/stable polynomials and matroids was first realized in [11]. A polynomial is *multiaffine* provided that each variable occurs at most to the first power. Choe *et al.* [11] proved that if

$$P(\mathbf{x}) = \sum_{B \subseteq [m]} a(B) \prod_{i \in B} x_i \in \mathbb{C}[x_1, \ldots, x_m] \tag{3.1}$$

is a homogeneous, multiaffine and stable polynomial, then its *support*

$$\mathcal{B} = \{B : a(B) \neq 0\}$$

is the set of bases of a matroid, $\mathcal{M}$, on $[m]$. Such matroids are called *weak half–plane property matroids* (abbreviated WHPP–matroids). If further $P(\mathbf{x})$ can be chosen so that $a(B) \in \{0, 1\}$, then $\mathcal{M}$ is called a *half–plane property matroid* (abbreviated HPP–matroid). If so, then $P(\mathbf{x})$ is the *bases generating polynomial* of $\mathcal{M}$. Here are a few known facts regarding WHPP or HPP matroids.

- All matroids representable over $\mathbb{C}$ are WHPP, [11].
- A binary matroid is WHPP if and only if it is HPP, and if and only if it is regular, [9, 11].
- No finite projective geometry PG(r, n) is WHPP, [9, 11].
- The Vámos matroid V_8 is HPP (but not representable over any field), [41].

We shall now see how weak half-plane property matroids may conveniently be described in terms of hyperbolic polynomials.

Let E be a finite set. A *polymatroid* is a function $r : 2^E \to \mathbb{N}$ satisfying

(i) $r(\emptyset) = 0$,
(ii) $r(S) \leq r(T)$ whenever $S \subseteq T \subseteq E$,
(iii) r is *semimodular*, i.e.,

$$r(S) + r(T) \geq r(S \cap T) + r(S \cup T),$$

for all $S, T \subseteq E$.

Recall that rank functions of matroids on E coincide polymatroids r on E with $r(\{i\}) \leq 1$ for all $i \in E$.

Let $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a tuple of vectors in $\Lambda_+(h, \mathbf{e})$, where $\mathbf{e} \in \mathbb{R}^n$. The *(hyperbolic) rank*, $\mathrm{rk}(\mathbf{x})$, of $\mathbf{x} \in \mathbb{R}^n$ is defined to be the number of non-zero eigenvalues of $\mathbf{x}$, i.e., $\mathrm{rk}(\mathbf{x}) = \deg h(\mathbf{e} + t\mathbf{x})$. Define a function $r_{\mathcal{V}} : 2^{[m]} \to \mathbb{N}$, where $[m] := \{1, 2, \ldots, m\}$, by

$$r_{\mathcal{V}}(S) = \mathrm{rk}\left(\sum_{i \in S} \mathbf{v}_i\right).$$

It follows from [20] (see also [7]) that $r_{\mathcal{V}}$ is a polymatroid. We call such polymatroids *hyperbolic polymatroids*. Hence if the vectors in $\mathcal{V}$ have rank at most one, then we obtain the hyperbolic rank function of a *hyperbolic matroid*.

Example 3.1. Let $A_1 = \mathbf{u}_1\mathbf{u}_1^*, \ldots, A_m = \mathbf{u}_m\mathbf{u}_m^*$ be PSD matrices of rank at most one in $\mathbb{C}^n$. By Example 2.1 the function $r : 2^{[m]} \to \mathbb{N}$ defined by

$$r(S) = \mathrm{rk}\left(\sum_{i \in S} A_i\right)$$

is the rank function of a hyperbolic matroid. It is not hard to see that $r(S)$ is equal to the dimension of the subspace of $\mathbb{C}^n$ spanned by $\{\mathbf{u}_i : i \in S\}$. Hence r is the rank function of the linear matroid defined by $\mathbf{u}_1, \ldots, \mathbf{u}_m$.

Proposition 3.2. *A matroid is hyperbolic if and only it has the weak half–plane property.*

Proof. Suppose $\mathcal{B}$ is the set of bases of a matroid, $\mathcal{M}$, with the weak half–plane property realized by (3.1). By Lemma 2.7 we may assume that $a(B)$ is a nonnegative real number for all $B \subseteq [m]$. Then $P(\mathbf{x})$ is hyperbolic with hyperbolicity cone containing the positive orthant

by Lemma 2.7. Let $\mathcal{V} = (\delta_1, \dots, \delta_m)$, be the standard basis of $\mathbb{R}^m$, and let $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^m$ be the all ones vector. Then

$$\begin{aligned} r_{\mathcal{V}}(S) &= \operatorname{rk}\left(\sum_{i\in S}\delta_i\right) = \deg P\left(\mathbf{1} + t\sum_{i\in S}\delta_i\right) \\ &= \deg \sum_B a(B)(1+t)^{|B\cap S|} = \max\{|B\cap S| : B \in \mathcal{B}\}, \end{aligned}$$

and hence $r_{\mathcal{V}}$ is the rank function of $\mathcal{M}$.

Conversely, assume that h is hyperbolic, that $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_m) \in \Lambda_+(h, \mathbf{e})^m$, and that $r_{\mathcal{V}}$ is the rank function of a hyperbolic matroid of rank r. We may assume $h(\mathbf{e}) > 0$. The polynomial $g(x_0, x_1, \dots, x_m) = h(x_0\mathbf{e} + x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m)$ is stable by Lemma 2.8 and has only nonnegative coefficients by Lemma 2.7. Since $\mathbf{v}_i$ has rank at most one for each i we see that g has degree at most one in x_i for all $i \geq 1$. It follows that

$$g(\mathbf{x}) = x_0^{d-r}\sum_{i=0}^{r} g_i(x_1, \dots, x_m)x_0^{r-i},$$

where $g_i(\mathbf{x})$ is a homogeneous and multiaffine polynomial of degree i for $0 \leq i \leq r \leq d = \deg h$. By dividing by x_0^{d-r} and setting $x_0 = 0$, we see that $g_r(\mathbf{x})$ is stable by Lemma 2.6. Moreover B is a basis of the matroid defined by $\mathcal{V}$ if and only if $|B| = r$ and $g(\delta_0 + t\sum_{i\in B}\delta_i)$ has degree d. This happens if and only if $g_r(\sum_{i\in B}\delta_i) \neq 0$, that is, if and only if B is in the support of $g_r(\mathbf{x})$. □

4. Projections and face lattices of hyperbolicity cones

Let C be a closed convex cone in $\mathbb{R}^n$. If $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{y} - \mathbf{x} \in C$, we write $\mathbf{x} \leq \mathbf{y}$. Recall that a *face* F of a convex cone C is a convex subcone of C with the property that $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \in F$ implies $\mathbf{x} \in F$. Equivalently a face is a convex subcone of C such that for each open line segment in C that intersects F, the closure of the segment is contained in F. The collection of all faces of C is a lattice, $L(C)$, under containment with smallest element $\{0\}$ and largest element C. Clearly $F \wedge G = F \cap G$ and $F \vee G = \bigcap_H H$, where H ranges over all faces containing F and G. The collection of all relative interiors of faces of C partitions C. If $F_{\mathbf{x}}$ is the unique face that contains $\mathbf{x} \in C$ in its relative interior, then $F_{\mathbf{x}} \vee F_{\mathbf{y}} = F_{\mathbf{x}+\mathbf{y}}$. See [37] for more on the face lattices of convex cones.

The *rank* of a face F of the hyperbolicity cone Λ_+ is defined by

$$\operatorname{rk}(F) = \max_{\mathbf{x}\in F}\operatorname{rk}(\mathbf{x}).$$

Note that if $L(\Lambda_+)$ is a graded lattice, then the above hyperbolic rank function is not necessarily the rank function of $L(\Lambda_+)$.

Lemma 4.1 (Thm 26, [36])**.** *Let F be a face of Λ_+ and let $\mathbf{x} \in F$. Then $\operatorname{rk}(\mathbf{x}) = \operatorname{rk}(F)$ if and only if $\mathbf{x}$ is in the relative interior of F.*

By Lemma 4.1 and the semimodularity of hyperbolic polymatroids we see that $\operatorname{rk} : L(\Lambda_+) \to \mathbb{N}$ is *semimodular*, that is,

$$\operatorname{rk}(F \vee G) + \operatorname{rk}(F \wedge G) \leq \operatorname{rk}(F) + \operatorname{rk}(G)$$

for all $F, G \in L(\Lambda_+)$. We may therefore equivalently define a hyperbolic polymatroid in terms of the face lattice of the hyperbolicity cone as follows: If $\mathcal{F} = (F_1, \ldots, F_m)$ is a tuple of elements of the face lattice $L(\Lambda_+)$, then the function $r_{\mathcal{F}} : 2^{[m]} \to \mathbb{N}$ defined by

$$r_{\mathcal{F}}(S) = \operatorname{rk}\left(\bigvee_{i \in S} F_i\right)$$

is a hyperbolic polymatroid.

The following theorem collects a few fundamental facts about hyperbolic polynomials and their hyperbolicity cones. For proofs see [21, 36].

Theorem 4.2 (Gårding, [21])**.** *Suppose h is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$.*

(i) *$\Lambda_+(\mathbf{e})$ and $\Lambda_{++}(\mathbf{e})$ are convex cones.*
(ii) *$\Lambda_{++}(\mathbf{e})$ is the connected component of*

$$\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$$

which contains $\mathbf{e}$.
(iii) *$\lambda_{\min} : \mathbb{R}^n \to \mathbb{R}$ is a concave function, and $\lambda_{\max} : \mathbb{R}^n \to \mathbb{R}$ is a convex function.*
(iv) *If $\mathbf{e}' \in \Lambda_{++}(\mathbf{e})$, then h is hyperbolic with respect to $\mathbf{e}'$ and $\Lambda_{++}(\mathbf{e}') = \Lambda_{++}(\mathbf{e})$.*

Recall that the *lineality space* of a convex cone C is $C \cap -C$, i.e., the largest linear space contained in C. It follows that the lineality space of a hyperbolicity cone is $\{\mathbf{x} : \lambda_i(\mathbf{x}) = 0 \text{ for all } i\}$, see e.g. [36]. Also if $\mathbf{x}$ is in the lineality space, then $\lambda_i(\mathbf{x} + \mathbf{y}) = \lambda_i(\mathbf{y})$ for all $1 \leq i \leq d$ and $\mathbf{y} \in \mathbb{R}^n$ [36].

By homogeneity of h

$$\lambda_j(s\mathbf{x} + t\mathbf{e}) = \begin{cases} s\lambda_j(\mathbf{x}) + t & \text{if } s \geq 0 \text{ and} \\ s\lambda_{d-j+1}(\mathbf{x}) + t & \text{if } s \leq 0 \end{cases}, \tag{4.1}$$

for all $s, t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

In analogy with the eigenvalue characterization of matrix projections we define projections in Λ_+ as follows.

Definition 4.3. An element in Λ_+ is a *projection* if its eigenvalues are contained in $\{0, 1\}$.

Remark 4.4. Note that $\mathbf{0}$, $\mathbf{e}$ and appropriate multiples of rank one vectors in Λ_+ are always projections.

Lemma 4.5. *Suppose $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are such that $F_{\mathbf{x}} \leq F_{\mathbf{y}}$ and $\operatorname{rk}(\mathbf{y}) = r$. If $\lambda_1(\mathbf{x}) \leq \lambda_r(\mathbf{y})$, then $\mathbf{x} \leq \mathbf{y}$.*

In particular if $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are projections, then $F_{\mathbf{x}} \leq F_{\mathbf{y}}$ if and only if $\mathbf{x} \leq \mathbf{y}$.

Proof. Suppose $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are such that $F_{\mathbf{x}} \leq F_{\mathbf{y}}$, $\operatorname{rk}(\mathbf{y}) = r$ and $\lambda_1(\mathbf{x}) \leq \lambda_r(\mathbf{y})$. Consider the polynomial

$$g(u, s, t) = h(u\mathbf{e} + s\mathbf{y} + t\mathbf{x}),$$

which is hyperbolic with respect to $(1, 0, 0)$ and whose hyperbolicity cone contains the positive orthant. Since $\mathbf{x} \in F_{\mathbf{y}}$ we know that $\operatorname{rk}(a\mathbf{x} + b\mathbf{y}) = r$ for all $a, b > 0$. Since all non-zero Taylor coefficients of $g(u, s, t)$ have the same sign, by Lemma 2.7, we may write

$$g(u, s, t) = u^{d-r} g_0(u, s, t), \quad d = \deg h,$$

where $g_0(u,s,t)$ is hyperbolic with respect to $(1,0,0)$ and also $(0,1,0)$, and its hyperbolicity cone contains the positive orthant. Let $\lambda'_j(a,b,c)$, $j=1,\dots,r$, denote the eigenvalues of g_0 (with respect to $(1,0,0)$). Then by (4.1) and the concavity of λ'_r (Theorem 4.2):

$$\lambda'_r(0,1,-1) \geq \lambda'_r(0,1,0) + \lambda'_r(0,0,-1) = \lambda_r(\mathbf{y}) - \lambda_1(\mathbf{x}) \geq 0.$$

By construction $\lambda_{\min}(\mathbf{y}-\mathbf{x}) = \min\{0, \lambda'_r(0,1,-1)\}$, and the lemma follows. □

Lemma 4.6. *If $\mathbf{x},\mathbf{y} \in \Lambda_+$ are projections with $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{y}-\mathbf{x}$ is a projection with*

$$\mathrm{rk}(\mathbf{y}-\mathbf{x}) = \mathrm{rk}(\mathbf{y}) - \mathrm{rk}(\mathbf{x}).$$

Proof. Suppose first that $F_{\mathbf{y}} = \Lambda_+ = F_{\mathbf{e}}$. Then $\mathbf{y}-\mathbf{e}, \mathbf{e}-\mathbf{y} \in \Lambda_+$ by Lemma 4.5, and hence $\mathbf{y}-\mathbf{e}$ is in the lineality space of Λ_+. Then

$$\lambda_i(\mathbf{y}-\mathbf{x}) = \lambda_i(\mathbf{e}-\mathbf{x}) = 1 - \lambda_{d-i+1}(\mathbf{x}),$$

for all $1 \leq i \leq d = \deg h$, and hence $\mathbf{y}-\mathbf{x}$ is a projection of rank $d - \mathrm{rk}(\mathbf{x})$.

If $F_{\mathbf{y}} \neq F_{\mathbf{e}}$, then $r := \mathrm{rk}(\mathbf{y}) < d$. Consider the hyperbolic polynomial

$$g(u,s,t) = h(u\mathbf{e} + s\mathbf{x} + t\mathbf{y}) = u^{d-r} g_0(u,s,t),$$

where g_0 is hyperbolic with respect to $\mathbf{e}' = (1,0,0)$. It follows that $\mathbf{x}' = (0,1,0)$ and $\mathbf{y}' = (0,0,1)$ are projections with $F_{\mathbf{e}'} = F_{\mathbf{y}'}$. The lemma now follows from the first case considered. □

Remark 4.7. Note that if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{x}$, then $\mathbf{y}-\mathbf{x}$ is in the lineality space of Λ_+. Moreover $\mathbf{x} \leq \mathbf{y}$ if and only if $\mathbf{e}-\mathbf{y} \leq \mathbf{e}-\mathbf{x}$. Since $F_{\mathbf{e}} = \Lambda_+$ we have by Lemma 4.5 that $\mathbf{x} \leq \mathbf{e}$ for all projections $\mathbf{x} \in \Lambda_+$. Hence by Lemma 4.6 it follows that $\mathbf{x}$ is a projection if and only if $\mathbf{e}-\mathbf{x}$ is a projection.

The following proposition gives a sufficient condition for two faces in Λ_+ to be modular with respect to the hyperbolic rank function.

Proposition 4.8. *If $\mathbf{x},\mathbf{y} \in \Lambda_+$ are projections such that $F_{\mathbf{x}} \wedge F_{\mathbf{y}}$, $F_{\mathbf{x}} \vee F_{\mathbf{y}}$, $F_{\mathbf{e}-\mathbf{x}} \wedge F_{\mathbf{e}-\mathbf{y}}$ and $F_{\mathbf{e}-\mathbf{x}} \vee F_{\mathbf{e}-\mathbf{y}}$ all contain a projection in their relative interiors, then*

$$\mathrm{rk}(F_{\mathbf{x}}) + \mathrm{rk}(F_{\mathbf{y}}) = \mathrm{rk}(F_{\mathbf{x}} \wedge F_{\mathbf{y}}) + \mathrm{rk}(F_{\mathbf{x}} \vee F_{\mathbf{y}}).$$

Proof. Let $\mathbf{v},\mathbf{w},\mathbf{v}',\mathbf{w}'$ be the projections in the relative interiors of $F_{\mathbf{x}} \wedge F_{\mathbf{y}}$, $F_{\mathbf{x}} \vee F_{\mathbf{y}}$, $F_{\mathbf{e}-\mathbf{x}} \wedge F_{\mathbf{e}-\mathbf{y}}$ and $F_{\mathbf{e}-\mathbf{x}} \vee F_{\mathbf{e}-\mathbf{y}}$, respectively. Then $\mathbf{e}-\mathbf{w} \leq \mathbf{e}-\mathbf{x}$ and $\mathbf{e}-\mathbf{w} \leq \mathbf{e}-\mathbf{y}$, so that $\mathbf{e}-\mathbf{w} \in F_{\mathbf{e}-\mathbf{x}} \wedge F_{\mathbf{e}-\mathbf{y}}$ by Lemma 4.5. By Lemma 4.5 again, $\mathbf{e}-\mathbf{w} \leq \mathbf{v}'$. We also have $\mathbf{e}-\mathbf{v}' \geq \mathbf{x}$ and $\mathbf{e}-\mathbf{v}' \geq \mathbf{y}$ so that $\mathbf{e}-\mathbf{v}' \geq \mathbf{w}$, that is, $\mathbf{e}-\mathbf{w} \geq \mathbf{v}'$. Thus $F_{\mathbf{v}'} = F_{\mathbf{e}-\mathbf{w}}$ and analogously $F_{\mathbf{w}'} = F_{\mathbf{e}-\mathbf{v}}$. Since $\mathrm{rk} : L(\Lambda_+) \to \mathbb{N}$ is semimodular we have

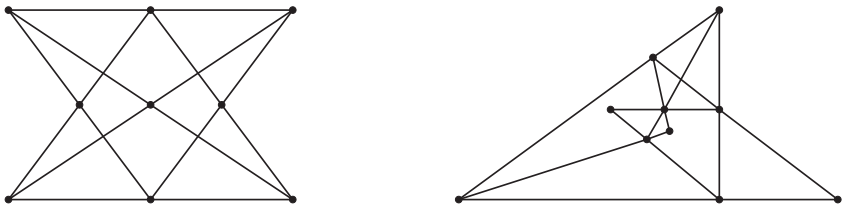
$$\mathrm{rk}(\mathbf{x}) + \mathrm{rk}(\mathbf{y}) \geq \mathrm{rk}(\mathbf{v}) + \mathrm{rk}(\mathbf{w}),$$

and also

$$\mathrm{rk}(\mathbf{e}-\mathbf{x}) + \mathrm{rk}(\mathbf{e}-\mathbf{y}) \geq \mathrm{rk}(\mathbf{e}-\mathbf{w}) + \mathrm{rk}(\mathbf{e}-\mathbf{v}),$$

and so the proposition follows from Lemma 4.6. □

Corollary 4.9. *Let $\Lambda_+(h,\mathbf{e})$ be a hyperbolicity cone with trivial lineality space. Suppose all extreme rays of Λ_+ have the same hyperbolic rank, and that each face of Λ_+ contains a projection in its relative interior. Then $L(\Lambda_+)$ is a modular geometric lattice.*



(A) Non-Pappus (B) Non-Desargues

FIGURE 1. The Non-Pappus and Non-Desargues configurations.

Proof. Since each face of $L(\Lambda_+)$ except $\{0\}$ is generated by extreme rays, see e.g. [37, Cor. 18.5.2], it follows that $L(\Lambda_+)$ is atomic with all atoms (extreme rays) having the same hyperbolic rank by hypothesis. Suppose $\mathrm{rk}(\mathbf{a}) = c$ for all atoms $\mathbf{a} \in L(\Lambda_+)$. By modularity of the hyperbolic rank function (Proposition 4.8) and induction we see that c divides $\mathrm{rk}(F)$ for all $F \in L(\Lambda_+)$. It follows that the function defined by $\mathrm{rk}(F)/c$ is the proper rank function of $L(\Lambda_+)$, since it is modular and equal to one on each atom. □

5. Hyperbolic matroids and Euclidean Jordan algebras

In light of the generalized Lax conjecture it is of interest to find hyperbolic but non-linear (poly-) matroids. Until present the only known instances of non-linear hyperbolic matroids are the Vámos matroid [41] and a generalization of it [10]. The generalized Vámos matroids introduced in the following section provide an infinite family of such matroids. In this section we identify two further types of matroids that are hyperbolic but not linear through a connection with Euclidean Jordan algebras and projective geometry.

Some classical examples of non-linear matroids are obtained by relaxing a circuit hyperplane in a matroid that comes from a geometric configuration. In fact the Non-Fano, Non-Pappus and Non-Desargues matroids (see Fig 1) are all derived from the family n_3 of symmetric configurations on n points and n lines, arranged such that 3 lines pass through each point and 3 points lie on each line [19]. Note that such configurations need not be unique up to incidence isomorphism for given n. The Non-Fano, Non-Pappus and Non-Desargues matroids are all rank three matroids corresponding respectively to instances of the configurations $7_3, 9_3$ and 10_3 after removing one line. It is interesting to note how representability diminishes as we move upwards in this hierarchy: The Non-Fano matroid is representable over all fields that do not have characteristic 2 [34]. The Non-Pappus matroid is skew-linear but not linear [23], which is to say that it only admits representations over non-commutative division rings e.g. the quaternions $\mathbb{H}$. Moreover it is known that the Non-Desargues matroid is not even skew-linear [23]. On the other hand, it is known that the Non-Desargues matroid can be coordinatized by rank one projections over the octonions $\mathbb{O}$, see e.g. [18]. The octonions form a non-commutative and non-associative division ring over the reals.

An algebra $(A, \circ)$ over a field $\mathbb{K}$ is said to be a *Jordan algebra* if for all $a, b \in A$

$$a \circ b = b \circ a \quad \text{and} \quad a \circ ((a \circ a) \circ b) = (a \circ a) \circ (a \circ b).$$

Note in particular that Jordan algebras are not necessarily associative. A Jordan algebra is *Euclidean* if

$$a_1^2 + \cdots + a_k^2 = 0 \text{ implies } a_1 = \cdots = a_k = 0$$

for all $a_1, \ldots, a_k \in A$. By a theorem of Jordan, von Neumann and Wigner [25] the simple finite dimensional real Euclidean Jordan algebras classify into four infinite families and one exceptional algebra (the Albert algebra) as follows:

(i) $H_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) - the algebra of Hermitian $n \times n$ matrices over $\mathbb{K}$ with Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.
(ii) $\mathbb{R}^n \oplus \mathbb{R}$ - the real inner product space with inner product $(u \oplus \lambda, v \oplus \mu) = (u, v)_{\mathbb{R}^n} + \lambda\mu$ and Jordan product $(u \oplus \lambda) \circ (v \oplus \mu) = (\mu u + \lambda v) \oplus ((u, v)_{\mathbb{R}^n} + \lambda\mu)$.
(iii) $H_3(\mathbb{O})$ - the algebra of octonionic Hermitian 3×3 matrices with Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

Let A be a real Euclidean Jordan algebra of rank r with identity e. A *Jordan frame* is a complete system of orthogonal idempotents of rank one, that is, rank one elements $c_1, \ldots, c_r \in A$ such that $c_i^2 = c_i$, $c_i \circ c_j = 0$ for $i \neq j$ and $c_1 + \cdots + c_r = e$. A characteristic property of finite dimensional real Euclidean Jordan algebras is the following spectral theorem, see [17, Theorem III.1.2].

Theorem 5.1. *Let A be a real Euclidean Jordan algebra of rank r. Then for each $x \in A$ there exists a Jordan frame $c_1, \ldots, c_r \in A$ and unique real numbers $\lambda_1(x), \ldots, \lambda_r(x)$ (the eigenvalues) such that*

$$x = \sum_{j=1}^{r} \lambda_j(x) c_j.$$

Moreover

$$\sum_{j:\lambda_j=\lambda} c_j$$

is uniquely determined for each eigenvalue λ.

A finite dimensional real Euclidean Jordan algebra is equipped with a hyperbolic determinant polynomial $\det : A \to \mathbb{R}$ given by

$$\det(x) = \prod_{j=1}^{r} \lambda_j(x).$$

Let P be a set of *points* and L a set of *lines*. Recall that a pair $G = (P, L)$ is a *projective geometry* if the following axioms are satisfied:

(i) For any two distinct points $a, b \in P$ there is a unique line $ab \in L$ containing a and b.
(ii) Any line contains at least three points.
(iii) If $a, b, c, d \in P$ are distinct points such that $ab \cap cd \neq \emptyset$ then $ac \cap bd \neq \emptyset$.

Each projective geometry is a (simple) modular geometric lattice, and each modular geometric lattice is a direct product of a Boolean algebra with projective geometries, see [2, p. 93]. The following proposition is essentially a known connection between Jordan algebras and projective geometries, which we prove here in the theory of hyperbolic polynomials.

Proposition 5.2. *Let A be a finite dimensional real Euclidean Jordan algebra and let Λ_+ denote the hyperbolicity cone of* $\det : A \to \mathbb{R}$. *Then $L(\Lambda_+)$ is a modular geometric lattice.*

In particular if A is simple, then $L(\Lambda_+)$ is a projective geometry.

Proof. By Theorem 5.1 the extreme rays of Λ_+ are multiples of rank one idempotents. Also, a face F_x contains the projection

$$c = \sum_{j:\lambda_j(x)\neq 0} c_j$$

in its relative interior. The proposition now follows from Corollary 4.9. □

The Non-Pappus and Non-Desargues configurations are depicted in Fig 1. The configurations give rise to rank 3 matroids where three points are dependent if and only if they are collinear. The Non-Pappus and Non-Desargues matroids are not linear but may be represented over the projective geometries associated to the Euclidean Jordan algebras $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$, respectively. This may be deduced from the coordinatizations in [34, Example 1.5.14] and [18]. Hence by Proposition 5.2 we have

Theorem 5.3. *The Non-Pappus and Non-Desargues matroids are hyperbolic.*

6. Generalized Vámos Matroids with the (weak) half–plane property

In this section we provide an infinite family of hyperbolic matroids that do not arise from modular geometric lattices. Suppose that L is a lattice with a smallest element $\hat{0}$, and $f : L \to \mathbb{N}$ is a function satisfying

(i) $f(\hat{0}) = 0$,
(ii) if $x \leq y$, then $f(x) \leq f(y)$,
(iii) for any $x, y \in L$,

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y).$$

If $x_1, \ldots, x_m \in L$, then the function $r : 2^{[m]} \to \mathbb{N}$ defined by

$$r(S) = f\left(\bigvee_{i\in S} x_i\right)$$

defines a polymatroid. All polymatroids arise in this manner. Indeed if $r : 2^{[m]} \to \mathbb{N}$ is a polymatroid, we may take $L = 2^{[m]}$, $f = r$, and $x_i = \{i\}$ for each $i \in [m]$. However if f is modular, i.e.,

$$f(x) + f(y) = f(x \vee y) + f(x \wedge y), \qquad \text{for all } x, y \in L,$$

we say that r is *modularly represented*. Hence all linear matroids as well as all projective geometries are modularly represented. Although Ingleton's proof [23] of the next lemma only concerns linear matroids it extends verbatim to modularly represented matroids.

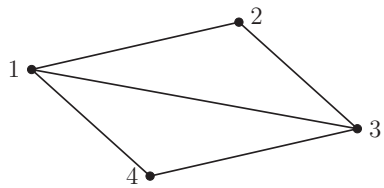
Lemma 6.1 (Ingleton's Inequality, [23])**.** *Suppose $r : 2^E \to \mathbb{N}$ is a modularly represented polymatroid and $A, B, C, D \subseteq E$. Then*

$$r(A \cup B) + r(A \cup C \cup D) + r(C) + r(D) + r(B \cup C \cup D) \leq$$
$$r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D) + r(C \cup D).$$

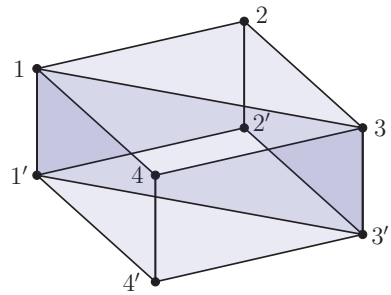
The *Vámos matroid* V_8 is the rank-four matroid on $E = [8]$ having set of bases

$$\mathcal{B}(V_8) = \binom{E}{4} \setminus \{\{1,2,3,4\}, \{1,2,5,6\}, \{1,2,7,8\}, \{3,4,5,6\}, \{5,6,7,8\}\}.$$

The rank function of the Vámos matroid fails to satisfy Ingleton's inequality (see [23]), and hence it is not modularly represented. Nevertheless Wagner and Wei [41] proved that



(A) G (Diamond graph)



(B) $V_G \cong V_8$ (Vámos matroid)

V_8 has the half-plane property, and hence V_8 is hyperbolic. This was used in [7] to provide counterexamples to stronger algebraic versions of the generalized Lax conjecture.

Burton, Vinzant and Youm [10] studied an infinite family of generalized Vámos matroids, $\{V_{2n}\}_{n\geq 4}$, and conjectured that all members of the family have the half-plane property. They proved their conjecture for $n = 5$. Below we generalize their construction and construct a family of matroids; one matroid for each uniform hypergraph. We prove that all matroids corresponding to simple graphs are HPP, and that all matroids corresponding to uniform hypergraphs are WHPP. In particular this will prove the conjecture of Burton *et al.*

Recall that a rank r *paving matroid* is a matroid such that all its circuits have size at least r. Paving matroids may be characterized in terms of d-partition. A *d-partition* of a set E is a collection $\mathcal{S}$ of subsets of E all of size at least d, such that every d-subset of E lies in a unique member of $\mathcal{S}$. The d-partition $\mathcal{S} = \{E\}$ is the *trivial d-partition.* For a proof of the next proposition see [34, Prop. 2.1.21].

Proposition 6.2. *The hyperplanes of any rank $d+1 \geq 2$ paving matroid form a non-trivial d-partition.*

Conversely, the elements of a non-trivial d-partition form the set of hyperplanes of a paving matroid of rank $d+1$.

A paving matroid of rank r is *sparse* if its hyperplanes all have size $r-1$ or r.

Recall that a *hypergraph* H consists of a set $V(H)$ of *vertices* together with a set $E(H) \subseteq 2^{V(H)}$ of *hyperedges.* We say that a hypergraph H is *d-uniform* if all hyperedges have size d.

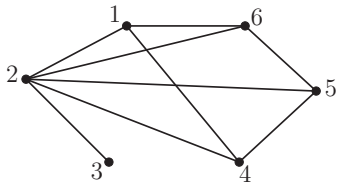
Theorem 6.3. *Let H be an d-uniform hypergraph on $[n]$, and let $E = \{1, 1', \ldots, n, n'\}$. Let*

$$\mathcal{B}(V_H) = \binom{E}{2d} \setminus \{e \cup e' : e \in E(H)\},$$

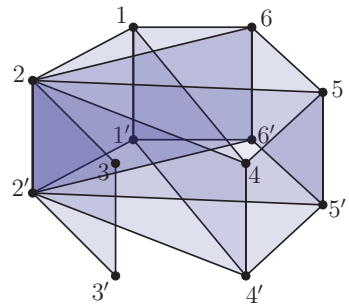
in which $e' := \{i' : i \in e\}$ for each $e \in E(H)$. Then $\mathcal{B}(V_H)$ is the set of bases of a sparse paving matroid V_H of rank $2d$.

Proof. Let

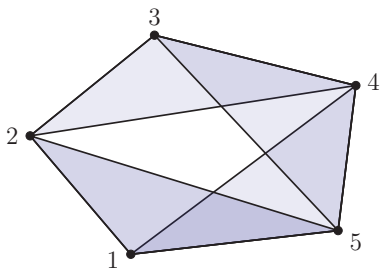
$$\mathcal{S} = \{e \cup e' : e \in E(H)\} \cup \left\{ S \in \binom{E}{2d-1} : S \subset e \cup e' \text{ for no } e \in E(H) \right\}.$$



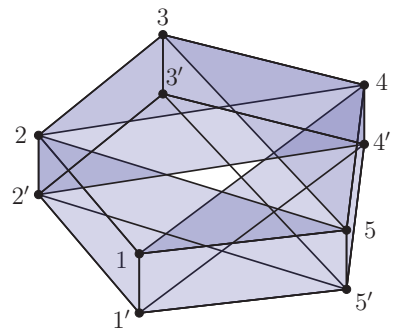
(A) A simple graph G



(B) The matroid V_G



(A) A 3-uniform hypergraph H



(B) The matroid V_H

Then $\mathcal{S}$ is a $(2d-1)$-partition, and so it defines a sparse paving matroid with set of bases $\binom{E}{2d} \setminus \{e \cup e' : e \in E(H)\}$ by Proposition 6.2. □

Let $\mathcal{V} = \{V_H : H \text{ is a } d\text{-uniform hypergraph on } [n] \text{ with } 0 < d \leq n, n \in \mathbb{N}\}$.

Example 6.4. If G is the diamond graph (Fig 2a) then $V_G = V_8$, the Vámos matroid. Moreover $V_{\bar{K}_n} = U_{4,2n}$, where $\bar{K}_n$ denotes the complement of the complete graph on n vertices and $U_{4,2n}$ denotes the uniform rank 4 matroid on $2n$ elements. The family $\{V_{2n}\}_{n\geq 4}$ studied by Burton *et al.* [10] corresponds to V_{G_n} where G_n is an n-cycle with edges $\{1, i\}$, $i = 2, \ldots, n$, adjoined.

We postpone the proofs of the next two theorems until Section 9.

Theorem 6.5. *All matroids in $\mathcal{V}$ are hyperbolic, i.e., they all have the weak half-plane property.*

Theorem 6.6. *For each simple graph G, V_G has the half–plane property.*

If G contains the Diamond graph as an induced subgraph then the rank function of V_G fails to satisfy Ingleton's inequality, and thus V_G is hyperbolic but not modularly represented.

There is no full analogue of Theorem 6.6 in the hypergraph setting. To see this let H be the complete 3-uniform hypergraph on [6]. The bases generating polynomial of V_H is given

by

$$h_{V_H}(\mathbf{x}) = e_6(x_1, x_{1'}, \dots, x_6, x_{6'}) - e_3(x_1x_{1'}, \dots, x_6x_{6'}).$$

By Lemma 2.7 we have that $h_{V_H}(\mathbf{x})$ is stable if and only if $h_{V_H}(\mathbf{x})$ is hyperbolic with $\mathbb{R}^{12}_+ \subseteq \Lambda_+(h_{V_H})$. Take $\mathbf{e} = (1, 1, 0, \dots, 0) \in \mathbb{R}^{12}_+$ and $\mathbf{x} \in \mathbb{R}^{12}$ with $x_1 = x_{1'} = 0$, $x_2 = x_{2'} = x_3 = x_{3'} = 2$ and $x_i = x_{i'} = -1$ for $i > 3$. Then

$$h_{V_H}(t\mathbf{e} - \mathbf{x}) = -4t^2 - 36t - 160$$

is a quadratic polynomial with non-real zeros $t = \frac{9}{2} \pm \frac{1}{2}\sqrt{79}i$. Hence V_H does not have the half-plane property. Clearly if V_8 is a minor of V_H then V_H cannot be representable. Below we give an example of a non-representable matroid V_H with no Vámos minor. Hence this constitutes a genuinely new instance of a hyperbolic matroid in the family which is not representable.

Example 6.7. The following linear rank inequality in six variables was identified by Dougherty *et al.* [15]

$$\begin{aligned}
&r(A \cup D) + r(B \cup C) + r(C \cup E) + r(E \cup F) + r(B \cup D \cup F) + r(A \cup B \cup C \cup D) + \\
&r(A \cup B \cup C \cup E) + r(A \cup C \cup E \cup F) + r(A \cup D \cup E \cup F) \leq \\
&r(A \cup B \cup C) + r(A \cup B \cup D) + r(A \cup C \cup E) + r(A \cup D \cup F) + r(A \cup E \cup F) + \\
&r(B \cup C \cup D) + r(B \cup C \cup E) + r(C \cup E \cup F) + r(D \cup E \cup F).
\end{aligned}$$

This inequality is satisfied by all polymatroids r representable over some field, where $r : 2^{[n]} \to \mathbb{N}$, $n \in \mathbb{N}$ and $A, B, C, D, E, F \subseteq [n]$. We proceed by designing a 3-uniform hypergraph H on [6] such that V_H violates the above inequality. Let

$$A = \{1, 1'\},\ B = \{2, 2'\},\ C = \{3, 3'\},\ D = \{4, 4'\},\ E = \{5, 5'\},\ F = \{6, 6'\}.$$

By taking the hypergraph H with edges

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 5, 6\}, \{4, 5, 6\}$$

we see that V_H violates the above inequality. One checks that V_8 is not a minor of V_H.

7. Consequences for the generalized Lax conjecture

Helton and Vinnikov [22] conjectured that if a polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic with respect to $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{R}^n$, then there exist positive integers M, N and a linear polynomial $\ell(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ which is positive on $\Lambda_{++}(h, \mathbf{e})$ such that

$$\ell(\mathbf{x})^{M-1} h(\mathbf{x})^N = \det\left(\sum_{i=1}^n x_i A_i\right)$$

for some symmetric matrices $A_1, \dots, A_n$ such that $e_1A_1 + \cdots + e_nA_n$ is positive definite. In [7] the second author used the bases generating polynomial h_{V_8} of the Vámos matroid to prove that there is no linear polynomial $\ell(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ which is nonnegative on the hyperbolicity cone of h_{V_8} and positive integers M, N such that

$$\ell(\mathbf{x})^{M-1} h_{V_8}(\mathbf{x})^N = \det\left(\sum_{i=1}^8 x_i A_i\right)$$

for some symmetric matrices $A_1, \dots, A_8$ with $e_1A_1 + \cdots + e_8A_8$ positive definite. We will here construct further "counterexamples" that preclude more general factors $q(\mathbf{x})$ in (2.1). First we prove two lemmata of matroid theoretic nature. If $r : 2^E \to \mathbb{N}$ is a polymatroid and

$A \subseteq E$, we say that A is *spanning* if $r(A) = r(E)$. Moreover $A \subset E$ is a *hyperplane* if it is a maximal non–spanning set.

Lemma 7.1. *For $n, r, c \geq 1$, let $\mathcal{P}(n, r, c)$ be the family of all polymatroids of rank at most r on n elements such that each hyperplane has at most $r - 1 + c$ elements. If $\alpha(n, r, c)$ denotes the maximal number of non-spanning sets of size r taken over all polymatroids in $\mathcal{P}(n, r, c)$, then*

$$\alpha(n, r, c) \leq c\binom{n}{r-1}. \tag{7.1}$$

Proof. If $r = 1$, then each hyperplane has at most c elements, i.e., there are at most c loops so that $\alpha(n, r, c) = c$ as desired. The proof is by induction over $n \geq 1$ where $r \geq 1$. The lemma is trivially true for $n = 1$.

Let $\mathcal{P} \in \mathcal{P}(n, r, c)$, where $n, r \geq 2$. If $n \leq r$, then (7.1) is trivially true. Assume $n > r$. Let i be a non-loop of $\mathcal{P}$. If $r(E \setminus i) < r(E)$, then $E \setminus i$ is a hyperplane and hence $n-1 \leq r-1+c$, so that $\binom{n}{r} \leq c\binom{n}{r-1}$. Hence we may assume $r(E \setminus i) = r(E) > 0$.

If S is a non-spanning r-set of $\mathcal{P}$, then either S is a non-spanning r-set of $\mathcal{P} \setminus i$, or $S \setminus i$ is a non-spanning $(r-1)$-set of $\mathcal{P}/i$. Hence $\mathcal{P} \setminus i \in \mathcal{P}(n-1, r, c)$ and $\mathcal{P}/i \in \mathcal{P}(n-1, r-1, c)$, and thus

$$\begin{aligned}\alpha(n, r, c) &\leq \alpha(n-1, r, c) + \alpha(n-1, r-1, c)\\ &\leq c\binom{n-1}{r-1} + c\binom{n-1}{r-2} = c\binom{n}{r-1},\end{aligned}$$

by induction. □

Lemma 7.2. *Let $\mathcal{P}_i$, $i = 1, \ldots, s$, be polymatroids on $[n]$ of rank at most $k-1$ such that no hyperplane has more than k elements. If $n \geq (2s+1)k - 1$, then there is a set S of size k such that there are at least two $(k-1)$-subsets of S that are spanning in all $\mathcal{P}_i$, $i = 1, \ldots, s$.*

Proof. Suppose the conclusion is not true. Let

$$A = \left\{(S, T) \in \binom{[n]}{k-1} \times \binom{[n]}{k} : S \subset T \text{ and } S \text{ is not spanning in } \mathcal{P}_i \text{ for some } i \in [s]\right\}.$$

Then

$$|A| \geq (k-1)\binom{n}{k}.$$

Furthermore by Lemma 7.1 we have

$$\begin{aligned}|A| &= \#\left\{S \in \binom{[n]}{k-1} : S \text{ is not spanning in } \mathcal{P}_i \text{ for some } i \in [s]\right\} \cdot (n-k+1)\\ &\leq s\alpha(n, k-1, 2)(n-k+1)\\ &\leq 2s\binom{n}{k-2}(n-k+1).\end{aligned}$$

Hence

$$(k-1)\binom{n}{k} \leq 2s\binom{n}{k-2}(n-k+1).$$

Solving for n gives $n \leq (2s+1)k - 2$, which proves the lemma. □

Given positive integers n and k, consider the k-uniform hypergraph $H_{n,k}$ on $[n+2]$ containing all hyperedges $e \in \binom{[n+2]}{k}$ except those for which $\{n+1, n+2\} \subseteq e$. By Theorem 6.5 the matroid $V_{H_{n,k}}$ is hyperbolic and therefore has a stable weighted bases generating polynomial $h_{V_{H_{n,k}}}$ by Proposition 3.2. The polynomial $h_{n,k} \in \mathbb{R}[x_1, \dots, x_{n+2}]$ obtained from the multi-affine polynomial $h_{V_{H_{n,k}}}$ by identifying the variables x_i and $x_{i'}$ pairwise for all $i \in [n+2]$ is stable. Hence by Lemma 2.7 we have $\mathbb{R}_+^{n+2} \subseteq \Lambda_+(h_{n,k})$ so $h_{n,k}$ is hyperbolic with respect to $\mathbf{1}$.

Theorem 7.3. *Let n and k be a positive integers. Suppose there exists a positive integer N and a hyperbolic polynomial $q(\mathbf{x})$ such that*

$$q(\mathbf{x})h_{n,k}(\mathbf{x})^N = \det\left(\sum_{i=1}^{n+2} x_i A_i\right) \tag{7.2}$$

with $\Lambda_+(h_{n,k}) \subseteq \Lambda_+(q)$ for some symmetric matrices $A_1, \dots, A_{n+2}$ such that $A_1 + \cdots + A_{n+2}$ is positive definite and

$$q(\mathbf{x}) = \prod_{i=1}^{s} p_j(\mathbf{x})^{\alpha_i}$$

for some irreducible hyperbolic polynomials $p_1, \dots, p_s \in \mathbb{R}[x_1, \dots, x_{n+2}]$ of degree at most $k-1$ where $\alpha_1, \dots, \alpha_s$ are positive integers. Then

$$n < (2s+1)k - 1.$$

Proof. Suppose the hypotheses are satisfied and $n \geq (2s+1)k - 1$. Let $r_0 : 2^{[n]} \to \mathbb{N}$ be the hyperbolic polymatroid defined by $h_{n,k}$ and $\mathcal{V} = (\delta_1, \dots, \delta_n)$, where δ_i, $i \in [n]$ are the standard basis vectors. Hence $r_0(S)$ is the rank of $S \cup \{i' : i \in S\}$ in the matroid $V_{H_{n,k}}$. Moreover, for $i \in [s]$, let $r_i : 2^{[n]} \to \mathbb{N}$ be the hyperbolic polymatroid defined by p_i and $\mathcal{V} = (\delta_1, \dots, \delta_n)$. Any subset S of $[n]$ of size at least $k+1$ is spanning for r_0, and thus $\sum_{i \in S} \delta_i \in \Lambda_{++}(h_{n,k})$. Hence $\sum_{i \in S} \delta_i \in \Lambda_{++}(p_i)$, and thus S is spanning with respect to r_i for all $i \in [s]$. By Lemma 7.2, since $n \geq (2s+1)k - 1$, there exists a subset $T \subseteq [n]$ of size k containing at least two distinct subsets X, Y of size $k-1$ with full rank with respect to all hyperbolic polymatroids r_i, $i = 1, \dots, s$. Let $x, y \in T$ be the unique elements in X, Y, respectively, not contained in $Z = X \cap Y$. Define

$$A = Z \cup \{n+1\}, \quad B = Z \cup \{n+2\}, \quad C = Z \cup \{x\}, \quad D = Z \cup \{y\}.$$

Now $A \cup B, A \cup C \cup D$ and $B \cup C \cup D$ have full rank with respect to r_0. Since $\Lambda_{++}(h_{n,k}) \subseteq \Lambda_{++}(p_i)$ for all i, we see that $A \cup B, A \cup C \cup D$ and $B \cup C \cup D$ have full rank with respect to r_i for all i. Hence the rank of each set to the left in the Ingleton inequality have full rank with respect to r_i, so that

$$\begin{aligned} &r_i(A \cup B) + r_i(A \cup C \cup D) + r_i(C) + r_i(D) + r_i(B \cup C \cup D) \geq \\ &r_i(A \cup C) + r_i(A \cup D) + r_i(B \cup C) + r_i(B \cup D) + r_i(C \cup D) \end{aligned}$$

for $i = 1, \dots, s$. Note also that

$$\begin{aligned} &r_0(A \cup B) + r_0(A \cup C \cup D) + r_0(C) + r_0(D) + r_0(B \cup C \cup D) = \\ &2k + 2k + (2k-2) + (2k-2) + 2k > (2k-1) + (2k-1) + (2k-1) + (2k-1) + (2k-1) = \\ &r_0(A \cup C) + r_0(A \cup D) + r_0(B \cup C) + r_0(B \cup D) + r_0(C \cup D). \end{aligned}$$

Thus r_0 violates the Ingleton inequality. Let $\mathcal{R}$ denote the representable polymatroid with rank function

$$r_{\mathcal{R}}(S) = \operatorname{rank}\left(\sum_{i\in S} A_i\right).$$

for all $S \subseteq [n]$. Then, by (7.2),

$$r_{\mathcal{R}}(S) = \operatorname{rank}\left(\sum_{i\in S} A_i\right) = \sum_{i=1}^{s} \alpha_i r_i(S) + N r_0(S).$$

Hence $r_{\mathcal{R}}$ violates Ingleton's inequality, a contradiction. □

Hence, for n sufficiently large, q in (2.1) either has an irreducible factor of large degree or is the product of many factors of low degree.

Example 7.4. Consider

$$h_{2,2} = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4)(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4).$$

The polynomial $h_{2,2}$ comes from the bases generating polynomial of the Vámos matroid under the restriction $x_i = x_{i'}$ for $i = 1, \ldots, 4$. Kummer [26] found real symmetric matrices A_i, $i = 1, \ldots, 4$ with $A_1 + A_2 + A_3 + A_4$ positive definite and a hyperbolic polynomial q of degree 3 with $\Lambda_+(h_{2,2}) \subseteq \Lambda_+(q)$ such that

$$q(\mathbf{x})h_{2,2}(\mathbf{x}) = \det\left(x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4\right),$$

where

$$q(\mathbf{x}) = 32(2x_1 + 3x_2 + 3x_3 + 4x_4)(x_1x_2 + x_1x_3 + 2x_1x_4 + x_2x_4 + x_3x_4).$$

If $s = 2$ and $k = 3$ in Theorem 7.3 it follows that there exists no linear and quadratic hyperbolic polynomials $\ell(\mathbf{x}), q(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_{16}]$ respectively such that $h_{14,3}(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_{16}]$ has a positive definite representation of the form

$$\ell(\mathbf{x})q(\mathbf{x})h_{14,3}(\mathbf{x}) = \det\left(\sum_{i=1}^{16} x_iA_i\right)$$

with $\Lambda_+(h_{14,3}) \subseteq \Lambda_+(\ell q)$.

8. Nonnegative symmetric polynomials

Recall that a polynomial $P(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is *nonnegative* if $P(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, and it is *symmetric* if it is invariant under the action (permuting the variables) of the symmetric group of order n. In this section we prove that certain symmetric polynomials are nonnegative. This is needed for the proof of Theorem 6.5. The results are interesting in their own right, and they generalize several well known inequalities in the literature.

Recall that a *partition* of a natural number d is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of natural numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = d$. We write $\lambda \vdash d$ to denote that λ is a partition of d. The *length*, $\ell(\lambda)$, of λ is the number of nonzero entries of λ. If λ is a partition and $\ell(\lambda) \leq n$, then the *monomial symmetric polynomial*, m_α, is defined as

$$m_\lambda(x_1, \ldots, x_n) = \sum x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n},$$

where the sum is over all distinct permutations $(\beta_1, \beta_2, \ldots, \beta_n)$ of $(\lambda_1, \ldots, \lambda_n)$. If $\ell(\lambda) > n$, we set $m_\lambda(\mathbf{x}) = 0$. If $k_1, \ldots, k_\ell$ are distinct positive integers and $a_1, \ldots, a_\ell \in \mathbb{N}$ we denote by $k_1^{a_1} k_2^{a_2} \cdots k_\ell^{a_\ell}$ the unique partition of $a_1 + \cdots + a_\ell$ with exactly a_j coordinates equal to k_j for

$1 \leq j \leq \ell$. The dth *elementary symmetric polynomial* is $e_d(\mathbf{x}) = m_{1^d}(\mathbf{x})$, and the dth *power symmetric polynomial* is $p_d(\mathbf{x}) = m_d(\mathbf{x})$.

Nonnegative symmetric polynomials have been studied in several areas of mathematics, see [3, 12, 16] and the references therein. We will initially concentrate on nonnegative polynomials of the form

$$\sum_{k=0}^{2r} a_k e_k(\mathbf{x}) e_{2r-k}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^m, \tag{8.1}$$

where r is a positive integer and $a_k \in \mathbb{R}$ for $0 \leq k \leq 2r$. Hence these are the nonnegative symmetric polynomials spanned by $\{m_{2^k 1^{2(r-k)}} : 0 \leq k \leq r\}$. A classical family of such nonnegative and symmetric polynomials was found already by Newton [33]:

$$\frac{e_r(\mathbf{x})^2}{\binom{n}{r}^2} - \frac{e_{r-1}(\mathbf{x})}{\binom{n}{r-1}} \frac{e_{r+1}(\mathbf{x})}{\binom{n}{r+1}} \geq 0,$$

for $\mathbf{x} \in \mathbb{R}^m$ with $m \leq n$ and $1 \leq r \leq n-1$. Letting $n \to \infty$ in Newton's inequalities we obtain the Laguerre–Turán inequalities (see e.g. [14]):

$$r e_r(\mathbf{x})^2 - (r+1) e_{r-1}(\mathbf{x}) e_{r+1}(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^m, m \geq 1.$$

A different but equivalent view on nonnegative symmetric polynomial is that of inequalities satisfied by the derivatives of a real–rooted polynomial: Let $\{a_k\}_{k=0}^{2m}$ be a sequence of real numbers. Then the polynomial (8.1) is nonnegative if and only if

$$\sum_{k=0}^{2r} a_k \binom{2r}{k} f^{(k)}(t) f^{(2r-k)}(t) \geq 0, \quad t \in \mathbb{R}, \tag{8.2}$$

holds for all real–rooted polynomials f of degree at most m. Indeed by translation invariance (8.2) holds for all real–rooted polynomials f of degree at most m if and only if (8.2) holds at $t = 0$ for all real–rooted polynomials f of degree at most m. Hence if $f(t) = \prod_{j=1}^m (1 + x_j t)$, then the left–hand–side of (8.2) at $t = 0$ is the same as (8.1) up to a constant factor $(2r)!$. The following inequality is due to Jensen.

Theorem 8.1 (Jensen [24])**.**

$$\sum_{k=0}^{2r} (-1)^{r+j} \binom{2r}{k} f^{(k)}(t) f^{(2r-k)}(t) \geq 0, \quad t \in \mathbb{R}, \tag{8.3}$$

for all real–rooted polynomials f.

The inequality (8.3) follows easily from a symmetric function identity as follows

$$\begin{aligned}
\sum_{r=0}^{n} m_{2^r}(\mathbf{x}) t^{2r} &= \sum_{r=0}^{n} e_r(x_1^2, \ldots, x_n^2) t^{2r} \\
&= \prod_{j=1}^{n} (1 + x_j^2 t^2) = \prod_{j=1}^{n} (1 + i x_j t) \prod_{j=1}^{n} (1 - i x_j t) \\
&= \left(\sum_{k=0}^{n} i^k e_k(\mathbf{x}) t^k \right) \left(\sum_{k=0}^{n} (-i)^k e_k(\mathbf{x}) t^k \right) \\
&= \sum_{r=0}^{n} \left(\sum_{k=0}^{2r} (-1)^{k+r} e_k(\mathbf{x}) e_{2r-k}(\mathbf{x}) \right) t^{2r}.
\end{aligned}$$

Clearly $m_{2^r}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, so that inequality (8.3) follows from

$$m_{2^r}(\mathbf{x}) = \sum_{k=0}^{2r} (-1)^{k+r} e_k(\mathbf{x}) e_{2r-k}(\mathbf{x}). \tag{8.4}$$

Lemma 8.2. *If r is a positive integer and $0 \leq t \leq 2/r$, then*

$$m_{2^r}(\mathbf{x}) + t m_{2^{r-1}1^2}(\mathbf{x})$$

is a sum of squares (sos for short), and in particular nonnegative.

Proof. Since $m_{2^r}(\mathbf{x})$ is a sos it suffices to consider $t = 2/r$, by convexity. Note that

$$m_{2^{r-1}1^2}(\mathbf{x}) = \sum_{|S|=r-1} e_2(\mathbf{x}^S) \prod_{i \in S} x_i^2,$$

where $\mathbf{x}^S = \mathbf{x} \setminus \{x_i : i \in S\}$. Using $e_2(\mathbf{x}) = e_1(\mathbf{x})^2/2 - p_2(\mathbf{x})/2$

$$\begin{aligned} m_{2^{r-1}1^2}(\mathbf{x}) &= \frac{1}{2} \sum_{|S|=r-1} e_1(\mathbf{x}^S)^2 \prod_{i \in S} x_i^2 - \frac{1}{2} \sum_{|S|=r-1} p_2(\mathbf{x}^S) \prod_{i \in S} x_i^2 \\ &= S(\mathbf{x}) - \frac{r}{2} m_{2^r}(\mathbf{x}), \end{aligned}$$

where $S(\mathbf{x})$ is a sum of squares. Indeed

$$\frac{1}{2} \sum_{|S|=r-1} p_2(\mathbf{x}^S) \prod_{i \in S} x_i^2 = C m_{2^r}(\mathbf{x})$$

for some C, and setting $\mathbf{x} = (1, \ldots, 1)$ one sees that

$$\frac{1}{2} \binom{n}{r-1} (n - r + 1) = C \binom{n}{r},$$

so that $C = r/2$. The lemma follows. □

Let $P(\mathbf{x})$ be a symmetric polynomial. Suppose $P(\mathbf{x}) = Q(e_1(\mathbf{x}), \ldots, e_m(\mathbf{x}))$ is the unique expression of P in terms of the elementary symmetric polynomials. If Q is of degree d, let $H(x_0, x_1, \ldots, x_m) = x_0^d Q(x_1/x_0, \ldots, x_m/x_0)$ be its homogenization, and let

$$L(P) := H(e_1(\mathbf{x}), 2e_2(\mathbf{x}), \ldots, (m+1)e_{m+1}(\mathbf{x}))$$

be the *lift* of P. This operation enables us to lift symmetric nonnegative polynomial inequalities to higher degrees.

Lemma 8.3. *If P is a symmetric and nonnegative polynomial, then so is its lift $L(P)$.*

Proof. Note first that if P is nonnegative and symmetric, then the degree of Q above is even. Indeed if $\mathbf{x}(t) = (t, x_2, \ldots, x_n)$ where $x_2, \ldots, x_n \in \mathbb{R}$ are generic and t is a variable, then we obtain a univariate nonnegative polynomial $t \mapsto P(\mathbf{x}(t))$ of degree d. Hence d is even. Now if $\mathbf{x} \in \mathbb{R}^n$ is such that $e_1(\mathbf{x}) \neq 0$, then there is a $\mathbf{y} \in \mathbb{R}^n$ such that $e_k(\mathbf{y}) = (k+1)e_{k+1}(\mathbf{x})/e_1(\mathbf{x})$ for all k. Indeed

$$\frac{1}{e_1(\mathbf{x})} \frac{d}{dt} \prod_{k=1}^{n} (1 + x_k t) = \prod_{k=0}^{n} (1 + y_k t), \qquad \text{where } y_n = 0,$$

since the operator d/dt preserves real–rootedness. Thus

$$L(P)(\mathbf{x}) = P(\mathbf{y})$$

and the proof follows. □

Lemma 8.4. *The lift of* $m_{2^{r-1}}(\mathbf{x})$ *is*

$$r^2 m_{2^r}(\mathbf{x}) + 2m_{2^{r-1}1^2}(\mathbf{x}).$$

Proof. By (8.4), the lift of $m_{2^{r-1}}(\mathbf{x})$ is

$$\begin{aligned} L &:= \sum_{j=0}^{2r-2} (-1)^{j+r-1}(j+1)(2r-1-j)e_{j+1}(\mathbf{x})e_{2r-1-j}(\mathbf{x}) \\ &= \sum_{j=0}^{2r} (-1)^{j+r} j(2r-j)e_j(\mathbf{x})e_{2r-j}(\mathbf{x}). \end{aligned}$$

The coefficient in front of $m_{2^k 1^{2(r-k)}}(\mathbf{x})$ in the expansion of $e_j(\mathbf{x})e_{2r-j}(\mathbf{x})$ in the monomial bases is seen to be $\binom{2r-2k}{j-k}$. (Look at how many times we get the monomial $x_1^2 x_2^2 \cdots x_k^2 x_{k+1} x_{k+2} \cdots$ in the expansion of the $e_j(\mathbf{x})e_{2r-j}(\mathbf{x})$.) Hence the coefficient infront of $m_{2^k 1^{2(r-k)}}(\mathbf{x})$ in the expansion of L in the monomial basis is

$$a_k = \sum_{j=0}^{2r} (-1)^{j+r} j(2r-j)\binom{2r-2k}{j-k}.$$

Now $a_r = r^2, a_{r-1} = 2$, and $a_k = 0$ otherwise. This follows from the fact that if p is a polynomial of degree d, then

$$\sum_{j=0}^{n} (-1)^j p(j)\binom{n}{j} = 0$$

whenever $n > d$. □

Our next lemma is a refinement of the Laguerre–Turán inequalities and may be formulated as *the Laguerre–Turán inequalities beat Jensen's inequalities* (8.3). Lemma 8.5 is also a generalization of [16, Theorem 3], where the case $r = 2$ was proved. If $P, Q \in \mathbb{R}[\mathbf{x}]$, we write $P \leq Q$ if $Q - P$ is a nonnegative polynomial.

Lemma 8.5. *If* $r \geq 1$, *then*

$$m_{2^r}(\mathbf{x}) \leq r e_r(\mathbf{x})^2 - (r+1)e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}).$$

Proof. The proof is by induction over r. For $r = 1$ we have equality. Let $r \geq 2$ and assume the inequality in the case $r - 1$ by induction. Taking the lift of this inequality and applying Lemmas 8.3 and 8.4, we find that

$$r^2 m_{2^r}(\mathbf{x}) + 2m_{2^{r-1}1^2}(\mathbf{x}) \leq (r-1)r^2 e_r(\mathbf{x})^2 - r(r-1)(r+1)e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}).$$

By Lemma 8.2

$$(r^2 - r)m_{2^r}(\mathbf{x}) \leq r^2 m_{2^r}(\mathbf{x}) + 2m_{2^{r-1}1^2}(\mathbf{x}),$$

and the lemma follows. □

Lemma 8.6. *If* $r \geq 2$ *is an integer, then*

$$(a_r e_{r-1}(\mathbf{x})e_r(\mathbf{x}) - e_{r-2}(\mathbf{x})e_{r+1}(\mathbf{x}))^2 \geq C_r e_{r-2}(\mathbf{x})e_r(\mathbf{x})m_{2^r}(\mathbf{x}), \tag{8.5}$$

where

$$a_r = 3\frac{r-1}{r+1} \quad \textit{and} \quad C_r = 9\frac{r-1}{(r+1)^2}.$$

Proof. We prove the inequality by induction over $r \geq 2$. Assume $r = 2$. The polynomial $(t, \mathbf{x}) \mapsto e_4(t, t, x_1, x_1, \ldots, x_n, x_n)$ is stable, this may for instance be deduced from the Grace-Walsh-Szegő theorem (see Remark 9.3). It specializes to a real-rooted (or identically zero) polynomial when we set $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$(4e_2(\mathbf{x}) + p_2(\mathbf{x}))t^2 + 4(e_1(\mathbf{x})e_2(\mathbf{x}) + e_3(\mathbf{x}))t + m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x}).$$

Hence its discriminant is nonnegative, which gives

$$(e_3(\mathbf{x}) + e_1(\mathbf{x})e_2(\mathbf{x}))^2 \geq (e_2(\mathbf{x}) + p_2(\mathbf{x})/4)(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x})).$$

To prove (8.5) for $r = 2$ we may assume $e_2(\mathbf{x}) > 0$. Rewriting (8.5) as

$$(e_3(\mathbf{x}) + e_1(\mathbf{x})e_2(\mathbf{x}))^2 \geq e_2(\mathbf{x})(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x})),$$

we may assume also $m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x}) > 0$. Then, since $p_2(\mathbf{x}) \geq 0$,

$$\begin{aligned}(e_3(\mathbf{x}) + e_1(\mathbf{x})e_2(\mathbf{x}))^2 &\geq (e_2(\mathbf{x}) + p_2(\mathbf{x})/4)(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x})) \\ &\geq e_2(\mathbf{x})(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x}))\end{aligned}$$

which proves the lemma for $r = 2$.

Assume that the inequality is true for a given $r \geq 2$. We lift the inequality for r and use Lemma 8.4 to get

$$\begin{aligned}&(a_r r(r+1)e_r(\mathbf{x})e_{r+1}(\mathbf{x}) - (r-1)(r+2)e_{r-1}(\mathbf{x})e_{r+2}(\mathbf{x}))^2 \geq \\ &C_r(r-1)(r+1)^3 e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x})\left(m_{2^{r+1}(\mathbf{x})} + \frac{2}{(r+1)^2}m_{2^r1^2}(\mathbf{x})\right)\end{aligned}$$

We may exchange the factor $m_{2^{r+1}} + (2/(r+1)^2)m_{2^r1^2}$ by something nonnegative and smaller and still get a valid inequality. By Lemma 8.2 we obtain the inequality

$$\begin{aligned}&(a_r r(r+1)e_r(\mathbf{x})e_{r+1}(\mathbf{x}) - (r-1)(r+2)e_{r-1}(\mathbf{x})e_{r+2}(\mathbf{x}))^2 \geq \\ &C_r(r-1)(r+1)^3 e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x})\frac{r}{r+1}m_{2^{r+1}}(\mathbf{x}).\end{aligned}$$

Dividing through by $(r-1)^2(r+2)^2$ we obtain

$$\begin{aligned}&\left(a_r\frac{r(r+1)}{(r-1)(r+2)}e_r(\mathbf{x})e_{r+1}(\mathbf{x}) - e_{r-1}(\mathbf{x})e_{r+2}(\mathbf{x})\right)^2 \geq \\ &C_r\frac{r(r+1)^2}{(r-1)(r+2)^2}e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x})m_{2^{r+1}}(\mathbf{x}),\end{aligned}$$

which simplifies to the desired inequality for $r + 1$. □

9. Proof of Theorem 6.6

The next tool for the proof of Theorem 6.6 is a lemma that enables us to prove hyperbolicity of a polynomial by proving real-rootedness along a few (degenerate) directions.

Lemma 9.1. *Let $h \in \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^n$. Define d to be the maximum degree of the polynomial $t \mapsto h(t\mathbf{v}_2 + \mathbf{y})$, where the maximum is taken over all $\mathbf{y} \in \mathbb{C}^n$. Let further*

$$P(\mathbf{x}) := \lim_{t\to\infty} t^{-d}h(t\mathbf{v}_2 + \mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n].$$

Suppose $S \subseteq \mathbb{C}^n$ and $\mathbf{x}_0 \in S$ are such that

(i) *$S + \mathbb{R}\mathbf{v}_2 = S$, i.e., S is closed under translations by real multiples of $\mathbf{v}_2$.*

(ii) *S is pathwise connected.*
(iii) *The polynomial $(s,t) \mapsto h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x}_0)$ is stable and not identically zero.*
(iv) *For each $\mathbf{x} \in S$, the polynomials $s \mapsto h(s\mathbf{v}_1 + \mathbf{x})$ and $s \mapsto P(s\mathbf{v}_1 + \mathbf{x})$ are stable and not identically zero.*

Then the polynomial $(s,t) \mapsto h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x})$ is stable for all $\mathbf{x} \in S$.

Proof. The proof is by contradiction. Suppose $\mathbf{x}_1 \in S$ and $\xi, \eta \in \mathbb{C}$ are such that $\mathrm{Im}(\xi) > 0$, $\mathrm{Im}(\eta) > 0$ and

$$h(\xi\mathbf{v}_1 + \eta\mathbf{v}_2 + \mathbf{x}_1) = 0.$$

Let $\mathbf{x}(\theta) : [0,1] \to S$ be a continuous path such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(1) = \mathbf{x}_1$ and let

$$p_\theta(t) = h(\xi\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x}(\theta)) = t^d P(\xi\mathbf{v}_1 + \mathbf{x}(\theta)) + O(t^{d-1}),$$

where $P(\xi\mathbf{v}_1 + \mathbf{x}(\theta)) \neq 0$ by (iv). By assumption all zeros of $p_0(t)$ are in the closed lower half-plane, while $p_1(\eta) = 0$ where $\mathrm{Im}(\eta) > 0$. Hence, by continuity using Hurwitz' theorem on the continuity of zeros (see e.g., [11, Footnote 3, p. 96]), a zero will cross the real axis as θ runs from 0 to 1. In other words

$$0 = p_\theta(\alpha) = h(\xi\mathbf{v}_1 + \alpha\mathbf{v}_2 + \mathbf{x}(\theta)),$$

for some $\alpha \in \mathbb{R}$ and $\theta \in [0,1]$. Since $\alpha\mathbf{v}_2 + \mathbf{x}(\theta) \in S$, by (i), this contradicts (iv). □

The next theorem is a version of the Grace-Walsh-Szegő coincidence theorem, see [4, Prop. 3.4].

Theorem 9.2 (Grace-Walsh-Szegő)**.** *Suppose $P(x_1, \dots, x_n) \in \mathbb{C}[\mathbf{x}]$ is a polynomial of degree at most d in the variable x_1:*

$$P(x_1, \dots, x_n) = \sum_{k=0}^{d} P_k(x_2, \dots, x_n) x_1^k.$$

Let Q be the polynomial in the variables $x_2, \dots, x_n, y_1, \dots, y_n$

$$Q = \sum_{k=0}^{d} P_k(x_2, \dots, x_n) \frac{e_k(y_1, \dots, y_d)}{\binom{d}{k}}.$$

Then P is stable if and only if Q is stable.

Remark 9.3. Note that $e_k(x_1, \dots, x_n)$ is stable, by the Grace–Walsh–Szegő theorem applied to the polynomial x_1^k considered as a polynomial of degree at most n.

Lemma 9.4. *If $\mathbf{x} \in \mathbb{R}^n$ and $e_k(\mathbf{x}) = e_{k+1}(\mathbf{x}) = 0$ where $0 < k < n$, then $\mathbf{x}$ has at most $k-1$ nonzero coordinates.*

Proof. It is well known that if $e_k(\mathbf{x}) = e_{k+1}(\mathbf{x}) = 0$, then $e_j(\mathbf{x}) = 0$ for all $k \leq j \leq n$, see e.g., [6, Example 3.6]. Hence the number of non-zero coordinates of $\mathbf{x}$ is equal to

$$\max\{0 \leq i \leq n : e_i(\mathbf{x}) \neq 0\} < k.$$

□

The following theorem provides families of stable polynomials which are closed under convex sums.

Theorem 9.5. *Let $r \geq 2$ be an integer, and let*

$$M(\mathbf{x}) = \sum_{|S|=r} a(S) \prod_{i \in S} x_i^2 \in \mathbb{R}[x_1, \ldots, x_n],$$

where $0 \leq a(S) \leq 1$ for all $S \subseteq [n]$, where $|S| = r$. Then the polynomial

$$4e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x}) + \frac{3}{r+1}M(\mathbf{x}) \tag{9.1}$$

is stable.

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}' = (x_2, \ldots, x_n)$ where $n \geq 2r+4$. Suppose M is of the form described in the statement of the theorem and that additionally no x_i, $i = 1, \ldots, 2r+3$ appears in M. We will prove, by applying Lemma 9.1, that polynomials of the form

$$h(\mathbf{x}) := 4(a_r x_1 e_r(\mathbf{x}') + e_{r+1}(\mathbf{x}'))(x_1 e_{r-2}(\mathbf{x}') + e_{r-1}(\mathbf{x}')) + \frac{3}{r+1}M \tag{9.2}$$

are stable, where a_r is defined as in Lemma 8.6. Since any polynomial of the form (9.1) may be obtained from some polynomial of the form (9.2) by setting variables to zero and relabelling the indices, the stability of polynomials of the form (9.1) follows from Lemma 2.6.

Recall the notation of Lemma 9.1. Let $\mathbf{v}_1 = \delta_1$, $\mathbf{v}_2 = \delta_2 + \delta_3 + \cdots + \delta_{r+2}$, and let S be the set of all $\mathbf{x} \in \mathbb{R}^n$ such that at least $r+1$ of the coordinates $\{x_{r+3}, \ldots, x_n\}$ are nonzero. Note that S is pathwise connected and $S + \mathbb{R}\mathbf{v}_2 = S$. Let $\mathbf{x}_0 = \delta_{r+3} + \cdots + \delta_{2r+3}$. The polynomial

$$\begin{aligned} q(\mathbf{x}) &:= 4(a_r x_1 e_r(\mathbf{x}') + e_{r+1}(\mathbf{x}'))(x_1 e_{r-2}(\mathbf{x}') + e_{r-1}(\mathbf{x}')) \\ &= 4e_{r+1}(a_r x_1, x_2, \ldots, x_n)e_{r-1}(x_1, x_2, \ldots, x_n) \end{aligned}$$

is stable by Remark 9.3 and Lemma 2.6. Since $q(\mathbf{x}_0) > 0$ we know, by Lemma 2.8, that the bivariate polynomial $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x}_0) = q(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x}_0)$ is stable and not identically zero. This verifies (iii) of Lemma 9.1. Note that $P(\mathbf{x})$ is a non-zero constant. Consider

$$\begin{aligned} h(s\mathbf{v}_1 + \mathbf{x}) &= 4a_r e_r(\mathbf{x}')e_{r-2}(\mathbf{x}')(s + x_1)^2 \\ &\quad + 4(a_r e_r(\mathbf{x}')e_{r-1}(\mathbf{x}') + e_{r+1}(\mathbf{x}')e_{r-2}(\mathbf{x}'))(s + x_1) \\ &\quad + 4e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}M. \end{aligned}$$

We now prove that $h(s\mathbf{v}_1 + \mathbf{x}) \not\equiv 0$ for each $\mathbf{x} \in S$, as a polynomial in s. Assume $h(s\mathbf{v}_1 + \mathbf{x}) \equiv 0$ for some $\mathbf{x} \in S$. Then $e_r(\mathbf{x}')e_{r-2}(\mathbf{x}') = 0$, so suppose first $e_r(\mathbf{x}') = 0$. If $e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') = 0$, then either $e_{r-1}(\mathbf{x}') = e_r(\mathbf{x}') = 0$ or $e_{r+1}(\mathbf{x}') = e_r(\mathbf{x}') = 0$, which implies $\mathbf{x}'$ has at most $r-1$ non-zero coordinates by Lemma 9.4. This contradicts $\mathbf{x} \in S$. Hence $e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') < 0$ by Lemma 8.5. Then $h(-x_1\mathbf{v}_1 + \mathbf{x})$ is equal to

$$\begin{aligned} 4e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}M &< 3e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}M \\ &\leq 3e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}m_{2^r}(\mathbf{x}') \leq 0, \end{aligned}$$

by Lemma 8.5, a contradiction. If $e_r(\mathbf{x}') \neq 0$, then $e_{r-2}(\mathbf{x}') = 0$. But then also $e_{r-1}(\mathbf{x}') = 0$, since $h(s\mathbf{v}_1 + \mathbf{x}) \equiv 0$. Hence $\mathbf{x}'$ has at most $r-3$ non-zero coordinates Lemma 9.4, which contradicts $\mathbf{x} \in S$. We conclude that $h(s\mathbf{v}_1 + \mathbf{x}) \not\equiv 0$ for $\mathbf{x} \in S$.

To apply Lemma 9.1 and prove that $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x})$ is stable for all $\mathbf{x} \in S$ it remains to prove that $h(s\mathbf{v}_1 + \mathbf{x})$ is real–rooted. However $h(s\mathbf{v}_1 + \mathbf{x})$ is of degree at most two so it suffices to show that its discriminant Δ is nonnegative. Now

$$\frac{\Delta}{16} = (a_r e_{r-1}(\mathbf{x}')e_r(\mathbf{x}') - e_{r-2}(\mathbf{x}')e_{r+1}(\mathbf{x}'))^2 - \frac{3}{r+1}a_r e_r(\mathbf{x}')e_{r-2}(\mathbf{x}')M.$$

If $e_r(\mathbf{x}')e_{r-2}(\mathbf{x}') < 0$, then clearly $\Delta \geq 0$, so assume $e_r(\mathbf{x}')e_{r-2}(\mathbf{x}') \geq 0$. Then, since $M(\mathbf{x}') \leq m_{2^r}(\mathbf{x}')$, it follows that $\Delta \geq 0$ by Lemma 8.6.

Since S is dense in $\mathbb{R}^n$ we have by Hurwitz' theorem that $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x})$ is stable or identically zero for all $\mathbf{x} \in \mathbb{R}^n$. However $h(\mathbf{v}_2) \neq 0$ so that $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x})$ is stable for all $\mathbf{x} \in \mathbb{R}^n$. In particular h is hyperbolic with respect to $\mathbf{v}_2$. Since all Taylor coefficients of h are nonnegative we see that the hyperbolicity cone contains the positive orthant, i.e., h is stable, by Lemma 2.7.

□

Lemma 9.6. *Let $r \geq 2$. Then*

$$e_{2r}(x_1, x_1, \ldots, x_n, x_n) - m_{2^r}(\mathbf{x}) =$$
$$4(e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}) + e_{r-3}(\mathbf{x})e_{r+3}(\mathbf{x}) + e_{r-5}(\mathbf{x})e_{r+5}(\mathbf{x}) + \cdots).$$

Proof. Note that

$$\sum_{k=0}^{2n} e_k(x_1, x_1, \ldots, x_n, x_n)t^k = \prod_{j=1}^{n}(1 + x_j t)^2 = \left(\sum_{k=0}^{2n} e_k(\mathbf{x})t^k\right)^2.$$

The coefficient of t^{2r} is

$$e_{2r}(x_1, x_1, \ldots, x_n, x_n) = \sum_{j=0}^{2r} e_j(\mathbf{x})e_{2r-j}(\mathbf{x}).$$

The proof follows by combining this with (8.4). □

We are now in a position to prove Theorem 6.6 and Theorem 6.5.

Proof of Theorem 6.5. By definition the bases generating polynomial of $V_H \in \mathcal{V}$ is given by

$$h_{V_H} = \sum_{B \in \mathcal{B}(V_H)} \prod_{i \in B} x_i = e_{2r}(x_1, x_{1'}, \ldots, x_n, x_{n'}) - e_r(x_1 x_{1'}, \ldots, x_n x_{n'}) + N(\mathbf{x}).$$

where

$$N(\mathbf{x}) = \sum_{(i_1,\ldots,i_r) \notin E(H)} \prod_{j=1}^{r} x_{i_j} x_{i'_j}.$$

The polynomial h_{V_H} is clearly multiaffine and symmetric pairwise in $x_i, x_{i'}$ for all $i \in [n]$. Set $x_{i'} = x_i$ for all $1 \leq i \leq n$ and obtain the polynomial

$$f_{V_H} = e_{2r}(x_1, x_1, \ldots, x_n, x_n) - e_r(x_1^2, \ldots, x_n^2) + N(x_1, x_1, \ldots, x_n, x_n).$$

By Lemma 9.6

$$f_{V_H} = 4 \sum_{j=0}^{\lceil r/2 \rceil - 1} e_{r+2j+1}(\mathbf{x})e_{r-2j-1}(\mathbf{x}) + N(x_1, x_1, \ldots, x_n, x_n).$$

The support of $e_{r+j}(\mathbf{x})e_{r-j}(\mathbf{x})$ is contained in the support of $e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x})$ for each $1 \leq j \leq r$. Hence f_{V_H} has the same support as the polynomial

$$W_{V_H} = 4e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x}) + \frac{3}{r+1}N(x_1, x_1, \ldots, x_n, x_n)$$

which in turn is stable by Theorem 9.5. Hence if we replace x_i^k for $k = 0, 1, 2$, in W_{V_H} with $e_k(x_i, x_{i'})/\binom{2}{k}$, we obtain a polynomial which is stable by the Grace-Walsh-Szegő theorem (Theorem 9.2), and has the same support as h_{V_H}. Hence V_H is a WHPP-matroid so V_H is hyperbolic by Proposition 3.2. □

Proof of Theorem 6.6. Recall the notation in the proof of Theorem 6.5. If $r = 2$, then $W_{V_G} = f_{V_G}$, so that V_G has the half-plane property by the proof of Theorem 6.5. □

10. Representability and minor closure

The following amended section is not part of the published research article. We begin the section by proving some facts related to the representability of the matroids in the class $\mathcal{V}$. The class $\mathcal{V}$ is not closed under taking minors. Therefore in the final part of the section we determine the minor closure of $\mathcal{V}$.

Let H be a d-uniform hypergraph on $[n]$. Let H_j denote the hypergraph with vertex set $V(H_j) = V(H) \setminus j$ and edges $E(H_j) = \{e \setminus j : e \in E(H),\ j \in e\}$. Moreover let H^* denote the $(n-d)$-uniform hypergraph on $[n]$ with $E(H^*) = \{[n] \setminus e : e \in E(H)\}$. Recall that the *dual* of a matroid M on E, denoted M^*, is the matroid on E whose bases are the complements of the bases of M. The *free extension* of a matroid M of rank k with rank function $r_M : 2^{E(M)} \to \mathbb{N}$, by an element $e \notin E(M)$, is given by the matroid, denoted $M + e$, on $E(M) \sqcup e$ with rank function

$$r_{M+e}(S) = \begin{cases} r_M(S), & \text{if } e \notin S, \\ r_M(S \setminus e) + 1, & \text{if } e \in S \text{ and } r_M(S \setminus e) < k, \\ k, & \text{if } r_M(S \setminus e) = k. \end{cases}$$

The *free coextension* of M by e is given by the matroid $M \times e = (M^* + e)^*$. For $i \in [n]$ we shall also use the convention that $(i')' = i$. Below we list some basic properties of the matroid class $\mathcal{V}$.

Proposition 10.1. *If H, H_1 and H_2 are d-uniform hypergraphs on $[n]$, then*

(i) $V_{H_1} \cong V_{H_2}$ *if and only if* $H_1 \cong H_2$,
(ii) $([n] \cup [n]', \mathcal{I}(V_{H_1}) \cap \mathcal{I}(V_{H_2})) = V_{H_1 \cup H_2}$,
(iii) $([n] \cup [n]', \mathcal{I}(V_{H_1}) \cup \mathcal{I}(V_{H_2})) = V_{H_1 \cap H_2}$,
(iv) $V_H^* = V_{H^*}$. *In particular* V_H *is self-dual if and only if* $H \cong H^*$,
(v) $V_H \setminus j = V_{H \setminus j} + j'$,
(vi) $V_H / j = V_{H_j} \times j'$,
(vii) V_H *has Tutte-polynomial,*

$$T_{V_H}(x, y) = |E(H)|(xy - x - y) + \sum_{i=1}^{2d} \binom{2n-i-1}{2n-2d-1} x^i + \sum_{j=1}^{2n-2d} \binom{2n-j-1}{2d-1} y^j.$$

Proof. For (i), suppose that $\Phi : E(V_{H_1}) \to E(V_{H_2})$ is a an isomorphism between V_{H_1} and V_{H_2}. Then we have $\Phi(i') = \Phi(i)'$ for all $i \in [n]$ since i is included in a circuit hyperplane of V_{H_1} if and only if i' is included in a circuit hyperplane of V_{H_1}. Hence

$$\begin{aligned} \phi : V(H_1) &\to V(H_2) \\ i &\mapsto \begin{cases} j, & \text{if } \Phi(i) = j, \\ j, & \text{if } \Phi(i) = j', \end{cases} \end{aligned}$$

is a hypergraph isomorphism between H_1 and H_2 since $\{i_1, \dots, i_r\} \in E(H_1)$ if an only if $\{i_1, i_1', \dots, i_r, i_r'\}$ is a circuit hyperplane in V_{H_1} if and only if $\{\Phi(i_1), \Phi(i_1'), \dots,$ $\Phi(i_r), \Phi(i_r')\}$ is a circuit hyperplane in V_{H_2} if and only if $\{\phi(i_1), \dots, \phi(i_r)\} \in E(H_2)$. Conversely if $\phi : V(H_1) \to V(H_2)$ is a hypergraph isomorphism, then clearly

$$\begin{aligned} \Phi : E(V_{H_1}) &\to E(V_{H_2}) \\ i &\mapsto \phi(i) \\ i' &\mapsto \phi(i)' \end{aligned}$$

is an isomorphism between the sparse paving matroids V_{H_1} and V_{H_2} since Φ bijectively maps circuit hyperplanes to circuit hyperplanes.

The statements (ii) and (iii) are clear since both sides are readily seen to have the same independent sets.

For (iv), we use the circuit-hyperplane correspondence between a matroid M on E and its dual M^*. Namely, C is a circuit in M if and only if $E - C$ is a hyperplane in M^* and H is a hyperplane in M if and only if $E \setminus C$ is a circuit in M^* (see [34, Prop. 2.16]). This shows that sparse paving matroids are closed under duality. In particular C is a circuit hyperplane in V_H if and only if $E(V_H) \setminus C$ is a circuit hyperplane in the sparse paving matroid V_H^*. Hence $V_H^* = V_{H^*}$ which proves (iv).

For (v), let $S \subseteq E(V_H) \setminus j$. If $j' \notin S$, then

$$r_{V_{H\setminus j}+j'}(S) = r_{V_{H\setminus j}}(S) = r_{V_H\setminus j}(S).$$

Suppose therefore $j' \in S$. If $r_{V_{H\setminus j}+j'}(S \setminus j') < k$, then

$$r_{V_{H\setminus j}+j'}(S) = r_{V_{H\setminus j}}(S \setminus j') + 1 = r_{V_H\setminus j}(S \setminus j') + 1 = r_{V_H\setminus j}(S).$$

Otherwise if $r_{V_{H\setminus j}+j'}(S \setminus j') = k$, then

$$k = r_{V_{H\setminus j}+j'}(S) = r_{V_{H\setminus j}+j'}(S \setminus j') = r_{V_{H\setminus j}}(S \setminus j') = r_{V_H\setminus j}(S \setminus j'),$$

so $r_{V_H\setminus j}(S) = k$. Hence $r_{V_{H\setminus j}+j'}(S) = r_{V_H\setminus j}(S)$ for all $S \subseteq E(V_H) \setminus j$ which establishes (v). Statement (vi) now follows from (iv) and (v) via

$$V_H/j = (V_H^* \setminus j)^* = (V_{H^*} \setminus j)^* = (V_{H^*\setminus j} + j')^* = V_{H^*\setminus j}^* \times j' = V_{(H^*\setminus j)^*} \times j' = V_{H_j} \times j'.$$

For (vii), note that the uniform matroid $U_{2n,2d}$ on $2n$ elements of rank $2d$ is obtained from V_H via a sequence of $|E(H)|$ circuit hyperplane relaxations. Hence (vii) follows from the fact that

$$T_M(x, y) = xy - x - y + T_{M'}(x, y)$$

where M' is the matroid obtained from M by relaxing a circuit hyperplane, and

$$T_{U_{n,d}}(x,y) = \sum_{i=1}^{d} \binom{n-i-1}{n-d-1} x^i + \sum_{j=1}^{n-d} \binom{n-j-1}{d-1} y^j,$$

see [30]. □

Remark 10.2. The class $\mathcal{V}$ is seen not to be closed under minors, direct sums and matroid unions.

Although V_H does not have the half-plane property for hypergraphs H in general, a simple consequence of Proposition 10.1 (iv) is the following.

Proposition 10.3. *Let H be a d-uniform hypergraph on $[n]$. If $n \leq d+2$, then V_H has half-plane property.*

Proof. Suppose $n = d+2$. By Proposition 10.1 (iv) we have that $V_H^* = V_{H^*}$ where H^* is a 2-uniform hypergraph on $[n]$ (i.e. a graph). Thus V_H has half-plane property if and only if V_{H^*} has half-plane property by closure under duality [11]. Hence V_H has half-plane property by Theorem 6.6. Finally if $n < d+2$ then via free extension V_H is a minor of a matroid $V_{H'}$ with $|V(H')| = d+2$. Since the half–plane property is closed under taking minors [11], the result follows. □

Clearly any matroid $V_H \in \mathcal{V}$ in which the Vámos matroid V_8 is a minor cannot be representable (and necessarily fails to satisfy Ingleton's inequality). Hence Theorem 6.5 provides an infinite family of hyperbolic matroids which are not representable. By Proposition 10.1 we have that $V_H \setminus j$ and V_H/j are representable if and only if $V_{H\setminus j}$ and V_{H_j} are respectively representable. It follows that every non-representable matroid V_H has a minimal excluded minor for representability of the form $V_{H'}$ in its minor hierarchy for some hypergraph H'.

A natural question is which matroids in $\mathcal{V}$ are representable/non-representable? Below we identify a class of matroids in $\mathcal{V}$ which are guaranteed to be representable over any infinite field.

Theorem 10.4. *Let H be a d-uniform hypergraph on $[n]$ and let $\mathbb{F}$ be an infinite field. Suppose $j \in [n]$ such that $j \in e$ for at most one $e \in E(H)$. Then V_H is $\mathbb{F}$-representable if and only if $V_{H\setminus j}$ is $\mathbb{F}$-representable.*

Proof. If V_H is $\mathbb{F}$-representable, then $V_{H\setminus j}$ is $\mathbb{F}$-representable by Proposition (v) since representability is closed under taking minors. Conversely suppose $V_{H\setminus j}$ is representable over $\mathbb{F}$. If $j \notin e$ for all $e \in E(H)$, then $V_H = (V_{H\setminus j} + j) + j'$ so the statement follows since representability is closed under taking free extensions. Therefore suppose $j \in [n]$ belongs to a unique $e \in E(H)$. If $n \leq d$ then the proposition is clear so we may assume $n > d$. By relabelling if necessary we may assume $j = n$. Let $u_1, u_{1'} \ldots, u_{n-1}, u_{(n-1)'}$ be vectors in a $2d$-dimensional vector space V over $\mathbb{F}$ representing the elements $E_{n-1} = \{1, 1', \ldots, n-1, (n-1)'\}$ of $V_{H\setminus j}$. Since $\mathbb{F}$ is infinite, V cannot be a union of finitely many proper subspaces, so we may choose

$$u_n \in V \setminus \bigcup_{i_1,\ldots,i_{2d-2} \in E_{n-1}} \langle u_{i_1}, \ldots, u_{i_{2d-2}} \rangle,$$

where $\langle u_{i_1}, \ldots, u_{i_{2d-2}} \rangle$ denotes the linear span of the vectors $u_{i_1}, \ldots, u_{i_{2d-2}}$ over $\mathbb{F}$. Let

$$U = \langle u_n \rangle \oplus \langle u_i, u_{i'} : i \in e \setminus n \rangle.$$

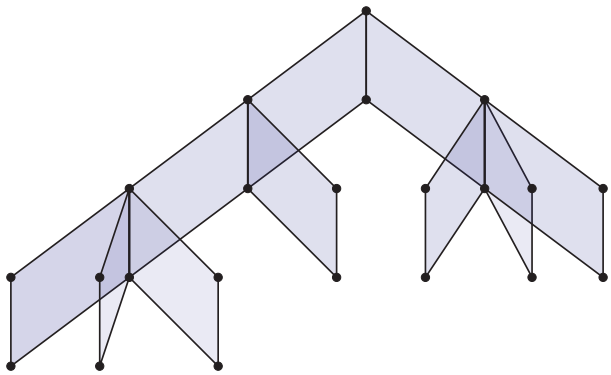


FIGURE 5. The matroid V_H in the figure is representable since H is a tree graph.

By modularity we have

$$\begin{aligned}\dim\Big(U \cap \langle u_{i_1}, \dots, u_{i_{2d-1}}\rangle\Big) &= \dim(U) + \dim(\langle u_{i_1}, \dots, u_{i_{2d-1}}\rangle) - \dim(U + \langle u_{i_1}, \dots, u_{i_{2d-1}}\rangle)\\ &\le (2d-1) + (2d-1) - 2d < 2d-1.\end{aligned}$$

for all $i_1, \dots, i_{2d-1} \in E$. Thus $U \neq \langle u_{i_1}, \dots, u_{i_{2d-1}}\rangle$ for all $\{i_1, \dots, i_{2d-1}\} \neq \{n\} \cup \{i, i' : i \in e \setminus n\}$. Therefore we may choose

$$u_{n'} \in U \setminus \bigcup_{\substack{i_1,\dots,i_{2d-1} \in E_{n-1} \cup \{n\} \\ \{i_1,\dots,i_{2d-1}\} \neq \{n\} \cup \{i,i' : i \in e \setminus n\}}} \langle u_{i_1}, \dots, u_{i_{2d-1}}\rangle.$$

It follows that $\langle u_i, u_{i'} : i \in e\rangle$ has rank $2d-1$, and for any other subset containing u_n or $u_{n'}$ of size at most $2r$ the rank is equal to the cardinality of the spanning set. Hence $u_1, u_{1'}, \dots, u_n, u_{n'}$ are vectors in V representing V_H. □

Corollary 10.5. *If G is a forest then V_G is $\mathbb{F}$-representable for any infinite field $\mathbb{F}$.*

Proof. If G is the empty graph forest then $V_G \cong U_{2n,4}$ which is representable over $\mathbb{F}$. If G is non-empty then the statement follows by removing a leaf in any tree-component and arguing by induction applying Theorem 10.4. □

If $\mathcal{A}$ is a class of matroids, then denote by $\overline{\mathcal{A}}$ the *minor closure* of the class $\mathcal{A}$, that is, the smallest class that contains all minors of matroids in $\mathcal{A}$. Consider the following class:

Definition 10.6. Let $\mathcal{C}$ denote the class of sparse paving matroids M with $E(M) \subseteq [n] \cup [n]'$ having circuit hyperplanes $C_1, \dots, C_k$ and a set $S \subseteq \bigcap_{t=1}^k C_t$ such that

(i) $i \in S$ implies $i' \notin E(M)$,
(ii) $i \in C_t \setminus S$ if and only if $i' \in C_t \setminus S$ for all $t = 1, \dots, k$.

Remark 10.7. Note that the definition implies that S is unique. Indeed suppose $S_1, S_2 \subseteq \bigcap_{t=1}^k C_t$ both satisfy conditions (i) and (ii). If $j \in S_1 \setminus S_2$, then on one hand $j' \notin E(M)$, and on the other hand $j \in C_t \setminus S_2$ for all $t = 1, \dots, k$, so $j' \in C_t \setminus S_2 \subseteq E(M)$ for all $t = 1, \dots, k$. This gives a contradiction. Hence $S_1 = S_2$.

Lemma 10.8. *The matroid class $\mathcal{C}$ is minor closed, i.e., $\mathcal{C} = \overline{\mathcal{C}}$.*

Proof. Let $M \in \mathcal{C}$ be a matroid with circuit hyperplanes $C_1, \ldots, C_k$ and let $i \in E(M)$. We distinguish between three cases:

Case 1. Suppose $i \in E(M) \setminus \bigcup_{t=1}^{k} C_t$. Clearly $M \setminus i$ is a matroid with the same set of circuits and therefore belongs to $\mathcal{C}$. The matroid M/i has no circuit hyperplanes since i belongs to no circuit hyperplane of M. Hence M/i is a uniform matroid which again belongs to $\mathcal{C}$.

Case 2. Suppose $i \in S$. Then $M \setminus i$ has no circuit hyperplanes since i belongs to every circuit hyperplane of M. Thus $M \setminus i$ is a uniform matroid and therefore belongs to $\mathcal{C}$. Since $i \in S$ the circuit hyperplanes of M/i are given by $C_t \setminus i$ for $t = 1, \ldots, k$. Hence the axioms of $\mathcal{C}$ remain intact with $S \setminus i \subseteq \bigcap_{t=1}^{k} C_t \setminus i$.

Case 3. Suppose $i \in C_t \setminus S$ for $t \in I$ where $I \subseteq [k]$. The remaining circuit hyperplanes in $M \setminus i$ are given by $\{C_t : t \in [k] \setminus I\}$ which are easily seen to satisfy the axioms of the class $\mathcal{C}$. The circuit hyperplanes in M/i are given by $\{C_t \setminus i : t \in I\}$ and the axioms of $\mathcal{C}$ are satisfied with $S \cup \{i'\} \subseteq \bigcap_{i \in I} C_t \setminus i$.

Hence $\mathcal{C}$ is minor closed. □

Theorem 10.9. *The minor closure of $\mathcal{V}$ is $\mathcal{C}$, i.e., $\overline{\mathcal{V}} = \mathcal{C}$.*

Proof. Certainly $\mathcal{V} \subseteq \mathcal{C}$ since the matroids in $\mathcal{V}$ are instances of matroids in $\mathcal{C}$ with $S = \emptyset$. Hence by Lemma 10.8 we have $\overline{\mathcal{V}} \subseteq \overline{\mathcal{C}} = \mathcal{C}$. Conversely let $M \in \mathcal{C}$ and suppose M has circuit hyperplanes $C_1, \ldots, C_k$. Let $T = (S \cup S') \cap [n]$ and $e_t = (C_t \cup C_t') \cap [n]$ for $t = 1, \ldots, k$. By definition $|C_t \setminus S| = 2l$ for all $t = 1, \ldots, k$ for some $l \in \mathbb{N}$. Thus $|e_1| = \cdots = |e_k| = |T| + l$. Consider the matroid $V_H \in \mathcal{V}$ where H is the $(|T| + l)$-uniform hypergraph on $[n]$ with edges $e_1, \ldots, e_k$. It follows that M is a minor of V_H. Indeed since $S \cup S' \subseteq \bigcap_{t=1}^{k} (e_t \cup e_t')$, we find that the circuit hyperplanes of V_H/S' are given by $C_1, \ldots, C_k$. Finally delete all elements in V_H/S' belonging to $U = [n] \cup [n]' \setminus E(M)$. Then $M = (V_H/S') \setminus U$. Hence $\overline{\mathcal{V}} \supseteq \mathcal{C}$. □

Remark 10.10. We remark that $\mathcal{C}$ is also closed under taking duals. Indeed if $M \in \mathcal{C}$ with circuit hyperplanes $C_1, \ldots C_k$, then M^* has circuit hyperplanes $E(M) \setminus C_t$ for $t = 1, \ldots, k$ which together with the unique maximal subset $S \subseteq E(M) \setminus \bigcup_{t=1}^{k} C_t$ such that $i \in S \Rightarrow i' \notin E(M)$ satisfies the axioms of $\mathcal{C}$.

Corollary 10.11. *The class $\mathcal{C}$ consists of hyperbolic matroids.*

Proof. By Theorem 6.5 the class $\mathcal{V}$ consists of hyperbolic matroids. Since the class of hyperbolic matroids is minor closed the statement follows by Theorem 10.9. □

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Paper B

SPECTRAHEDRALITY OF HYPERBOLICITY CONES OF MULTIVARIATE MATCHING POLYNOMIALS

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Abstract. The generalized Lax conjecture asserts that each hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices. We prove the conjecture for a multivariate generalization of the matching polynomial. This is further extended (albeit in a weaker sense) to a multivariate version of the independence polynomial for simplicial graphs. As an application we give a new proof of the conjecture for elementary symmetric polynomials (originally due to Brändén). Finally we consider a hyperbolic convolution of determinant polynomials generalizing an identity of Godsil and Gutman.

1. Introduction

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$, and if for all $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(t\mathbf{e} - \mathbf{x})$ has only real zeros. Note that if h is a hyperbolic polynomial of degree d, then we may write

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x})),$$

where

$$\lambda_{\max}(\mathbf{x}) = \lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x}) = \lambda_{\min}(\mathbf{x})$$

are called the *eigenvalues* of $\mathbf{x}$ with respect to $\mathbf{e}$. The *(hyperbolic) rank* of $\mathbf{x} \in \mathbb{R}^n$ with respect to $\mathbf{e}$ is defined as $\mathrm{rk}(\mathbf{x}) = \#\{\lambda_i(\mathbf{x}) \neq 0\}$. The *hyperbolicity cone* of h with respect to $\mathbf{e}$ is the set $\Lambda_+(h, \mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \lambda_{\min}(\mathbf{x}) \geq 0\}$. If $\mathbf{v} \in \Lambda_+(h, \mathbf{e})$, then h is hyperbolic with respect to $\mathbf{v}$ and $\Lambda_+(h, \mathbf{v}) = \Lambda_+(h, \mathbf{e})$. For this reason we usually abbreviate and write $\Lambda_+(h)$ if there is no risk for confusion. We denote by $\Lambda_{++}(h)$ the interior of $\Lambda_+(h)$. The cone $\Lambda_{++}(h)$ is convex and can be characterized as the connected component of the set $\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$ containing $\mathbf{e}$. These are all facts due to Gårding [17].

Example 1.1. An important example of a hyperbolic polynomial is $\det(X)$, where $X = (x_{ij})_{i,j=1}^n$ is a matrix of variables where we impose $x_{ij} = x_{ji}$. Note that $t \mapsto \det(tI - X)$ where $I = \mathrm{diag}(1, \ldots, 1)$, is the characteristic polynomial of a symmetric matrix so it has only real zeros. Hence $\det(X)$ is a hyperbolic polynomial with respect to I, and its hyperbolicity cone is the cone of positive semidefinite matrices. Note that the hyperbolic rank of a symmetric matrix X with respect to I coincides with the usual notion of rank for matrices.

Denote the directional derivative of $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ with respect to $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$ by

$$D_{\mathbf{v}} h(\mathbf{x}) = \sum_{k=1}^{n} v_k \frac{\partial h}{\partial x_k}(\mathbf{x}).$$

The following lemma is well-known and essentially follows from the identity $D_{\mathbf{v}} h(t) = \frac{d}{dt} h(t\mathbf{v} + \mathbf{x})|_{t=0}$ together with Rolle's theorem (see [17] [35]).

Lemma 1.2. *Let h be a hyperbolic polynomial and let $\mathbf{v} \in \Lambda_+$ be such that $D_{\mathbf{v}}h \not\equiv 0$. Then $D_{\mathbf{v}}h$ is hyperbolic with $\Lambda_+(h, \mathbf{v}) \subseteq \Lambda_+(D_{\mathbf{v}}h, \mathbf{v})$.*

A class of polynomials which is intimately connected to hyperbolic polynomials is the class of stable polynomials. A polynomial $P(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ is *stable* if $P(z_1, \ldots, z_n) \neq 0$ whenever $\mathrm{Im}(z_j) > 0$ for all $1 \leq j \leq n$. A stable polynomial $P(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is said to be *real stable*. Hyperbolic and stable polynomials are related as follows, see [3, Prop. 1.1].

Lemma 1.3. *Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogenous polynomial. Then P is stable if and only if P is hyperbolic with $\mathbb{R}^n_+ \subseteq \Lambda_+(P)$.*

The next theorem which follows (see [27]) from a theorem of Helton and Vinnikov [21] proved the Lax conjecture (after Peter Lax 1958 [25]).

Theorem 1.4 (Helton-Vinnikov [21])**.** *Suppose that $h(x, y, z)$ is of degree d and hyperbolic with respect to $e = (e_1, e_2, e_3)^T$. Suppose further that h is normalized such that $h(e) = 1$. Then there are symmetric $d \times d$ matrices A, B, C such that $e_1A + e_2B + e_3C = I$ and*

$$h(x, y, z) = \det(xA + yB + zC).$$

Remark 1.5. The exact analogue of Theorem 1.4 fails for $n > 3$ variables. This may be seen by comparing dimensions. The set of polynomials on $\mathbb{R}^n$ of the form $\det(x_1A_1 + \cdots x_nA_n)$ with A_i a $d \times d$ symmetric matrix for $1 \leq i \leq n$, has dimension at most $n\binom{d+1}{2}$ (as an algebraic image $(A_1, \ldots, A_n) \mapsto \det(x_1A_1 + \cdots x_nA_n)$ of a vector space of the same dimension) whereas the set of hyperbolic polynomials of degree d on $\mathbb{R}^n$ has non-empty interior in the space of homogeneous polynomials of degree d in n variables (see [34]) and therefore has the same dimension $\binom{n+d-1}{d}$.

A convex cone in $\mathbb{R}^n$ is *spectrahedral* if it is of the form

$$\left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_iA_i \ \text{ is positive semidefinite} \right\}$$

where A_i, $i = 1, \ldots, n$ are symmetric matrices such that there exists a vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n y_iA_i$ positive definite. It is easy to see that spectrahedral cones are hyperbolicity cones. A major open question asks if the converse is true.

Conjecture 1.6 (Generalized Lax conjecture [21, 37])**.** *All hyperbolicity cones are spectrahedral.*

Remark 1.7. An important consequence of Conjecture 1.6 in the field of optimization is that hyperbolic programming [35] is the same as semidefinite programming.

We may reformulate Conjecture 1.6 as follows, see [21, 37]. The hyperbolicity cone of $h(\mathbf{x})$ with respect to $\mathbf{e} = (e_1, \ldots, e_n)$ is spectrahedral if there is a homogeneous polynomial $q(\mathbf{x})$ and real symmetric matrices $A_1, \ldots, A_n$ of the same size such that

$$q(\mathbf{x})h(\mathbf{x}) = \det\left(\sum_{i=1}^n x_iA_i\right) \tag{1}$$

where $\Lambda_{++}(h, \mathbf{e}) \subseteq \Lambda_{++}(q, \mathbf{e})$ and $\sum_{i=1}^n e_iA_i$ is positive definite. If we can choose $q(\mathbf{x}) \equiv 1$, then we say that $h(\mathbf{x})$ admits a *definite determinantal representation*.

- Conjecture 1.6 is true for $n = 3$ by Theorem 1.4,
- Conjecture 1.6 is true for homogeneous cones [9], i.e., cones for which the automorphism group acts transitively on its interior,
- Conjecture 1.6 is true for quadratic polynomials, see e.g. [33],
- Conjecture 1.6 is true for elementary symmetric polynomials, see [5],
- Weaker versions of Conjecture 1.6 are true for smooth hyperbolic polynomials, see [23, 32].
- Stronger algebraic versions of Conjecture 1.6 are false, see [1, 4].

The paper is organized as follows. In Section 2 we prove Conjecture 1.6 for a multivariate generalization of the matching polynomial (Theorem 2.16). We also show that this implies Conjecture 1.6 for elementary symmetric polynomials (Theorem 2.19). Our result may therefore be viewed as a generalization of [5]. In Section 3 we generalize further to a multivariate version of the independence polynomial using a recent divisibility relation of Leake and Ryder [26] (Theorem 3.9). The variables of the homogenized independence polynomial do not fully correspond combinatorially (under the line graph operation) to the more refined homogeneous matching polynomial. The restriction of Theorem 3.9 to line graphs is therefore weaker than Theorem 2.16. Finally, in Section 4 we consider a hyperbolic convolution of determinant polynomials generalizing an identity of Godsil and Gutman [14] which asserts that the expected characteristic polynomial of a random signing of the adjacency matrix of a graph is equal to its matching polynomial.

Unless stated otherwise, $G = (V(G), E(G))$ denotes a simple undirected graph. We shall adopt the following notational conventions.

- $\mathrm{Sym}(S)$ denotes the symmetric group on the set S. Write $\mathfrak{S}_n = \mathrm{Sym}([n])$.
- $N_G(u) = \{v \in V(G) : (u, v) \in E(G)\}$ (resp. $N_G[u] = N_G(u) \cup \{u\}$) denotes the open (resp. closed) neighbourhood of $u \in V(G)$.
- If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S.
- $G \sqcup H$ denotes the disjoint union of the graphs G and H.
- $\mathbb{R}^S = \{(a_s)_{s\in S} : a_s \in \mathbb{R}\} \cong \mathbb{R}^{|S|}$.
- $\mathbb{R}^G = \mathbb{R}^{V(G)} \times \mathbb{R}^{E(G)}$.

2. Hyperbolicity cones of multivariate matching polynomials

A *k-matching* in G is a subset $M \subseteq E(G)$ of k edges, no two of which have a vertex in common. Let $\mathcal{M}(G)$ denote the set of all matchings in G and let $m(G, k)$ denote the number of k-matchings in G. By convention $m(G, 0) = 1$. We denote by $V(M)$ the set of vertices contained in the matching M. If $|V(M)| = |V(G)|$, then we call M a *perfect matching*. The (univariate) *matching polynomial* is defined by

$$\mu(G, t) = \sum_{k\geq 0} (-1)^k m(G, k) t^{|V(G)|-2k}.$$

Note that this is indeed a polynomial since $m(G, k) = 0$ for $k > \frac{|V(G)|}{2}$. Heilmann and Lieb [20] studied the following multivariate version of the matching polynomial with variables $\mathbf{x} = (x_i)_{i\in V}$ and non-negative weights $\boldsymbol{\lambda} = (\lambda_e)_{e\in E}$,

$$\mu_{\boldsymbol{\lambda}}(G, \mathbf{x}) = \sum_{M\in\mathcal{M}(G)} (-1)^{|M|} \prod_{ij\in M} \lambda_{ij} x_i x_j.$$

Remark 2.1. Note that $t^{|V(G)|}\mu_{\mathbf{1}}(G, t^{-1}\mathbf{1}) = \mu(G, t)$, where $\mathbf{1} = (1, \ldots, 1)$.

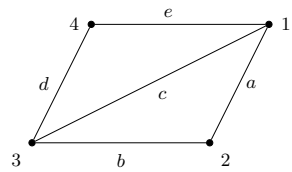


FIGURE 1.

Theorem 2.2 (Heilmann-Lieb [20])**.**
If $\boldsymbol{\lambda} = (\lambda_e)_{e\in E}$ is a sequence of non-negative edge weights, then $\mu_{\boldsymbol{\lambda}}(G, \mathbf{x})$ is stable.

Remark 2.3. A quick way to see Theorem 2.2 is to observe that

$$\mathrm{MAP}\left(\prod_{e=(i,j)\in E(G)} (1 - \lambda_e x_i x_j)\right) = \mu_{\boldsymbol{\lambda}}(G, \mathbf{x})$$

where $\mathrm{MAP} : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ is the stability preserving linear map taking a multivariate polynomial to its multiaffine part (see [2]). Since real stable univariate polynomials are real-rooted the Heilmann-Lieb theorem (together with Remark 2.1) implies the real-rootedness of $\mu(G, t)$.

We will consider the following homogeneous multivariate version of the matching polynomial.

Definition 2.4. Let $\mathbf{x} = (x_v)_{v\in V}$ and $\mathbf{w} = (w_e)_{e\in E}$ be indeterminates. Define the *homogeneous multivariate matching polynomial* $\mu(G, \mathbf{x} \oplus \mathbf{w}) \in \mathbb{R}[\mathbf{x}, \mathbf{w}]$ by

$$\mu(G, \mathbf{x} \oplus \mathbf{w}) = \sum_{M\in\mathcal{M}(G)} (-1)^{|M|} \prod_{v\notin V(M)} x_v \prod_{e\in M} w_e^2.$$

Example 2.5. The homogeneous multivariate matching polynomial of the graph G in Figure 1 is given by

$$\mu(G, \mathbf{x} \oplus \mathbf{w}) = x_1x_2x_3x_4 - x_3x_4w_a^2 - x_1x_4w_b^2 - x_2x_4w_c^2 - x_1x_2w_d^2 - x_2x_3w_e^2 + w_a^2w_d^2 + w_b^2w_e^2.$$

Remark 2.6. Note that $\mu(G, t\mathbf{1} \oplus \mathbf{1}) = \mu(G, t)$ and that $\mu(G, \mathbf{0} \oplus \mathbf{w})$ is the multivariate matching polynomial restricted to perfect matchings.

In this section we prove Conjecture 1.6 in the affirmative for the polynomials $\mu(G, \mathbf{x} \oplus \mathbf{w})$. We first assert that $\mu(G, \mathbf{x} \oplus \mathbf{w})$ is indeed a hyperbolic polynomial.

Lemma 2.7. *The polynomial $\mu(G, \mathbf{x} \oplus \mathbf{w})$ is hyperbolic with respect to $\mathbf{e} = \mathbf{1} \oplus \mathbf{0}$.*

Proof. Clearly $\mu(G, \mathbf{1} \oplus \mathbf{0}) = 1 \neq 0$. Let $\mathbf{x} \oplus \mathbf{w} \in \mathbb{R}^G$ and $\lambda_e = w_e^2$ for all $e \in E(G)$. Then

$$\mu(G, t\mathbf{e} - \mathbf{x} \oplus \mathbf{w}) = \left(\prod_{v\in V}(t - x_v)\right) \mu_{\boldsymbol{\lambda}}(G, (t\mathbf{1} - \mathbf{x})^{-1}).$$

Since $\mu_{\boldsymbol{\lambda}}(G, \mathbf{x})$ is real stable by Heilmann-Lieb theorem it follows that the right hand side is real-rooted. Hence $\mu(G, \mathbf{x} \oplus \mathbf{w})$ is hyperbolic with respect to $\mathbf{e} = \mathbf{1} \oplus \mathbf{0}$.

□

Analogues of the standard recursions for the univariate matching polynomial (see [13, Thm 1.1]) also hold for $\mu(G, \mathbf{x} \oplus \mathbf{w})$. In particular the following recursion is used frequently so we give details.

Lemma 2.8. *Let $u \in V(G)$. Then the homogeneous multivariate matching polynomial satisfies the recursion*

$$\mu(G, \mathbf{x} \oplus \mathbf{w}) = x_u \mu(G \setminus u, \mathbf{x} \oplus \mathbf{w}) - \sum_{v \in N(u)} w_{uv}^2 \mu((G \setminus u) \setminus v, \mathbf{x} \oplus \mathbf{w}).$$

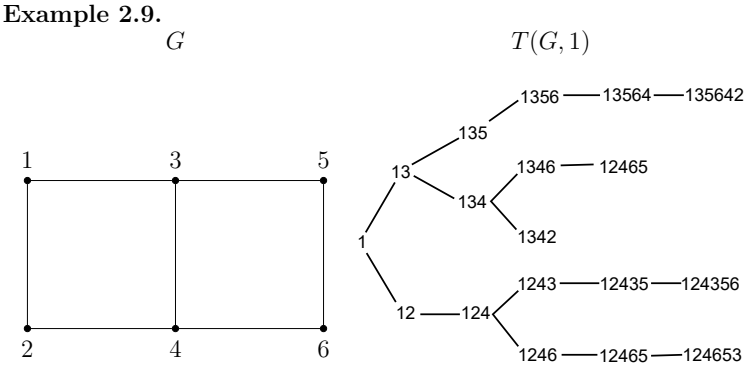
Proof. The identity follows by partitioning the matchings $M \in \mathcal{M}(G)$ into two parts depending on whether $u \in V(M)$ or $u \notin V(M)$. Let $f_G(M) = \prod_{v \notin V(M)} x_v \prod_{e \in M} w_e^2$. Then

$$\begin{aligned}
\mu(G, \mathbf{x} \oplus \mathbf{w}) &= \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} f_G(M) \\
&= \sum_{\substack{M \in \mathcal{M}(G) \\ u \notin V(M)}} (-1)^{|M|} f_G(M) + \sum_{\substack{M \in \mathcal{M}(G) \\ u \in V(M)}} (-1)^{|M|} f_G(M) \\
&= x_u \sum_{M \in M(G \setminus u)} (-1)^{|M|} f_{G \setminus u}(M) + \sum_{v \in N(u)} \sum_{\substack{M \in \mathcal{M}(G) \\ uv \in M}} (-1)^{|M|} f_G(M) \\
&= x_u \mu(G \setminus u, \mathbf{x} \oplus \mathbf{w}) - \sum_{v \in N(u)} w_{uv}^2 \sum_{M \in M((G \setminus u) \setminus v)} (-1)^{|M|} f_{(G \setminus u) \setminus v}(M) \\
&= x_u \mu(G \setminus u, \mathbf{x} \oplus \mathbf{w}) - \sum_{v \in N(u)} w_{uv}^2 \, \mu((G \setminus u) \setminus v, \mathbf{x} \oplus \mathbf{w}).
\end{aligned}$$

□

Let G be a graph and $u \in V(G)$. The *path tree* $T(G, u)$ is the tree with vertices labelled by simple paths in G (i.e. paths with no repeated vertices) starting at u and where two vertices are joined by an edge if one vertex is labelled by a maximal subpath of the other.

Example 2.9.



Definition 2.10. Let G be a graph and $u \in V(G)$. Let $\phi : \mathbb{R}^{T(G,u)} \to \mathbb{R}^G$ denote the linear change of variables defined by

$$x_p \mapsto x_{i_k},$$
$$w_{pp'} \mapsto w_{i_k i_{k+1}},$$

where $p = i_1 \cdots i_k$ and $p' = i_1 \cdots i_k i_{k+1}$ are adjacent vertices in $T(G,u)$. For every subforest $T \subseteq T(G,u)$, define the polynomial

$$\eta(T, \mathbf{x} \oplus \mathbf{w}) = \mu(T, \phi(\mathbf{x}' \oplus \mathbf{w}'))$$

where $\mathbf{x}' = (x_p)_{p \in V(T)}$ and $\mathbf{w}' = (w_e)_{e \in E(T)}$.

Remark 2.11. Note that $\eta(T, \mathbf{x} \oplus \mathbf{w})$ is a polynomial in variables $\mathbf{x} = (x_v)_{v \in V(G)}$ and $\mathbf{w} = (w_e)_{e \in E(G)}$.

For the univariate matching polynomial we have the following rather unexpected divisibility relation due to Godsil [12],

$$\frac{\mu(G \setminus u, t)}{\mu(G, t)} = \frac{\mu(T(G,u) \setminus u, t)}{\mu(T(G,u), t)}.$$

Below we prove a multivariate analogue of this fact. A similar multivariate analogue was also noted independently by Leake and Ryder [26]. In fact they were able to find a further generalization to independence polynomials of simplicial graphs. We will revisit their results in Section 3. The arguments all closely resemble Godsil's proof for the univariate matching polynomial. For the convenience of the reader we provide the details in our setting.

Lemma 2.12. *Let $u \in V(G)$. Then*

$$\frac{\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w})}{\mu(G, \mathbf{x} \oplus \mathbf{w})} = \frac{\eta(T(G,u) \setminus u, \mathbf{x} \oplus \mathbf{w})}{\eta(T(G,u), \mathbf{x} \oplus \mathbf{w})}.$$

Proof. If G is a tree, then $\mu(G, \mathbf{x} \oplus \mathbf{w}) = \eta(T(G,u), \mathbf{x} \oplus \mathbf{w})$ and $\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w}) = \eta(T(G,u) \setminus u, \mathbf{x} \oplus \mathbf{w})$ so the lemma holds. In particular the lemma holds for all graphs with at most two vertices. We now argue by induction on the number of vertices of G. We first claim that

$$\frac{\eta(T(G,u) \setminus \{u, uv\}, \mathbf{x} \oplus \mathbf{w})}{\eta(T(G,u) \setminus u, \mathbf{x} \oplus \mathbf{w})} = \frac{\eta(T(G \setminus u, v) \setminus v, \mathbf{x} \oplus \mathbf{w})}{\eta(T(G \setminus u, v), \mathbf{x} \oplus \mathbf{w})}.$$

Let $v \in N(u)$. By examining the path tree $T(G,u)$ we note the following isomorphisms

$$T(G,u) \setminus u \cong \bigsqcup_{n \in N(u)} T(G \setminus u, n),$$

$$T(G,u) \setminus \{u, uv\} \cong \left(\bigsqcup_{\substack{n \in N(u) \\ n \neq v}} T(G \setminus u, n) \right) \sqcup T(G \setminus u, v) \setminus v,$$

following from the fact that $T(G \setminus u, n)$ is isomorphic to the connected component of $T(G,u) \setminus u$ which contains the path un in G. By the definition of ϕ and the general multiplicative identity

$$\mu(G \sqcup H, \mathbf{x} \oplus \mathbf{w}) = \mu(G, \mathbf{x} \oplus \mathbf{w}) \mu(H, \mathbf{x} \oplus \mathbf{w}),$$

the above isomorphisms translate to the following identities

$$\eta(T(G,u)\setminus u,\mathbf{x}\oplus\mathbf{w}) = \prod_{n\in N(u)} \eta(T(G\setminus u,n),\mathbf{x}\oplus\mathbf{w}),$$

$$\eta(T(G,u)\setminus\{u,uv\},\mathbf{x}\oplus\mathbf{w}) = \eta(T(G\setminus u,v)\setminus v,\mathbf{x}\oplus\mathbf{w}) \prod_{\substack{n\in N(u)\\ n\neq v}} \eta(T(G\setminus u,n),\mathbf{x}\oplus\mathbf{w}),$$

from which the claim follows. By Lemma 2.8, induction, above claim and the definition of ϕ we finally get

$$\begin{aligned}
\frac{\mu(G,\mathbf{x}\oplus\mathbf{w})}{\mu(G\setminus u,\mathbf{x}\oplus\mathbf{w})} &= \frac{x_u\mu(G\setminus u,\mathbf{x}\oplus\mathbf{w}) - \sum_{v\in N(u)} w_{uv}^2\mu(G\setminus\{u,v\},\mathbf{x}\oplus\mathbf{w})}{\mu(G\setminus u,\mathbf{x}\oplus\mathbf{w})}\\
&= x_u - \sum_{v\in N(u)} w_{uv}^2 \frac{\mu((G\setminus u)\setminus v,\mathbf{x}\oplus\mathbf{w})}{\mu(G\setminus u,\mathbf{x}\oplus\mathbf{w})}\\
&= x_u - \sum_{v\in N(u)} w_{uv}^2 \frac{\eta(T(G\setminus u,v)\setminus v,\mathbf{x}\oplus\mathbf{w})}{\eta(T(G\setminus u,v),\mathbf{x}\oplus\mathbf{w})}\\
&= x_u - \sum_{v\in N(u)} w_{uv}^2 \frac{\eta(T(G,u)\setminus\{u,uv\},\mathbf{x}\oplus\mathbf{w})}{\eta(T(G,u)\setminus u,\mathbf{x}\oplus\mathbf{w})}\\
&= \frac{\eta(T(G,u),\mathbf{x}\oplus\mathbf{w})}{\eta(T(G,u)\setminus u,\mathbf{x}\oplus\mathbf{w})}
\end{aligned}$$

which is the reciprocal of the desired identity.

□

Lemma 2.13. *Let $u\in V(G)$. Then $\mu(G,\mathbf{x}\oplus\mathbf{w})$ divides $\eta(T(G,u),\mathbf{x}\oplus\mathbf{w})$.*

Proof. The argument is by induction on the number of vertices of G. Deleting the root u of $T(G,u)$ we get a forest with $|N(u)|$ disjoint components isomorphic to $T(G\setminus u,v)$ respectively for $v\in N(u)$. This gives

$$\eta(T(G,u)\setminus u,\mathbf{x}\oplus\mathbf{w}) = \prod_{v\in N(u)} \eta(T(G\setminus u,v),\mathbf{x}\oplus\mathbf{w}). \tag{2}$$

Therefore $\eta(T(G\setminus u,v),\mathbf{x}\oplus\mathbf{w})$ divides $\eta(T(G,u)\setminus u,\mathbf{x}\oplus\mathbf{w})$ for all $v\in N(u)$. By induction $\mu(G\setminus u,\mathbf{x}\oplus\mathbf{w})$ divides $\eta(T(G\setminus u,v),\mathbf{x}\oplus\mathbf{w})$ for all $v\in N(u)$. Hence $\mu(G\setminus u,\mathbf{x}\oplus\mathbf{w})$ divides $\eta(T(G,u)\setminus u,\mathbf{x}\oplus\mathbf{w})$, so by Lemma 2.12, $\mu(G,\mathbf{x}\oplus\mathbf{w})$ divides $\eta(T(G,u),\mathbf{x}\oplus\mathbf{w})$.

□

In [14] Godsil and Gutman proved the following relationship between the univariate matching polynomial $\mu(G,t)$ of a graph G and the characteristic polynomial $\chi(A,t)$ of its adjacency matrix A

$$\chi(A,t) = \sum_C (-2)^{\mathrm{comp}(C)}\mu(G\setminus C,t),$$

where the sum ranges over all subgraphs C (including $C=\emptyset$) in which each component is a cycle of degree 2 and $\mathrm{comp}(C)$ is the number of connected components of C. In particular if

T is a tree, then the only such subgraph is $C = \emptyset$ and therefore

$$\chi(A, t) = \mu(T, t).$$

Next we will derive a multivariate analogue of this relationship for trees.

Lemma 2.14. *Let $T = (V, E)$ be a tree. Then $\mu(T, \mathbf{x} \oplus \mathbf{w})$ has a definite determinantal representation.*

Proof. Let $X = \operatorname{diag}(\mathbf{x})$ and $A = (A_{ij})$ be the matrix

$$A_{ij} = \begin{cases} w_{ij} & \text{if } ij \in E(T) \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in V(T)$. If $\sigma \in \operatorname{Sym}(V(T))$ is an involution (i.e $\sigma^2 = id$), then clearly $A_{j\sigma(j)} = w_{j\sigma(j)} = A_{\sigma(j)\sigma^2(j)}$ since A is symmetric. Hence by acyclicity of trees we have that

$$\begin{aligned}
\det(X + A) &= \sum_{\sigma \in \operatorname{Sym}(V(T))} \operatorname{sgn}(\sigma) \prod_{i \in V(T)} (X_{i\sigma(i)} + A_{i\sigma(i)}) \\
&= \sum_{S \subseteq V(T)} \prod_{i \in V(T) \setminus S} x_i \sum_{\substack{\sigma \in \operatorname{Sym}(S) \\ \sigma(j) \neq j \, \forall j \in S \\ \sigma^2 = \mathrm{id}}} \operatorname{sgn}(\sigma) \prod_{j \in S} A_{j\sigma(j)} \\
&= \sum_{S \subseteq V(T)} \prod_{i \in V(T) \setminus S} x_i \sum_{\substack{M \in \mathcal{M}(T[S]) \\ M \text{ perfect}}} (-1)^{|M|} \prod_{jk \in M} w_{jk}^2 \\
&= \sum_{M \in \mathcal{M}(T)} (-1)^{|M|} \prod_{i \notin V(M)} x_i \prod_{jk \in M} w_{jk}^2 \\
&= \mu(T, \mathbf{x} \oplus \mathbf{w}).
\end{aligned}$$

Write

$$X + A = \sum_{i \in V(T)} x_i E_{ii} + \sum_{ij \in E(T)} w_{ij}(E_{ij} + E_{ji}),$$

where $\{E_{ij} : i, j \in V(T)\}$ denotes the standard basis for the vector space of all real $|V(T)| \times |V(T)|$ matrices. Evaluated at $\mathbf{e} = \mathbf{1} \oplus \mathbf{0}$ we obtain the identity matrix I which is positive definite.

□

Remark 2.15. The proof of Lemma 2.14 is not dependent on T being connected so the statement remains valid for arbitrary undirected acyclic graphs (i.e. forests).

We now have all the ingredients to prove our main theorem.

Theorem 2.16. *The hyperbolicity cone of $\mu(G, \mathbf{x} \oplus \mathbf{w})$ is spectrahedral.*

Proof. The proof is by induction on the number of vertices of G. For the base case we have $\mu(G, \mathbf{x} \oplus \mathbf{w}) = x_v$, so $\Lambda_+ = \{x \in \mathbb{R} : x \geq 0\}$ which is clearly spectrahedral. Assume G contains more than one vertex. If $G = G_1 \sqcup G_2$ for some non-empty graphs G_1, G_2, then

$\Lambda_{++}(\mu(G_i, \mathbf{x} \oplus \mathbf{w}))$ is spectrahedral by induction for $i = 1, 2$. Therefore

$$\begin{aligned}\Lambda_{++}(\mu(G, \mathbf{x} \oplus \mathbf{w})) &= \Lambda_{++}(\mu(G_1 \sqcup G_2, \mathbf{x} \oplus \mathbf{w})) \\ &= \Lambda_{++}\left(\mu(G_1, \mathbf{x} \oplus \mathbf{w})\mu(G_1, \mathbf{x} \oplus \mathbf{w})\right) \\ &= \Lambda_{++}(\mu(G_1, \mathbf{x} \oplus \mathbf{w})) \cap \Lambda_{++}(\mu(G_2, \mathbf{x} \oplus \mathbf{w}))\end{aligned}$$

showing that $\Lambda_{++}(\mu(G, \mathbf{x} \oplus \mathbf{w}))$ is spectrahedral. We may therefore assume G is connected. Let $u \in V(G)$. Since G is connected and has size greater than one, $N(u) \neq \emptyset$. By Lemma 2.13 we may define the polynomial

$$q_{G,u}(\mathbf{x} \oplus \mathbf{w}) = \frac{\eta(T(G, u), \mathbf{x} \oplus \mathbf{w})}{\mu(G, \mathbf{x} \oplus \mathbf{w})}$$

for each graph G and $u \in V(G)$. We want to show that

$$\Lambda_{++}(\mu(G, \mathbf{x} \oplus \mathbf{w})) \subseteq \Lambda_{++}(q_{G,u}(\mathbf{x} \oplus \mathbf{w})).$$

By Lemma 2.12 we have that

$$q_{G,u}(\mathbf{x} \oplus \mathbf{w})\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w}) = \eta(T(G, u) \setminus u, \mathbf{x} \oplus \mathbf{w}).$$

Fixing $v \in N(u)$ it follows using (2) that

$$\begin{aligned}\frac{q_{G,u}(\mathbf{x} \oplus \mathbf{w})}{q_{G\setminus u,v}(\mathbf{x} \oplus \mathbf{w})} &= \frac{q_{G,u}(\mathbf{x} \oplus \mathbf{w})\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w})}{q_{G\setminus u,v}(\mathbf{x} \oplus \mathbf{w})\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w})} \\ &= \frac{\eta(T(G, u) \setminus u, \mathbf{x} \oplus \mathbf{w})}{\eta(T(G \setminus u, v), \mathbf{x} \oplus \mathbf{w})} \\ &= \prod_{w \in N(u)\setminus v} \eta(T(G \setminus u, w), \mathbf{x} \oplus \mathbf{w}) \\ &= \prod_{w \in N(u)\setminus v} q_{G\setminus u,w}(\mathbf{x} \oplus \mathbf{w})\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w}).\end{aligned}$$

Note that

$$\frac{\partial}{\partial x_u}\mu(G, \mathbf{x} \oplus \mathbf{w}) = \mu(G \setminus u, \mathbf{x} \oplus \mathbf{w}).$$

Therefore by Lemma 1.2,

$$\Lambda_{++}(\mu(G, \mathbf{x} \oplus \mathbf{w})) \subseteq \Lambda_{++}(\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w})) \subseteq \Lambda_{++}(q_{G\setminus u,w}(\mathbf{x} \oplus \mathbf{w}))$$

for all $w \in N(u)$ where the last inclusion follows by inductive hypothesis. Hence

$$\begin{aligned}\Lambda_{++}(\mu(G, \mathbf{x} \oplus \mathbf{w})) &\subseteq \bigcap_{w \in N(u)} \Lambda_{++}(q_{G\setminus u,w}(\mathbf{x} \oplus \mathbf{w})) \cap \Lambda_{++}\left(\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w})\right) \\ &= \Lambda_{++}\left(q_{G\setminus u,v}(\mathbf{x} \oplus \mathbf{w}) \prod_{w \in N(u)\setminus v} q_{G\setminus u,w}(\mathbf{x} \oplus \mathbf{w})\mu(G \setminus u, \mathbf{x} \oplus \mathbf{w})\right) \\ &= \Lambda_{++}(q_{G,u}(\mathbf{x} \oplus \mathbf{w})).\end{aligned}$$

Finally by Lemma 2.14, $\eta(T(G, u), \mathbf{x} \oplus \mathbf{w})$ has a definite determinantal representation. Hence the theorem follows by induction.

□

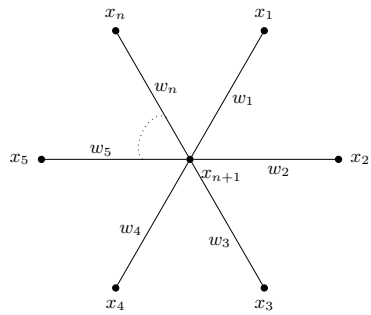


FIGURE 2. The star graph S_n labelled by vertex and edge variables

Remark 2.17. To show that a hyperbolic polynomial h has a spectrahedral hyperbolicity cone it is by Theorem 2.16 sufficient to show that h can be realized as a factor of a matching polynomial $\mu(G, \mathbf{x} \oplus \mathbf{w})$ with $\Lambda_{++}(h, \mathbf{e}) \subseteq \Lambda_{++}\left(\frac{\mu(G,\mathbf{x}\oplus\mathbf{w})}{h}, \mathbf{e}\right)$ (possibly after a linear change of variables).

The *elementary symmetric polynomial* $e_d(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d in n variables is defined by

$$e_d(\mathbf{x}) = \sum_{\substack{S \subseteq [n] \\ |S|=d}} \prod_{i \in S} x_i.$$

The polynomials $e_d(\mathbf{x})$ are hyperbolic (in fact stable) as a consequence of e.g Grace-Walsh-Szegő theorem (see [31, Thm 15.4]).

Example 2.18. The star graph, denoted S_n, is given by the complete bipartite graph $K_{1,n}$ with $n+1$ vertices. As an application of Theorem 2.16 we show that several well-known instances of hyperbolic polynomials have spectrahedral hyperbolicity cones by realizing them as factors of the multivariate matching polynomial of S_n under some linear change of variables. With notation as in Figure 2, using the recursion in Lemma 2.8, the multivariate matching polynomial of S_n is given by

$$\mu(S_n, \mathbf{x} \oplus \mathbf{w}) = \prod_{i=1}^{n+1} x_i - \sum_{i=1}^{n} w_i^2 \prod_{\substack{j=1 \\ j \neq i}}^{n} x_j.$$

(i) For $h(\mathbf{x}) = e_{n-1}(\mathbf{x})$ consider the linear change of variables $x_n \mapsto -x_n$ and $w_i \mapsto x_n$ for $i = 1, \ldots, n-1$. Then $\mu(S_{n-1}, \mathbf{x} \oplus \mathbf{w}) \mapsto -x_n e_{n-1}(\mathbf{x})$. Clearly $\Lambda_{++}(e_{n-1}(\mathbf{x}), \mathbf{1}) \subseteq \Lambda_{++}(x_n, \mathbf{1})$. The spectrahedrality of $\Lambda_{++}(e_{n-1}(\mathbf{x}), \mathbf{1})$ was first proved by Sanyal in [36].

(ii) For $h(\mathbf{x}) = e_2(\mathbf{x})$ consider the linear change of variables $x_i \mapsto e_1(x_1, \ldots, x_n)$ and $w_i \mapsto x_i$ for $i = 1, \ldots, n+1$. Then $\mu(S_n, \mathbf{x} \oplus \mathbf{w}) \mapsto 2e_1(\mathbf{x})^{n-1} e_2(\mathbf{x})$. Since $D_{\mathbf{1}} e_2(\mathbf{x}) = (n-1)e_1(\mathbf{x})$, Lemma 1.2 implies that $\Lambda_{++}(e_2(\mathbf{x}), \mathbf{1}) \subseteq \Lambda_{++}(e_1(\mathbf{x}), \mathbf{1})$. Hence $\Lambda_{++}(e_2(\mathbf{x}), \mathbf{1})$ is spectrahedral.

(iii) Let $h(\mathbf{x}) = x_n^2 - x_{n-1}^2 - \cdots - x_1^2$. Recall that $\Lambda_{++}(h, \mathbf{e})$ is the *Lorentz cone* where $\mathbf{e} = (0, \ldots, 0, 1)$. Consider the linear change of variables $x_i \mapsto x_n$ and $w_i \mapsto x_i$

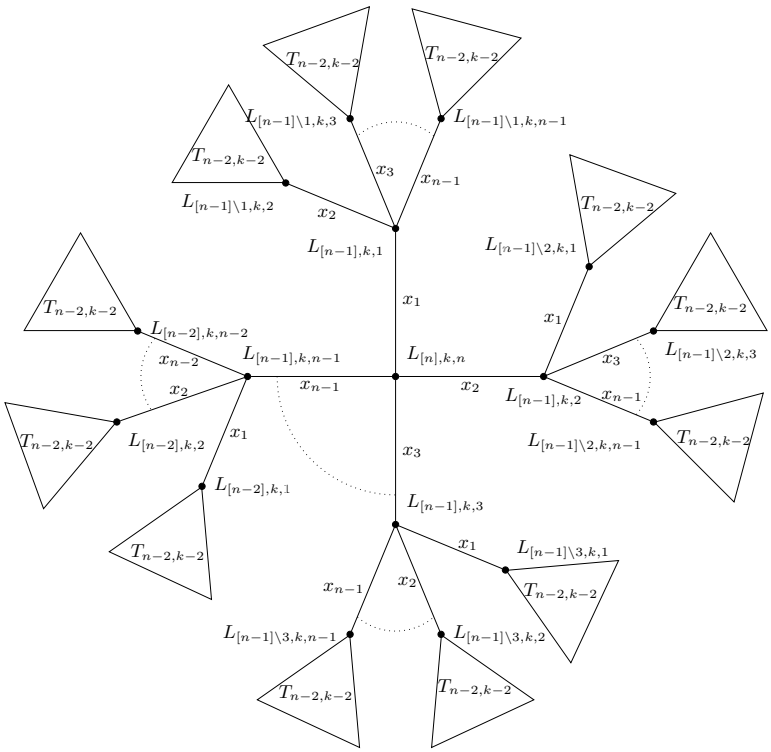


FIGURE 3. The length k-truncated path tree $T_{n,k}$ of K_n labelled by linear change of variables.

for $i = 1, \ldots, n$. Then $\mu(S_{n-1}, \mathbf{x} \oplus \mathbf{w}) \mapsto x_n^n - \sum_{i=1}^{n-1} x_i^2 x_n^{n-2} = x_n^{n-2} h(\mathbf{x})$. Clearly $\Lambda_{++}(h, \mathbf{e}) \subseteq \Lambda_{++}(x_n^{n-2}, \mathbf{e})$. Hence the Lorentz cone is spectrahedral. Of course this (and the preceding example) also follow from the fact that all quadratic hyperbolic polynomials have spectrahedral hyperbolicity cone [33].

Hyperbolicity cones of elementary symmetric polynomials have been studied by Zinchenko [39], Sanyal [36] and Brändén [5]. Brändén proved that all hyperbolicity cones of elementary symmetric polynomials are spectrahedral. As an application of Theorem 2.16 we give a new proof of this fact using matching polynomials.

Theorem 2.19. *Hyperbolicity cones of elementary symmetric polynomials are spectrahedral.*

Proof. For a subset $S \subseteq [n]$ we shall use the notation

$$e_k(S) = \sum_{\substack{T \subseteq S \\ |T|=k}} \prod_{j \in T} x_j.$$

We show that $e_k(\mathbf{x}) = e_k([n])$ divides the multivariate matching polynomial of the length k-truncated path tree $T_{n,k}$ of the complete graph K_n rooted at a vertex v after a linear change

of variables. Let $(C_k)_{k\geq 0}$ denote the real sequence defined by

$$C_0 = 1,\ C_1 = 1,\ C_k = \prod_{j=0}^{\lfloor k/2\rfloor -1} \frac{k-2j}{k-2j-1} \quad \text{for } k \geq 2,$$

so that

$$C_k C_{k-1} = k \text{ for all } k \geq 1.$$

Consider the family

$$M_{S,k,i} = \mu(T_{|S|,k}, \phi_{S,k,i}(\mathbf{x} \oplus \mathbf{w}))$$

of multivariate matching polynomials where $i \in S$, $k \in \mathbb{N}$ and $\phi_{S,k,i}$ is the linear change of variables defined recursively (see Fig 3) via

(i) $\phi_{S,0,i}$ is the map $x_v \mapsto e_1(S)$ for all $S \subseteq [n]$ and $i \in S$.
(ii) $x_v \mapsto L_{S,k,i}$ if $k \geq 1$ where

$$L_{S,k,i} = \frac{1}{C_{k-1}} e_1(S \setminus i) + C_k x_i$$

and x_v is the variable corresponding to the root of $T_{n,k}$.
(iii) $w_{e_j} \mapsto x_j$ for $j \in S \setminus i$ where w_{e_j} are the variables corresponding to the edges e_j incident to the root of $T_{n,k}$.
(iv) For each $j \in S \setminus i$ make recursively the linear substitutions $\phi_{S\setminus i,k-1,j}$ respectively to the variables corresponding to the j-indexed copies of the subtrees of $T_{n,k}$ isomorphic to $T_{n-1,k-1}$.

We claim

$$\begin{aligned} M_{S,0,i} &= e_1(S), \\ M_{S,k,i} &= \frac{C_k e_k(S)}{e_{k-1}(S \setminus \{i\})} \prod_{j \in S\setminus\{i\}} M_{S\setminus i,k-1,j} \end{aligned}$$

for all $S \subseteq [n]$, $i \in S$ and $k \in \mathbb{N}$ by induction on k. Clearly $M_{S,0,i} = e_1(S)$ since $\mu(T_{n,0}, \mathbf{x} \oplus \mathbf{w}) = x_v$. By Lemma 2.8 and induction we have

$$\begin{aligned} &M_{S,k,i} \\ &= L_{S,k,i} \prod_{s \in S\setminus\{i\}} M_{S\setminus i,k-1,s} - \sum_{j \in S\setminus i} x_j^2 \prod_{s \in S\setminus\{i,j\}} M_{S\setminus i,k-1,s}\, M_{S\setminus\{i,j\},k-2,s} \\ &= \left(\frac{1}{C_{k-1}} e_1(S \setminus i) + C_k x_i - \sum_{j \in S\setminus\{i\}} x_j^2 \frac{e_{k-2}(S \setminus \{i,j\})}{C_{k-1} e_{k-1}(S \setminus i)} \right) \prod_{s \in S\setminus\{i\}} M_{S\setminus i,k-1,s} \\ &= \frac{1}{e_{k-1}(S \setminus i)} \left(\left(\frac{1}{C_{k-1}} e_1(S \setminus i) + C_k x_i \right) e_{k-1}(S \setminus i) - \frac{1}{C_{k-1}} \sum_{j \in S\setminus i} x_j^2 e_{k-2}(S \setminus \{i,j\}) \right) \\ &\times \prod_{s \in S\setminus\{i\}} M_{S\setminus i,k-1,s} \\ &= \frac{1}{e_{k-1}(S \setminus i)} \left(\frac{k}{C_{k-1}} e_k(S \setminus i) + C_k x_i e_{k-1}(S \setminus i) \right) \prod_{s \in S\setminus\{i\}} M_{S\setminus i,k-1,s} \end{aligned}$$

$$= \frac{C_k e_k(S)}{e_{k-1}(S \setminus i)} \prod_{s \in S \setminus \{i\}} M_{S \setminus i, k-1, s}.$$

Unwinding the above recursion it follows that $M_{S,k,i}$ is of the form

$$M_{S,k,i} = C e_k(S) \prod_{\substack{T \subseteq S \setminus i \\ |T| > |S| - k}} e_{k+|T|-|S|}(T)^{\alpha_T}$$

for some constant C and exponents $\alpha_T \in \mathbb{N}$. Taking $S = [n]$ we thus see that $e_k(\mathbf{x})$ is a factor of the multivariate matching polynomial $M_{[n],k,n}$. It remains to show that

$$\Lambda_{++}(e_k(\mathbf{x}), \mathbf{1}) \subseteq \Lambda_{++}\left(\frac{M_{[n],k,n}}{e_k(\mathbf{x})}, \mathbf{1}\right).$$

for all $k \leq n$. By Lemma 1.2 above inclusion follows from the fact that

$$\Lambda_{++}(e_k(S), \mathbf{1}) \subseteq \Lambda_{++}(e_{k-1}(S), \mathbf{1})$$

for all $k \geq 1$ since $D_{\mathbf{1}} e_k(S) = (|S| - k) e_{k-1}(S)$, and from the fact that

$$\Lambda_{++}(e_k(S), \mathbf{1}) \subseteq \Lambda_{++}\left(e_k(T), \mathbf{1}\right)$$

for all $T \subseteq S$ since $e_k(T) = \left(\prod_{i \in S \setminus T} \frac{\partial}{\partial x_i}\right) e_k(S)$. Hence $\Lambda_{++}(e_k(\mathbf{x}), \mathbf{1})$ is spectrahedral by Theorem 2.16.

□

3. Hyperbolicity cones of multivariate independence polynomials

A subset $I \subseteq V(G)$ is *independent* if no two vertices of I are adjacent in G. Let $\mathcal{I}(G)$ denote the set of all independent sets in G and $i(G,k)$ denote the number of independent sets in G of size k. By convention $i(G,0) = 1$. The (univariate) *independence polynomial* is defined by

$$I(G,t) = \sum_{k \geq 0} i(G,k) t^k.$$

The *line graph* $L(G)$ of G is the graph having vertex set $E(G)$ and where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident. It follows that $\mu(G,t) = t^{|V(G)|} I(L(G), -t^{-2})$. Therefore the independence polynomial can be viewed as a generalization of the matching polyomial. In contrast to the matching polynomial, the independence polynomial of a graph is not real-rooted in general. However Chudnovsky and Seymour [10] proved that $I(G,t)$ is real-rooted if G is claw-free, that is, if G has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. The theorem was later generalized by Engström to graphs with weighted vertices.

Theorem 3.1 (Engström [11])**.** *Let G be a claw-free graph and $\boldsymbol{\lambda} = (\lambda_v)_{v \in V(G)}$ a sequence of non-negative vertex weights. Then the polynomial*

$$I_{\boldsymbol{\lambda}}(G,t) = \sum_{I \in \mathcal{I}(G)} \left(\prod_{v \in I} \lambda_v\right) t^{|I|}$$

is real-rooted.

A full characterization of the graphs for which $I(G,t)$ is real-rooted remains an open problem.

A natural multivariate analogue of the independence polynomial is given by

$$I(G,\mathbf{x}) = \sum_{I\in\mathcal{I}(G)} \prod_{v\in I} x_v.$$

Leake and Ryder [26] define a strictly weaker notion of stability which they call same-phase stability. A polynomial $p(\mathbf{z}) \in \mathbb{R}[z_1,\dots,z_n]$ is *(real) same-phase stable* if for every $\mathbf{x} \in \mathbb{R}^n_+$, the univariate polynomial $p(t\mathbf{x})$ is real-rooted. The authors prove that $I(G,\mathbf{x})$ is same-phase stable if and only if G is claw-free. In fact the same-phase stability of $I(G,\mathbf{x})$ is an immediate consequence of Theorem 3.1.

The added variables in a homogeneous multivariate independence polynomial should preferably have labels carrying combinatorial meaning in the graph. For line graphs it is additionally desirable to maintain a natural correspondence with the homogeneous multivariate matching polynomial $\mu(G,\mathbf{x}\oplus\mathbf{w})$. Unfortunately we have not found a hyperbolic definition that satisfies both of the above properties. We have thus settled for the following definition.

Definition 3.2. Let $\mathbf{x} = (x_v)_{v\in V}$ and t be indeterminates. Define the *homogeneous multivariate independence polynomial* $I(G,\mathbf{x}\oplus t) \in \mathbb{R}[\mathbf{x},t]$ by

$$I(G,\mathbf{x}\oplus t) = \sum_{I\in\mathcal{I}(G)} (-1)^{|I|} \left(\prod_{v\in I} x_v^2\right) t^{2|V(G)|-2|I|}.$$

Lemma 3.3. *If G is a claw-free graph, then $I(G,\mathbf{x}\oplus t)$ is a hyperbolic polynomial with respect to $\mathbf{e} = (0,\dots,0,1) \in \mathbb{R}^{V(G)} \times \mathbb{R}$.*

Proof. First note that $I(G,\mathbf{e}) = 1 \neq 0$. Let $\mathbf{x}\oplus t \in \mathbb{R}^{V(G)} \times \mathbb{R}$ and $\lambda_v = x_v^2$ for all $v \in V(G)$. Then

$$I(G, s\mathbf{e} - \mathbf{x}\oplus t) = (s-t)^{2|V(G)|} I_{\boldsymbol{\lambda}}(G, -(s-t)^{-2}).$$

By Theorem 3.1 the polynomial $I_{\boldsymbol{\lambda}}(G,s)$ is real-rooted. Clearly all roots are negative which implies $I_{\boldsymbol{\lambda}}(G,-s^{-2})$ is real-rooted. Hence the univariate polynomial $s \mapsto I(G, s\mathbf{e} - \mathbf{x}\oplus t)$ is real-rooted which shows that $I(G,\mathbf{x}\oplus t)$ is hyperbolic with respect to $\mathbf{e}$.

□

An induced clique K in G is called a *simplicial clique* if for all $u \in K$ the induced subgraph $N[u] \cap (G \setminus K)$ of $G \setminus K$ is a clique. In other words the neighbourhood of each $u \in K$ is a disjoint union of two induced cliques in G. Furthermore, a graph G is said to be *simplicial* if G is claw-free and contains a simplicial clique.

In this section we prove Conjecture 1.6 for the polynomial $I(G,\mathbf{x}\oplus t)$ when G is simplicial. The proof unfolds in a parallel manner to Theorem 2.16 by considering a different kind of path tree. Before the results can be stated we must outline the necessary definitions from [26].

A connected graph G is a *block graph* if each 2-connected component is a clique. Given a simplicial graph G with a simplicial clique K we recursively define a block graph $T^{\boxtimes}(G,K)$ called the *clique tree* associated to G and rooted at K (see Figure 4).

We begin by adding K to $T^{\boxtimes}(G,K)$. Let $K_u = N[u] \setminus K$ for each $u \in K$. Attach the disjoint union $\bigsqcup_{u\in K} K_u$ of cliques to $T^{\boxtimes}(G,K)$ by connecting $u \in K$ to every $v \in K_u$. Finally recursively attach $T^{\boxtimes}(G\setminus K, K_u)$ to the clique K_u in $T^{\boxtimes}(G,K)$ for every $u \in K$. Note that the recursion is made well-defined by the following lemma.

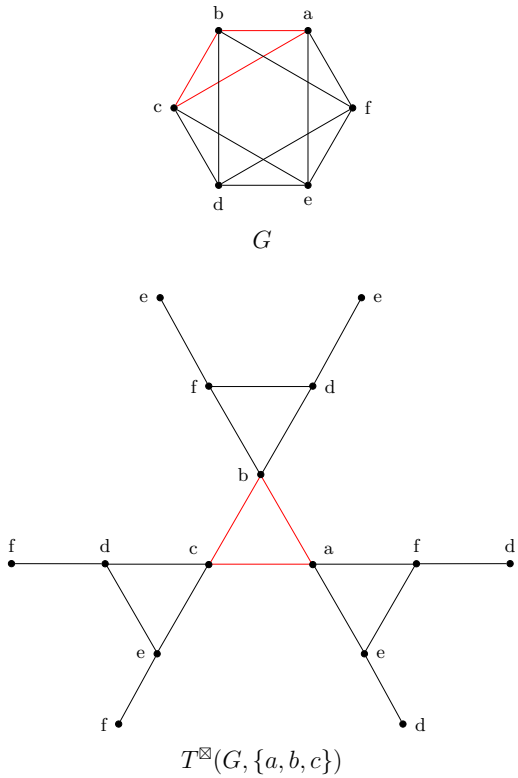


FIGURE 4. A simplicial graph G and its associated relabelled clique tree $T^{\boxtimes}(G, K)$ rooted at $K = \{a, b, c\}$ (highlighted in red).

Lemma 3.4 (Chudnovsky-Seymour [10])**.** *Let G be a clawfree graph and let K be a simplicial clique in G. Then $N[u] \setminus K$ is a simplicial clique in $G \setminus K$ for all $u \in K$.*

It is well-known that a graph is the line graph of a tree if and only if it is a claw-free block graph [19, Thm 8.5]. In [26] it was demonstrated that the block graph $T^{\boxtimes}(G, K)$ is the line graph of a certain *induced path tree* $T^{\angle}(G, K)$. Its precise definition is not important to us, but we remark that it is a subtree of the usual path tree defined in Section 2 that avoids traversed neighbours. This enables us to find a definite determinantal representation of $I(T^{\boxtimes}(G, K), \mathbf{x} \oplus t)$ via Lemma 2.14. The second important fact is that $I(G, \mathbf{x})$ divides $I(T^{\boxtimes}(G, K), \mathbf{x})$ where $T^{\boxtimes}(G, K)$ is relabelled according to the natural graph homomorphism $\phi_K : T^{\boxtimes}(G, K) \to G$. Hence using the recursion provided by the simplicial structure of G we have almost all the ingredients to finish the proof of Conjecture 1.6 for $I(G, \mathbf{x} \oplus t)$.

Lemma 3.5 (Leake-Ryder [26]))**.**
For any simplicial graph G, and any simplicial clique $K \leq G$, we have

$$L(T^{\angle}(G, K)) \cong T^{\boxtimes}(G, K).$$

The following theorem is a generalization of Godsil's divisibility theorem for matching polynomials. It can be proved in a similar manner by induction using the recursive structure of simplicial graphs and removing cliques instead of vertices. For the proof to go through in the homogeneous setting we must replace the usual recursion by

$$I(G, \mathbf{x} \oplus t) = t^{2|K|} I(G \setminus K, \mathbf{x} \oplus t) - \sum_{v \in K} t^{2|N(v)|} x_v^2 I(G \setminus N[v], \mathbf{x} \oplus t).$$

Theorem 3.6 (Leake-Ryder [26])). *Let K be a simplicial clique of the simplicial graph G. Then*

$$\frac{I(G, \mathbf{x} \oplus t)}{I(G \setminus K, \mathbf{x} \oplus t)} = \frac{I(T^{\boxtimes}(G, K), \mathbf{x} \oplus t)}{I(T^{\boxtimes}(G, K) \setminus K, \mathbf{x} \oplus t)},$$

where $T^{\boxtimes}(G, K)$ is relabelled according to the natural graph homomorphism $\phi_K : T^{\boxtimes}(G, K) \to G$. Moreover $I(G, \mathbf{x} \oplus t)$ divides $I(T^{\boxtimes}(G, K), \mathbf{x} \oplus t)$.

The following lemma ensures the hyperbolicity cones behave well under vertex deletion.

Lemma 3.7. *Let $v \in V(G)$. Then $\Lambda_{++}(I(G, \mathbf{x} \oplus t)) \subseteq \Lambda_{++}(I(G \setminus v, \mathbf{x} \oplus t))$.*

Proof. Let $\mathbf{x} \oplus t \in \mathbb{R}^{V(G)} \times \mathbb{R}$ and $\mathbf{e} = (0, \ldots, 0, 1)$. By Lemma 3.3 the polynomials $s \mapsto I(G, s\mathbf{e} - \mathbf{x} \oplus t)$ and $s \mapsto I(G \setminus v, s\mathbf{e} - \mathbf{x} \oplus t)$ are both real-rooted. Denote their roots by $\alpha_1, \ldots, \alpha_{2n}$ and $\beta_1, \ldots, \beta_{2n-2}$ respectively where $n = |V(G)|$. We claim that

$$\min_i \alpha_i \leq \min_i \beta_i \leq \max_i \beta_i \leq \max_i \alpha_i$$

by induction on the number of vertices of G. Indeed the claim is vacuously true if $|V(G)| = 1$. Suppose therefore $|V(G)| > 1$. If G is not connected, then $G = G_1 \sqcup G_2$ for some non-empty graphs G_1, G_2. Without loss assume $v \in G_1$. Then $G \setminus v = (G_1 \setminus v) \sqcup G_2$. By induction the claim holds for the pair G_1 and $G_1 \setminus v$. This implies the claim for G and $G \setminus v$ since $I(G, \mathbf{x} \oplus t)$ is multiplicative with respect to disjoint union. We may therefore assume G is connected. Thus $G \setminus N[v]$ is of strictly smaller size than $G \setminus v$. We have

$$I(G, \mathbf{x} \oplus t) = t^2 I(G \setminus v, \mathbf{x} \oplus t) - x_v^2 t^{2|N(v)|} I(G \setminus N[v], \mathbf{x} \oplus t). \tag{3}$$

By induction, the maximal root γ of $I(G \setminus N[v], s\mathbf{e} - \mathbf{x} \oplus t)$ is less than the maximal root β of $I(G \setminus v, s\mathbf{e} - \mathbf{x} \oplus t)$. Since $I(G \setminus N[v], s\mathbf{e} - \mathbf{x} \oplus t)$ is an even degree polynomial with positive leading coefficient we have that $I(G \setminus N[v], s\mathbf{e} - \mathbf{x} \oplus t) \geq 0$ for all $s \geq \gamma$. By (3) this implies that $I(G, \beta\mathbf{e} - \mathbf{x} \oplus t) \leq 0$. Hence $\max_i \beta_i \leq \max_i \alpha_i$ since $I(G, s\mathbf{e} - \mathbf{x} \oplus t) \to \infty$ as $s \to \infty$. Since each of the terms involved in the polynomials $I(G, s\mathbf{e} - \mathbf{x} \oplus t)$ and $I(G \setminus v, s\mathbf{e} - \mathbf{x} \oplus t)$ have even degree in $s - t$, their respective roots are symmetric about $s = t$. Hence $\min_i \alpha_i \leq \min_i \beta_i$ proving the claim. Finally if $\mathbf{x}_0 \oplus t_0 \in \Lambda_{++}(I(G, \mathbf{x} \oplus t))$, then $\min_i \alpha_i > 0$ so by the claim $\min_i \beta_i > 0$ showing that $\mathbf{x}_0 \oplus t_0 \in \Lambda_{++}(I(G \setminus v, \mathbf{x} \oplus t))$. This proves the lemma.

□

Remark 3.8. Since

$$I(G, \mathbf{x} \oplus t)\Big|_{x_v = 0} = t^2 I(G \setminus v, \mathbf{x} \oplus t),$$

we see by Lemma 3.7 that setting vertex variables equal to zero relaxes the hyperbolicity cone.

Theorem 3.9. *If G is a simplicial graph, then the hyperbolicity cone of $I(G, \mathbf{x} \oplus t)$ is spectrahedral.*

Proof. Let K be a simplicial clique of G. Arguing by induction as in Theorem 2.16, using the clique tree $T^{\boxtimes}(G,K)$ instead of the path tree $T(G,u)$, and invoking Theorem 3.6 we get a factorization

$$q_{G,K}(\mathbf{x}\oplus t) = q_{G\setminus K,K_v}(\mathbf{x}\oplus t)\prod_{w\in K\setminus v} q_{G\setminus K,K_w}(\mathbf{x}\oplus t)I(G\setminus K,\mathbf{x}\oplus t), \tag{4}$$

where $v\in K$ is fixed, $K_w = N[w]\setminus K$ and

$$q_{G,K}(\mathbf{x}\oplus t)I(G,\mathbf{x}\oplus t) = I(T^{\boxtimes}(G,K),\mathbf{x}\oplus t),$$
$$q_{G\setminus K,K_w}(\mathbf{x}\oplus t)I(G\setminus K,\mathbf{x}\oplus t) = I(T^{\boxtimes}(G\setminus K,K_w),\mathbf{x}\oplus t)$$

for $w\in K$. Repeated application of Lemma 3.7 gives

$$\Lambda_{++}(I(G,\mathbf{x}\oplus t)) \subseteq \Lambda_{++}(I(G\setminus K,\mathbf{x}\oplus t)).$$

By the factorization (4) and induction we hence get the desired cone inclusion

$$\Lambda_{++}(I(G,\mathbf{x}\oplus t)) \subseteq \Lambda_{++}(q_{G,K}(\mathbf{x}\oplus t)).$$

Since $L(T^{\angle}(G,K)) \cong T^{\boxtimes}(G,K)$ by Lemma 3.5 we see that

$$I(T^{\boxtimes}(G,K),\mathbf{x}\oplus t) = \mu(T^{\angle}(G,K),t\mathbf{1}\oplus\mathbf{x}).$$

Hence $I(T^{\boxtimes}(G,K),\mathbf{x}\oplus t)$ has a definite determinantal representation by Lemma 2.14 proving the theorem.

□

4. Convolutions

If G is a simple undirected graph with adjacency matrix $A=(a_{ij})$, then we may associate a signing $\mathbf{s}=(s_{ij})\in\{\pm1\}^{E(G)}$ to its edges. The symmetric adjacency matrix $A^{\mathbf{s}}=(a^{s}_{ij})$ of the resulting graph is given by $a^{\mathbf{s}}_{ij}=s_{ij}a_{ij}$ for $ij\in E(G)$ and $a^{\mathbf{s}}_{ij}=0$ otherwise. Godsil and Gutman [15] proved that

$$\mathop{\mathbb{E}}_{\mathbf{s}\in\{\pm1\}^{E(G)}} \det(tI-A^{\mathbf{s}}) = \mu(G,t). \tag{5}$$

In other words, the expected characteristic polynomial of an independent random signing of the adjacency matrix of a graph is equal to its matching polynomial. Therefore the expected characteristic polynomial is real-rooted. This was one of the facts used by Marcus, Spielman and Srivastava [28] in proving that there exist infinite families of regular bipartite Ramanujan graphs. Since then, several other families of characteristic polynomials have been identified with real-rooted expectation (see e.g. [30][18]). Such families are called *interlacing families*, based on the fact that there exists a common root interlacing polynomial if and only if every convex combination of the family is real-rooted. The method of interlacing families have been successfully applied to other contexts, in particular to the affirmative resolution of the Kadison-Singer problem [29].

In this section we define a convolution of multivariate determinant polynomials and show that it is hyperbolic as a direct consequence of a more general theorem by Brändén [6]. In particular this convolution can be viewed as a generalization of the fact that the expectation in (5) is real-rooted. Namely, we show that the expected characteristic polynomial over any finite set of independent random edge weightings is real-rooted barring certain adjustments to the weights of the loop edges.

Recall that every symmetric matrix may be identified with the adjacency matrix of an undirected weighted graph (with loops).

Definition 4.1. Let $W \subseteq \mathbb{R}$ be a finite set. Given a real symmetric matrix A and a vector $\mathbf{w} \in W^{\binom{n}{2}}$, define a *weighting* of A to be a symmetric matrix $A^{\mathbf{w}} = (a_{ij}^{\mathbf{w}})$ given by

$$a_{ij}^{\mathbf{w}} = \begin{cases} w_{ij} a_{ij} & \text{if } i < j \\ \sum_{k=1}^{i} a_{ik} + \sum_{k=i+1}^{n} w_{ik}^2 a_{ik} & \text{if } i = j \end{cases}.$$

Definition 4.2. Let $X = (x_{ij})_{i,j=1}^n$ and $Y = (y_{ij})_{i,j=1}^n$ be symmetric matrices in variables $\mathbf{x} = (x_{ij})_{i \leq j}$ and $\mathbf{y} = (y_{ij})_{i \leq j}$ respectively. Let $W \subseteq \mathbb{R}$ be a finite set. We define the convolution

$$\det(X) *_W \det(Y) = \mathop{\mathbb{E}}_{\mathbf{w}_1, \mathbf{w}_2 \in W^{\binom{n}{2}}} \det(X^{\mathbf{w}_1} + Y^{\mathbf{w}_2}) \in \mathbb{R}[\mathbf{x}, \mathbf{y}].$$

We have the following general fact about hyperbolic polynomials.

Theorem 4.3 (Brändén [6])**.** *Let $h(\mathbf{x})$ be a hyperbolic polynomial with respect to $\mathbf{e} \in \mathbb{R}^n$, let $V_1, \ldots, V_m$ be finite sets of vectors of rank at most one in Λ_+. For $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m) \in V_1 \times \cdots \times V_m$, let*

$$g(\mathbf{V}; t) = h(t\mathbf{e} + \mathbf{u} - \alpha_1 \mathbf{v}_1 - \cdots - \alpha_m \mathbf{v}_m)$$

where $\mathbf{u} \in \mathbb{R}^n$ and $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. Then $\mathop{\mathbb{E}}_{\mathbf{V} \in V_1 \times \cdots \times V_m} g(\mathbf{V}; t)$ is real-rooted.

Proposition 4.4. *Let $W \subseteq \mathbb{R}$ be a finite subset. Then $\det(X) *_W \det(Y)$ is hyperbolic with respect to $\mathbf{e} = I \oplus \mathbf{0}$ where I denotes the identity matrix.*

Proof. Let $h(X \oplus Y) = \det(X) *_W \det(Y)$. We note that $h(\mathbf{e}) = 1 \neq 0$. Let $\delta_1, \ldots, \delta_n$ denote the standard basis of $\mathbb{R}^n$. Put

$$V_{ij} = \{\mathbf{v}_{ijw} : w \in W\}$$

where $\mathbf{v}_{ijw} = (\delta_i + w\delta_j)(\delta_i + w\delta_j)^T$ for $i < j$ and $w \in W$. Note that $\mathbf{v}_{ijw}$ is a rank one matrix belonging to the hyperbolicity cone of positive semidefinite matrices (with non-zero eigenvalue $w^2 + 1$). Letting $\mathbf{u} = \mathbf{0}$ and $\alpha_{ij}^X = x_{ij}, \alpha_{ij}^Y = y_{ij}$ for $i < j$ we see that

$$\begin{aligned} h(t\mathbf{e} - X \oplus Y) &= \mathop{\mathbb{E}}_{\mathbf{w}_1, \mathbf{w}_2} \det(tI - X^{\mathbf{w}_1} - Y^{\mathbf{w}_2}) \\ &= \mathop{\mathbb{E}}_{\substack{\mathbf{v}_{ijw_1}, \mathbf{v}_{ijw_2} \in V_{ij} \\ i<j}} \det\left(tI + \mathbf{u} - \sum_{i<j} (\alpha_{ij}^X \mathbf{v}_{ijw_1} + \alpha_{ij}^Y \mathbf{v}_{ijw_2})\right), \end{aligned}$$

where the right hand side is a real-rooted polynomial in t by Theorem 4.3. Hence $\det(X) *_W \det(Y)$ is hyperbolic with respect to $\mathbf{e}$.

□

Remark 4.5. Taking $W = \{\pm 1\}$ we see that $a_{ii}^{\mathbf{w}} = \sum_{k=1}^n a_{ik}$ for all $\mathbf{w} \in W$ and $i = 1, \ldots, n$. Therefore setting $\mathbf{u} = \operatorname{diag}(d_1, \ldots, d_n)$ where $d_i = \sum_{j \neq i} (x_{ij} + y_{ij})$ in the proof of Proposition 4.4, we get that

$$\mathop{\mathbb{E}}_{\mathbf{s}_1, \mathbf{s}_2} \det(X^{\mathbf{s}_1} + Y^{\mathbf{s}_2}) \tag{6}$$

is hyperbolic, where the expectation is taken over independent random signings of the matrices X and Y as in (5) without weighting the diagonal. This shows in particular that the expectation in (5) is real-rooted.

Corollary 4.6. *Let $W \subseteq \mathbb{R}$ be a finite subset and A a real symmetric $n \times n$ matrix. Then*

$$\mathop{\mathbb{E}}_{\mathbf{w} \in W^{\binom{n}{2}}} \det(tI - A^{\mathbf{w}})$$

is real-rooted.

Proof. By Corollary 4.4 the polynomial $\det(Y) *_W \det(X)$ is hyperbolic, so in particular $t \mapsto \mathop{\mathbb{E}}_{\mathbf{w}} \det(tI - A^{\mathbf{w}})$ is real-rooted with $X = \mathbf{0}$ and $Y = A$.

□

Next we see that the convolution (6) over independent random signings can be realized as a convolution of multivariate matching polynomials. The proof is similar to that of the univariate identity (5) (cf [15]). Let G_X and G_Y denote the weighted graphs corresponding to the symmetric matrices X and Y.

Proposition 4.7. *Let $X = (x_{ij})_{i,j=1}^n$ and $Y = (y_{ij})_{i,j=1}^n$ be symmetric matrices in variables $\mathbf{x} = (x_{ij})_{i \le j}$ and $\mathbf{y} = (y_{ij})_{i \le j}$. Then*

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\mathbf{s}^{(1)}, \mathbf{s}^{(2)}} \det(X^{\mathbf{s}^{(1)}} + Y^{\mathbf{s}^{(2)}}) \\
&= \sum_{S \subseteq [n]} (-1)^{|S|/2} \prod_{i \notin S} (x_{ii} + y_{ii}) \sum_{S_1 \sqcup S_2 = S} \mu(G_X[S_1], \mathbf{0} \oplus \mathbf{x}) \mu(G_Y[S_2], \mathbf{0} \oplus \mathbf{y})
\end{aligned}$$

where the expectation is taken over independent random signings as in (5).

Proof. Expanding the convolution from the definition of the determinant we have

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\mathbf{s}^{(1)}, \mathbf{s}^{(2)}} \det(X^{\mathbf{s}^{(1)}} + Y^{\mathbf{s}^{(2)}}) \\
&= \mathop{\mathbb{E}}_{\mathbf{s}^{(1)}, \mathbf{s}^{(2)}} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \left(X^{\mathbf{s}^{(1)}} + Y^{\mathbf{s}^{(2)}} \right)_{i\sigma(i)} \\
&= \mathop{\mathbb{E}}_{\mathbf{s}^{(1)}, \mathbf{s}^{(2)}} \sum_{S \subseteq [n]} \prod_{i \notin S} (x_{ii} + y_{ii}) \sum_{\substack{\sigma \in \operatorname{Sym}(S) \\ \sigma(j) \neq j \ \forall j \in S}} \operatorname{sgn}(\sigma) \prod_{j \in S} \left(s^{(1)}_{j\sigma(j)} x_{j\sigma(j)} + s^{(2)}_{j\sigma(j)} y_{j\sigma(j)} \right) \\
&= \sum_{S \subseteq [n]} \prod_{i \notin S} (x_{ii} + y_{ii}) \sum_{\substack{\sigma \in \operatorname{Sym}(S) \\ \sigma(j) \neq j \ \forall j \in S}} \operatorname{sgn}(\sigma) \sum_{S_1 \sqcup S_2 = S} \mathop{\mathbb{E}}_{\mathbf{s}^{(1)}} \prod_{j \in S_1} s^{(1)}_{j\sigma(j)} x_{j\sigma(j)} \mathop{\mathbb{E}}_{\mathbf{s}^{(2)}} \prod_{j \in S_2} s^{(2)}_{j\sigma(j)} y_{j\sigma(j)}
\end{aligned}$$

Note the following regarding the random variables $s^{(k)}_{ij}$, $k = 1, 2$:

(i) $s^{(k)}_{ij}$ appears with power at most two in each of the products.
(ii) The random variables $s^{(k)}_{ij}$ are independent.
(iii) $\mathbb{E}\, s^{(k)}_{ij} = 0$.
(iv) $\mathbb{E}(s^{(k)}_{ij})^2 = 1$.

As a consequence, permutations with the following characteristics may be eliminated since they produce factors $s^{(k)}_{ij}$ of power one making the term vanish:

(i) $\sigma \in \mathfrak{S}_n$ having no factorization $\sigma = \sigma_1\sigma_2$ for $\sigma_i \in \mathrm{Sym}(S_i)$, $i = 1, 2$.
(ii) $\sigma \in \mathfrak{S}_n$ such that σ is not a complete product of disjoint transpositions.

This leaves us with products of fixed-point-free involutions in $\mathrm{Sym}(S_1)$ and $\mathrm{Sym}(S_2)$. Thus the non-vanishing terms are those corresponding to perfect matchings on $G_X[S_1]$ and $G_Y[S_2]$. Hence

$$\mathop{\mathbb{E}}_{\mathbf{s}^{(1)},\mathbf{s}^{(2)}} \det(X^{\mathbf{s}^{(1)}} + Y^{\mathbf{s}^{(2)}}) = \sum_{S\subseteq[n]} \prod_{i\notin S}(x_{ii} + y_{ii}) \sum_{S_1\sqcup S_2=S} P_1(\mathbf{x})P_2(\mathbf{y})$$

where

$$\begin{aligned} P_1(\mathbf{x}) &= \sum_{\substack{\sigma_1\in\mathrm{Sym}(S_1)\\ \sigma_1(j)\neq j\ \forall j\in S_1}} \mathrm{sgn}(\sigma_1) \mathop{\mathbb{E}}_{\mathbf{s}^{(1)}} \prod_{i\in S_1} s^{(1)}_{i\sigma_1(i)} x_{i\sigma_1(i)} \\ &= \sum_{\substack{M\in\mathcal{M}(G_X[S_1])\\ M \text{ perfect}}} (-1)^{|S_1|/2} \prod_{ij\in M} \mathop{\mathbb{E}}_{\mathbf{s}^{(1)}} (s^{(1)}_{ij})^2 x^2_{ij} \\ &= (-1)^{|S_1|/2} \sum_{\substack{M\in\mathcal{M}(G_X[S_1])\\ M \text{ perfect}}} \prod_{ij\in M} x^2_{ij} \\ &= (-1)^{|S_1|/2} \mu(G_X[S_1], \mathbf{0}\oplus\mathbf{x}) \end{aligned}$$

and similarly for $P_2(\mathbf{y})$.

□

Remark 4.8. The expression in Proposition 4.7 may also be written

$$\mathop{\mathbb{E}}_{\mathbf{s}^{(1)},\mathbf{s}^{(2)}} \det(X^{\mathbf{s}^{(1)}} + Y^{\mathbf{s}^{(2)}}) = \sum_{M\in\mathcal{M}(K_n)} (-1)^{|M|} \prod_{i\notin V(M)} (x_{ii} + y_{ii}) \prod_{jk\in M} (x^2_{jk} + y^2_{jk}).$$

Example 4.9.

(i) Let A be the adjacency matrix of a simple undirected graph G. Under the specialization $X = tI$ and $Y = -A$ in Proposition 4.7 we recover the identity (5) of Godsil and Gutman.
(ii) Let A and B both be adjacency matrices of the complete graph K_n. It is well-known (see e.g. [13]) that the number of perfect matchings in K_n is given by $(n-1)!!$ if n is even and 0 otherwise, where $(n)!! = n(n-2)(n-4)\cdots$. By Proposition 4.7 and a simple calculation it follows that

$$\begin{aligned} \mathop{\mathbb{E}}_{\mathbf{s}^{(1)},\mathbf{s}^{(2)}} \det(tI + A^{\mathbf{s}^{(1)}} + B^{\mathbf{s}^{(2)}}) &= \sum_{k=0}^{\lfloor n/2\rfloor} t^{n-2k}(-1)^k \binom{n}{2k} \sum_{i+j=k} \binom{2k}{2i} (2i-1)!!(2j-1)!! \\ &= \sum_{k=0}^{\lfloor n/2\rfloor} t^{n-2k}(-1)^k \binom{n}{2k} (2k-1)!! \left(\frac{3}{2}\right)^k \\ &= t^n \mu_{\frac{3}{2}\mathbf{1}}(K_n, t^{-1}\mathbf{1}). \end{aligned}$$

Final remarks

In Theorem 3.9 we proved Conjecture 1.6 for $I(G, \mathbf{x} \oplus t)$ whenever G is a simplicial graph. An extension of the divisibility relation in Theorem 3.6 to all claw-free graphs would immediately extend Theorem 3.9 to all claw-free graphs.

An interesting extension of this work would be to study a family of stable graph polynomials introduced by Wagner [38] in a general effort to prove Heilmann-Lieb type theorems. Let $G = (V, E)$ be a graph. For $H \subseteq E$, let $\deg_H : V \to \mathbb{N}$ denote the degree function of the subgraph (V, H). Furthermore let

$$\mathbf{u}^{(v)} = (u_0^{(v)}, u_1^{(v)}, \ldots, u_d^{(v)})$$

denote a sequence of *activities* at each vertex $v \in V$ where $d = \deg_G(v)$. Define the polynomial

$$Z(G, \boldsymbol{\lambda}, \mathbf{u}; \mathbf{x}) = \sum_{H \subseteq E} (-1)^{|H|} \boldsymbol{\lambda}^H \mathbf{u}_{\deg_H} \mathbf{x}^{\deg_H}$$

where $\boldsymbol{\lambda} = \{\lambda_e\}_{e \in E}$ are edge weights and

$$\boldsymbol{\lambda}^H = \prod_{e \in H} \lambda_e, \quad \mathbf{u}_{\deg_H} = \prod_{v \in V} u^{(v)}_{\deg_H(v)}, \quad \mathbf{x}^{\deg_H} = \prod_{v \in .V} x_v^{\deg_H(v)}.$$

Wagner proves that $Z(G, \boldsymbol{\lambda}, \mathbf{u}, \mathbf{x})$ is stable whenever $\lambda_e \geq 0$ for all $e \in E$ and the univariate *key-polynomial* $K_v(z) = \sum_{j=0}^{d} \binom{d}{j} u_j^{(v)} z^j$ is real-rooted for all $v \in V$ (cf [38, Thm 3.2]). We note in particular that if $u_0^{(v)} = u_1^{(v)} = 1$, $u_k^{(v)} = 0$ for all $k > 1$ and $v \in V$, then $Z(G, \boldsymbol{\lambda}, \mathbf{u}; \mathbf{x}) = \mu_{\boldsymbol{\lambda}}(G, \mathbf{x})$ where $\mu_{\boldsymbol{\lambda}}(G, \mathbf{x})$ is the weighted multivariate matching polynomial studied by Heilmann and Lieb [20]. An appropriate homogenization of $Z(G, \boldsymbol{\lambda}, \mathbf{u}; \mathbf{x})$ could be defined as

$$W(G, \mathbf{u}; \mathbf{x} \oplus \mathbf{w}) = \sum_{H \subseteq E} (-1)^{|H|} \mathbf{u}_{\deg_H} \mathbf{w}^{2H} \mathbf{x}^{\deg_G - \deg_H}.$$

Since $W(G, \mathbf{u}; \mathbf{x} \oplus \mathbf{w}) = \mathbf{x}^{\deg_G} Z(G, \mathbf{w}^2, \mathbf{u}; \mathbf{x}^{-1})$ we see that $W(G, \mathbf{u}; \mathbf{x} \oplus \mathbf{w})$ is hyperbolic with respect to $\mathbf{e} = \mathbf{1} \oplus \mathbf{0}$ whenever $K_v(z)$ is real-rooted for all $v \in V$. We also note the following edge and node recurrences for $e \in E$ and $v \in V$,

$$\begin{aligned}
&W(G, \mathbf{u}; \mathbf{x} \oplus \mathbf{w}) \\
&= \mathbf{x}^e W(G \setminus e, \mathbf{u}; \mathbf{x} \oplus \mathbf{w}) - \mathbf{w}^{2e} W(G \setminus e, \mathbf{u} \ll e; \mathbf{x} \oplus \mathbf{w}) \\
&= \sum_{S \subseteq N(v)} (-1)^{|S|} u^{(v)}_{|S|} \mathbf{w}^{2E(S,v)} x_v^{\deg_G(v) - |S|} \mathbf{x}^{N(v) \setminus S} W(G \setminus v, \mathbf{u} \ll S; \mathbf{x} \oplus \mathbf{w})
\end{aligned}$$

where $E(S, v) = \{sv \in E : s \in S\}$ and $(\mathbf{u} \ll S)^{(v)} = \begin{cases} (u_1^{(v)}, \ldots, u_d^{(v)}), & v \in S \\ \mathbf{u}^{(v)}, & v \notin S \end{cases}$

Although it is not clear in general how to find a definite determinantal representation of $W(G, \mathbf{u}; \mathbf{x} \oplus \mathbf{w})$, it may be possible to consider special form activity vectors and obtain a reduction by constructing divisibility relations in the spirit of Lemma 2.12 and Theorem 3.6. This may also be of independent interest for studying root bounds of their univariate specializations.

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Paper C

STABLE MULTIVARIATE GENERALIZATIONS OF MATCHING POLYNOMIALS

NIMA AMINI

Abstract. The first part of this note concerns stable averages of multivariate matching polynomials. In proving the existence of infinite families of bipartite Ramanujan d-coverings, Hall, Puder and Sawin introduced the d-matching polynomial of a graph G, defined as the uniform average of matching polynomials over the set of d-sheeted covering graphs of G. We prove that a natural multivariate version of the d-matching polynomial is stable, consequently giving a short direct proof of the real-rootedness of the d-matching polynomial. Our theorem also includes graphs with loops, thus answering a question of said authors. Furthermore we define a weaker notion of matchings for hypergraphs and prove that a family of natural polynomials associated to such matchings are stable. In particular this provides a hypergraphic generalization of the classical Heilmann-Lieb theorem.

1. Introduction

The real-rootedness of the matching polynomial of a graph is a well-known result in algebraic graph theory due to Heilmann and Lieb [12]. Slightly less quoted is its stronger multivariate counterpart (see [12]) which proclaims that the multivariate matching polynomial is non-vanishing when its variables are restricted to the upper complex half-plane, a property known as *stability*. Other stable polynomials occurring in combinatorics include e.g. multivariate Eulerian polynomials [10], several bases generating polynomials of matroids (including multivariate spanning tree polynomials) [6] and certain multivariate subgraph polynomials [22]. In the present note we consider several different stable generalizations of multivariate matching polynomials. Hall, Puder and Sawin prove in [11] that every connected bipartite graph has a Ramanujan d-covering of every degree for each $d \geq 1$, generalizing seminal work of Marcus, Spielman and Srivastava [16, 18] for the case $d = 2$. An important object in their proof is a certain generalization of the matching polynomial of a graph G, called the *d-matching polynomial*, defined by taking averages of matching polynomials over the set of d-sheeted covering graphs of G. The authors prove (via an indirect method) that the d-matching polynomial of a multigraph is real-rooted provided that the graph contains no loops. We prove in Theorem 3.7 that the latter hypothesis is redundant by establishing a stronger result, namely that the multivariate d-matching polynomial is stable for any multigraph (possibly with loops). In the final section we consider a hypergraphic generalization of the Heilmann-Lieb theorem. The hypergraphic matching polynomial is not real-rooted in general (see [24]) so it does not admit a natural stable multivariate refinement. However by relaxing the notion of matchings in hypergraphs we prove in Theorem 5.6 that an associated "relaxed" multivariate matching polynomial is stable.

2. Preliminaries

2.1. **Graph coverings and group labelings.** In this subsection we outline relevant definitions from [11]. Let $G = (V(G), E(G))$ be a finite, connected, undirected graph on $[n]$.

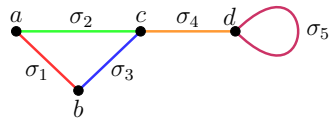

FIGURE 1. A S_4-labeling γ of a graph G with $\gamma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = ((1\ 2),\ (1\ 2)(3\ 4),\ (1\ 3\ 2),\ (1\ 2\ 3\ 4),\ (1\ 2\ 3))$.

In particular we allow G to have multiple edges between vertices and contain edges from a vertex to itself, i.e., G is a multigraph with loops.

A graph homomorphism $f : H \to G$ is called a *local isomorphism* if for each vertex v in H, the restriction of f to the neighbours of v in H is an isomorphism onto the neighbours $f(v)$ in G. We call f a *covering map* if it is a surjective local isomorphism, in which case we say that H *covers* G. If the image of H under the covering map f is connected, then each fiber $f^{-1}(v)$ of $v \in V(G)$ is an independent set of vertices in H of the same size d. If so, we call H a *d-sheeted covering* (or *d-covering* for short) of G.

Although G is undirected we shall dually view it as an oriented graph, containing two edges with opposite orientation for each undirected edge. We denote the edges with positive (resp. negative) orientation by $E^+(G)$ (resp. $E^-(G)$) and identify $E(G)$ with the disjoint union $E^+(G) \sqcup E^-(G)$. If $e \in E^\pm(G)$, then we write $-e$ for the corresponding edge in $E^\mp(G)$ with opposite orientation. Moreover we denote by $h(e)$ and $t(e)$, the head and tail of the edge $e \in E(G)$ respectively. A d-covering H of a graph G can be constructed via the following model, introduced in [1, 7]. The vertices of H are defined by $V(H) := \{v_i : v \in V(G), 1 \leq i \leq d\}$. The edges of H are determined, as described below, by a labeling $\sigma : E(G) \to S_d$ (see Figure 1) satisfying $\sigma(-e) = \sigma(e)^{-1}$. For notational purposes we write $\sigma(e) = \sigma_e$. For every positively oriented edge $e \in E^+(G)$ we introduce d (undirected) edges in H connecting $h(e)_i$ to $t(e)_{\sigma_e(i)}$ for $1 \leq i \leq d$, that is, we replace each undirected edge e in G by the perfect matching induced by σ_e, see Figure 2. We shall interchangeably refer to the map σ and the covering graph H which it determines, as a covering of G. Let $\mathcal{C}_{d,G}$ denote the probability space of all d-coverings of G endowed with the uniform distribution.

Instead of labeling each edge in G by a permutation in S_d we may label the edges with elements coming from an arbitrary finite group Γ. A *Γ-labeling* of a graph G is a function $\gamma : E(G) \to \Gamma$ satisfying $\gamma(-e) = \gamma(e)^{-1}$. Let $\mathcal{C}_{\Gamma,G}$ denote the probability space of all Γ-labelings of G endowed with the uniform distribution. Let $\pi : \Gamma \to \mathrm{GL}_d(\mathbb{C})$ be a representation of Γ. For any Γ-labeling γ of G, let $A_{\gamma,\pi}$ denote the $nd \times nd$ matrix obtained from the adjacency matrix A_G of G by replacing the (i, j) entry in A_G with the $d \times d$ block $\sum_{e \in E(G)} \pi(\gamma(e))$ (where the sum runs over all oriented edges from i to j) and by a zero block if there are no edges between i and j. The matrix $A_{\gamma,\pi}$ is called a *(Γ, π)-covering* of G.

Consider the d-dimensional representation $\pi : S_d \to \mathrm{GL}_d(\mathbb{C})$ of the symmetric group S_d mapping every $\sigma \in S_d$ to its corresponding permutation matrix. The representation π is reducible since the 1-dimensional space $\langle \mathbf{1} \rangle \leq \mathbb{C}^d$, where $\mathbf{1} = (1, \ldots, 1)$, is invariant under the action of π. The action of π on the orthogonal complement $\langle \mathbf{1} \rangle^\perp$ is an irreducible $(d-1)$-dimensional representation called the *standard representation*, denoted $\mathrm{std} : S_d \to \mathrm{GL}_{d-1}(\mathbb{C})$.

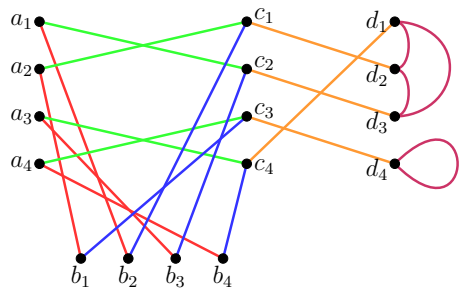

FIGURE 2. The 4-sheeted covering graph H corresponding to the S_4-labeling γ of G in Figure 1.

As outlined in [11], every d-covering H of G corresponds uniquely to a (S_d, std)-covering of G.

2.2. **Stable polynomials.** A polynomial $f(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be *stable* if $f(x_1, \ldots, x_n) \neq 0$ whenever $\mathrm{Im}(x_j) > 0$ for all $j = 1, \ldots, n$. By convention we also regard the zero polynomial to be stable. A stable polynomial with only real coefficients is said to be *real stable*. Note that univariate real stable polynomials are precisely the real-rooted polynomials (i.e. real polynomials in one variable with all zeros in $\mathbb{R}$). Thus stability may be regarded as a multivariate generalization of real-rootedness. Below we collect a few facts about stable polynomials which are relevant for the forthcoming sections. For a more comprehensive background we refer to the survey by Wagner [21] and references therein.

A common technique for proving that a polynomial $f(\mathbf{x})$ is stable is to realize $f(\mathbf{x})$ as the image of a known stable polynomial under a stability preserving linear transformation. Stable polynomials satisfy several basic closure properties, among them are diagonalization $f \mapsto f(\mathbf{x})|_{x_i = x_j}$ for $i, j \in [n]$ and differentiation $f \mapsto \partial_i f(\mathbf{x})$ where $\partial_i := \frac{\partial}{\partial x_i}$. The following theorem by Lieb and Sokal provides the construction for a large family of stability preserving linear transformations.

Theorem 2.1 (Lieb-Sokal [14])**.** *If $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is a stable polynomial, then $f(\partial_1, \ldots, \partial_n)$ is a stability preserving linear operator.*

Borcea and Brändén [3] gave a complete characterization of the linear operators preserving stability. The following is the transcendental characterization of stability preservers on infinite-dimensional complex polynomial spaces. Define the *complex Laguerre-Pólya class* to be the class of entire functions in n variables that are limits, uniformly on compact sets of stable polynomials in n variables. Throughout we will use the following multi-index notation

$$\mathbf{x}^S := \prod_{i \in S}^{n} x_i \qquad \mathbf{x}^{\boldsymbol{\alpha}} := \prod_{i=1}^{n} x_i^{\alpha_i}, \qquad \boldsymbol{\alpha}! := \prod_{i=1}^{n} \alpha_i!,$$

where $S \subseteq [n]$ and $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{N}^n$.

Theorem 2.2 (Borcea-Brändén [3])**.** *Let $T : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ be a linear operator. Then T preserves stability if and only if either*

(i) *T has range of dimension at most one and is of the form*

$$T(f) = \alpha(f)P,$$

where α is a linear functional on $\mathbb{C}[x_1, \ldots, x_n]$ and P is a stable polynomial, or

(ii)

$$G_T(\mathbf{x}, \mathbf{y}) \coloneqq \sum_{\alpha \in \mathbb{N}^n} (-1)^\alpha T(\mathbf{x}^{\boldsymbol{\alpha}}) \frac{\mathbf{y}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}$$

belongs to the Laguerre-Pólya class.

A polynomial $f(x_1, \ldots, x_n)$ is said to be *multiaffine* if each variable x_i occurs with degree at most one in $f(x_1, \ldots, x_n)$, and is called *symmetric* if $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$ for all $\sigma \in S_n$. The Grace-Walsh-Szegö coincidence theorem is a cornerstone in the theory of stable polynomials frequently used to depolarize symmetries before checking stability. One version of it is stated below, see [3, 21] for modern references and alternative proofs.

Theorem 2.3 (Grace-Walsh-Szegö [9, 20, 23])**.** *Let $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ be a symmetric and multiaffine polynomial. Then $f(x_1, \ldots, x_n)$ is stable if and only if $f(x, \ldots, x)$ is stable.*

3. Stability of multivariate d-matching polynomials

A *matching* of an undirected graph G is a subset $M \subseteq E(G)$ such that no two edges in M share a common vertex. Let $V(M) \coloneqq \bigcup_{\{i,j\} \in M} \{i, j\}$ denote the set of vertices in the matching M. For $d \in \mathbb{Z}_{\geq 1}$, the *$d$-matching polynomial* of G is defined by

$$\mu_{d,G}(x) \coloneqq \mathbb{E}_{H \in \mathcal{C}_{d,G}} \mu_H(x),$$

where

$$\mu_G(x) \coloneqq \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i m_i x^{n-2i} \in \mathbb{Z}[x]$$

and m_i denotes the number of matchings in G of size i with $m_0 = 1$. In particular if $d = 1$, then $\mu_{d,G}(x)$ coincides with the conventional matching polynomial $\mu_G(x)$. The following results are proved in [11].

Theorem 3.1 (Hall-Puder-Sawin [11])**.** *Let Γ be a finite group and $\pi : \Gamma \to GL_d(\mathbb{C})$ be an irreducible representation such that $\pi(\Gamma)$ is a complex reflection group, i.e., $\pi(\Gamma)$ is generated by pseudo-reflections. If G is a finite connected multigraph, then*

$$\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma,G}} \det(xI - A_{\gamma,\pi}) = \mu_{d,G}(x). \tag{3.1}$$

Remark 3.2. Remarkably the expected characteristic polynomial in (3.1) depends only on the dimension d of the irreducible representation π and not on the particular choice of group Γ, nor the specifics of the map π. Real-rooted expected characteristic polynomials have seen a surge of interest recently in light of the Kadison-Singer problem and Ramanujan coverings, see e.g. [2, 11, 16, 17, 18, 19].

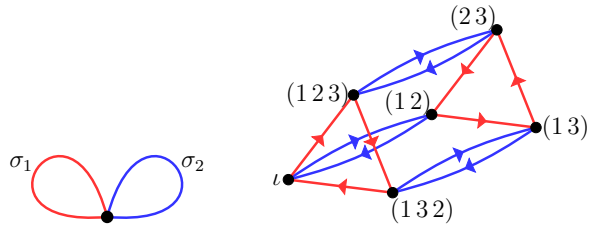

FIGURE 3. A S_3-labeling $\gamma = (\sigma_1, \sigma_2) = ((1\,2\,3), (1\,2))$ of the bouquet graph B_2 (left) and the Cayley graph of S_3 with respect to $\{(1\,2\,3), (1\,2)\}$ (right).

Example 3.3. A classical result due to Godsil and Gutman [8] states that if $A = (a_{ij})$ is the adjacency matrix of a finite simple undirected graph G, then

$$\mathbb{E}_{\mathbf{s}} \det(xI - A^{\mathbf{s}}) = \mu_G(x),$$

where $A^{\mathbf{s}}_{ij} := s_e a_{ij}$ for all $e = \{i, j\} \in E(G)$ and $\mathbf{s} = (s_e)_{e \in E(G)} \in \{\pm 1\}^{E(G)}$. In other words, the expected characteristic polynomial over all signings of G equals the matching polynomial of G. In the language of Hall, Puder and Sawin this corresponds to taking $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $\pi = \mathrm{sgn} : \mathbb{Z}/2\mathbb{Z} \to \mathrm{GL}_1(\mathbb{C})$ to be the sign representation in Theorem 3.1.

Generalizing and extending the following theorem will be the main focus of this section.

Theorem 3.4 (Hall-Puder-Sawin [11])**.** *If G is a finite multigraph with no loops, then $\mu_{d,G}(x)$ is real-rooted.*

Remark 3.5. Hall, Puder and Sawin also showed that the roots of $\mu_{d,G}(x)$ are contained inside the Ramanujan interval of G (see [11]).

Define the *multivariate d-matching polynomial* of G by

$$\mu_{d,G}(\mathbf{x}) := \mathbb{E}_{H \in \mathcal{C}_{d,G}} \mu_H(\mathbf{x}),$$

where

$$\mu_G(\mathbf{x}) := \sum_M (-1)^{|M|} \prod_{i \in [n] \setminus V(M)} x_i,$$

and the sum runs over all matchings in G.

The real-rootedness of $\mu_{d,G}(x)$ was proved indirectly in [11] by considering a limit of interlacing families converging to the left-hand side in Theorem 3.1. In this section we use a more direct approach for proving the real-rootedness of $\mu_{d,G}(x)$. In fact we prove something stronger, namely that $\mu_{d,G}(\mathbf{x})$ is stable. Our proof also holds for graphs with loop edges, thus removing the redundant hypothesis in Theorem 3.4.

Coverings of graphs with loop edges have interesting properties. In particular, consider the $|\Gamma|$-dimensional *regular representation* $\mathrm{reg} : \Gamma \to \mathrm{GL}_{|\Gamma|}(\mathbb{C})$ sending an element $g \in \Gamma$ to the permutation matrix afforded by g acting on Γ through left translation $h \mapsto gh$. The *bouquet graph* B_r is the graph consisting of a single vertex with r loop edges. A (Γ, reg)-covering $A_{\gamma,\mathrm{reg}}$ of B_r is equivalent to the Cayley graph of Γ with respect to the set $\gamma(E(B_r))$. In this sense (Γ, reg)-coverings of finite multigraphs with loops generalize Cayley graphs of finite groups.

Example 3.6. Let $\Gamma = S_3$, $G = B_2$ and $\pi = \mathrm{reg} : S_3 \to \mathrm{GL}_6(\mathbb{C})$. Consider the S_3-labeling $\gamma = (\sigma_1, \sigma_2) = ((1\,2\,3), (1\,2))$ of B_2 as in Figure 3 (left). Then

$$A \coloneqq \mathrm{reg}(\sigma_1) + \mathrm{reg}(\sigma_2) = \begin{array}{c} \\ \iota \\ (1\,2) \\ (1\,3) \\ (2\,3) \\ (1\,2\,3) \\ (1\,3\,2) \end{array} \begin{array}{c} \begin{array}{cccccc} \iota & (1\,2) & (1\,3) & (2\,3) & (1\,2\,3) & (1\,3\,2) \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

is the adjacency matrix of the Cayley graph of S_3 with respect to the set $\{\sigma_1, \sigma_2\}$, see Figure 3 (right), and the (S_3, reg)-covering $A_{\gamma,\mathrm{reg}}$ is given by $A + A^T$.

Choe, Oxley, Sokal and Wagner [6] (see also [4]) consider the *multi-affine part* operator

$$\begin{aligned} \mathrm{MAP} : \mathbb{C}[x_1, \ldots, x_n] &\to \mathbb{C}[x_1, \ldots, x_n] \\ \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a(\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\alpha}} &\mapsto \sum_{\boldsymbol{\alpha} : \alpha_i \leq 1, i \in [n]} a(\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\alpha}} \end{aligned}$$

and note that it is a stability preserving linear operator. Indeed the symbol

$$G_{\mathrm{MAP}}(\mathbf{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} (-1)^{\boldsymbol{\alpha}} \mathrm{MAP}(\mathbf{x}^{\boldsymbol{\alpha}}) \frac{\mathbf{y}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^n \\ \alpha_i \leq 1}} (-1)^{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\alpha}} = \prod_{i=1}^{n} (1 - x_i y_i),$$

is stable being a product of stable polynomials. Since the range of MAP has dimension greater than one, it follows that MAP preserves stability by Theorem 2.2. Given the identity

$$P_G(\mathbf{x}) \coloneqq \sum_{E \subseteq E(G)} (-1)^{|E|} \prod_{i=1}^{n} x_i^{\deg_{G[E]}(i)} = \prod_{\{i,j\} \in E(G)} (1 - x_i x_j), \tag{3.2}$$

where $G[E]$ is the subgraph of G induced by $E \subseteq E(G)$ and $\deg_{G[E]}(i)$ denotes the degree of i in $G[E]$, we have that

$$\mathrm{MAP}[P_G(\mathbf{x})] = \mu_G(\mathbf{x}),$$

and hence that $\mu_G(\mathbf{x})$ is stable being the image of a stable polynomial under MAP. This result is also known as the Heilmann-Lieb theorem [12].

By using Theorem 2.3 and the stability preserving linear operator MAP we will show below that $\mu_{d,G}(\mathbf{x})$ is stable.

Theorem 3.7. *If G is a finite multigraph (possibly with loops), then $\mu_{d,G}(\mathbf{x})$ is stable for all $d \geq 1$.*

Proof. For a d-covering $\sigma : E(G) \to S_d$, let

$$P_{\sigma,G}(\mathbf{x}) \coloneqq \left(\prod_{e \in E^+(G) \setminus E_\circ^+(G)} \prod_{k=1}^{d} (1 - x_{h(e)k} x_{t(e)\sigma_e(k)}) \right) \times \left(\prod_{e \in E_\circ^+(G)} \prod_{k : \sigma_e(k) \neq k} (1 - x_{h(e)k} x_{t(e)\sigma_e(k)}) \right),$$

where $E_\circ^+(G) \coloneqq \{e \in E^+(G) : h(e) = t(e)\}$ denotes the set of positively oriented loops in G. Since no matching may contain loops we have excluded the factors $(1 - x_{ik}^2)$ from the subgraph generating polynomial $P_H(\mathbf{x})$ in (3.2) where H is the covering graph corresponding to σ. This explains the form of $P_{\sigma,G}(\mathbf{x})$. It follows that

$$\mathrm{MAP}[P_{\sigma,G}(\mathbf{x})] = \mu_H(\mathbf{x}).$$

We have

$$\begin{aligned}\mathbb{E}_{\sigma\in\mathcal{C}_{d,G}} P_{\sigma,G}(\mathbf{x}) &= \sum_{\sigma\in\mathcal{C}_{d,G}} \frac{1}{|\mathcal{C}_{d,G}|} P_{\sigma,G}(\mathbf{x}) \\ &= \frac{1}{|\mathcal{C}_{d,G}|} \left(\prod_{e\in E^+(G)\setminus E_\circ^+(G)} \sum_{\sigma_e\in S_d} \prod_{k=1}^{d} (1 - x_{h(e)k} x_{t(e)\sigma_e(k)}) \right) \times \\ &\qquad \left(\prod_{e\in E_\circ^+(G)} \sum_{\sigma_e\in S_d} \prod_{k:\sigma_e(k)\neq k} (1 - x_{h(e)k} x_{t(e)\sigma_e(k)}) \right).\end{aligned}$$

For $e \in E^+(G) \setminus E_\circ^+(G)$ the polynomials

$$f_e(\mathbf{x}) = \sum_{\sigma_e\in S_d} \prod_{k=1}^{d} (1 - x_{h(e)k} x_{t(e)\sigma_e(k)}),$$

are symmetric and multiaffine polynomials in the two sets of variables

$$\{x_{h(e)k} : 1 \le k \le d\} \quad \text{and} \quad \{x_{t(e)k} : 1 \le k \le d\},$$

respectively. By Theorem 2.3 we have that $f_e(\mathbf{x})$ is stable if and only if

$$\sum_{\sigma_e\in S_d} \prod_{k=1}^{d} (1 - xy) = d!(1 - xy)^d,$$

is stable, the latter of which is clear. Similary if $e \in E_\circ^+(G)$, then $f_e(\mathbf{x})$ is symmetric and multiaffine in the set of variables $\{x_{h(e)k} : 1 \le k \le d\}$, so checking stability of $f_e(\mathbf{x})$ reduces by Theorem 2.3 to checking the stability of $d!(1 - x^2)^d$, which is again clear. Hence $\mathbb{E}_{\sigma\in\mathcal{C}_{S_d,G}} P_{\sigma,G}(\mathbf{x})$ is stable being a product of stable polynomials. Finally we have

$$\begin{aligned}\mathrm{MAP}\left[\mathbb{E}_{\sigma\in\mathcal{C}_{d,G}} P_{\sigma,G}(\mathbf{x})\right] &= \mathbb{E}_{\sigma\in\mathcal{C}_{d,G}} \mathrm{MAP}[P_{\sigma,G}(\mathbf{x})] \\ &= \mathbb{E}_{H\in\mathcal{C}_{d,G}} \mu_H(\mathbf{x}) \\ &= \mu_{d,G}(\mathbf{x}).\end{aligned}$$

Hence $\mu_{d,G}(\mathbf{x})$ is stable. □

Corollary 3.8. *If G is a finite multigraph (possibly with loops), then $\mu_{d,G}(x)$ is real-rooted for all $d \ge 1$.*

Proof. Follows by putting $\mathbf{x} = (x, \ldots, x)$ in Theorem 3.7 □

4. Stable expected matching polynomials over induced subgraphs

In the previous section we considered stable averages of multivariate matching polynomials over the set of d-sheeted covering graphs of G. In this section we consider stable averages over (vertex-) induced subgraphs of G. To this end, if $S \subseteq [n]$, let $G[S]$ denote the subgraph of G induced by the vertices in S. Let $\mathbb{P}$ be a probability distribution on the power set $\mathcal{P}([n]) := \{S : S \subseteq [n]\}$. The polynomial

$$Z_{\mathbb{P}}(\mathbf{x}) = \sum_{S \subseteq [n]} \mathbb{P}(S)\mathbf{x}^S \in \mathbb{R}[x_1, \ldots, x_n],$$

is called the *partition function* of $\mathbb{P}$. The probability distribution $\mathbb{P}$ is called *Rayleigh* if

$$Z_{\mathbb{P}}(\mathbf{x})\frac{\partial^2 Z_{\mathbb{P}}(\mathbf{x})}{\partial x_i \partial x_j} \leq \frac{\partial Z_{\mathbb{P}}(\mathbf{x})}{\partial x_i}\frac{\partial Z_{\mathbb{P}}(\mathbf{x})}{\partial x_j} \tag{4.1}$$

for all $\mathbf{x} \in \mathbb{R}^n_+$, $1 \leq i, j \leq n$ and is called *strong Rayleigh* if (4.1) holds for all $\mathbf{x} \in \mathbb{R}^n$, $1 \leq i, j \leq n$.

Theorem 4.1 (Brändén [5])**.** *A probability distribution $\mathbb{P}$ is strong Rayleigh if and only if $Z_{\mathbb{P}}(\mathbf{x})$ is stable.*

Proposition 4.2. *Let $G = (V(G), E(G))$ be a finite undirected graph on $[n]$ and let $\mathbb{P}$ be a probability distribution on $\mathcal{P}([n])$. If $\mathbb{P}$ is strong Rayleigh, then $\mathbb{E}^{\mathbb{P}}_{S \subseteq [n]} \mu_{G[S]}(\mathbf{x})$ is stable.*

Proof. By Theorem 2.1 the linear operator

$$T_G := \prod_{\{i,j\} \in E(G)} (1 - \partial_i \partial_j)$$

preserves stability. Moreover it is easy to see that for $S \subseteq [n]$,

$$T_G(\mathbf{x}^S) = \mu_{G[S]}(\mathbf{x}).$$

If $\mathbb{P}$ is strong Rayleigh, then $Z_{\mathbb{P}}(\mathbf{x})$ is stable by Theorem 4.1. Hence

$$\begin{aligned} T_G(Z_{\mathbb{P}}(\mathbf{x})) &= \sum_{S \subseteq [n]} \mathbb{P}(S) T_G(\mathbf{x}^S) \\ &= \sum_{S \subseteq [n]} \mathbb{P}(S) \mu_{G[S]}(\mathbf{x}) \\ &= \mathbb{E}^{\mathbb{P}}_{S \in \subseteq [n]} \mu_{G[S]}(\mathbf{x}), \end{aligned}$$

is stable. □

Corollary 4.3. *If $\mathbb{P}$ is a strong Rayleigh probability distribution, then $\mathbb{E}^{\mathbb{P}}_{S \subseteq [n]} \mu_{G[S]}(x)$ is real-rooted.*

Example 4.4. The following example demonstrates that the converse to Proposition 4.2 is false. Consider the graph $G = \bullet\!-\!\bullet$ on two vertices and one edge. If $\mathbb{P}$ is a probability distribution with $\mathbb{P}(\{1,2\}) = a$, $\mathbb{P}(\{1\}) = b$, $\mathbb{P}(\{1,2\}) = c$ and $\mathbb{P}(\emptyset) = d$, then

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}_{S \subseteq [2]} \mu_{G[S]}(\mathbf{x}) &= a\mu_G(\mathbf{x}) + b\mu_{G[1]}(\mathbf{x}) + c\mu_{G[2]}(\mathbf{x}) + d\mu_{G[\emptyset]}(\mathbf{x}) \\ &= a(-1 + x_1 x_2) + bx_1 + cx_2 + d, \end{aligned}$$

which is stable if and only of $bc - a(-a + d) \geq 0$. On the other hand

$$Z_{\mathbb{P}}(\mathbf{x}) = ax_1x_2 + bx_1 + cx_2 + d,$$

is stable if and only if $bc - ad \geq 0$. Hence there exists probability distributions $\mathbb{P}$ which are not strong Rayleigh for which $\mathbb{E}^{\mathbb{P}}_{S\subseteq[n]}\mu_{G[S]}(\mathbf{x})$ is stable. An interesting question would be to characterize the probability distributions for which $\mathbb{E}^{\mathbb{P}}_{S\subseteq[n]}\mu_{G[S]}(\mathbf{x})$ is stable.

Example 4.5. A natural probability distribution $\mathbb{P}$ on the set of induced subgraphs of G is the Bernoulli distribution $\mathbb{B}$ where a vertex $i \in [n]$ is selected independently with probability p_i and not selected with probability $1 - p_i$. Note that $\mathbb{B}$ is a strong Rayleigh probability distribution since

$$Z_{\mathbb{B}}(\mathbf{x}) = \sum_{S\subseteq[n]} \mathbb{B}(S)\mathbf{x}^S = \sum_{S\subseteq[n]} \prod_{i\in S} p_i \prod_{i\in[n]\setminus S} (1 - p_i)\mathbf{x}^S = \prod_{i=1}^{n}((1 - p_i) + p_ix_i),$$

is stable. Hence $\mathbb{E}^{\mathbb{B}}_{S\subseteq[n]}\mu_{G[S]}(\mathbf{x})$ is stable by Proposition 4.2.

Next we shall provide bounds for the real roots of $\mathbb{E}^{\mathbb{P}}_{S\subseteq[n]}\mu_{G[S]}(x)$.

Let $i \in V(G)$ and define a graph $U_i(G)$ with vertex set being the set of all non-backtracking walks in G starting from i, i.e., sequences $(i_0, i_1, \ldots, i_k)$ such that $i_0 = i$, i_r and i_{r+1} are adjacent and $i_{r+1} \neq i_{r-1}$. Two such walks are connected by an edge in $U_i(G)$ if one walk extends the other by one vertex, i.e., $(i_0, \ldots, i_k, i_{k+1})$ is adjacent to $(i_0, \ldots, i_k)$. The graph thus constructed is a tree that covers G. It is called the *universal covering tree* of G. The universal covering tree $U_i(G)$ of G is unique up to isomorphism and has the property that it covers every other covering of G. Thus we henceforth remove reference to the root i and write $U(G)$ for the universal covering tree of G. The tree $U(G)$ is countably infinite, unless G is a finite tree, in which case $U(G) = G$.

The *spectral radius* $r(G)$ of a finite graph G is the largest absolute eigenvalue of the adjacency matrix A_G of G. By a theorem of Mohar [15] the spectral radius of an infinite graph U can be defined as follows,

$$r(U) \coloneqq \sup\{r(G) : G \text{ is a finite induced subgraph of } U\}.$$

If G is a finite undirected graph, then let $\rho(G) \coloneqq r(U(G))$ denote the spectral radius of its universal covering tree. Say that a probability distribution $\mathbb{P}$ on $\mathcal{P}([n])$ has *constant parity* if the set $\{|S| : S \subseteq [n],\ \mathbb{P}(S) > 0\}$ consists of numbers with the same parity (i.e. are either all odd or all even).

Proposition 4.6. *Let G be a finite undirected graph with n vertices and $\mathbb{P}$ a probability distribution on $\mathcal{P}([n])$. Then the real roots of $\mathbb{E}^{\mathbb{P}}_{S\subseteq[n]}\mu_{G[S]}(x)$ are bounded above by $\rho(G)$. Moreover if $\mathbb{P}$ has constant parity, then the real roots of $\mathbb{E}^{\mathbb{P}}_{S\subseteq[n]}\mu_{G[S]}(x)$ are contained in $[-\rho(G), \rho(G)]$.*

Proof. Let $S \subseteq [n]$. There is a clear injective embedding of $U(G[S])$ into $U(G)$ such that any finite induced subgraph of $U(G[S])$ is an induced subgraph of $U(G)$. Therefore $\rho(G[S]) \leq \rho(G)$. Heilmann and Lieb [12] showed that for any finite graph G, the roots of $\mu_G(x)$ are contained in $[-\rho(G), \rho(G)]$. Therefore $\mathbb{E}^{\mathbb{P}}_{S\subseteq[n]}\mu_{G[S]}(x) > 0$ for all $x \in (\rho(G), \infty)$, being a convex combination of monic polynomials with the same property. Hence the real roots of the expectation are bounded above by $\rho(G)$. If $\mathbb{P}$ also has constant parity, then $\mathbb{E}^{\mathbb{P}}_{S\subseteq[n]}\mu_{G[S]}(x)$ is a convex combination of monic polynomials with same degree parity and are therefore, by

above, strictly positive or strictly negative on the interval $(-\infty, -\rho(G))$. Hence the real roots are contained in $[-\rho(G), \rho(G)]$. □

Corollary 4.7. *Let G be a finite undirected graph on n vertices and $k \in [n]$. Then the uniform average of all matching polynomials over the set of induced size k-subgraphs of G is a real-rooted polynomial with all roots contained in the interval $[-\rho(G), \rho(G)]$.*

Proof. Let $\mathbb{P}$ be the probability distribution on $\mathcal{P}([n])$ with uniform support on $\binom{[n]}{k}$. Then

$$Z_{\mathbb{P}}(\mathbf{x}) = \frac{1}{\binom{n}{k}} e_k(\mathbf{x}),$$

where $e_k(\mathbf{x})$ denotes the elementary symmetric polynomial of degree k. The polynomial $e_k(\mathbf{x})$ is stable, e.g. by Theorem 2.3. Therefore $\mathbb{P}$ is a strong Rayleigh probability distribution by Theorem 4.1, so the statement follows by Corollary 4.3 and Proposition 4.6. □

5. Stable relaxed matching polynomials

A hypergraph $H = (V(H), E(H))$ is a set of vertices $V(H) = [n]$ together with a family of subsets $E(H)$ of $V(H)$ called *hyperedges* (or *edges* for short). The *degree* of a vertex $i \in V(H)$ is defined as $\deg_H(i) \coloneqq |\{e \in E(H) : i \in e\}|$. In analogy with graph matchings, a matching in a hypergraph consists of a subset of edges with empty pairwise intersection. Although the matching polynomial $\mu_G(x)$ of a graph G is real-rooted, the analogous polynomial for hypergraphs is not real-rooted in general, see e.g. [24]. From the point of view of real-rootedness we consider a weaker notion of matchings that provide a natural generalization of the real-rootedness property of $\mu_G(x)$ to hypergraphs.

Let $H = (V(H), E(H))$ be a hypergraph. Define a *relaxed matching* in H to be a collection $M = (S_e)_{e \in E}$ of edge subsets such that $E \subseteq E(H)$, $S_e \subseteq e$, $|S_e| > 1$ and $S_e \cap S_{e'} = \emptyset$ for all pairwise distinct $e, e' \in E$ (see Figure 4).

Remark 5.1. If H is a graph then the concept of relaxed matching coincides with the conventional notion of graph matching. Note also that a conventional hypergraph matching is a relaxed matching $M = (S_e)_{e \in E}$ for which $S_e = e$ for all $e \in E$.

Remark 5.2. The subsets S_e in the relaxed matching are labeled by the edge they are chosen from in order to avoid ambiguity. However if H is a *linear hypergraph*, that is, the edges pairwise intersect in at most one vertex, then the subsets uniquely determine the edges they belong to and therefore no labeling is necessary. Graphs and finite projective geometries (viewed as hypergraphs) are examples of linear hypergraphs.

Let $V(M) \coloneqq \bigcup_{S_e \in M} S_e$ denote the set of vertices in the relaxed matching. Moreover let $m_k(M) \coloneqq |\{S_e \in M : |S_e| = k\}|$ denote the number of subsets in the relaxed matching of size k. Define the *multivariate relaxed matching polynomial* of H by

$$\eta_H(\mathbf{x}) \coloneqq \sum_M (-1)^{|M|} W(M) \prod_{i \in [n] \setminus V(M)} x_i,$$

where the sum runs over all relaxed matchings of H and

$$W(M) \coloneqq \prod_{k=1}^{n-1} k^{m_{k+1}(M)}.$$

Let $\eta_H(x) \coloneqq \eta_H(x\mathbf{1})$ denote the *univariate relaxed matching polynomial.*

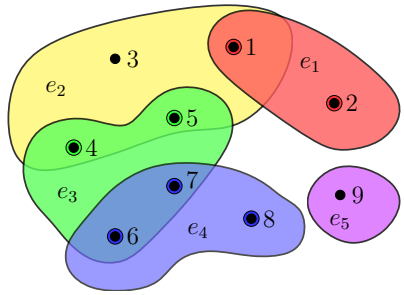

FIGURE 4. A relaxed matching $M = (S_{e_1}, S_{e_3}, S_{e_4})$ in a hypergraph H with $S_{e_1} = \{1, 2\}$, $S_{e_3} = \{4, 5\}$ and $S_{e_4} = \{6, 7, 8\}$.

Remark 5.3. Note that $\eta_H(x) = \mu_H(x)$ if H is a graph.

Our aim is to prove that $\eta_H(\mathbf{x})$ is a stable polynomial. In fact we shall prove the stability of a more general polynomial accommodating for arbitrary degree restrictions on each vertex.

Define a *relaxed subgraph* of H to be a hypergraph $K = (E(K), V(K))$ with edges $E(K) := (S_e)_{e\in E}$ such that $E \subseteq E(H)$, $S_e \subseteq e$ and $|S_e| > 1$ for $e \in E$ with $V(K) := \bigcup_{e\in E} S_e$. Again if H is a graph, then the notion of a relaxed subgraph coincides with the conventional notion of a (edge-induced) subgraph of H. Let $\boldsymbol{\kappa} = (\kappa_1, \dots \kappa_n) \in \mathbb{N}^n$. Define a *relaxed $\boldsymbol{\kappa}$-subgraph* of H to be a relaxed subgraph $K^{\boldsymbol{\kappa}}$ of H such that $\deg_{K^{\boldsymbol{\kappa}}}(i) \le \kappa_i$ for $i \in V(K^{\boldsymbol{\kappa}})$. Let $m_k(K^{\boldsymbol{\kappa}}) := |\{S_e \in E(K^{\boldsymbol{\kappa}}) : |S_e| = k\}|$ and let $(n)_k = n(n-1)\cdots(n-k+1)$ denote the *Pochhammer symbol*.

Define the *multivariate relaxed $\boldsymbol{\kappa}$-subgraph polynomial* of H by

$$\eta_H^{\boldsymbol{\kappa}}(\mathbf{x}) := \sum_{K^{\boldsymbol{\kappa}}} (-1)^{|E(K^{\boldsymbol{\kappa}})|} W(K^{\boldsymbol{\kappa}}) \prod_{i\in[n]\setminus V(K^{\boldsymbol{\kappa}})} x_i^{\kappa_i - \deg_{K^{\boldsymbol{\kappa}}}(i)},$$

where the sum runs over all relaxed $\boldsymbol{\kappa}$-subgraphs $K^{\boldsymbol{\kappa}}$ of H and

$$W(K^{\boldsymbol{\kappa}}) := \prod_{k=1}^{n-1} k^{m_{k+1}(K^{\boldsymbol{\kappa}})} \prod_{i\in V(K^{\boldsymbol{\kappa}})} (\kappa_i)_{\deg_{K^{\boldsymbol{\kappa}}}(i)}.$$

Remark 5.4. Note that a relaxed matching in H is the same as a relaxed $(1, \dots, 1)$-subgraph of H and that $\eta_H^{(1,\dots,1)}(\mathbf{x}) = \eta_H(\mathbf{x})$.

In the rest of this section we will adopt the following notation,

$$\boldsymbol{\partial}_S := \sum_{i\in S} \partial_i, \quad \boldsymbol{\partial}^S := \prod_{i\in S} \partial_i, \quad \boldsymbol{\partial}^{\boldsymbol{\alpha}} := \prod_{i=1}^{n} \partial_i^{\alpha_i},$$

where $S \subseteq [n]$ and $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{N}^n$.

With abuse of notation we shall let the multiaffine part operator MAP act analogously on polynomial spaces of differential operators as follows,

$$\begin{aligned} \mathrm{MAP} : \mathbb{C}[\partial_1, \dots, \partial_n] &\to \mathbb{C}[\partial_1, \dots, \partial_n] \\ \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a(\boldsymbol{\alpha})\boldsymbol{\partial}^{\boldsymbol{\alpha}} &\mapsto \sum_{\boldsymbol{\alpha}:\alpha_i\le 1, i\in[n]} a(\boldsymbol{\alpha})\boldsymbol{\partial}^{\boldsymbol{\alpha}}. \end{aligned}$$

The following lemma follows from Theorem 2.1.

Lemma 5.5. *If $P(\boldsymbol{\partial}) \in \mathbb{C}[\partial_1, \dots, \partial_n]$ is a linear operator such that $P(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_n]$ is stable, then $MAP[P(\boldsymbol{\partial})]$ preserves stability.*

Proof. Write $P(\boldsymbol{\partial}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}} a(\boldsymbol{\alpha}) \boldsymbol{\partial}^{\boldsymbol{\alpha}}$. Since $\mathrm{MAP} : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$ is a stability preserver we have that $\mathrm{MAP}\left[\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a(\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\alpha}}\right] = \sum_{\boldsymbol{\alpha}: \alpha_i \leq 1, i \in [n]} \mathbf{x}^{\boldsymbol{\alpha}}$ is stable and hence by Theorem 2.1 that $\sum_{\boldsymbol{\alpha}: \alpha_i \leq 1, i \in [n]} \boldsymbol{\partial}^{\boldsymbol{\alpha}} = \mathrm{MAP}\left[\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a(\boldsymbol{\alpha}) \boldsymbol{\partial}^{\boldsymbol{\alpha}}\right]$ is a stability preserving linear operator. □

Theorem 5.6. *Let $H = (V(H), E(H))$ be a hypergraph and $\boldsymbol{\kappa} = (\kappa_i) \in \mathbb{N}^n$. Then the multivariate relaxed $\boldsymbol{\kappa}$-subgraph polynomial $\eta_H^{\boldsymbol{\kappa}}(\mathbf{x})$ is stable with*

$$\eta_H^{\boldsymbol{\kappa}}(\mathbf{x}) = \prod_{e \in E(H)} MAP\left[(1 - \boldsymbol{\partial}_e) \prod_{i \in e} (1 + \partial_i)\right] \mathbf{x}^{\boldsymbol{\kappa}}.$$

Proof. Let $e \in E(H)$. Then

$$\begin{aligned}
(1 - \boldsymbol{\partial}_e) \prod_{i \in e} (1 + \partial_i) &= (1 - \boldsymbol{\partial}_e) \left(1 + \sum_{\emptyset \neq S \subseteq e} \boldsymbol{\partial}^S\right) \\
&= 1 + \sum_{\substack{\emptyset \neq S \subseteq e \\ |S| > 1}} \boldsymbol{\partial}^S - \sum_{\emptyset \neq S \subseteq e} \boldsymbol{\partial}_e \boldsymbol{\partial}^S \\
&= 1 + \sum_{\substack{\emptyset \neq S \subseteq e \\ |S| > 1}} \boldsymbol{\partial}^S - \sum_{i \in e} \left(\sum_{\substack{\emptyset \neq S \subseteq e \\ i \in S}} \partial_i \boldsymbol{\partial}^S + \sum_{\substack{\emptyset \neq S \subseteq e \\ i \notin S}} \boldsymbol{\partial}^{S \cup i}\right) \\
&= 1 - \sum_{\substack{\emptyset \neq S \subseteq e \\ |S| > 1}} (|S| - 1) \boldsymbol{\partial}^S - \sum_{i \in e} \sum_{\substack{\emptyset \neq S \subseteq e \\ i \in S}} \partial_i \boldsymbol{\partial}^S.
\end{aligned}$$

Thus since

$$\left(1 - \sum_{i \in e} x_i\right) \prod_{i \in e} (1 + x_i),$$

is a stable polynomial, being a product of stable linear factors, it follows by Lemma 5.5 that

$$\mathrm{MAP}\left[(1 - \boldsymbol{\partial}_e) \prod_{i \in e} (1 + \partial_i)\right] = 1 - \sum_{\substack{\emptyset \neq S \subseteq e \\ |S| > 1}} (|S| - 1) \boldsymbol{\partial}^S,$$

is stability preserving. Hence

$$\begin{aligned}\prod_{e\in E(H)} \mathrm{MAP}\left[(1-\boldsymbol{\partial}_e)\prod_{i\in e}(1+\partial_i)\right]\mathbf{x}^{\boldsymbol{\kappa}} &= \prod_{e\in E(H)}\left(1-\sum_{\substack{\emptyset\neq S\subseteq e\\ |S|>1}}(|S|-1)\boldsymbol{\partial}^S\right)\mathbf{x}^{\boldsymbol{\kappa}}\\ &= \sum_{E\subseteq E(H)}(-1)^{|E|}\sum_{\substack{(S_e)_{e\in E}\\ S_e\subseteq e\\ |S_e|>1}}\prod_{e\in E}(|S_e|-1)\boldsymbol{\partial}^{S_e}\mathbf{x}^{\boldsymbol{\kappa}}\\ &= \sum_{K^{\boldsymbol{\kappa}}}(-1)^{|E(K^{\boldsymbol{\kappa}})|}W(K^{\boldsymbol{\kappa}})\prod_{i\in[n]\setminus V(K^{\boldsymbol{\kappa}})} x_i^{\kappa_i-\deg_{K^{\boldsymbol{\kappa}}}(i)}\\ &= \eta_H^{\boldsymbol{\kappa}}(\mathbf{x}),\end{aligned}$$

is a stable polynomial. □

The following corollary is immediate from Theorem 5.6.

Corollary 5.7. *The multivariate relaxed matching polynomial $\eta_H(\mathbf{x})$ is stable with*

$$\eta_H(\mathbf{x}) = \prod_{e\in E(H)}(1-\boldsymbol{\partial}_e)\prod_{i=1}^{n}(1+\partial_i)^{deg_H(i)}\mathbf{x}^{\mathbf{1}}.$$

In particular the univariate relaxed matching polynomial

$$\eta_H(x) = \sum_M (-1)^{|M|}W(M)x^{n-|V(M)|},$$

is a real-rooted polynomial for any hypergraph H.

Below follows a generalization of the standard identities for the multivariate matching polynomial $\mu_G(\mathbf{x})$. Let $i\in V(H)$. Recall that the *(weak) vertex-deletion* $H\setminus i$ is the hypergraph with vertex set $V(H)\setminus i$ and edges $\{e\cap(V(H)\setminus i) : e\in E(H)\}$. Let $e\in E(H)$. The *edge-deletion* $H\setminus e$ is the subhypergraph of H with vertex set $V(H)$ and edges $E(H)\setminus e$. Let $I_H(i) := \{e\in E(H) : i\in e\}$ denote the *incidence set* of $i\in V(H)$. The following identities are straightforward to verify.

Proposition 5.8. *Let $H=(V(H),E(H))$ be a hypergraph, $i\in V(H)$ and $e\in E(H)$. Then $\eta_H(\mathbf{x})$ satisfies the following identities:*

(i) $\eta_H(\mathbf{x}) = \eta_{H\setminus e}(\mathbf{x}) - \sum_{\substack{S\subseteq e\\ |S|>1}}(|S|-1)\eta_{(H\setminus e)\setminus S}(\mathbf{x})$,

(ii) $\eta_H(\mathbf{x}) = x_i\eta_{H\setminus i}(\mathbf{x}) - \sum_{e\in I_H(i)}\sum_{\substack{S\subseteq e\\ i\in S\\ |S|>1}}(|S|-1)\eta_{(H\setminus e)\setminus S}(\mathbf{x})$,

(iii) $\eta_{H_1\sqcup H_2}(\mathbf{x}) = \eta_{H_1}(\mathbf{x})\eta_{H_2}(\mathbf{x})$,

(iv) $\partial_i\eta_H(\mathbf{x}) = \eta_{H\setminus i}(\mathbf{x})$.

It would be interesting to understand what parts of the matching theory for graphs can be extended to relaxed matchings.

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Paper D

EQUIDISTRIBUTIONS OF MAHONIAN STATISTICS OVER PATTERN AVOIDING PERMUTATIONS

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Abstract. A Mahonian d-function is a Mahonian statistic that can be expressed as a linear combination of vincular pattern functions of length at most d. Babson and Steingrímsson classified all Mahonian 3-functions up to trivial bijections and identified many of them with well-known Mahonian statistics in the literature. We prove a host of Mahonian 3-function equidistributions over permutations in $\mathcal{S}_n$ avoiding a single classical pattern in $\mathcal{S}_3$. Tools used include block decomposition, Dyck paths and generating functions.

Contents

1. Introduction

A *combinatorial statistic* on a set S is a map stat : $S \to \mathbb{N}$. The *distribution* of stat over S is given by the coefficients of the generating function $\sum_{\sigma\in S} q^{\mathrm{stat}(\sigma)}$. Let $\mathcal{S}_n$ be the set of permutations $\sigma = a_1a_2\cdots a_n$ of the letters $[n] = \{1, 2, \ldots, n\}$ and let $\sigma(k)$ denote the entry a_k. Let $\mathcal{S} = \bigcup_{n\geq 0}\mathcal{S}_n$. The *inversion set* of $\sigma \in \mathcal{S}_n$ is defined by $\mathrm{Inv}(\sigma) = \{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$. A particularly well-studied statistic on $\mathcal{S}_n$ is inv : $\mathcal{S}_n \to \mathbb{N}$, given by $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$. An elegant formula for the distribution of the inversion statistic was found in 1839 by Rodrigues [27]

$$\sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!,$$

where $[n]_q! = [1]_q[2]_q\cdots[n]_q$ and $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. The *descent set* of σ is defined by $\mathrm{Des}(\sigma) = \{i : \sigma(i) > \sigma(i + 1)\}$. In 1915 MacMahon [25] showed that inv has the same distribution as another statistic, now called the *major index* (due to MacMahon's profession as a major in the british army) [17], given by $\mathrm{maj}(\sigma) = \sum_{i\in\mathrm{Des}(\sigma)} i$. We also write $\mathrm{imaj}(\sigma) = \mathrm{maj}(\sigma^{-1})$. In honor of MacMahon any permutation statistic with the same distribution as maj is called *Mahonian*. Mahonian statistics are well-studied in the literature. Since MacMahon's initial work, many new Mahonian statistics have been identified. Babson and Steingrímsson [1] showed that almost all (at the time) known Mahonian statistics can be expressed as linear combinations of statistics counting occurrences of vincular patterns. They

made several further conjectures regarding new vincular pattern-based Mahonian statistics. These have since been proved and reproved at various levels of refinement by a number of authors (see e.g., [4, 7, 18, 33]). Two sequences of integers $a_1a_2\cdots a_n$ and $b_1b_2\cdots b_n$ are said to be *order isomorphic* provided $a_i < a_j$ if and only if $b_i < b_j$ for all $1 \leq i < j \leq n$. A *vincular pattern* (also known as *generalized pattern*) of length m is a pair (π, X) where π is a permutation in $\mathcal{S}_m$ and $X \subseteq \{0, 1, \ldots, m\}$ is a set of adjacencies. Adjacencies are indicated by underlining the adjacent entries in π (see Example 1.1). If $0 \in X$ (respectively, $m \in X$), then we denote this by adding a square bracket at the beginning (respectively, end) of the pattern π. If $X = \emptyset$, then (π, X) coincides with the definition of a *classical pattern*. A permutation $\sigma = a_1a_2\cdots a_n \in \mathcal{S}_n$ *contains the vincular pattern* (π, X) if there is an m-tuple $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that the following criteria are satisfied

- $a_{i_1}a_{i_2}\cdots a_{i_m}$ is order-isomorphic to π,
- $i_{j+1} = i_j + 1$ for each $j \in X \setminus \{0, m\}$ and
- $i_1 = 1$ if $0 \in X$ and $i_m = n$ if $m \in X$.

We also say that $a_{i_1}a_{i_2}\cdots a_{i_m}$ is an *occurrence* of π in σ. We say that σ *avoids* π if σ contains no occurrences of π. We denote the set of permutations in $\mathcal{S}_n$ avoiding the pattern π by $\mathcal{S}_n(\pi)$. Moreover if Π is a set of patterns, then we set $\mathcal{S}_n(\Pi) = \bigcap_{\pi\in\Pi} \mathcal{S}_n(\pi)$.

In this paper we shall also need an additional generalization of vincular patterns, allowing us to restrict occurrences to particular value requirements. Let $\upsilon = (\upsilon_1, \ldots, \upsilon_m)$ where $\upsilon_i \in \mathbb{N} \sqcup \{\text{-}\}$. Define a *value-restricted vincular pattern* $(\pi, X)\big|_\upsilon$ to be a triple (π, X, υ) where (π, X) is a vincular pattern. We say that $a_{i_1}a_{i_2}\cdots a_{i_m}$ is an *occurrence* of $(\pi, X)\big|_\upsilon$ in σ if it is an occurrence of the vincular pattern (π, X) and $a_{i_j} = \upsilon_j$ whenever $\upsilon_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Note in particular that $(\pi, X)\big|_{(\text{-},\ldots,\text{-})} = (\pi, X)$. Every value-restricted vincular pattern $(\pi, X)\big|_\upsilon$ gives rise to a permutation statistic $(\pi, X)\big|_\upsilon : \mathcal{S}_n \to \mathbb{N}$ called a *pattern function* counting the number of occurrences of $(\pi, X)\big|_\upsilon$ in a given permutation $\sigma \in \mathcal{S}_n$ (see Example 1.1). The *length* of $(\pi, X)\big|_\upsilon : \mathcal{S}_n \to \mathbb{N}$ is defined as the length of the underlying vincular pattern (π, X).

Example 1.1. Let $\sigma = 246153$.

Pattern π	X	Occurrences in σ
231	$\emptyset$	$241, 261, 461, 463, 453$
[231	$\{0\}$	$241, 261$
$\underline{23}1$	$\{1\}$	$241, 461, 463$
$2\underline{31}$	$\{2\}$	$261, 461, 453$
$\underline{231}$	$\{1, 2\}$	461
$2\underline{31}]$	$\{2, 3\}$	453
$\underline{23}1\big\|_{(\text{-},6,\text{-})}$	$\{1\}$	$461, 463$

We also have $(231)\sigma = 5$, $[231)\sigma = 2$, $(\underline{23}1)\sigma = 3$, $(2\underline{31})\sigma = 3$, $(\underline{231})\sigma = 1$, $(2\underline{31}]\sigma = 1$ and $(\underline{23}1)\big|_{(\text{-},6,\text{-})}\sigma = 2$. On the other hand, the permutation $\sigma = 215346$ avoids the pattern $\pi = 231$ (and hence all the patterns in the table above).

In this paper we mainly study equidistributions of the form

$$\sum_{\sigma\in\mathcal{S}_n(\Pi_1)} q^{\mathrm{stat}_1(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(\Pi_2)} q^{\mathrm{stat}_2(\sigma)} \tag{1.1}$$

where Π_1, Π_2 are sets of patterns and $\mathrm{stat}_1, \mathrm{stat}_2$ are permutation statistics. We will almost exclusively focus on the case where Π_i consists of a single classical pattern of length three and stat_i is a Mahonian statistic. The equidistributions we prove are summarized in §5, Table 2. Although Mahonian statistics are equidistributed over $\mathcal{S}_n$, they need not be equidistributed over pattern avoiding sets of permutations. For instance maj and inv are not equidistributed over $\mathcal{S}_n(\pi)$ for any classical pattern $\pi \in \mathcal{S}_3$. Neither do the existing bijections in the literature for proving equidistribution over $\mathcal{S}_n$ necessarily restrict to bijections over $\mathcal{S}_n(\pi)$ (cf. [1, 4, 7, 18, 33]). Therefore whenever such an equidistribution is present, we must usually seek a new bijection which simultaneously preserves statistic and pattern avoidance. Another motivation for studying equidistributions over permutations avoiding a classical pattern of length three is that $|\mathcal{S}_n(\pi)| = C_n$ for all $\pi \in \mathcal{S}_3$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number (see [22]). Therefore equidistributions of this kind induce equidistributions between statistics on other Catalan objects (and vice versa) whenever we have bijections where the statistics translate in an appropriate fashion. We prove several results in this vein where an exchange between statistics on $\mathcal{S}_n(\pi)$, Dyck paths and polyominoes takes place. In general, studying the generating function (1.1) provides a rich source of interesting q-analogues to well-known sequences enumerated by pattern avoidance and raises new questions about the coefficients of such polynomials.

Equidistributions such as (1.1) has been studied in the past. For instance, Burstein and Elizalde proved the following result involving the Mahonian *Denert statistic*

$$\mathrm{den}(\sigma) = \mathrm{inv}(\mathrm{Exc}(\sigma)) + \mathrm{inv}(\mathrm{NExc}(\sigma)) + \sum_{\substack{i\in[n]\\ \sigma(i)>i}} i,$$

where $\mathrm{Exc}(\sigma) = (\sigma(i))_{\sigma(i)>i}$ and $\mathrm{NExc}(\sigma) = (\sigma(i))_{\sigma(i)\leq i}$.

Theorem 1.1 (Burstein-Elizalde [5]). *For any $n \geq 1$,*

$$\sum_{\sigma\in\mathcal{S}_n(231)} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(321)} q^{\mathrm{den}(\sigma)}.$$

Two sets of patterns Π_1 and Π_2 are said to be *Wilf-equivalent* if $|\mathcal{S}_n(\Pi_1)| = |\mathcal{S}_n(\Pi_2)|$ for all $n \geq 0$. Sagan and Savage [28] coined a q-analogue of this concept. Two sets of patterns Π_1 and Π_2 are said to be *st-Wilf equivalent* with respect to the statistic $\mathrm{st} : \mathcal{S} \to \mathbb{N}$ if (1.1) holds with $\mathrm{stat}_1 = \mathrm{st} = \mathrm{stat}_2$ for any $n \geq 0$. Let $[\Pi]_{\mathrm{st}}$ denote the st-Wilf class of the set Π. This concept have been studied at several places in the literature. An overview of the st-Wilf classification of single and multiple classical patterns of length three can be found in the table below.

st	Reference
maj, inv	Dokos-Dwyer-Johnson-Sagan-Selsor [14]
charge	Killpatrick [20]
fp, exc, des	Elizalde [15, 16]
peak, valley	Baxter [2]
peak, valley, head, last, lir, rir, lrmin, rank, comp, ldr	Claesson-Kitaev [11]

In particular it was shown in [14] that $I_n(132;q) = I_n(213;q) = C_n(q)$ and $I_n(231;q) =$

$I_n(312;q) = \tilde{C}_n(q)$ where

$$I_n(\pi;q) = \sum_{\sigma\in\mathcal{S}_n(\pi)} q^{\mathrm{inv}(\sigma)},$$

$$C_n(q) = \sum_{k=0}^{n-1} q^{(k+1)(n-k)} C_k(q) C_{n-k-1}(q),\ \ C_0(q) = 1,$$

$$\tilde{C}_n(q) = \sum_{k=0}^{n-1} q^{k} \tilde{C}_k(q) \tilde{C}_{n-k-1}(q),\ \ \tilde{C}_0(q) = 1.$$

The polynomial $C_n(q)$ is known as the *Carlitz-Riordan q-analogue of the Catalan numbers* and have been studied by numerous authors (though no explicit formula is known). Similar recursions for maj have been studied in [8, 14].

To decompose pattern avoiding permutations we will require some notation. Given permutations $\tau \in \mathcal{S}_k$ and $\sigma_1, \sigma_2 \ldots, \sigma_k \in \mathcal{S}$, the *inflation* of τ by $\sigma_1, \sigma_2 \ldots, \sigma_k$ is the permutation $\tau[\sigma_1, \sigma_2, \ldots, \sigma_k]$ obtained by replacing each entry $\tau(i)$ by a block of length $|\sigma_i|$ order isomorphic to σ_i for $i = 1, \ldots, k$ such that the blocks are externally order-isomorphic to τ.

Example 1.2. $231[21, 1, 213] = 546213$.

Let $\sigma \in \mathcal{S}_n$. Recall that the *descent set* of σ is given by $\mathrm{Des}(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$. The set of *descent bottoms* (resp. *descent tops*) of σ is given by $\mathrm{DB}(\sigma) = \{\sigma(i+1) : i \in \mathrm{Des}(\sigma)\}$ (resp. $\mathrm{DT}(\sigma) = \{\sigma(i) : i \in \mathrm{Des}(\sigma)\}$). Likewise the *ascent set* of σ is given by $\mathrm{Asc}(\sigma) = \{i : \sigma(i) < \sigma(i+1)\}$ and we define the set of *ascent bottoms* (resp. *ascent tops*) of σ to be $\mathrm{AB}(\sigma) = \{\sigma(i) : i \in \mathrm{Asc}(\sigma)\}$ (resp. $\mathrm{AT}(\sigma) = \{\sigma(i+1) : i \in \mathrm{Asc}(\sigma)\}$). An entry $\sigma(j)$ is called a *left-to-right maxima* if $\sigma(j) > \sigma(i)$ for all $i < j$. Let $\mathrm{LRMax}(\sigma)$ denote the set of left-to-right maxima in σ and let $\mathrm{lrmax}(\sigma) = |\mathrm{LRMax}(\sigma)|$. Similarly an entry $\sigma(j)$ is called a *left-to-right minima* if $\sigma(j) < \sigma(i)$ for all $i < j$. Let $\mathrm{LRMin}(\sigma)$ denote the set of left-to-right minima in σ and let $\mathrm{lrmin}(\sigma) = |\mathrm{LRMin}(\sigma)|$. We call $\sigma(i)$ a *pinnacle* if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ and $\sigma(i)$ a *trough* if $\sigma(i-1) > \sigma(i) < \sigma(i+1)$.

Example 1.3. Let $\sigma = 271985346$. Then $\mathrm{Des}(\sigma) = \{2, 4, 5, 6\}$, $\mathrm{DB}(\sigma) = \{1, 3, 5, 8\}$, $\mathrm{DT}(\sigma) = \{5, 7, 8, 9\}$, $\mathrm{Asc}(\sigma) = \{1, 3, 7, 8\}$, $\mathrm{AB}(\sigma) = \{1, 2, 3, 4\}$, $\mathrm{AT}(\sigma) = \{4, 6, 7, 9\}$, $\mathrm{LRMax}(\sigma) = \{2, 7, 9\}$, $\mathrm{LRMin}(\sigma) = \{2, 1\}$. The pinnacles of σ are given by $\{7, 9\}$ and the troughs of σ by $\{1, 3\}$.

If $\sigma = a_1 a_2 \cdots a_{n-1} a_n$, then the *reverse* of σ is given by $\sigma^r = a_n a_{n-1} \cdots a_2 a_1$ and the *complement* of σ by $\sigma^c = (n - a_1 + 1)(n - a_2 + 1) \cdots (n - a_{n-1} + 1)(n - a_n + 1)$. The *inverse* of σ (in the group theoretical sense) is denoted by σ^{-1}. The operations complement, reverse and inverse are often referred to as *trivial bijections* and together they generate a group isomorphic to the Dihedral group D_4 of order 8 acting on $\mathcal{S}_n$. If π is a classical pattern and $g \in D_4$, then it is not difficult to see that $\sigma \in \mathcal{S}_n(\pi)$ if and only if $\sigma^g \in \mathcal{S}_n(\pi^g)$. However if π is a non-classical pattern, then avoidance is not necessarily closed under inverse in any similar way. E.g. $\sigma = 6274251$ avoids the vincular pattern $\pi = \underline{12}3$, but $\sigma^{-1} = 7254613$ avoids no vincular pattern (π, X) of length three with $X = \{1\}$ or $X = \{2\}$. Therefore taking the inverse should not be viewed as a 'trivial bijection' in the same sense as complement and reverse when it comes to vincular patterns.

In Table 1 we list the vincular pattern definitions of the Mahonian statistics that we shall consider from [1]. The references in Table 1 indicate where the Mahonian nature of the

Name	Vincular pattern definition	Reference
maj	$(1\underline{32}) + (2\underline{31}) + (3\underline{21}) + (\underline{21})$	MacMahon [25]
inv	$(\underline{23}1) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	MacMahon [25]
mak	$(1\underline{32}) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Foata-Zeilberger [19]
makl	$(1\underline{32}) + (2\underline{31}) + (\underline{32}1) + (\underline{21})$	Clarke-Steingrímsson-Zeng [13]
mad	$(2\underline{31}) + (2\underline{31}) + (\underline{31}2) + (\underline{21})$	Clarke-Steingrímsson-Zeng [13]
bast	$(\underline{13}2) + (\underline{21}3) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[1]
bast′	$(\underline{13}2) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[1]
bast″	$(1\underline{32}) + (3\underline{12}) + (3\underline{21}) + (\underline{21})$	Babson-Steingrímsson[1]
foze	$(\underline{21}3) + (3\underline{21}) + (\underline{13}2) + (\underline{21})$	Foata-Zeilberger [18]
foze′	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Foata-Zeilberger [18]
foze″	$(\underline{23}1) + (\underline{31}2) + (\underline{31}2) + (\underline{21})$	Foata-Zeilberger [18]
sist	$(\underline{13}2) + (\underline{13}2) + (2\underline{13}) + (\underline{21})$	Simion-Stanton [28]
sist′	$(\underline{13}2) + (\underline{13}2) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [28]
sist″	$(\underline{13}2) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [28]

Table 1. Mahonian 3-functions.

statistics was first proved. Some of these statistics where originally defined in a slightly different form. See [1] for their translation into vincular pattern functions.

For example, Foata and Zeilberger introduced the Mahonian statistic mak in [18] where it was essentially defined as

$$\mathrm{mak}(\sigma) = \sum_{\alpha \in \mathrm{DB}(\sigma)} \alpha + (\underline{31}2)\sigma. \tag{1.2}$$

It is easy to see that

$$\sum_{\alpha \in \mathrm{DB}(\sigma)} \alpha = ((1\underline{32}) + (\underline{32}1) + (\underline{21}))\sigma.$$

The statistic mad introduced by Clarke-Steingrímsson-Zeng in [13] is defined similarly by replacing the sum of descent bottoms by the sum of descent differences, i.e., the sum of the differences between the two letters of a descent.

According to [1], Table 1 is the complete list of *Mahonian 3-functions* (up to trivial bijections), i.e., Mahonian statistics that can be written as a sum of vincular pattern functions of length at most three. Since some of these statistics have received no conventional name in the literature, we will take the liberty of naming them according to the initials of the authors who first proved their Mahonian nature.

2. Equidistributions via direct bijection

The equidistributions proved in this section are shown by directly exhibiting a bijection. The bijections are based on standard decompositions of pattern avoiding permutations, or rely on specifying data by which pattern avoiding permutations are uniquely determined. In

many cases we are able to find a more refined equidistribution. We begin by proving that maj and mak are related via the inverse map over certain pattern avoiding sets of permutations. This may seem unexpected given that vincular patterns do not behave as straightforwardly under the inverse map as they do under complement and reverse.

Proposition 2.1. *Let $\sigma \in \mathcal{S}_n(\pi)$ where $\pi \in \{132, 213, 231, 312\}$. Then*

$$\mathrm{mak}(\sigma) = \mathrm{imaj}(\sigma).$$

Moreover for any $n \geq 1$,

$$\sum_{\sigma \in \mathcal{S}_n(\pi)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\pi^{-1})} q^{\mathrm{mak}(\sigma)} t^{\mathrm{des}(\sigma)}.$$

Proof. Let $\sigma \in \mathcal{S}_n(231)$. If $\mathrm{Des}(\sigma) = \{i_1, \ldots, i_k\}$, then by [32, Lemma 3.1] we have that

$$\mathrm{Des}(\sigma^{-1}) = \{\sigma(i_1) - 1, \ldots, \sigma(i_k) - 1\}.$$

In particular $\mathrm{des}(\sigma) = \mathrm{des}(\sigma^{-1})$. Note that

$$\sigma(i_j) = \sigma(i_j + 1) + (\underline{31}2)\big|_{(\sigma(i_j), \sigma(i_j+1), \text{-})} \sigma + 1, \tag{2.1}$$

for $j = 1, \ldots, k$. Indeed if $\sigma(i_j + 1) < \alpha < \sigma(i_j)$, then α must appear to the right of the descent i_j in σ, otherwise $\alpha\sigma(i_j)\sigma(i_j + 1)$ is an occurrence of 231 (which is forbidden). Therefore $\sigma(i_j)\sigma(i_j + 1)\alpha$ is an occurrence of $(\underline{31}2)\big|_{(\sigma(i_j), \sigma(i_j+1), \text{-})}$ in σ for every α such that $\sigma(i_j + 1) < \alpha < \sigma(i_j)$. Thus (2.1) follows.

Hence by (2.1) and (1.2) we have

$$\begin{aligned}
\mathrm{imaj}(\sigma) &= \sum_{j=1}^{k} (\sigma(i_j) - 1) \\
&= \sum_{j=1}^{k} \Big(\sigma(i_j + 1) + (\underline{31}2)\big|_{(\sigma(i_j), \sigma(i_j+1), \text{-})} \Big) \\
&= \sum_{\alpha \in \mathrm{DB}(\sigma)} \alpha + (\underline{31}2)\sigma \\
&= \mathrm{mak}(\sigma).
\end{aligned}$$

The statement is proved similarly for remaining choices of π and those analogous arguments are omitted. □

Remark 2.2. By Proposition 2.1 and [32, Corollary 4.1] it follows that

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{maj}(\sigma) + \mathrm{mak}(\sigma)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \tag{2.2}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$. The right hand side of (2.2) is known as *MacMahon's q-analogue of the Catalan numbers* [26].

The following lemma regarding the structure of $\mathcal{S}_n(321)$ is part of the folklore of pattern avoidance (see e.g., [22]).

Lemma 2.3. *We have $\sigma \in \mathcal{S}_n(321)$ if and only if the elements of $[n] \setminus \mathrm{LRMax}(\sigma)$ form an increasing subsequence of σ.*

Theorem 2.4. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{DB}(\sigma)} \mathbf{y}^{\mathrm{DT}(\sigma)},$$

$$\sum_{\sigma \in \mathcal{S}_n(123)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{AB}(\sigma)} \mathbf{y}^{\mathrm{AT}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(123)} q^{\mathrm{mak}(\sigma)} \mathbf{x}^{\mathrm{AB}(\sigma)} \mathbf{y}^{\mathrm{AT}(\sigma)}.$$

Proof. Let $\sigma \in \mathcal{S}_n(321)$. By Lemma 2.3 we may decompose σ as

$$\sigma = u_1 v_1 u_2 v_2 \cdots u_t v_t,$$

where $u_1, \ldots, u_t$ are non-empty factors of left-to-right maxima in σ and $v_1, \ldots, v_t$ are non-empty factors (except possibly v_t) such that $v_1 v_2 \cdots v_t$ is an increasing subword. Assume first that $v_t \neq \emptyset$. Let $M_i = \max(u_i)$ and $m_i = \min(v_i)$ for $i = 1, \ldots, t$. Clearly $\mathrm{DB}(\sigma) = \{m_i : 1 \leq i \leq t\}$ and $\mathrm{DT}(\sigma) = \{M_i : 1 \leq i \leq t\}$. Let $\bar{u}_i = u_i \setminus M_i$ and $\bar{v}_i = v_i \setminus m_i$ for $i = 1, \ldots, t$. Write $\bar{u} = \bar{u}_1 \cdots \bar{u}_t$ and $\bar{v} = \bar{v}_1 \cdots \bar{v}_t$.

We now define an involution

$$\phi : \mathcal{S}_n(321) \to \mathcal{S}_n(321) \tag{2.3}$$

such that $\mathrm{maj}(\phi(\sigma)) = \mathrm{mak}(\sigma)$, preserving all pairs of descent top and descent bottoms. For convenience, set $M_0 = -\infty$ and $M_{t+1} = \infty$. Let u'_k denote the unique increasing word of the letters in the set

$$\{\alpha \in \bar{v} : M_{k-1} < \alpha < M_k\},$$

with M_k adjoined at the end and let v'_k denote the unique increasing word of the letters in the set

$$\{\beta \in \bar{u} : m_k < \beta < M_{k+1}\},$$

with m_k adjoined at the beginning for $k = 1, \ldots, t$. Define

$$\phi(\sigma) = \begin{cases} u'_1 v'_1 \cdots u'_t v'_t & \text{if } v_t \neq \emptyset \\ \phi(u_1 v_1 \cdots u_{t-1} v_{t-1}) u_t & \text{if } v_t = \emptyset \end{cases}.$$

Thus ϕ effectively swaps $\bar{u} = \mathrm{LRMax}(\sigma) \setminus \mathrm{DT}(\sigma)$ with $\bar{v} = [n] \setminus (\mathrm{LRMax}(\sigma) \cup \mathrm{DB}(\sigma))$ (when $v_t \neq \emptyset$) and $\mathrm{DB}(\phi(\sigma)) = \mathrm{DB}(\sigma)$, $\mathrm{DT}(\phi(\sigma)) = \mathrm{DT}(\sigma)$. Hence ϕ is an involution. We have

$$\begin{aligned}
(2\underline{31})\sigma &= \sum_{\beta \in \bar{u}} (2\underline{31})\big|_{(\beta,-,-)} \sigma \\
&= \sum_{\beta \in \bar{u}} (\max\{k : m_k < \beta\} - \min\{k : M_k > \beta\} + 1) \\
&= \sum_{\beta \in \bar{u}} (\max\{k : m_k < \phi(\beta)\} - \min\{k : M_k > \phi(\beta)\} + 1) \\
&= \sum_{\beta \in \bar{u}} (\underline{31}2)\big|_{(-,-,\phi(\beta))} \phi(\sigma) \\
&= (\underline{31}2)\phi(\sigma),
\end{aligned}$$

since under the involution ϕ, each $\beta \in \mathrm{LRMax}(\sigma) \setminus \mathrm{DT}(\sigma)$ precisely passes the number of descent bottoms that are less than it to its right. Therefore β is involved in the same number

of $2\underline{31}$ occurrences in σ as $\phi(\beta)$ is involved in $\underline{31}2$ occurrences in $\phi(\sigma)$. Hence

$$\begin{aligned}\mathrm{mak}(\phi(\sigma)) &= ((1\underline{32}) + (\underline{32}1) + (\underline{21}))\phi(\sigma) + (\underline{31}2)\phi(\sigma)\\ &= \sum_{\alpha\in \mathrm{DB}(\phi(\sigma))} \alpha + (\underline{31}2)\phi(\sigma)\\ &= \sum_{\alpha\in \mathrm{DB}(\sigma)} \alpha + (2\underline{31})\sigma\\ &= \mathrm{maj}(\sigma).\end{aligned}$$

The statement is proved analogously over $\mathcal{S}(123)$. □

Example 2.1. Let ϕ be the involution (2.3) in Theorem 2.4 and let $\sigma = 561237948 \in \mathcal{S}_9(321)$. Then

$$561237948 \overset{\phi}{\mapsto} 236189457,$$

where the black letters indicate the fixed pairs of descent tops and descent bottoms, red letters denote non-descent top left-to-right maxima and blue letters denote non-descent bottom non-left-to-right maxima. The involution swaps the role of red and blue letters while keeping consecutive pairs of black letters together in the same relative order.

Proposition 2.5. *We have*

$$\begin{aligned}[123]_{\mathrm{mak}} &= \{123\},\\ [321]_{\mathrm{mak}} &= \{321\},\\ [132]_{\mathrm{mak}} &= \{132,\, 312\} = [312]_{\mathrm{mak}},\\ [213]_{\mathrm{mak}} &= \{213,\, 231\} = [231]_{\mathrm{mak}}.\end{aligned}$$

Proof. As shown in [14, Theorem 2.6] the map $\phi : \mathcal{S}_n(132) \to \mathcal{S}_n(231)$ recursively defined by

$$\phi(231[\sigma_1, 1, \sigma_2]) = 132[\phi(\sigma_1), 1, \phi(\sigma_2)],$$

is a descent preserving bijection implying that $[132]_{\mathrm{maj}} = [231]_{\mathrm{maj}}$. Thus by Proposition 2.1 we have

$$\sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{mak}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(231)} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(312)} q^{\mathrm{mak}(\sigma)}.$$

Hence $[132]_{\mathrm{mak}} = [312]_{\mathrm{mak}}$. The remaining mak-Wilf equivalence is proved similarly invoking Proposition 2.1. The inequivalences between the four classes is easily verified by hand or with computer. □

Remark 2.6. The charge statistic is also a Mahonian statistic related to maj via trivial bijections by $\mathrm{maj}(\sigma) = \mathrm{charge}(((\sigma^r)^c)^{-1})$ (see [20]). It is worth noting that the mak-Wilf classes in Proposition 2.5 coincide with the charge-Wilf classes identified in [20].

Remark 2.7. It can be checked that maj, inv and mak are the only statistics in Table 1 with non-singleton st-Wilf classes for single classical patterns of length three.

The bijection (2.3) in Theorem 2.4 induces an interesting equidistribution on shortened polyominoes. A *shortened polyomino* is a pair (P, Q) of N (north), E (east) lattice paths $P = (P_i)_{i=1}^n$ and $Q = (Q_i)_{i=1}^n$ satisfying

(i) P and Q begin at the same vertex and end at the same vertex.

(ii) P stays weakly above Q and the two paths can share E-steps but not N-steps.

Denote the set of shortened polyominoes with $|P| = |Q| = n$ by $\mathcal{H}_n$. For $(P, Q) \in \mathcal{H}_n$, let $\mathrm{Proj}_P^Q(i)$ denote the step $j \in [n]$ of P that is the *projection of the ith step of Q on P*. Let

$$\mathrm{Valley}(Q) = \{i : Q_i Q_{i+1} = EN\}$$

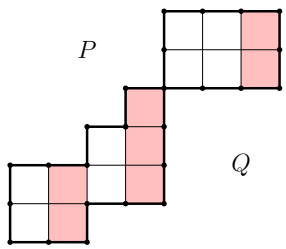
denote the set of indices of the *valleys* in Q and let $\mathrm{nval}(Q) = |\mathrm{Valley}(Q)|$. Moreover for each $i \in [n]$ define

$$\mathrm{area}_{(P,Q)}(i) = \#\text{squares between the } i^{\text{th}} \text{ step of } Q \text{ and the } j^{\text{th}} \text{ step of } P,$$

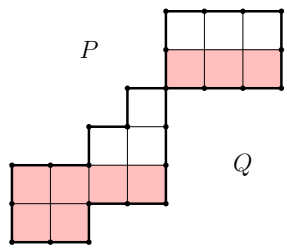
where $j = \mathrm{Proj}_P^Q(i)$. Consider the statistics *valley-column area* and *valley-row area* of (P, Q) given by

$$\mathrm{vcarea}(P, Q) = \sum_{i \in \mathrm{Valley}(Q)} \mathrm{area}_{(P,Q)}(i),$$

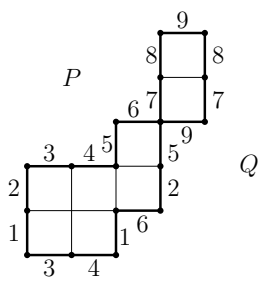
$$\mathrm{vrarea}(P, Q) = \sum_{i \in \mathrm{Valley}(Q)} \mathrm{area}_{(P,Q)}(i + 1).$$



(a) $\mathrm{vcarea}(P, Q) = 2 + 3 + 2 = 7$



(b) $\mathrm{vrarea}(P, Q) = 2 + 4 + 3 = 9$



The bijection Υ. Here $\Upsilon(P, Q) = 341625978 \in \mathcal{S}_9(321)$.

Theorem 2.8. *For any $n \geq 1$,*

$$\sum_{(P,Q) \in \mathcal{H}_n} q^{\mathrm{vcarea}(P,Q)} t^{\mathrm{nval}(Q)} = \sum_{(P,Q) \in \mathcal{H}_n} q^{\mathrm{vrarea}(P,Q)} t^{\mathrm{nval}(Q)}.$$

Proof. We begin by recalling a bijection $\Upsilon : \mathcal{H}_n \to \mathcal{S}_n(321)$ due to Cheng-Eu-Fu [9]. Given $(P, Q) \in \mathcal{H}_n$, set $\mathrm{Label}_P(i) = i$ and $\mathrm{Label}_Q(i) = \mathrm{Label}_P(\mathrm{Proj}_P^Q(i))$. Then

$$\Upsilon(P, Q) = \mathrm{Label}_Q(1) \cdots \mathrm{Label}_Q(n) \in \mathcal{S}_n(321)$$

is a bijection.

Let $(P,Q) \in \mathcal{H}_n$ and $i \in \mathrm{Valley}(Q)$. The definition of Υ immediately gives

$$\mathrm{Valley}(P,Q) = \mathrm{Des}(\Upsilon(P,Q)).$$

In particular $\mathrm{Label}_Q(i+1) < \mathrm{Label}_Q(i)$. Let $s = \mathrm{Proj}_P^Q(i+1)$ and $t = \mathrm{Proj}_P^Q(i)$. Then $s < t$ and

$$\begin{aligned}\mathrm{area}_{(P,Q)}(i) &= |\{j : P_j = N,\ s \le j \le t\}| \\ &= |\{j : \mathrm{Label}_Q(i+1) \le \mathrm{Label}_Q(j) < \mathrm{Label}_Q(i),\ j > i\}| \\ &= 1 + (\underline{31}2)\big|_{(\mathrm{Label}_Q(i),\,\mathrm{Label}_Q(i+1),\,\text{-})}\Upsilon(P,Q).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{area}_{(P,Q)}(i+1) &= |\{j : P_j = E,\ s \le j \le t\}| \\ &= |\{j : \mathrm{Label}_Q(i+1) < \mathrm{Label}_Q(j) \le \mathrm{Label}_Q(i),\ j \le i\}| \\ &= 1 + (2\underline{31})\big|_{(\text{-},\,\mathrm{Label}_Q(i),\,\mathrm{Label}_Q(i+1))}\Upsilon(P,Q).\end{aligned}$$

Let $\phi : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ be the bijection (2.3) from Theorem 2.4. Recall that $(\underline{31}2)\phi(\sigma) = (2\underline{31})\sigma$ and $\mathrm{des}(\phi(\sigma)) = \mathrm{des}(\sigma)$ for all $\sigma \in \mathcal{S}_n(321)$. Let $\Phi : \mathcal{H}_n \to \mathcal{H}_n$ be the bijection

$$\Phi = \Upsilon^{-1} \circ \phi \circ \Upsilon,$$

and set $(P',Q') = \Phi(P,Q)$. Then

$$\begin{aligned}\mathrm{vcarea}(\Phi(P,Q)) &= \sum_{i \in \mathrm{Valley}(Q')} \mathrm{area}_{(P',Q')}(i) \\ &= \sum_{i \in \mathrm{Valley}(Q')} \Big(1 + (\underline{31}2)\big|_{(\mathrm{Label}_{Q'}(i),\,\mathrm{Label}_{Q'}(i+1),\,\text{-})}\Upsilon(P',Q')\Big) \\ &= \sum_{i \in \mathrm{Des}(\phi(\Upsilon(P,Q)))} \Big(1 + (\underline{31}2)\big|_{(\phi(\mathrm{Label}_Q(i)),\,\phi(\mathrm{Label}_Q(i+1)),\,\text{-})}\phi(\Upsilon(P,Q))\Big) \\ &= (\mathrm{des} + (\underline{31}2))\phi(\Upsilon(P,Q)) \\ &= (\mathrm{des} + (2\underline{31}))\Upsilon(P,Q) \\ &= \sum_{i \in \mathrm{Valley}(Q)} \Big(1 + (2\underline{31})\big|_{(\text{-},\,\mathrm{Label}_Q(i),\,\mathrm{Label}_Q(i+1))}\Upsilon(P,Q)\Big) \\ &= \sum_{i \in \mathrm{Valley}(Q)} \mathrm{area}_{(P,Q)}(i+1) \\ &= \mathrm{vrarea}(P,Q).\end{aligned}$$

Since $\mathrm{Valley}(P,Q) = \mathrm{Des}(\Upsilon(P,Q))$ and $\mathrm{des}(\phi(\sigma)) = \mathrm{des}(\sigma)$ it follows that $\mathrm{nval}(Q') = \mathrm{nval}(Q)$. This concludes the proof. □

Below we provide a brief account for a well-known lemma due to Simion and Schmidt which will be used to justify the bijection in the next theorem.

Lemma 2.9 (Simion-Schmidt [29])**.** *A permutation $\sigma \in \mathcal{S}(132)$ is uniquely determined by the values and positions of its left-to-right minima.*

Proof. It is clear that the left-to-right minima are positioned in decreasing order relative to each other. Now fill in the remaining numbers from left to right, for each empty position i choosing the smallest remaining entry that is larger than the closest left-to-right minima m in position before i. If the remaining numbers are not entered in this unique way and y is placed before x where $y > x$, then myx is an occurrence of the pattern 132. □

Theorem 2.10. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}(132)} q^{\mathrm{maj}(\sigma)} \mathbf{x}^{\mathrm{LRMin}(\sigma)} = \sum_{\sigma \in \mathcal{S}(132)} q^{\mathrm{foze}(\sigma)} \mathbf{x}^{\mathrm{LRMin}(\sigma)}$$

Proof. Let $\sigma \in \mathcal{S}_n(132)$. It is not difficult to see that $\mathrm{LRMin}(\sigma) = \mathrm{DB}(\sigma) \cup \{\sigma(1)\}$. Indeed if $\sigma(i) \in \mathrm{DB}(\sigma)$ and $\sigma(j) < \sigma(i)$ for some $j < i$, then $\sigma(j)\sigma(i-1)\sigma(i)$ is an occurrence of 132. Hence by Lemma 2.9 we have that σ is uniquely determined equivalently by its first letter, $\mathrm{Des}(\sigma)$ and $\mathrm{DB}(\sigma)$. We define a map $\phi : \mathcal{S}_n(132) \to \mathcal{S}_n(132)$ by requiring

$$\begin{aligned} \phi(\sigma)(1) &= \sigma(1), \\ \mathrm{DB}(\phi(\sigma)) &= \mathrm{DB}(\sigma), \\ \mathrm{Des}(\phi(\sigma)) &= \{n - \sigma(i) + 1 : i \in \mathrm{Des}(\sigma)\}. \end{aligned}$$

We claim that a permutation $\phi(\sigma) \in \mathcal{S}_n(132)$ with the above requirements exists. If the claim holds, then the image of σ is uniquely determined by the data above and therefore ϕ is well-defined. It also immediately follows that ϕ is a bijection.

Let $i_1 < \cdots < i_m$ be the descents of σ. Suppose

$$n - \sigma(i_{j_1}) + 1 < \cdots < n - \sigma(i_{j_m}) + 1.$$

To show that ϕ is well-defined we show that the insertion procedure from Lemma 2.9 is always valid. Given a descent bottom (i.e. left-to-right minima) $\sigma(i_k + 1)$ in position $n - \sigma(i_{j_k}) + 2$ we must show that there exists enough remaining numbers greater than $\sigma(i_k + 1)$ to fill in the gap to the next descent bottom $\sigma(i_{k+1} + 1)$. Within the filling procedure, next after the descent bottom $\sigma(i_k + 1)$, there exists

$$n - \sigma(i_k + 1) - (n - \sigma(i_{j_k}) + 1) = \sigma(i_{j_k}) - \sigma(i_k + 1) - 1$$

numbers remaining that are greater than $\sigma(i_k + 1)$. There are

$$(n - \sigma(i_{j_{k+1}}) + 2) - (n - \sigma(i_{j_k}) + 2) - 1 = \sigma(i_{j_k}) - \sigma(i_{j_{k+1}}) - 1$$

positions to fill in the gap between the descent bottoms $\sigma(i_k + 1)$ and $\sigma(i_{k+1} + 1)$. By minimality

$$\sigma(i_{j_k}) - \sigma(i_{j_{k+1}}) \leq \sigma(i_{j_k}) - \sigma(i_k) \leq \sigma(i_{j_k}) - \sigma(i_k + 1),$$

so there are enough numbers remaining to fill in the gap. Hence ϕ is well-defined. Finally,

$$\begin{aligned}\mathrm{maj}(\phi(\sigma)) &= \sum_{i\in\mathrm{Des}(\phi(\sigma))} i \\ &= \sum_{i\in\mathrm{Des}(\sigma)} (n-\sigma(i)+1) \\ &= \sum_{\alpha\in\mathrm{DT}(\sigma)} (n-\alpha) + \mathrm{des}(\sigma) \\ &= ((\underline{21}3)+(3\underline{21}))\sigma + (\underline{21})\sigma \\ &= \mathrm{foze}(\sigma).\end{aligned}$$

Since also $\phi(\mathrm{LRMin}(\sigma)) = \mathrm{LRMin}(\sigma)$, the theorem follows. □

Below we provide an additional list of information uniquely determining permutations in $\mathcal{S}_n(231)$.

Lemma 2.11. *A permutation $\sigma \in \mathcal{S}_n(231)$ is uniquely determined by any of the following data:*

(i) The values and positions of right-to-left minima.
(ii) The last letter, ascents and ascent bottoms.
(iii) The pairs $P(\sigma) = \{(p,t) : p$ pinnacle and t its following trough$\}$.
(iv) The pairs $Q(\sigma) = \{(\alpha,\beta) : \alpha$ descent top and β its following descent bottom$\}$.
(v) The pairs $R(\sigma) = \left\{\left(\alpha,\ (\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma\right) : \alpha \text{ descent top}\right\}$.

Proof.

(i) Suppose the values and positions of right-to-left minima are fixed in σ. Then $\sigma^r \in \mathcal{S}_n(132)$ and the values and positions of the left-to-right minima in σ^r are fixed. By Lemma 2.9 this information uniquely determines σ^r. Hence σ is uniquely determined.

(ii) Follows directly from (i) since the positions and values of right-to-left minima are given by the positions and values of the ascents and ascent bottoms together with the last letter.

(iii) Consider the pinnacle-trough decomposition

$$\sigma = a_1p_1d_1t_1\cdots a_{m-1}p_{m-1}d_{m-1}t_{m-1}a_mp_kd_m$$

where p_i and t_i are pinnacles resp. troughs and a_i and d_i are (possibly empty) increasing resp. decreasing words for $i = 1,\dots,m$.

We claim that the pairs in P are relatively positioned in increasing order of the valleys. Indeed let $(p,t),(p',t') \in P(\sigma)$. Without loss assume $t < t'$. Suppose (for a contradiction) that (p',t') is ordered before (p,t) in σ. Note that $t' < p$, otherwise $t'\alpha p$ is an occurrence of 231, where α is the ascent top following t'. This in turn implies that $t'pt$ is an occurrence of 231 giving a contradiction. Therefore (p,t) is ordered before (p',t') proving the claim.

Next we claim that the decreasing words d_j are uniquely determined. Going from right to left, let d_j be the unique decreasing word of all remaining letters (in value) between p_j and t_j for $j = m,\dots,1$. If we do not insert the letters this way and $t_j < \sigma_i < p_j$, where σ_i is positioned before p_j (and hence t_j) then $\sigma_ip_jt_j$ is an occurrence of 231 which is forbidden.

Finally we show that the increasing words a_j are uniquely determined. Suppose a_j contains a letter α such that $\alpha > t_j$. Since $\alpha < p_j$ it follows that $\alpha p_j t_j$ is an occurrence of 231. Therefore all letters of a_i are smaller than t_j. Hence a_j is given by the unique increasing word of all letters α such that $t_{j-1} < \alpha < t_j$ for $j = 1, \ldots, m$. Hence σ is uniquely determined.

(iv) Partition the letters in $\mathrm{DB}(\sigma) \cup \mathrm{DT}(\sigma)$ into maximal consecutive decreasing subwords $d_1, \ldots, d_m$ based on the pairs in $Q(\sigma)$. The top element of each decreasing subword d_i must be a pinnacle and the bottom element trough. This information uniquely determines σ as per part (iii).

(v) Note that $\alpha \in \mathrm{DT}(\sigma)$ is the largest letter in an occurrence of $\underline{13}2$ in σ if and only if α is a pinnacle. Therefore the pinnacles are the descent tops α with $(\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma > 0$. Given a pinnacle p and the closest trough t to its right, any letter σ_i such that $t < \sigma_i < p$ must be in position after v, otherwise $\sigma_i p t$ is an occurrence of 231. Hence $(\underline{13}2)\big|_{(\text{-},p,\text{-})}\sigma$ precisely represents the difference between p and t. In other words $t = p - (\underline{13}2)\big|_{(\text{-},p,\text{-})}\sigma$. Hence σ is uniquely determined by part (iii).

□

Theorem 2.12. *For $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mak}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{foze}(\sigma)} t^{\mathrm{des}(\sigma)}.$$

Proof. Let $\sigma \in \mathcal{S}_n(231)$. Note that for $\alpha \in \mathrm{DT}(\sigma)$ we have $(\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma \leq \alpha - 2$ since there are at most $\alpha - 2$ numbers between α and its immediately preceding ascent bottom (if present). Thus the function

$$\begin{aligned} f_\sigma : \mathrm{DT}(\sigma) &\to [n] \\ \alpha &\mapsto (n - \alpha + 2) + (\underline{13}2)\big|_{(\text{-},\alpha,\text{-})}\sigma \end{aligned}$$

is well-defined.

We claim that f_σ is injective by induction on n. Consider the inflation form $\sigma = 132[\sigma_1, 1, \sigma_2]$ where $\sigma_1 \in \mathcal{S}_k(231)$ and $\sigma_2 \in \mathcal{S}_{n-k-1}(231)$. Let $\mathrm{DT}_{\leq k}(\sigma) = \{\alpha \in \mathrm{DT}(\sigma) : \alpha \leq k\}$ and $\mathrm{DT}_{>k}(\sigma) = \{\alpha \in \mathrm{DT}(\sigma) : \alpha > k\}$. By induction $f_{\sigma_1} : \mathrm{DT}(\sigma_1) \to [k]$ is injective and $f_\sigma(\alpha) = n - k + f_{\sigma_1}(\alpha)$ for every $\alpha \in \mathrm{DT}_{\leq}(\sigma)$. Hence $f_\sigma\big|_{\mathrm{DT}_{\leq k}(\sigma)}$ is injective. By induction $f_{\sigma_2} : \mathrm{DT}(\sigma_2) \to [n - k - 1]$ is injective and $f_\sigma(\alpha) = 1 + f_{\sigma_2}(\alpha - k)$ for every $\alpha \in \mathrm{DT}_{>k}(\sigma)$. Hence $f_\sigma\big|_{\mathrm{DT}_{>k}(\sigma)}$ is injective. Finally note that $f_\sigma(n) = 2 + |\sigma_2|$ if $\sigma_1 \neq \emptyset$ and $f_\sigma(n) = 2$ if $\sigma_1 = \emptyset$. Therefore for all $\alpha \in \mathrm{DT}_{\leq k}(\sigma)$ and $\beta \in \mathrm{DT}_{>k}(\sigma)$ we have

$$f_\sigma(\alpha) \geq (n - k + 2) > f_\sigma(n) > n - k \geq f_\sigma(\beta),$$

if $\sigma_1 \neq \emptyset$ and

$$f_\sigma(\alpha) \geq (n - k + 2) > f_\sigma(\beta) > 2 = f_\sigma(n),$$

if $\sigma_1 = \emptyset$. Hence f_σ is injective on all of $\mathrm{DT}(\sigma)$.

Define a map $\phi : \mathcal{S}_n(231) \to \mathcal{S}_n(231)$ by setting the pairs of descent tops and descent bottoms in $\phi(\sigma)$ to $Q(\phi(\sigma)) = \{(f_\sigma(\alpha), n - \alpha + 1) : \alpha \in \mathrm{DT}(\sigma)\}$. By Lemma 2.11 (iv) this data uniquely determines $\phi(\sigma)$. Note that the pairs are well-defined since f_σ is injective and $f_\sigma(\alpha) > n - \alpha + 1$ for all $\alpha \in \mathrm{DT}(\sigma)$.

We claim that ϕ is a bijection. By Lemma 2.11 (iv) we may uniquely associate σ with a set of pairs $R(\sigma) = \left\{ \left(\alpha,\, (1\underline{3}2)\big|_{(\text{-},\alpha,\text{-})}\sigma \right) : \alpha \in \mathrm{DT}(\sigma) \right\}$. It suffices to show that ϕ is injective. Let $\pi_1, \pi_2 \in \mathcal{S}_n(231)$ such that $\pi_1 \neq \pi_2$. If $\mathrm{DT}(\pi_1) \neq \mathrm{DT}(\pi_2)$, then $\mathrm{DB}(\phi(\pi_1)) \neq \mathrm{DB}(\phi(\pi_2))$, so $\phi(\pi_1) \neq \phi(\pi_2)$. Assume therefore $\mathrm{DT}(\pi_1) = \mathrm{DT}(\pi_2)$. Since $\pi_1 \neq \pi_2$ we have by uniqueness that $R(\pi_1) \neq R(\pi_2)$. Therefore there exists $\alpha \in \mathrm{DT}(\pi_1) = \mathrm{DT}(\pi_2)$ such that $f_{\pi_1}(\alpha) \neq f_{\pi_2}(\alpha)$. Thus $Q(\phi(\pi_1)) \neq Q(\phi(\pi_2))$ which again implies that $\phi(\pi_1) \neq \phi(\pi_2)$. Hence ϕ is injective and therefore a bijection.

It remains to show that $\mathrm{mak}(\phi(\sigma)) = \mathrm{foze}(\sigma)$. Note that

$$((1\underline{32}) + (\underline{32}1) + (\underline{21}))\sigma = \sum_{\beta \in \mathrm{DB}(\sigma)} \beta.$$

Since there are no occurrences of $2\underline{31}$ in σ by assumption, the letters between each pair of descent top and descent bottom occur to the right of the pair. Therefore the number of occurrences of $\underline{31}2$ in σ is given precisely by

$$\sum_{(\alpha,\beta) \in Q(\sigma)} (\alpha - \beta - 1).$$

Hence

$$\mathrm{mak}(\sigma) = \sum_{\alpha \in \mathrm{DT}(\sigma)} (\alpha - 1).$$

On the other hand note that

$$((\underline{21}3) + (3\underline{21}) + (\underline{21}))\sigma = \sum_{\alpha \in \mathrm{DT}(\sigma)} (n - \alpha + 1).$$

Thus

$$\begin{aligned} \mathrm{foze}(\sigma) &= \sum_{\alpha \in \mathrm{DT}(\sigma)} (n - \alpha + 1) + (1\underline{32})\sigma \\ &= \sum_{\alpha \in \mathrm{DT}(\sigma)} \left(n - \alpha + 1 + (1\underline{32})\big|_{(\text{-},\alpha,\text{-})}\sigma \right) \\ &= \sum_{\alpha \in \mathrm{DT}(\sigma)} (f_\sigma(\alpha) - 1). \end{aligned}$$

Hence

$$\mathrm{mak}(\phi(\sigma)) = \sum_{\alpha' \in \mathrm{DT}(\phi(\sigma))} (\alpha' - 1) = \sum_{\alpha \in \mathrm{DT}(\sigma)} (f(\alpha) - 1) = \mathrm{foze}(\sigma).$$

Finally since $\mathrm{des}(\phi(\sigma)) = \mathrm{des}(\sigma)$, the theorem follows. □

Remark 2.13. By combining Theorem 2.12 with Proposition 2.1 we may deduce further equidistributions between maj and foze, see Table 2 in §5 for a summary.

3. Equidistributions via Dyck paths

A *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never go below the x-axis. For convenience we denote the up-steps by U and the down-steps by D enabling us to encode a Dyck path as a *Dyck word* (we will refer to the two notions interchangeably). Let $\mathcal{D}_n$ denote the set of all Dyck paths of length $2n$ and set $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$. For $P \in \mathcal{D}_n$, let $|P| = 2n$ denote the

length of P. There are many statistics associated with Dyck paths in the literature. Here we will consider several Dyck path statistics that are intimately related with the inv statistic on pattern avoiding permutations.

Let $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$. A *double rise* in P is a subword UU and a *double fall* in P a subword DD. Let $\mathrm{dr}(P)$ (resp. $\mathrm{df}(P)$) denote the number of double rises (resp. double falls) in P. A *peak* in P is an up-step followed by a down-step, in other words, a subword of the form UD. Let $\mathrm{Peak}(P) = \{p : s_p s_{p+1} = UD\}$ denote the set of indices of the peaks in P and $\mathrm{npea}(P) = |\mathrm{Peak}(P)|$. For $p \in \mathrm{Peak}(P)$ define the *position* of p, $\mathrm{pos}_P(p)$, resp. the *height* of p, $\mathrm{ht}_P(p)$, to be the x resp. y-coordinate of its highest point. A *valley* in P is a down step followed by an up step, in other words, a subword of the form DU. Let $\mathrm{Valley}(P) = \{v : s_v s_{v+1} = DU\}$ denote the set of indices of the valleys in P and $\mathrm{nval}(P) = |\mathrm{Valley}(P)|$. For $v \in \mathrm{Valley}(P)$ define the *position* of v, $\mathrm{pos}_P(v)$, resp. the *height* of v, $\mathrm{ht}_P(v)$, to be the x resp. y-coordinate of its lowest point. For each $v \in \mathrm{Valley}(P)$, there is a corresponding *tunnel* which is the subword $s_i \cdots s_v$ of P where i is the step after the first intersection of P with the line $y = \mathrm{ht}_P(v)$ to the left of step v (see Figure 2). The length, $v - i$, of a tunnel is always an even number. Let $\mathrm{Tunnel}(P) = \{(i, j) : s_i \cdots s_j \text{ tunnel in } P\}$ denote the set of pairs of beginning and end indices of the tunnels in P. Cheng et.al. [8] define the statistics *sumpeaks* and *sumtunnels* given respectively by

$$\mathrm{spea}(P) = \sum_{p \in \mathrm{Peak}(P)} (\mathrm{ht}_P(p) - 1),$$

$$\mathrm{stun}(P) = \sum_{(i,j) \in \mathrm{Tunnel}(P)} (j - i)/2.$$

Let $\mathrm{Up}(P) = \{i : s_i = U\}$ denote the indices of the set of U-steps in P and $\mathrm{Down}(P) = \{i : s_i = D\}$ the set of indices of the D-steps in P. Given $i \in [2n]$ define the *height* of the step i in P, $\mathrm{ht}_P(i)$, to be the y-coordinate of its lowest point. Define the statistics *sumups* and *sumdowns* by

$$\mathrm{sups}(P) = \sum_{i \in \mathrm{Up}(P)} \lceil \mathrm{ht}_P(i)/2 \rceil$$

$$\mathrm{sdow}(P) = \sum_{i \in \mathrm{Down}(P)} \lfloor \mathrm{ht}_P(i)/2 \rfloor$$

Define the *area* of P, denoted $\mathrm{area}(P)$, to be the number of complete $\sqrt{2} \times \sqrt{2}$ tiles that fit between P and the x-axis (cf [21]).

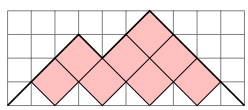


FIGURE 1. $\mathrm{area}(P) = 8$.

Burstein and Elizalde [5] define a statistic which they call the 'mass' of P. We will define two versions of it, one pertaining to the U-steps and one to the D-steps. For each $i \in \mathrm{Up}(P)$ define the *mass* of i, $\mathrm{mass}_P(i)$, as follows. If $s_{i+1} = D$, then $\mathrm{mass}_P(i) = 0$. If $s_{i+1} = U$, then P has a subword of the form $s_i U P_1 D P_2 D$ where P_1, P_2 are Dyck paths and we define $\mathrm{mass}_P(i) = |P_2|/2$. In other words, the mass is half the number of steps between the matching

D-steps of two consecutive U-steps. The part of the Dyck path P contributing to the mass of each of the first three U-steps is highlighted with matching colours in Figure 2. Define

$$\mathrm{mass}_{\mathrm{U}}(P) = \sum_{i \in \mathrm{Up}(P)} \mathrm{mass}_P(i).$$

The statistic $\mathrm{mass}_{\mathrm{U}}$ coincides with the mass statistic defined by Burstein and Elizalde [5]. Analogously if $i \in \mathrm{Down}(P)$, define $\mathrm{mass}_P(i) = 0$ if $s_{i-1} = U$. If $s_{i-1} = D$, then P has a subword of the form $UP_1UP_2Ds_i$ where P_1, P_2 are Dyck paths and we define $\mathrm{mass}_{\mathrm{D}}(s) = |P_1|/2$. In other words, the mass is half the number of steps between the matching U-steps of two consecutive D-steps. Define

$$\mathrm{mass}_{\mathrm{D}}(P) = \sum_{i \in \mathrm{Down}(P)} \mathrm{mass}_P(i).$$

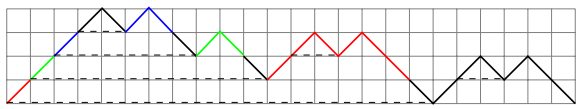


FIGURE 2. The tunnel lengths of a Dyck path (indicated with dashes) and the mass associated with the first three up-steps is highlighted with matching colours.

Next we give a description of the various Dyck path bijections that will be referenced. The *standard bijection* $\Delta : \mathcal{S}_n(231) \to \mathcal{D}_n$ can be defined recursively by

$$\Delta(\sigma) = U\Delta(\sigma_1)D\Delta(\sigma_2),$$

where $\sigma = 213[1, \sigma_1, \sigma_2]$. We will also (with abuse of notation) define the *standard bijection* $\Delta : \mathcal{S}_n(312) \to \mathcal{D}_n$ recursively by

$$\Delta(\sigma) = \Delta(\sigma_1)U\Delta(\sigma_2)D,$$

where $\sigma = 132[\sigma_1, \sigma_2, 1]$. There is also a non-recursive description of Δ due to Krattenthaler, see [23].

We now define another well-known map $\Gamma : \mathcal{S}_n(321) \to \mathcal{D}_n$ due to Krattenthaler [23] which also appears in a slightly different form in the work of Elizalde [15]. Let $\sigma \in \mathcal{S}_n(321)$ and consider an $n \times n$ array with crosses in positions (i, π_i) for $1 \leq i \leq n$, where the first coordinate is the column number, increasing from left to right, and the second coordinate is the row number, increasing from bottom to top. Consider the path with north and east steps from the lower-left corner to the upper-right corner of the array, whose right turns occur at

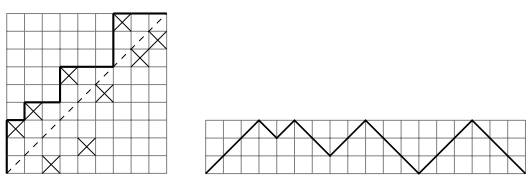


FIGURE 3. The Dyck path $\Gamma(\sigma)$ corresponding to $\sigma = 341625978$.

the crosses (i, σ_i) with $\sigma_i \geq i$. Define $\Gamma(\sigma)$ to be the Dyck path obtained from this path by reading a U-step for every north step and a D-step for every east step of the path. The bijection is illustrated in Figure 3.

Theorem 3.1 (Krattenthaler [23], Elizalde [15]).
For each $n \geq 1$ the map $\Gamma : \mathcal{S}_n(321) \to \mathcal{D}_n$ is a bijection.

Theorem 3.2 (Cheng-Elizalde-Kasraoui-Sagan [8]).
We have $\mathrm{inv}(\sigma) = \mathrm{spea}(\Gamma(\sigma))$ *and* $\mathrm{lrmax}(\sigma) = \mathrm{npea}(\Gamma(\sigma))$ *for all* $\sigma \in \mathcal{S}_n(321)$.

Next we define a Dyck path bijection $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ due to Cheng et.al. [8] that is weight preserving between the statistics spea and stun.

First we define a bijection $\delta : \bigsqcup_{k=0}^{n-1} \mathcal{D}_k \times \mathcal{D}_{n-k-1} \to \mathcal{D}_n$ as follows. Given two Dyck paths

$$Q = U^{a_1} D^{b_1} U^{a_2} D^{b_2} \cdots U^{a_s} D^{b_s} \in \mathcal{D}_k \text{ and } R = U^{c_1} D^{d_1} U^{c_2} D^{d_2} \cdots U^{c_t} D^{d_t} \in \mathcal{D}_{n-k-1}$$

where all exponents are positive, define $\delta(Q, R)$ by

$$\delta(Q, R) = U^{a_1+1} D^{b_1+1} U^{a_2} D^{b_2} \cdots U^{a_s} D^{b_s},$$

if $R = \emptyset$ and define

$$\delta(Q, R) = U^{a_1+1} D U^{a_2} D^{b_1} U^{a_3} D^{b_2} \cdots U^{a_s} D^{b_{s-1}} U^{c_1} D^{b_s+d_1} U^{c_2} D^{d_2} \cdots U^{c_t} D^{d_t},$$

if $R \neq \emptyset$. If $Q = \emptyset$ the same definition works with the convention that $a_1 = b_1 = 0$.

Let $P \in \mathcal{D}_n$ and $(Q, R) = \delta^{-1}(P)$. Define $\Psi(\emptyset) = \emptyset$ and for $n \geq 1$

$$\Psi(P) = \begin{cases} UD\Psi(Q) & \text{if } R = \emptyset \\ U\Psi(R)D & \text{if } Q = \emptyset, \\ U\Psi(Q)D\Psi(R) & \text{otherwise.} \end{cases}$$

Theorem 3.3 (Cheng-Elizalde-Kasraoui-Sagan [8]). *The map $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ is a bijection such that* $\mathrm{spea}(P) = \mathrm{stun}(\Psi(P))$ *and* $\mathrm{npea}(P) = n - \mathrm{nval}(\Psi(P))$ *for all $P \in \mathcal{D}_n$. In particular*

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)} t^{\mathrm{npea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{stun}(P)} t^{n - \mathrm{nval}(P)}$$

for all $P \in \mathcal{D}_n$.

We will now interpret mad over both $\mathcal{S}_n(231)$ and $\mathcal{S}_n(312)$ in terms of Dyck path statistics under Δ. The following theorem is a straightforward modification of Theorem 3.11 in [5].

Theorem 3.4. *For all $\sigma \in \mathcal{S}_n(231)$, $\pi \in \mathcal{S}_n(312)$ and $P \in \mathcal{D}_n$ we have*

(i) $\mathrm{mad}(\sigma) = \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma)) + \mathrm{dr}(\Delta(\sigma))$,
(ii) $\mathrm{mad}(\pi) = 2\,\mathrm{mass}_{\mathrm{D}}(\Delta(\pi)) + \mathrm{df}(\Delta(\pi))$,
(iii) a bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$ such that $\mathrm{sups}(P) = \mathrm{mass}_{\mathrm{U}}(\Theta(P)) + \mathrm{dr}(\Theta(P))$.

Proof.

(i) Let $\sigma \in \mathcal{S}_n(231)$ and decompose $\sigma = 213[1, \sigma_1, \sigma_2]$. If we assume $\sigma_1 \neq \emptyset$, then we may further decompose σ_1 and write $\sigma = 42135[1, 1, \sigma_3, \sigma_4, \sigma_2]$. In particular $(\underline{31}2)\big|_{(\sigma(1),\sigma(2),\text{-})}\sigma = |\sigma_4|$. Since

$$\Delta(\sigma) = UU\Delta(\sigma_3)D\Delta(\sigma_4)D\Delta(\sigma_2),$$

we have by induction that

$$\begin{aligned}\mathrm{mass}_{\mathrm{U}}(\Delta(\sigma)) &= \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma_3)) + \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma_4)) + \mathrm{mass}_{\mathrm{U}}(\Delta(\sigma_2)) + |\Delta(\sigma_4)|/2 \\ &= (\underline{31}2)\sigma_3 + (\underline{31}2)\sigma_4 + (\underline{31}2)\sigma_2 + |\sigma_4| \\ &= (\underline{31}2)\sigma.\end{aligned}$$

and

$$\begin{aligned}\mathrm{dr}(\Delta(\sigma)) &= \mathrm{dr}(\Delta(\sigma_1)) + \mathrm{dr}(\Delta(\sigma_2)) + 1 \\ &= \mathrm{des}(\sigma_1) + \mathrm{des}(\sigma_2) + 1 \\ &= \mathrm{des}(\sigma).\end{aligned}$$

Hence $\mathrm{mass}_{\mathrm{U}}(\Delta(\sigma)) + \mathrm{dr}(\Delta(\sigma)) = \mathrm{mad}(\sigma)$.

(ii) Let $\pi \in \mathcal{S}_n(312)$ and decompose $\pi = 132[\pi_1, \pi_2, 1]$. Assuming $\pi_2 \neq \emptyset$ we may write $\pi = 13542[\pi_1, \pi_3, \pi_4, 1, 1]$. In particular $(2\underline{31})\big|_{(-,\pi(n-1),\pi(n))}\pi = |\pi_3|$. Since

$$\Delta(\pi) = \Delta(\pi_1)U\Delta(\pi_3)U\Delta(\pi_4)DD,$$

it follows by an induction similar to part (i) that $\mathrm{mass}_{\mathrm{D}}(\Delta(\pi)) = (2\underline{31})\pi$ and $\mathrm{df}(\Delta(\pi)) = \mathrm{des}(\pi)$. Hence $2\,\mathrm{mass}_{\mathrm{D}}(\Delta(\pi)) + \mathrm{df}(\Delta(\pi)) = \mathrm{mad}(\pi)$.

(iii) Construct a recursive bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$ as follows. Let $P \in \mathcal{D}_n$. If $P = P_1 \cdots P_r$ where P_i is a Dyck path returning to the x-axis for the first time at its endpoint, then define $\Theta(P) = \Theta(P_1) \cdots \Theta(P_r)$. Assume therefore $r = 1$ and write

$$P = UUQ_1DUQ_2D \cdots UQ_sDD,$$

provided $P \neq UD$, where $Q_1, \ldots, Q_s$ are Dyck paths. Define

$$\Theta(P) = \begin{cases} \emptyset & \text{if } P = \emptyset, \\ UD & \text{if } P = UD, \\ U^{s+1}D\Theta(Q_1)D\Theta(Q_2)D \cdots \Theta(Q_s)D & \text{otherwise.} \end{cases}$$

The map Θ is clearly a bijection. Note that

$$\mathrm{sups}(P) = \sum_{i=1}^{s} \mathrm{sups}(Q_i) + \frac{1}{2}\sum_{i=1}^{s} |Q_i| + s,$$

$$\mathrm{mass}_{\mathrm{U}}(\Theta(P)) + \mathrm{dr}(\Theta(P)) = \sum_{i=1}^{s} (\mathrm{mass}_{\mathrm{U}}(\Theta(Q_i)) + \mathrm{dr}(\Theta(Q_i)) + \frac{1}{2}\sum_{i=1}^{s} |\Theta(Q_i)| + s.$$

Hence by induction it follows that $\mathrm{mass}_{\mathrm{U}}(\Theta(P)) + \mathrm{dr}(\Theta(P)) = \mathrm{sups}(P)$.

□

Theorem 3.5. *There exists a bijection* $\Phi : \mathcal{D}_n \to \mathcal{D}_n$ *such that* $\mathrm{stun}(P) = \mathrm{mass}_{\mathrm{U}}(\Phi(P)) + \mathrm{dr}(\Phi(P))$. *In particular, for any* $n \geq 1$,

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{mass}_{\mathrm{U}}(P) + \mathrm{dr}(P)}.$$

Proof. Let $P \in \mathcal{D}_n$ and consider the decomposition

$$P = UP_1D \cdots UP_{m-1}DUP_mD,$$

where $P_1, \ldots, P_{m-1}, P_m$ are (possibly empty) Dyck paths. Define $\Phi : \mathcal{D}_n \to \mathcal{D}_n$ recursively by

$$\Phi(P) = \begin{cases} \emptyset, & \text{if } P = \emptyset \\ UD\Phi(P_1), & \text{if } m = 1 \\ UUU^{m-2}D^{m-2}D\Phi(P_1)\cdots\Phi(P_{m-1})D\Phi(P_m), & \text{if } m > 1 \end{cases}$$

It is not difficult to verify by induction that Φ is a bijection from the recursion. It remains to show that $\mathrm{stun}(P) = \mathrm{mass}_{\mathrm{U}}(\Phi(P)) + \mathrm{dr}(\Phi(P))$. We argue by induction on n. The statement holds for $P = \emptyset$. If $m = 1$, then by induction

$$\begin{aligned} \mathrm{stun}(P) &= \mathrm{stun}(P_1) \\ &= \mathrm{mass}_{\mathrm{U}}(\Phi(P_1)) + \mathrm{dr}(\Phi(P_1)) \\ &= \mathrm{mass}_{\mathrm{U}}(UD\Phi(P_1)) + \mathrm{dr}(UD\Phi(P_1)) \\ &= \mathrm{mass}_{\mathrm{U}}(\Phi(P)) + \mathrm{dr}(\Phi(P)). \end{aligned}$$

Suppose $m > 1$. Note that

$$\mathrm{mass}_{\mathrm{U}}(UUP_0DP_1\cdots P_{m-1}DP_m) = \sum_{i=0}^{m} \mathrm{mass}_{\mathrm{U}}(P_i) + \sum_{i=1}^{m-1} |P_i|/2$$

and that $\mathrm{mass}_{\mathrm{U}}(U^kD^k) = 0$ for all $k \geq 0$. Hence by induction

$$\begin{aligned} \mathrm{stun}(P) &= \mathrm{stun}(P_m) + \sum_{i=1}^{m-1}(\mathrm{stun}(P_i) + (|P_i| + 2)/2) \\ &= \mathrm{mass}_{\mathrm{U}}(\Phi(P_m)) + \mathrm{dr}(\Phi(P_m)) + \sum_{i=1}^{m-1}\left[(\mathrm{mass}_{\mathrm{U}}(\Phi(P_i)) + \mathrm{dr}(\Phi(P_i)) + (|P_i| + 2)/2)\right] \\ &= \left(\mathrm{mass}_{\mathrm{U}}(U^{m-2}D^{m-2}) + \sum_{i=1}^{m} \mathrm{mass}_{\mathrm{U}}(\Phi(P_i)) + \sum_{i=1}^{m-1} |\Phi(P_i)|/2\right) \\ &\quad + \left((m-1) + \sum_{i=1}^{m} \mathrm{dr}(\Phi(P_i))\right) \\ &= \mathrm{mass}_{\mathrm{U}}(\Phi(P)) + \mathrm{dr}(\Phi(P)), \end{aligned}$$

as required. □

Corollary 3.6. *For any $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)}.$$

Proof. By Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 (i) and Theorem 3.5 we have the following diagram of weight preserving bijections

$$\begin{tikzcd} (\mathcal{S}_n(321), \mathrm{inv}) \arrow[r, "\Gamma"] \arrow[d, "\phi"] & (\mathcal{D}_n, \mathrm{spea}) \arrow[r, "\Psi"] & (\mathcal{D}_n, \mathrm{stun}) \arrow[d, "\Phi"] \\ (\mathcal{S}_n(231), \mathrm{mad}) \arrow[rr, "\Delta"] & & (\mathcal{D}_n, \mathrm{mass}_{\mathrm{U}} + \mathrm{dr}) \end{tikzcd}$$

Thus

$$\phi = \Delta^{-1} \circ \Phi \circ \Psi \circ \Gamma$$

is our sought bijection with $\mathrm{inv}(\sigma) = \mathrm{mad}(\phi(\sigma))$ for all $\sigma \in \mathcal{S}_n(321)$. □

The following corollary answers a question of Burstein and Elizalde in [5].

Corollary 3.7. *There exists a bijection* $\Lambda : \mathcal{D}_n \to \mathcal{D}_n$ *such that* $\mathrm{spea}(P) = \mathrm{sups}(\Lambda(P))$. *In particular for any* $n \geq 1$,

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\mathrm{sups}(P)}.$$

Proof. By Theorem 3.3, Theorem 3.4 (iii) and Theorem 3.5 we have the following diagram of weight preserving bijections

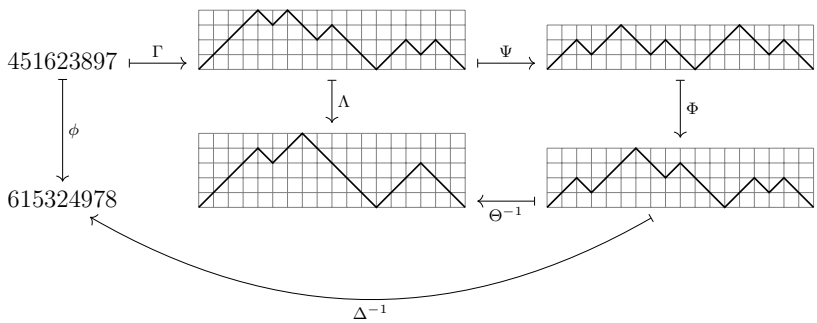
$$\begin{tikzcd}
(\mathcal{D}_n, \mathrm{spea}) \arrow[r, "\Psi"] \arrow[d, "\Lambda"] & (\mathcal{D}_n, \mathrm{stun}) \arrow[d, "\Phi"] \\
(\mathcal{D}_n, \mathrm{sups}) \arrow[r, "\Theta"] & (\mathcal{D}_n, \mathrm{mass_U} + \mathrm{dr})
\end{tikzcd}$$

Hence

$$\Lambda = \Theta^{-1} \circ \Phi \circ \Psi$$

is the required bijection. □

Example 3.1. The below diagram shows an example of the intermediate images under the bijections ϕ and Λ from Corollary 3.6 and Corollary 3.7.



For each Dyck path $P \in \mathcal{D}_n$, Kim et.al. [21] construct two bijections $\mathrm{DTS}(P, \cdot)$ and $\mathrm{DTR}(P, \cdot)$ from the set of linear extensions of the chord poset of P to the set of cover-inclusive Dyck tilings with lower path P (see [21] for terminology). In the special case where $P = (UD)^n$ and the set of linear extensions is restricted to $\mathcal{S}_n(312)$, it follows from [21, Theorem 2.3] that $\mathrm{DTS}(P, \cdot)$ and $\mathrm{DTR}(P, \cdot)$ induce bijections $\theta_{\mathrm{DTS}} : \mathcal{S}_n(312) \to \mathcal{D}_n$ and $\theta_{\mathrm{DTR}} : \mathcal{S}_n(312) \to \mathcal{D}_n$. We remark that the restriction is over $\mathcal{S}_n(231)$ in [21] due to difference in notation. By [21, Theorem 2.4] and [21, Theorem 6.1] it moreover follows that

$$\mathrm{inv}(\sigma) = \mathrm{area}(\theta_{\mathrm{DTS}}(\sigma)), \tag{3.1}$$

$$\mathrm{mad}(\sigma) = \mathrm{area}(\theta_{\mathrm{DTR}}(\sigma)) \tag{3.2}$$

for all $\sigma \in \mathcal{S}_n(312)$. Therefore we get a bijection $\theta : \mathcal{S}_n(312) \to \mathcal{S}_n(312)$ given by

$$\theta = \theta_{\mathrm{DTS}}^{-1} \circ \theta_{\mathrm{DTR}},$$

satisfying $\mathrm{mad}(\theta(\sigma)) = \mathrm{inv}(\sigma)$. Hence we obtain the following theorem.

Theorem 3.8 (Kim-Mésáros-Panova-Wilson [21])**.**
For any $n \geq 1$,

$$\sum_{\sigma \in \mathcal{S}_n(312)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(312)} q^{\mathrm{inv}(\sigma)}.$$

Corollary 3.9. *For any $n \geq 1$,*

$$\sum_{P \in \mathcal{D}_n} q^{\mathrm{area}(P)} = \sum_{P \in \mathcal{D}_n} q^{2\,\mathrm{mass_D}(P) + \mathrm{df}(P)}.$$

Proof. Combine Theorem 3.4 (ii) with (3.2). □

Below we find an interpretation of Theorem 1.1 in terms of Dyck path statistics. Part of the answer is given by a bijection $\Omega : \mathcal{S}_n(231) \to \mathcal{D}_n$ due to Stump [32] which we now define. Let $\sigma \in \mathcal{S}_n(231)$. Suppose $\mathrm{Des}(\sigma) = \{i_1, \ldots, i_k\}$ and $\mathrm{iDes} = \{i'_j \in \mathrm{Des}(\sigma^{-1})\}$ such that $i_1 < \cdots < i_k$ and $i'_1 < \cdots < i'_k$ (recall that $\mathrm{des}(\sigma) = \mathrm{des}(\sigma^{-1})$ via e.g. the argument in Proposition 2.1). For notational purposes set $i_{k+1} = n = i'_{k+1}$. Define a Dyck path $\Omega(\sigma)$ by starting with i'_1 U-steps, followed by i_1 D-steps, followed by $i'_2 - i'_1$ U-steps, followed by $i_2 - i_1$ D-steps, followed by $i'_3 - i'_2$ U-steps, and so on, ending with $i_{k+1} - i_k$ D-steps. Define the statistic

$$\beta(P) = \sum_{v \in \mathrm{Valley}(P)} |\{j \leq \mathrm{pos}_P(v) : s_j = D\}|,$$

for each Dyck path $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$.

Theorem 3.10 (Stump [32])**.** *The map $\Omega : \mathcal{S}_n(231) \to \mathcal{D}_n$ is a well-defined bijection such that $\mathrm{maj}(\sigma) = \beta(\Omega(\sigma))$ for all $\sigma \in \mathcal{S}_n(231)$.*

Proposition 3.11. *For all $\sigma \in \mathcal{S}_n(231)$ and $\pi \in \mathcal{S}_n(321)$ we have*

$$\mathrm{maj}(\sigma) = \sum_{v \in \mathrm{Valley}(\Omega(\sigma))} \frac{\mathrm{pos}_{\Omega(\sigma)}(v) - \mathrm{ht}_{\Omega(\sigma)}(v)}{2},$$

$$\mathrm{den}(\pi) = \mathrm{npea}(\Gamma(\pi)) + \sum_{p \in \mathrm{Peak}(\Gamma(\pi))} \frac{\mathrm{pos}_{\Gamma(\pi)}(p) - \mathrm{ht}_{\Gamma(\pi)}(p)}{2}.$$

Proof. As in [5, Theorem 2.5], observe that

$$\mathrm{den}(\pi) = \sum_{\substack{i \in [n] \\ \pi(i) > i}} i,$$

for all $\pi \in \mathcal{S}_n(321)$. In the definition of Krattenthaler's bijection Γ, each $i \in [n]$ such that $\pi(i) > i$ corresponds to a column i in the array containing a box above the main diagonal. In other words it corresponds to the number of east steps in the lattice path that occur to the left of the box. In the Dyck path $\Gamma(\pi) = s_1 \cdots s_{2n}$ this is reflected in the statistic

$$|\{j \leq \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = D\}| + 1,$$

associated with each $p \in \mathrm{Peak}(\Gamma(\pi))$. We have the following two obvious relations

$$|\{j \leq \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = U\}| - |\{j \leq \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = D\}| = \mathrm{ht}_{\Gamma(\pi)}(p),$$
$$|\{j \leq \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = U\}| + |\{j \leq \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = D\}| = \mathrm{pos}_{\Gamma(\pi)}(p),$$

for each $p \in \mathrm{Peak}(\Gamma(\pi))$. Hence

$$\begin{aligned}\mathrm{den}(\pi) &= \sum_{p\in \mathrm{Peak}(\Gamma(\pi))} (|\{j \le \mathrm{pos}_{\Gamma(\pi)}(p) : s_j = D\}| + 1) \\ &= \mathrm{npea}(\Gamma(\pi)) + \sum_{p\in \mathrm{Peak}(\Gamma(\pi))} \frac{\mathrm{pos}_{\Gamma(\pi)}(p) - \mathrm{ht}_{\Gamma(\pi)}(p)}{2}.\end{aligned}$$

The first statement in the proposition follows from Theorem 3.10 and a similar observation to above. □

Remark 3.12. By Theorem 1.1, the Dyck path statistics in Proposition 3.11 are equidistributed over $\mathcal{D}_n$.

4. Equidistributions via generating functions

In this section we use generating functions to derive the equidistributions (albeit non-bijectively) between Mahonian statistics over $\mathcal{S}_n(\pi)$. We also provide a recursion for a more general statistic involving arbitrary linear combinations of vincular pattern functions of length three. This recursion generalizes for instance the recursions in [14].

Theorem 4.1. *We have*

$$\sum_{\sigma\in\mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma\in\mathcal{S}(132)} q^{\mathrm{sist}(\sigma)} z^{|\sigma|} = \cfrac{1}{1-\cfrac{z}{1-\cfrac{qz}{1-\cfrac{qz}{1-\cfrac{q^2z}{1-\cfrac{q^2z}{\ddots}}}}}} \tag{4.1}$$

$$\sum_{\sigma\in\mathcal{S}(312)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma\in\mathcal{S}(213)} q^{\mathrm{sist}(\sigma)} z^{|\sigma|} = \cfrac{1}{1-\cfrac{z}{1-\cfrac{qz}{1-\cfrac{q^2z}{1-\cfrac{q^3z}{1-\cfrac{q^4z}{\ddots}}}}}}. \tag{4.2}$$

Proof. Note that over $\mathcal{S}(231)$ we have $\mathrm{mad} = (\underline{31}2) + (\underline{21})$. The reverse of sist (i.e. the statistic obtained by reversing all vincular patterns) is given by $\mathrm{rsist} = (\underline{31}2) + (\underline{12})$. Hence (4.1) is equivalent to proving

$$\sum_{\sigma\in\mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma\in\mathcal{S}(231)} q^{\mathrm{rsist}(\sigma)} z^{|\sigma|}.$$

Let $\sigma \in \mathcal{S}(231)$ and decompose $\sigma = 213[1, \sigma_1, \sigma_2]$. Then we obtain the recursion

$$\begin{aligned}\mathrm{rsist}(\sigma) &= [12)\sigma_1 + \delta_{\sigma_2\neq\emptyset} + \mathrm{rsist}(\sigma_1) + \mathrm{rsist}(\sigma_2), \\ [12)\sigma &= |\sigma_2|,\end{aligned}$$

where δ denotes the Kronecker delta. Let

$$F(q,t,z) = \sum_{\sigma \in \mathcal{S}(231)} q^{\mathrm{rsist}(\sigma)} t^{[12)\sigma} z^{|\sigma|}.$$

Then

$$\begin{aligned} F(q,t,z) &= 1 + z \left(\sum_{\sigma_1 \in \mathcal{S}(231)} q^{\mathrm{rsist}(\sigma_1)} q^{[12)\sigma_1} z^{|\sigma_1|} \right) \\ &\quad + qz \left(\sum_{\sigma_1 \in \mathcal{S}(231)} q^{\mathrm{rsist}(\sigma_1)} q^{[12)\sigma_1} z^{|\sigma_1|} \right) \left(\sum_{\sigma_2 \in \mathcal{S}(231)} q^{\mathrm{rsist}(\sigma_2)} (zt)^{|\sigma_2|} - 1 \right) \\ &= 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,zt) - 1). \end{aligned}$$

Substituting $t = 1$ and $t = q$ we obtain the equation system

$$\begin{cases} F(q,1,z) = 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,z) - 1) \\ F(q,q,z) = 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,qz) - 1) \end{cases}$$

Eliminating $F(q,q,z)$ and solving for $F(q,1,z)$ we obtain

$$F(q,1,z) = \cfrac{1}{1 - \cfrac{z}{1 - qzF(q,1,qz)}},$$

which gives the continued fraction in the theorem. Similarly letting

$$G(q,z,t) = \sum_{\sigma \in \mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} t^{[12)} z^{|\sigma|},$$

then we obtain the recursive relation

$$G(q,t,z) = 1 + zG(q,1,zt) + qzG(q,1,zt)(G(q,q,z) - 1).$$

Substituting $t = 1$ and $t = q$ as before and solving for $G(q,1,z)$ we obtain the same continued fraction expansion as above, proving the desired equidistribution.

The second statement in the theorem is proved similarly. Over $\mathcal{S}(312)$ we have $\mathrm{mad} = (2\underline{31}) + (2\underline{31}) + (\underline{21})$. Let $\sigma \in \mathcal{S}(312)$ and decompose $\sigma = 132[\sigma_1, \sigma_2, 1]$. Then we obtain the recursion

$$\begin{aligned} \mathrm{mad}(\sigma) &= 2 \cdot (12]\sigma_2 + \delta_{\sigma_2 \neq \emptyset} + \mathrm{mad}(\sigma_1) + \mathrm{mad}(\sigma_2), \\ (12]\sigma &= |\sigma_1|. \end{aligned}$$

Letting $F(q,t,z) = \sum_{\sigma \in \mathcal{S}(312)} q^{\mathrm{mad}(\sigma)} t^{(12]\sigma} z^{|\sigma|}$ we thus obtain

$$F(q,t,z) = 1 + zF(q,1,zt) + qzF(q,1,zt)(F(q,q^2,z) - 1).$$

Putting $t = 1$ and $t = q^2$, eliminating $F(q,q^2,z)$ from the resulting equation system and solving for $F(q,1,z)$ we obtain the continued fraction expansion in the theorem.

A similar argument for rsist over $\mathcal{S}(312)$ gives a matching continued fraction expansion. We leave the details to the reader. $\square$

Remark 4.2. In [8, Corollary 8.6] it was proved that the continued fraction expansion of the generating function of inv over $\mathcal{S}(321)$ matches that of (4.1). This gives an alternative proof of Corollary 3.6.

Remark 4.3. For mad, the continued fractions (4.1) and (4.2) may also be deduced from the following more refined continued fraction in [12, Theorem 22] by specializing $(x, y, p, q) = (1, q, 0, q) = 1$ resp. $(x, y, p, q) = (1, p, p^2, 0)$ and using the fact that $\sigma \in \mathcal{S}(2\underline{31})$ if and only if $\sigma \in \mathcal{S}(231)$ (see [10, Lemma 2]),

$$\sum_{\sigma\in\mathcal{S}} x^{\delta_{\sigma\neq\emptyset}+(\underline{12})\sigma} y^{(\underline{21})\sigma} p^{(2\underline{31})\sigma} q^{(\underline{312})\sigma} z^{|\sigma|} = \cfrac{1}{1-\cfrac{x[1]_{p,q}z}{1-\cfrac{y[1]_{p,q}z}{1-\cfrac{x[2]_{p,q}z}{1-\cfrac{y[2]_{p,q}z}{1-\cfrac{x[3]_{p,q}z}{\ddots}}}}}}$$

where $[n]_{p,q} = q^{n-1} + pq^{n-2} + \cdots + p^{n-2}q + p^{n-1}$ and δ denotes the Kronecker delta.

Using almost identical arguments to Theorem 4.1 we may moreover prove the following equidistributions.

Theorem 4.4. *For any* $n \geq 1$

$$\sum_{\sigma\in\mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{sist}'(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(231)} q^{\mathrm{sist}''(\sigma)},$$
$$\sum_{\sigma\in\mathcal{S}_n(312)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{foze}'(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(231)} q^{\mathrm{sist}'(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{sist}''(\sigma)}.$$

By combining Theorem 4.1 and Theorem 4.4 with Theorem 3.8 and Corollary 3.6 we may deduce further equidistributions between inv and the statistics $\mathrm{foze}', \mathrm{sist}, \mathrm{sist}'$ and sist'', see Table 2 in §5 for a summary.

For each $k \geq 1$, let $\iota_{k-1} = (12\cdots k)$ denote the statistic that counts the number of increasing subsequences of length k in a permutation. Define ι_{-1} by $\iota_{-1}(\sigma) = 1$ for all $\sigma \in \mathcal{S}$ (i.e. we declare all permutations to have exactly one subsequence of length 0). We will now find a statistic expressed as a linear combination of ι_k's which is equidistributed with the continued fraction (4.1). We will derive this statistic using the Catalan continued fraction framework of Brändén-Claesson-Steingrímsson[3]. Let

$$\mathcal{A} = \{A : \mathbb{N}\times\mathbb{N} \to \mathbb{Z} : A_{nk} = 0 \text{ for all but finitely many } k \text{ for each } n\}$$

be the ring of infinite matrices with a finite number of non-zero entries in each row. Note in particular that the matrices in $\mathcal{A}$ are indexed starting from 0. With each $A \in \mathcal{A}$ associate a family of statistics $\{\langle \boldsymbol{\iota}, A_k\rangle\}_{k\geq 0}$ where $\boldsymbol{\iota} = (\iota_0, \iota_1, \dots)$, A_k is the kth column of A, and

$$\langle \boldsymbol{\iota}, A_k\rangle = \sum_{i=0}^{\infty} A_{ik}\iota_i.$$

Let $\mathbf{q} = (q_0, q_1, \dots)$, where $q_0, q_1, \dots$ are indeterminates. For each $A \in \mathcal{A}$ define

$$F_A(\mathbf{q}) = \sum_{\sigma \in \mathcal{S}(132)} \prod_{k \geq 0} q_k^{\langle \boldsymbol{\iota}, A_k \rangle(\sigma)},$$

$$C_A(\mathbf{q}) = \cfrac{1}{1 - \cfrac{\prod q_k^{A_{0k}}}{1 - \cfrac{\prod q_k^{A_{1k}}}{1 - \cfrac{\prod q_k^{A_{2k}}}{1 - \cfrac{\prod q_k^{A_{3k}}}{1 - \cfrac{\prod q_k^{A_{4k}}}{\ddots}}}}}}$$

Theorem 4.5 (Brändén-Claesson-Steingrímsson[3])**.** *Let $A \in \mathcal{A}$ and $B = \left(\binom{i}{j}\right)_{i,j \geq 0}$. Then*

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}),$$

and conversely

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

In particular, all continued fractions $C_A(\mathbf{q})$ are generating functions of statistics on $\mathcal{S}(132)$ expressed as (possibly infinite) linear combinations of ι_k's.

Define the permutation statistic

$$\mathrm{inc} = \iota_1 + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \iota_k.$$

Note that inc is not a Mahonian statistic.

Proposition 4.6. *We have*

$$\sum_{\sigma \in \mathcal{S}(132)} q^{\mathrm{inc}(\sigma)} z^{|\sigma|} = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{qz}{1 - \cfrac{q^2 z}{1 - \cfrac{q^2 z}{\ddots}}}}}} \tag{4.3}$$

Proof. Comparing (4.3) with the definition of $C_A(\mathbf{q})$ we get

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $B^{-1} = \left((-1)^{i-j}\binom{i}{j}\right)_{i,j\geq 0}$. In $B^{-1}A$ we see that columns $2, 3, \dots$ are zero columns and that column 1 is equal to $(1, 0, 0, \dots)^T$ since $\sum_{k\geq 0}(-1)^{n-k}\binom{n}{k} = \delta_{n0}$ where δ_{ij} denotes the Kronecker delta. The entries $(B^{-1}A)_{k0}$ in column 0 are given by

$$(B^{-1}A)_{n0} = \sum_{i\geq 0}\lfloor (i+1)/2\rfloor(-1)^{k-i}\binom{k}{i} = \begin{cases} 0, & \text{if } k = 0 \\ 1, & \text{if } k = 1 \\ (-1)^{k-1}2^{k-2}, & \text{if } k > 1. \end{cases}$$

Hence the proposition follows from Theorem 4.5. □

Remark 4.7. Applying the same argument to the continued fraction (4.2) it is easy to see that Theorem 4.5 gives equidistribution with

$$\sum_{\sigma\in\mathcal{S}(132)} q^{\iota_1(\sigma)}z^{|\sigma|} = \sum_{\sigma\in\mathcal{S}(312)} q^{\mathrm{inv}(\sigma)}z^{|\sigma|}.$$

Corollary 4.8. *For any $n \geq 1$,*

$$\sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{inc}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)}$$

Proof. Follows by combining Corollary 3.6, Theorem 4.1 and Proposition 4.6. □

Proposition 4.9. *Let $\Delta : \mathcal{S}(132) \to \mathcal{D}$ denote the standard bijection defined by $\Delta(\sigma) = U\Delta(\sigma_1)D\Delta(\sigma_2)$ where $\sigma = 231[\sigma_1, 1, \sigma_2] \in \mathcal{S}(132)$. Then*

$$\mathrm{inc}(\sigma) = \mathrm{sdow}(\Delta(\sigma))$$

for all $\sigma \in \mathcal{S}(132)$.

Proof. In [23] (see also [3]) Krattenthaler shows that

$$\iota_k(\sigma) = \sum_{i\in\mathrm{Down}(\Delta(\sigma))}\binom{\mathrm{ht}_{\Delta(\sigma)}(i)-1}{k},$$

for all $\sigma \in \mathcal{S}(132)$. Hence

$$\begin{aligned}\mathrm{inc}(\sigma) &= \sum_{i\in\mathrm{Down}(\Delta(\sigma))}\left(\binom{\mathrm{ht}_{\Delta(\sigma)}(i)-1}{1} + \sum_{k=2}^{\infty}(-1)^{k-1}2^{k-2}\binom{\mathrm{ht}_{\Delta(\sigma)}(i)-1}{k}\right) \\ &= \sum_{i\in\mathrm{Down}(\Delta(\sigma))}\lfloor \mathrm{ht}_{\Delta(\sigma)}(i)/2\rfloor \\ &= \mathrm{sdow}(\Delta(\sigma)),\end{aligned}$$

for all $\sigma \in \mathcal{S}(132)$. □

Since the Mahonian statistics in Table 1 are linear combinations of vincular patterns of length at most three, it is natural to consider the following more general statistic.

Definition 4.1. Let $\mathcal{P} = \{a\underline{bc} : abc \in \mathcal{S}_3\} \cup \{\underline{ab}c : abc \in \mathcal{S}_3\} \cup \{\underline{21}\}$ and $\boldsymbol{\alpha} = (\alpha_\rho) \in \mathbb{N}^{\mathcal{P}}$. Define the statistic $\mathrm{stat}_{\boldsymbol{\alpha}} : \mathcal{S} \to \mathbb{N}$ by

$$\mathrm{stat}_{\boldsymbol{\alpha}}(\sigma) = \sum_{\rho\in\mathcal{P}}\alpha_\rho(\rho)\sigma,$$

for all $\sigma \in \mathcal{S}$.

Let *head* and *last* be the statistics defined by $\mathrm{head}(\sigma) = \sigma(1)$ and $\mathrm{last}(\sigma) = \sigma(n)$ for all $\sigma \in \mathcal{S}_n$. We associate to $\mathrm{stat}_{\boldsymbol{\alpha}}$ the following generating function for each set Π of patterns

$$F_n(\Pi, \boldsymbol{\alpha}; q, t, u, v) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\mathrm{stat}_{\boldsymbol{\alpha}}(\sigma)} t^{\mathrm{des}(\sigma)} u^{\mathrm{head}(\sigma)} v^{\mathrm{last}(\sigma)}.$$

Theorem 4.10. *We have*

$$\begin{aligned} &F_n(312, \boldsymbol{\alpha}; q, t, u, v) \\ &= q^{C(0)} uv F_{n-1}\left(312, \boldsymbol{\alpha}; q, q^{A_2(0)} t, q^{B_2}, v\right) + q^{C(n-1)} tuv F_{n-1}\left(312, \boldsymbol{\alpha}; q, q^{A_1(n-1)} t, u, q^{B_1}\right) \\ &\quad + \sum_{k=1}^{n-2} q^{C(k)} tuv^k F_k\left(312, \boldsymbol{\alpha}; q, q^{A_1(k)} t, u, q^{B_1}\right) F_{n-k-1}\left(312, \boldsymbol{\alpha}; q, q^{A_2(k)} t, q^{B_2}, v\right), \end{aligned}$$

where

$$\begin{aligned} A_1(k) &= \alpha_{\underline{32}1} - \alpha_{\underline{23}1} + (n-k-1)\left(\alpha_{\underline{21}3} - \alpha_{\underline{12}3}\right), \\ A_2(k) &= (k+1)\left(\alpha_{1\underline{32}} - \alpha_{1\underline{23}}\right), \\ B_1 &= \alpha_{2\underline{31}} - \alpha_{3\underline{21}}, \\ B_2 &= \alpha_{\underline{13}2} - \alpha_{\underline{12}3}, \\ C(k) &= (k\alpha_{\underline{12}3} - \alpha_{\underline{21}3})(n-k-1) - \delta_{k<n-1}\alpha_{\underline{13}2} + \delta_{k>0}(n-k-1)\alpha_{\underline{21}3} \\ &\quad + \delta_{k>0}(k-1)\alpha_{\underline{23}1} + \delta_{k<n-1}(k+1)(n-k-2)\alpha_{1\underline{23}} \\ &\quad + \delta_{k<n-1}k\alpha_{2\underline{13}} - \delta_{k>0}\alpha_{2\underline{31}} + k\alpha_{3\underline{21}} + \delta_{k>0}\alpha_{\underline{21}}, \end{aligned}$$

and δ denotes the Kronecker delta.

Proof. Let $\sigma \in \mathcal{S}_n(312)$ and consider the inflation form $\sigma = 213[\sigma_1, 1, \sigma_2]$ where $\sigma_1 \in \mathcal{S}_k(312)$ and $\sigma_2 \in \mathcal{S}_{n-k-1}(312)$. Then for each $\rho \in \mathcal{P}$ we get the recursive relations

$$(\rho)\sigma = (\rho)\sigma_1 + (\rho)\sigma_2 + m_\rho,$$

where

$$\begin{aligned} m_{1\underline{23}} &= [12)\sigma_2 + |\sigma_2|(\underline{12})\sigma_1, & m_{1\underline{23}} &= (|\sigma_1|+1)(\underline{12})\sigma_2, \\ m_{1\underline{32}} &= [21)\sigma_2, & m_{1\underline{32}} &= (|\sigma_1|+1)(\underline{21})\sigma_2, \\ m_{\underline{21}3} &= ((\underline{21}) + \delta_{\sigma_1 \neq \emptyset})|\sigma_2|, & m_{\underline{21}3} &= |\sigma_1|\delta_{\sigma_2 \neq \emptyset}, \\ m_{\underline{23}1} &= (\underline{12})\sigma_1, & m_{\underline{23}1} &= (12]\sigma_1, \\ m_{\underline{32}1} &= (\underline{21})\sigma_1, & m_{\underline{32}1} &= (21]\sigma_1 \end{aligned}$$

and $m_{\underline{21}} = \delta_{\sigma_1 \neq \emptyset}$. It follows that $\mathrm{stat}_{\boldsymbol{\alpha}}$ satisfies the following recursion

$$\mathrm{stat}_{\boldsymbol{\alpha}}(\sigma) = \mathrm{stat}_{\boldsymbol{\alpha}}(\sigma_1) + \mathrm{stat}_{\boldsymbol{\alpha}}(\sigma_1) + \sum_{\rho \in \mathcal{P}} m_\rho.$$

We note that $|\sigma_1| = k$, $|\sigma_2| = n-k-1$, $(\underline{21})\sigma = \mathrm{des}(\sigma)$, $(\underline{12})\sigma = \delta_{\sigma \neq \emptyset}(|\sigma|-1) - \mathrm{des}(\sigma)$, $[21)\sigma = \mathrm{head}(\sigma) - \delta_{\sigma \neq \emptyset}$, $[12)\sigma = |\sigma| - \mathrm{head}(\sigma)$, $(12]\sigma = \mathrm{last}(\sigma) - \delta_{\sigma \neq \emptyset}$ and $(21]\sigma = |\sigma| - \mathrm{last}(\sigma)$ for all $\sigma \in \mathcal{S}_n(312)$. Converting these statements into generating functions proves the theorem. □

Remark 4.11. If $\alpha_{\underline{23}1} = \alpha_{\underline{31}2} = \alpha_{\underline{32}1} = \alpha_{\underline{21}} = 1$ and $\alpha_\rho = 0$ otherwise, then $\mathrm{stat}_{\boldsymbol{\alpha}} = \mathrm{inv}$ and $F(312, \boldsymbol{\alpha}; q, 1, 1, 1) = I_n(q) = \widetilde{C}_n(q)$. Similarly if we choose $\boldsymbol{\alpha}$ such that $\mathrm{stat}_{\boldsymbol{\alpha}} = \mathrm{maj}$, then we recover the recursion in [14, Theorem 3.4] via the recursion for $F(312, \boldsymbol{\alpha}; q, t, 1, 1)$ in Theorem 4.10.

Recall the Simion-Schmidt bijection $\phi : \mathcal{S}_n(123) \to \mathcal{S}_n(132)$ which maps $\sigma \in \mathcal{S}_n(123)$ to the unique permutation in $\mathcal{S}_n(132)$ with the same left-to-right minima in the same positions as σ (cf Lemma 2.9). As explicitly noted by Claesson and Kitaev [11] this bijection clearly preserves the head statistic and hence $[123]_{\text{head}} = [132]_{\text{head}}$. Although head is not a Mahonian statistic we complete its st-Wilf classification below for all subsets of $\mathcal{S}_3$ of size at most three. Equivalences for subsets of larger size can easily be found using similar analysis on the inflation forms. These are less interesting and omitted for brevity. We note in particular that the single pattern distributions with respect to the head statistic are given by well-known refinements of the Catalan numbers.

Proposition 4.12. *We have*

$$\begin{aligned}
[123]_{\text{head}} &= \{123, 132\} = [132]_{\text{head}},\\
[321]_{\text{head}} &= \{321, 312\} = [312]_{\text{head}},\\
[231]_{\text{head}} &= \{213, 231\} = [213]_{\text{head}}\\
[123, 213]_{\text{head}} &= \{\{123, 213\}, \{132, 213\}, \{132, 231\}\}\\
[231, 321]_{\text{head}} &= \{\{231, 321\}, \{213, 312\}, \{231, 312\}\}\\
[213, 231, 321]_{\text{head}} &= \{\{213, 231, 321\}, \{213, 231, 312\}\}\\
[132, 213, 231]_{\text{head}} &= \{\{132, 213, 231\}, \{123, 213, 231\}\}\\
[132, 213, 321]_{\text{head}} &= \{\{132, 213, 321\}, \{132, 213, 312\}, \{132, 231, 321\},\\
&\qquad \{132, 231, 312\}, \{123, 213, 312\}\}.
\end{aligned}$$

Remaining subsets $\Pi \subseteq \mathcal{S}_3$ *of size at most three have singleton* head-*Wilf class. Moreover for any* $n \geq 1$

$$\sum_{\sigma \in \mathcal{S}_n(123)} q^{\text{head}(\sigma)} = \sum_{k=1}^{n} C_{n-1,k-1} q^k,$$

$$\sum_{\sigma \in \mathcal{S}_n(213)} q^{\text{head}(\sigma)} = \sum_{k=1}^{n} C_{k-1} C_{n-k} q^k,$$

$$\sum_{\sigma \in \mathcal{S}_n(123,213)} q^{\text{head}(\sigma)} = q + \sum_{k=2}^{n} 2^{k-2} q^k.$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ *and* $C_{n,k} = \frac{n-k+1}{n+1}\binom{n+k}{n}$ (`A009766` [31]).

Proof. The map $\psi : \mathcal{S}_n(321) \to \mathcal{S}_n(312)$ given by $\psi(\sigma) = \phi(\sigma^c)^c$, where $\phi : \mathcal{S}_n(123) \to \mathcal{S}_n(132)$ is the Simion-Schmidt bijection, clearly satisfies $\text{head}(\psi(\sigma)) = \text{head}(\sigma)$. Hence $[321]_{\text{head}} = [312]_{\text{head}}$. Let $\sigma = a_1 a_2 \cdots a_n \in \mathcal{S}_n(132)$. According to the non-recursive description of the standard bijection $\Delta : \mathcal{S}_n(132) \to \mathcal{D}_n$ (due to Krattenthaler [23]), when a_i is read from left to right we adjoin as many U-steps as necessary to the path obtained thus far to reach height $h_j + 1$, followed by a D-step to height h_j. Here h_j is the number of letters in $a_{j+1} \cdots a_n$ which are larger than a_j. As such, the number of permutations $\sigma \in \mathcal{S}_n(132)$ with $\text{head}(\sigma) = k$ is given by the number of Dyck paths starting with exactly $n-k+1$ number of U-steps. These are equivalently enumerated by the number of lattice paths with steps $(1, 0)$ and $(0, 1)$ from $(1, n-k+1)$ to (n, n) staying weakly above the line $y = x$. By [24, Theorem 10.3.1] the

number of such paths are given by

$$\binom{n+n-1-(n-k+1)}{n-(n-k+1)} - \binom{n+n-1-(n-k+1)}{n-1+1} = C_{n-1,k-1}.$$

The map $\varphi : \mathcal{S}_n(231) \to \mathcal{S}_n(213)$ recursively defined by

$$\varphi(213[1,\sigma_1,\sigma_2]) = 231[1,\varphi(\sigma_1),\varphi(\sigma_2)],$$

where $\sigma_1 \in \mathcal{S}_{k-1}(231)$ and $\sigma_2 \in \mathcal{S}_{n-k}(231)$, is clearly a head-preserving bijection. Hence $[231]_{\text{head}} = [213]_{\text{head}}$. Since $|\mathcal{S}_k(231)| = C_k$ it follows from the inflation form that there are $C_{k-1}C_{n-k}$ permutations $\sigma \in \mathcal{S}_n(231)$ with $\text{head}(\sigma) = k$.

If $\sigma \in \mathcal{S}_n(132,231)$, then σ is either decomposed as $12[\sigma_1,1]$ or as $21[1,\sigma_1]$ where $\sigma_1 \in \mathcal{S}_{n-1}(132,231)$. Thus the letters $1,2,\ldots,n$ are in reverse order recursively placed at the beginning or at the end of the permutation. For σ to have $\text{head}(\sigma) = k$, the letters $k+1,\ldots,n$ must be placed in increasing order at the end and k at the beginning. Remaining $k-1$ letters may be placed on either end giving two choices each (except for the last letter). Hence there exists 2^{k-2} permutations $\sigma \in \mathcal{S}_n(132,231)$ with $\text{head}(\sigma) = k$ for $k > 1$.

Let $\iota_k = 12\cdots k$ and $\delta_k = k\cdots 21$ for $k \geq 1$. If $\sigma \in \mathcal{S}_n(123,213)$ and $\text{head}(\sigma) = k$, then $\sigma = 231[1,\delta_{n-k},\sigma_1]$ for some $\sigma_1 \in \mathcal{S}_{k-1}(123,213)$. It is easy to see that $|\mathcal{S}_k(123,213)| = 2^{k-1}$ by induction. Hence $[132,231]_{\text{head}} = [123,213]_{\text{head}}$.

If $\sigma \in \mathcal{S}_n(132,213)$, then $\sigma = 231[1,\iota_{n-k},\sigma_1]$ where $\sigma_1 \in \mathcal{S}_{k-1}(132,213)$. The map $\chi : \mathcal{S}_n(132,213) \to \mathcal{S}_n(123,213)$ recursively given by

$$\chi(231[1,\iota_{n-k},\sigma_1]) = 231[1,\delta_{n-k},\chi(\sigma_1)],$$

is clearly a head-preserving bijection. Hence $[132,213]_{\text{head}} = [123,213]_{\text{head}}$. Remaining equivalences and their distributions may be deduced from the fact that $\text{head}(\sigma^c) = n - \text{head}(\sigma) + 1$. The equivalences between the size three subsets can be proved similarly via bijections between their corresponding inflation forms (the inflation forms can be referenced in [14]). The details for these are left to the reader. □

5. Summary and conjectures

In Table 2 we summarize the equidistributions proved in this paper (highlighted in black). In a given cell corresponding to stat_{row} and stat_{col}, a pair of patterns π_1, π_2 denotes the equidistribution

$$\sum_{\sigma\in\mathcal{S}_n(\pi_1)} q^{\text{stat}_{\text{row}}(\sigma)} \sum_{\sigma\in\mathcal{S}_n(\pi_2)} q^{\text{stat}_{\text{col}}(\sigma)}.$$

The equidistributions in Table 2 highlighted in blue were established in [14, 21]. The equidistributions between maj, bast′ and bast″ can be proved in a similar way to Proposition 2.1, since the inverse map is the right bijection in two of the cases and the rest can be deduced via the maj-Wilf equivalences from [14]. Remaining equidistributions were either proved directly or follow by combining equidistributions proved in this paper. For instance $\sum_{\sigma\in\mathcal{S}_n(213)} q^{\text{maj}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(231)} q^{\text{foze}(\sigma)}$ is deduced by combining Proposition 2.1 and Theorem 2.12.

Conjecture 5.1. *Table 2 is the complete table of Mahonian 3-function equidistributions over permutations avoiding a single classical pattern of length three.*

We have verified all entries in Table 2 by computer for $n \leq 10$. Other than than the entries in Table 2 there are no additional equidistributions (over permutations avoiding a single classical pattern of length three) between the statistics in Table 1.

	maj	inv	mak	makl	mad	bast	bast′	bast″	foze	foze′	foze″	sist	sist′	sist″
maj	132, 231 213, 312		123, 123 132, 132 132, 312 213, 213 213, 231 231, 132 231, 312 312, 213 312, 231 321, 321	132, 231 213, 312 231, 231 312, 312 321, 321		132, 213 213, 231 231, 213 312, 231	132, 132 231, 132	213, 231 312, 231	132, 132 213, 231 231, 132 312, 231					
inv	•	132, 213 231, 312			231, 312 312, 312 321, 231					231, 132 312, 132 321, 213	231, 132 312, 132 321, 213	231, 213 312, 213 321, 132	231, 231 312, 231 321, 132	231, 132 312, 132 321, 231
mak	•	•	132, 312 213, 231	132, 231 213, 312 231, 312 312, 231 321, 321		132, 213 213, 231 231, 231 312, 213	132, 132 312, 132	213, 231 231, 231	132, 132 213, 231 231, 231 312, 132					
makl	•	•	•			132, 132 231, 213 312, 231	231, 132	312, 231	132, 213 231, 132 312, 231					
mad	•	•	•	•						231, 213 312, 132	231, 213 312, 132	231, 132 312, 213	132, 213 231, 132 312, 231	213, 213 231, 231 312, 132
bast	•	•	•	•	•		213, 132	231, 231	123, 123 213, 132 132, 213 231, 231 312, 312 321, 321					
bast′	•	•	•	•	•	•			132, 132					
bast″	•	•	•	•	•	•	•		231, 231					
foze	•	•	•	•	•	•	•	•						
foze′	•	•	•	•	•	•	•	•	•		132, 132 213, 213	132, 213 213, 132	132, 231 213, 132	132, 132 213, 231
foze″	•	•	•	•	•	•	•	•	•	•		213, 132 132, 213	213, 132 132, 231	132, 132 213, 231
sist	•	•	•	•	•	•	•	•	•	•	•		132, 132 213, 231 312, 312	132, 231 213, 132 231, 312
sist′	•	•	•	•	•	•	•	•	•	•	•	•		132, 231 231, 132
sist″	•	•	•	•	•	•	•	•	•	•	•	•	•	

TABLE 2. Previously established equidistributions in blue, equidistributions proved in this paper in black and conjectured equidistributions in red.

Note. The conjectured equidistributions in Table 2 between maj and bast (and consequently between mak and bast) were recently established by J. N. Chen [6].

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Paper E

THE CONE OF CYCLIC SIEVING PHENOMENA

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Abstract. We study cyclic sieving phenomena (CSP) on combinatorial objects from an abstract point of view by considering a rational polyhedral cone determined by the linear equations that define such phenomena. Each lattice point in the cone corresponds to a non-negative integer matrix which jointly records the statistic and cyclic order distribution associated with the set of objects realizing the CSP. In particular we consider a *universal* subcone onto which every CSP matrix linearly projects such that the projection realizes a CSP with the same cyclic orbit structure, but via a *universal* statistic that has even distribution on the orbits.

Reiner et.al. showed that every cyclic action give rise to a unique polynomial (mod $q^n - 1$) complementing the action to a CSP. We give a necessary and sufficient criterion for the converse to hold. This characterization allows one to determine if a combinatorial set with a statistic give rise (in principle) to a CSP without having a combinatorial realization of the cyclic action. We apply the criterion to conjecture a new CSP involving stretched Schur polynomials and prove our conjecture for certain rectangular tableaux. Finally we study some geometric properties of the CSP cone. We explicitly determine its half-space description and in the prime order case we determine its extreme rays.

Contents

Key words and phrases. Cyclic sieving, universal, polytope, roots of unity.

1. Introduction

1.1. **Background on cyclic sieving phenomena.** The cyclic sieving phenomenon was introduced by Reiner, Stanton and White in [RSW04]. For a survey, see [Sag].

Definition 1.1. Let C_n be a cyclic group of order n generated by σ_n, X a finite set on which C_n acts and $f(q) \in \mathbb{N}[q]$. Let $X^g := \{x \in X : g \cdot x = x\}$ denote the fixed point set of X under $g \in C_n$. We say that the triple $(X, C_n, f(q))$ exhibits the *cyclic sieving phenomenon (CSP)* if

$$f(\omega_n^k) = |X^{\sigma_n^k}|, \text{ for all } k \in \mathbb{Z}, \tag{1.1}$$

where ω_n is any fixed primitive n^{th} root of unity.

Since $f(1)$ is always the cardinality of X, it is common that $f(q)$ is given as $f_\tau(q) := \sum_{x \in X} q^{\tau(x)}$ for some statistic on X. With this in mind, we say that the triple (X, C_n, τ) exhibits CSP if $(X, C_n, f_\tau(q))$ does.

Here is a short list of cyclic sieving phenomena found in the literature (see [RSW04, Sag] for a more comprehensive list):

- Words $X = W_{n,k}$ of length n over an alphabet of size k, C_n acting via cyclic shift,
$$f(q) := \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q = \sum_{w \in W_{n,k}} q^{\operatorname{maj} w}.$$
- Standard Young tableaux $X = \mathrm{SYT}(\lambda)$ of rectangular shape $\lambda = (n^m)$, C_n acting via jeu-de-taquin promotion [Rho10],
$$f(q) := \frac{[n]_q!}{\prod_{(i,j) \in \lambda} [h_{i,j}]_q} = q^{-n\binom{m}{2}} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\operatorname{maj}(T)},$$
this expression being the q-hook-length formula [Sta71].
- Triangulations X of a regular $(n+2)$-gon, C_{n+2} acting via rotation of the triangulation, $f(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$, MacMahon's q-analogue of the Catalan numbers [Mac16]. Note that through well-known bijections (see [Sta15]) we get induced CSPs with the sets $X = \mathrm{Dyck}(n)$, the set of Dyck paths of semi-length n, and $X = \mathfrak{S}_n(231)$, the set of permutations in $\mathfrak{S}_n$ avoiding the classical pattern 231. Moreover one has
$$f(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{P \in \mathrm{Dyck}(n)} q^{\operatorname{maj}(P)} = \sum_{\pi \in \mathfrak{S}_n(231)} q^{\operatorname{maj}(\pi) + \operatorname{maj}(\pi^{-1})},$$
where the last equality is due to Stump [Stu09].

1.2. **Outline of the paper.** The examples presented in the previous subsection have one or more of the following pair of common features:

- The action of C_n on X has a *natural* definition.
- The polynomial $f(q)$ is generated by a *natural* statistic on X.

What is *natural* largely lies in the eyes of the beholder, but broadly it could be taken to mean a definition with combinatorial substance.

The following equivalent condition for a triple $(X, C_n, f(q))$ to exhibit the cyclic sieving phenomenon was given by Reiner–Stanton–White in [RSW04]:

$$f(q) \equiv \sum_{\mathcal{O} \in \mathrm{Orb}_{C_n}(X)} \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1} \pmod{q^n - 1}, \tag{1.2}$$

where $\mathrm{Orb}_{C_n}(X)$ denotes the set of orbits of X under the action of C_n.

Therefore the coefficient of q^i in $f(q) \pmod{q^n - 1}$ is generically interpreted as the number of orbits whose stabilizer-order divides i. This alternative condition also means that every cyclic action of C_n on a finite set X give rise to a (not necessarily natural) polynomial $f(q)$, unique modulo $q^n - 1$, such that $(X, C_n, f(q))$ exhibits the cyclic sieving phenomenon.

In this paper we consider when the converse of the above property holds. *Given a combinatorial set X with a natural statistic $\tau : X \to \mathbb{N}$, when does it give rise to a (not necessarily natural) action of C_n on X such that (X, C_n, τ) exhibits the cyclic sieving phenomenon?*

Having a necessary and sufficient criteria for the existence of such a CSP adds a couple of benefits:

- Given a polynomial $f(q) = \sum_{x \in X} q^{\tau(x)}$ generated by a natural statistic $\tau : X \to \mathbb{N}$, we can determine if a CSP exists in principle without knowing a combinatorial realization of the cyclic action. The criteria thus serves as a tool for confirming or refuting the existence of cyclic sieving phenomena involving a candidate polynomial.
- Generic evidence that a CSP exists provides motivation to search for a combinatorially meaningful cyclic action on the set X.

The main result in Section 2 is the following: Theorem 2.7 provides the necessary and sufficient conditions for $(X, C_n, f(q))$ to exhibit CSP. The natural (necessary) condition is that $f(q) \in \mathbb{Z}[q]$ take non-negative integer values at all n^{th} roots of unity, which is evident from the definition of a cyclic sieving phenomena.

We prove the following: Define

$$S_k \coloneqq \sum_{j|k} \mu(k/j) f(\omega_n^j), \qquad \text{where } k|n.$$

Then $(X, C_n, f(q))$ exhibits CSP for some C_n acting on X if and only if $S_k \geq 0$ for all $k|n$. We warn that merely having a polynomial $f(q) \in \mathbb{N}[q]$ that takes non-negative integer values at all n^{th} roots of unity is no guarantee for the existence of a cyclic action complementing $f(q)$ to a CSP. A polynomial demonstrating this is given in Example 2.9.

In Section 3, we conjecture a new cyclic sieving phenomena involving stretched Schur polynomials. In a special case, we prove this conjecture by applying Theorem 2.7, see Theorem 3.7 below. That is, we prove existence of CSP without having to provide a natural cyclic group action.

Section 4 and onwards treat the cyclic sieving phenomenon from a more geometric perspective. We record the joint cyclic order and statistic distribution of the elements of X in a matrix and reformulate the CSP condition in terms of linear equations in the matrix entries. The set of matrices that satisfy these linear equations we call *CSP matrices* and we prove via Theorem 7.1 that they form a convex rational polyhedral cone whose integer lattice points correspond to realizable instances of CSP. Inspired by [AS17], we further proceed

to identify a certain subcone which we call the *universal CSP cone* containing all matrices corresponding to realizable instances of CSP with evenly distributed statistic on all its orbits. We prove that all integer CSP matrices can be obtained from a universal CSP matrix through a sequence of *swaps* without going outside of the CSP cone (Proposition 6.4). The swaps can be interpreted as a sequence of statistic interchanges between pairs of elements in the corresponding CSP-instance.

Finally we explicitly determine all extreme rays of the universal CSP cone (Corollary 7.4) and in Section 5 we prove some general properties for all CSP cones.

1.3. **Notation.** The following notation will be used throughout the paper.

- $[n] \coloneqq \{1, \dots, n\}$.
- $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers.
- $K^{n\times n}$ denotes the set of $n \times n$ matrices over the set K.
- $\mu(n) \coloneqq \begin{cases} 0, & \text{if } n \text{ is not square-free,} \\ (-1)^r, & \text{if } n \text{ is a product of } r \text{ distinct primes,} \end{cases}$
  denotes the classical Möbius function.
- ω_n denotes a primitive n^{th} root of unity.
- $\Phi_n(q) \coloneqq \prod_{\substack{1\leq k\leq n \\ \gcd(n,k)=1}} (q - \omega_n^k)$ denotes the nth cyclotomic polynomial.
- $[n]_q \coloneqq \dfrac{q^n - 1}{q - 1}$, $[n]_q! \coloneqq [n]_q[n-1]_q \cdots [1]_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q \coloneqq \dfrac{[n]_q!}{[k]_q![n-k]_q!}$,
  denotes the q-integer, q-factorial and q-binomial coefficients respectively.

2. Integer-valued polynomials at roots of unity

In the context of discovering cyclic sieving phenomena, one may sometimes have a candidate polynomial (*e.g.* a natural q-analogue of the enumeration formula for the underlying set) that takes integer values at all roots of unity, but the cyclic action complementing it to a CSP is unknown. In such situations one may like to know if a CSP could exist even in principle. In this section we characterize the set polynomials $f(q) \in \mathbb{Z}[x]$ of degree less than n such that $f(\omega_n^j) \in \mathbb{Z}$ for all $j = 1, \dots, n$ and show that they are indeed $\mathbb{Z}$-linear combinations of polynomials of the form

$$\frac{q^n - 1}{q^{n/d} - 1} = \sum_{i=0}^{n/d-1} q^{di} \qquad \text{for } d|n.$$

Using the characterization one can quickly determine if a CSP is present and get the count of the number of elements of each order in terms of evaluations of the polynomial at roots of unity. Often it is much simpler to determine the evaluations at roots of unity than it is to write the polynomial in terms of the above basis.

Finally note that not all polynomials $f(q) \in \mathbb{N}[q]$ such that $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \dots, n$ may necessarily be paired with a cyclic action to produce a CSP, see Example 2.9.

The set

$$M(n) \coloneqq \{f(q) \in \mathbb{Z}[q] : \deg(f) < n,\ f(\omega_n^j) \in \mathbb{Z} \text{ for } j = 1, \dots, n\}$$

forms a $\mathbb{Z}$-module. First we identify two useful bases for $M(n)$ using the following proposition and Lemma 2.2.

Proposition 2.1 (Désarménien [Dé89])**.** *Let $f(q) \in \mathbb{Z}[q]$ be a polynomial of degree less than n. Then the following two properties are equivalent:*

(i) For every $d|n$,

$$f(q) \equiv r_d \pmod{\Phi_d(q)} \text{ for some } r_d \in \mathbb{Z},$$

where $\Phi_d(q)$ denotes the dth cyclotomic polynomial.

(ii) The polynomial $f(q)$ has the form

$$f(q) = \sum_{j=0}^{n-1} a_j q^j, \quad \text{where } a_j = a_{gcd(n,j)}. \tag{2.1}$$

Lemma 2.2. *For each $n \in \mathbb{N}$, the following sets form $\mathbb{Z}$-bases for $M(n)$:*

(i) $\mathcal{B}_1(n) = \{g_d(q) : d|n\}$ where

$$g_d(q) = \sum_{\substack{0 \le j < n \\ \gcd(j,n)=d}} q^j,$$

(ii) $\mathcal{B}_2(n) = \{h_d(q) : d|n\}$ where

$$h_d(q) = \sum_{j=0}^{n/d-1} q^{dj}.$$

Proof. Let $f(q) \in M(n)$ and suppose $d|n$. Then $\omega_n^{n/d}$ is a dth root of unity. Note that $f(q) - f(\omega_n^{n/d})$ vanishes at $q = \omega_n^{n/d}$ so it is divisible by the minimal polynomial of $\omega_n^{n/d}$ over $\mathbb{Z}$, that is, $\Phi_d(q)$. Hence $f(q) \equiv r_d \pmod{\Phi_d(q)}$ where $r_d = f(\omega_n^{n/d}) \in \mathbb{Z}$. By Proposition 2.1 it follows that $f(q)$ has the form (2.1). Such polynomials are clearly spanned by $\mathcal{B}_1(n)$.

Now, the elements in $\mathcal{B}_2(n)$ are linearly independent, since the lowest-degree terms of $h_d(q) - 1$ are all different. By inclusion-exclusion we see that for each $d|n$,

$$g_d(q) = \sum_{d|r} \mu(r/d) h_r(q)$$

and hence $\mathcal{B}_1(n)$ and $\mathcal{B}_2(n)$ both form bases of $M(n)$. □

We may in fact extend the characterization in Lemma 2.2 to multivariate polynomials $f \in \mathbb{Z}[q_1, \ldots, q_m]$ of degree less than n_i in variable q_i for $i = 1, \ldots, m$ taking integer values at all points $(\omega_{n_1}^{j_1}, \ldots, \omega_{n_m}^{j_m}) \in \mathbb{C}^m$ for $j_i = 1, \ldots, n_i$, $i = 1, \ldots, m$.

Theorem 2.3. *Let $M(n_1, \ldots, n_m) = \{f \in \mathbb{Z}[q_1, \ldots, q_m] : deg_i f < n_i,\ f(\omega_{n_1}^{j_1}, \ldots, \omega_{n_m}^{j_m}) \in \mathbb{Z}$ for $j_i = 1, \ldots, n_i,\ i = 1, \ldots, m\}$ where $n_1, \ldots, n_m \in \mathbb{N}$ and $deg_i f$ denotes the degree of x_i in f. Then the following sets form $\mathbb{Z}$-bases for $M(n_1, \ldots, n_m)$:*

(i) $\mathcal{B}_1(n_1, \ldots, n_m) = \left\{\prod_{i=1}^m g_{d_i}^{(i)}(q_i) : d_i|n_i,\ i = 1, \ldots, m\right\}$ where

$$g_{d_i}^{(i)}(q_i) = \sum_{\substack{0 \le j < n_i \\ gcd(j,n_i)=d_i}} q_i^j,$$

(ii) $\mathcal{B}_2(n_1,\dots,n_m) = \left\{\prod_{i=1}^m h_{d_i}^{(i)}(q_i) : d_i|n_i,\ i=1,\dots,m\right\}$ *where*

$$h_{d_i}^{(i)}(q_i) = \sum_{j=0}^{n_i/d_i-1} q_i^{d_i j}.$$

Proof. We prove that $\mathcal{B}_1(n_1,\dots,n_m)$ is a $\mathbb{Z}$-basis of $M(n_1,\dots,n_m)$ by induction on m. The proof for $\mathcal{B}_2$ is similar and therefore omitted. The base case $m=1$ follows from Lemma 2.2. Let $f \in M(n_1,\dots,n_{m+1})$. Write

$$f = f_{n_{m+1}-1}(q_1,\dots,q_m)q_{m+1}^{n_{m+1}-1} + \cdots + f_1(q_1,\dots,q_m)q_{m+1} + f_0(q_1,\dots,q_m),$$

where $f_0, f_1, \dots, f_{n_{m+1}-1} \in \mathbb{Z}[q_1,\dots,q_m]$ with $f_k(\omega_{n_1}^{j_1},\dots,\omega_{n_m}^{j_m}) \in \mathbb{Z}$ for all $k=0,\dots,n_{m+1}-1$, $j_i = 1,\dots,n_i$ and $i=1,\dots,m$. The univariate polynomials

$$F_{\omega_{n_1}^{j_1},\dots,\omega_{n_m}^{j_m}}(q_{m+1}) = f(\omega_{n_1}^{j_1},\dots,\omega_{n_m}^{j_m},q_{m+1}) \in \mathbb{Z}[q_{m+1}],$$

take integer values at $q_{m+1} = \omega_{n_{m+1}}^j$ for all $j=1,\dots,n_{m+1}$. By Proposition 2.1 we therefore have that

$$f_k(\omega_{n_1}^{j_1},\dots,\omega_{n_m}^{j_m}) = f_{\gcd(n_{m+1},k)}(\omega_{n_1}^{j_1},\dots,\omega_{n_m}^{j_m}),$$

for all $(\omega_{n_1}^{j_1},\dots,\omega_{n_m}^{j_m}) \in \mathbb{C}^m$. Since the $\prod_{i=1}^m n_i$ points $(\omega_{n_1}^{j_1},\dots,\omega_{n_m}^{j_m}) \in \mathbb{C}^m$ lie in general position the polynomials must coincide on all points in $\mathbb{C}^m$. Hence

$$f_k(q_1,\dots,q_m) = f_{\gcd(n_{m+1},k)}(q_1,\dots,q_m)$$

for all $k=0,\dots,n_{m+1}-1$. It follows that f is uniquely spanned by $\mathcal{B}_1(n_{m+1})$ over $\mathbb{Z}[q_1,\dots,q_m]$. By induction $f_k(q_1,\dots,q_m)$ is uniquely spanned by $\mathcal{B}_1(n_1,\dots,n_m)$ over $\mathbb{Z}$ for all $k=0,\dots,n_{m+1}-1$. Hence f is uniquely spanned by $\mathcal{B}_1(n_1,\dots,n_{m+1})$ over $\mathbb{Z}$ completing the induction. □

Lemma 2.4. *Let $f(q) \in \mathbb{Z}[q]$ such that $f(\omega_n^j) \in \mathbb{Z}$ for all $j=1,\dots,n$. Then for each $m,p,e \in \mathbb{N}$ where p is prime we have*

$$f(\omega_n^{mp^e}) \equiv f(\omega_n^{mp^{e-1}}) \pmod{p^e}.$$

In particular if $p \nmid n$, then $f(\omega_n^{mp^{e-1}}) = f(\omega_n^{mp^e})$.

Proof. Since we are only concerned with evaluations of $f(q)$ at n^{th} roots of unity, we may assume $f(q) \in M(n)$. Furthermore by Lemma 2.2 and linearity it suffices to show the statement for the basis elements $\mathcal{B}_2$ of $M(n)$. For each $d|n$ and $k \in \mathbb{Z}$ we have

$$h_d(\omega_n^k) = \sum_{j=0}^{n/d-1} (\omega_{n/d}^k)^j = \begin{cases} n/d, & \text{if } k \equiv 0 \pmod{n/d}, \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose $k = mp^e$ for some $m,p,e \in \mathbb{N}$ with p prime, and consider the different cases: Suppose first $mp^{e-1} \equiv 0 \pmod{n/d}$. This implies that $mp^e \equiv 0 \pmod{n/d}$, so $h_d(\omega_n^{mp^e}) = n/d = h_d(\omega_n^{mp^{e-1}})$. Secondly, suppose $mp^{e-1} \not\equiv 0 \pmod{n/d}$. If $mp^e \not\equiv 0 \pmod{n/d}$, then $h_d(\omega_n^{mp^e}) = 0 = h_d(\omega_n^{mp^{e-1}})$. On the other hand if $mp^e \equiv 0 \pmod{n/d}$, then $n/d = p^f a$ for some $f \geq e$ and $a \in \mathbb{N}$. Therefore $h_d(\omega_n^{mp^e}) - h_d(\omega_n^{mp^{e-1}}) = p^f a - 0 \equiv 0 \pmod{p^e}$. Hence the lemma follows. □

Lemma 2.5. *Let $f(q) \in \mathbb{Z}[q]$ such that $f(\omega_n^j) \in \mathbb{Z}$ for all $j = 1, \ldots, n$. Then for each $k = 1, \ldots, n$ we have that*

$$\sum_{j|k} \mu(k/j) f(\omega_n^j) \equiv 0 \pmod{k}.$$

Moreover if $k \nmid n$, then $\sum_{j|k} \mu(k/j) f(\omega_n^j) = 0$.

Proof. Let $1 \leq k \leq n$ and write $k = mp^e$ where $p, m \in \mathbb{N}$, p prime and $p \nmid m$. By Lemma 2.4 we have

$$\begin{aligned}\sum_{j|k} \mu(k/j) f(\omega_n^j) &= \sum_{j|m} \mu(k/(jp^{e-1})) f(\omega_n^{jp^{e-1}}) + \sum_{j|m} \mu(k/(jp^e)) f(\omega_n^{jp^e}) \\ &\equiv \sum_{j|m} \mu(k/(jp^{e-1})) f(\omega_n^{jp^{e-1}}) + \sum_{j|m} \mu(k/(jp^e)) f(\omega_n^{jp^{e-1}}) \pmod{p^e} \\ &\equiv 0 \pmod{p^e}.\end{aligned}$$

If $k \nmid n$, then we may write $k = mp^e$ for some $m, p \in \mathbb{N}$ with p prime such that $p \nmid n$. Then by the second assertion in Lemma 2.4 the congruences above hold with equality and we are done. □

Construction 2.6. Let $X = \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \cdots \sqcup \mathcal{O}_m$ be a partition of a finite set X into m parts such that $|\mathcal{O}_i|$ divides n for $i = 1, \ldots, m$. Fix a total ordering on the elements of $\mathcal{O}_i$ for $i = 1, \ldots, m$. Let C_n act on X by permuting each element $x \in \mathcal{O}_i$ cyclically with respect to the total ordering on $\mathcal{O}_i$ for $i = 1, \ldots, m$.

This *ad-hoc* cyclic action in Construction 2.6 lacks combinatorial context and depends only on the choice of partition and total order.

Theorem 2.7. *Let $f(q) \in \mathbb{N}[q]$ and suppose $f(\omega_n^j) \in \mathbb{N}$ for each $j = 1, \ldots, n$. Let X be any set of size $f(1)$. Then there exists an action of C_n on X such that $(X, C_n, f(q))$ exhibits CSP if and only if for each $k|n$,*

$$\sum_{j|k} \mu(k/j) f(\omega_n^j) \geq 0. \tag{2.2}$$

Proof. The forward direction follows from [RSW04, Prop. 4.1]. Conversely if we put

$$S_k = \sum_{j|k} \mu(k/j) f(\omega_n^j) \tag{2.3}$$

for each $k = 1, \ldots, n$ and consider X of size $f(1)$, then by Möbius inversion

$$|X| = f(\omega_n^n) = \sum_{j|n} S_j.$$

Thus by hypothesis and Lemma 2.5, we may partition X into orbits, such that for each $k|n$, there are $\frac{1}{k} S_k$ orbits of size k. We then let C_n act on X as in Construction 2.6. The fixed points of X under $\sigma_n^k \in C_n$ are given by the elements of order dividing k. This gives (by Möbius inversion)

$$|X^{\sigma_n^k}| = \sum_{j|k} S_j = f(\omega_n^k).$$

Hence $(X, C_n, f(q))$ exhibits CSP. □

Remark 2.8. The sums S_k in (2.3) represent the number of elements with order k under the action of C_n.

Example 2.9. The following example demonstrates that even if $f(q) \in \mathbb{N}[q]$ satisfies $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \dots, n$, there might not be an associated cyclic action complementing $f(q)$ to a CSP.

Let $f(q) = q^5 + 3q^3 + q + 10$. Then $f(\omega_6^j)$ takes values $8, 12, 5, 12, 8, 15$ for $j = 1, \dots, 6$. On the other hand $S_k = \sum_{j|k} \mu(k/j) f(\omega_6^j)$ takes values $8, 4, -3, 0, 0, 6$ for $k = 1, \dots, 6$. Since we cannot have a negative number of elements of order 3, there is no action of C_6 on a set X of size $f(1) = 15$ such that $(X, C_6, f(q))$ is a CSP-triple.

Rao and Suk [RS17] generalized the notion of cyclic sieving to arbitrary groups with finitely generated representation ring, so called *G-sieving*. In particular, Berget, Eu and Reiner [BER11] considered the case where G is an Abelian group, whence $G \cong C_{n_1} \times \cdots \times C_{n_m}$, acting pointwise on a set $X_1 \times \cdots \times X_m$. Unfortunately G-sieving depends in general on the particular choices of representations ρ_i of G over $\mathbb{C}$ generating the representation ring. However, given the characterization in Theorem 2.3 it would be interesting to understand what conditions are necessary and sufficient for a polynomial $f \in M(n_1, \dots, n_m)$ to be complemented to a G-sieving phenomenon for an Abelian group $G \cong C_{n_1} \times \cdots \times C_{n_m}$ with respect to the canonical representations sending the generator σ_{n_i} of C_{n_i} to ω_{n_i}.

3. Applications

In this section we demonstrate how one can use Theorem 2.7 to find new cyclic sieving phenomena arising from natural polynomials.

By Theorem 2.7 any polynomial $f(q) \in \mathbb{N}[q]$ such that $f(\omega_n^j) \in \mathbb{N}$ for $j = 1, \dots, n$ satisfying the positivity condition (2.2), can be completed to a CSP with an ad-hoc cyclic action. Although this action lacks combinatorial context, it often helps to know that a CSP can exist even in principle, particularly if one is considering a combinatorial set where the cyclic action is not immediately apparent. The following example illustrates this point for the polynomial $C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ which is generated by statistics on multiple combinatorial (Catalan) objects, but where the naturalness of the action varies depending on the object under consideration.

Example 3.1. Stump [Stu09] showed that $C_n(q) = \sum_{\sigma \in \mathfrak{S}_n(231)} q^{\mathrm{maj}(\sigma) + \mathrm{maj}(\sigma^{-1})}$. There is no obvious natural cyclic action on $\mathfrak{S}_n(231)$ that is compatible with $C_n(q)$. However we can check the positivity condition (2.2) in Theorem 2.7 to reveal that a CSP is nevertheless present for $C_n(q)$ with an ad-hoc cyclic action on $\mathfrak{S}_n(231)$. Indeed rewriting $C_n(q) = \frac{1}{[2n+1]_q} \begin{bmatrix} 2n+1 \\ n+1 \end{bmatrix}_q$ and using [RSW04, Prop. 4.2 (iii)] we have for $j|n$,

$$C_n(\omega_n^j) = \begin{cases} \binom{2j}{j}, & \text{if } j < n, \\ \frac{1}{n+1}\binom{2n}{n}, & \text{if } j = n. \end{cases}$$

By Wallis formula, $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)=\frac{2}{\pi}$, the sequences

$$2n\left(\binom{2n}{n}\frac{1}{4^n}\right)^2=\frac{1}{2}\prod_{j=2}^{n}\left(1+\frac{1}{4j(j-1)}\right),$$
$$(2n+1)\left(\binom{2n}{n}\frac{1}{4^n}\right)^2=\prod_{j=1}^{n}\left(1-\frac{1}{4j^2}\right)$$

monotonically increase and decrease respectively towards $\frac{2}{\pi}$ as $n\to\infty$. Thus

$$\frac{4^n}{\sqrt{\pi(n+1/2)}}\leq\binom{2n}{n}\leq\frac{4^n}{\sqrt{\pi n}}.$$

A trivial bound for the number of divisors of n, excluding n, is given by $2\sqrt{n}-1$. Hence for each divisor $k<n$ we have

$$\begin{aligned}\sum_{j|k}\mu(k/j)C_n(\omega_n^j)&=\sum_{j|k}\mu(k/j)\binom{2j}{j}\\&\geq\binom{2k}{k}-\sum_{\substack{j|k\\j<k}}\binom{2j}{j}\\&\geq\frac{4^k}{\sqrt{\pi(k+1/2)}}-(2\sqrt{k}-1)\frac{4^{k/2}}{\sqrt{\pi(k/2)}}\geq 0.\end{aligned}$$

Moreover for $k=n$ we have by a similar calculation that

$$\sum_{j|n}\mu(n/j)C_n(\omega_n^j)\geq\frac{4^n}{(n+1)\sqrt{\pi(n+1/2)}}-(2\sqrt{n}-1)\frac{4^{n/2}}{\sqrt{\pi(n/2)}}\geq 0,$$

for $n\geq 5$. The required inequality can be verified explicitly by hand for $n<5$. Hence $C_n(q)$ exhibits CSP with an ad-hoc cyclic action on $\mathfrak{S}_n(231)$.

With this evidence one could now either proceed to search for a natural cyclic action on $\mathfrak{S}_n(231)$ matching the orbit structure of the ad-hoc cyclic action, or find a natural cyclic action on an object in bijection with $\mathfrak{S}_n(231)$. In this case there happens to exist known candidates *e.g.* the set of Dyck paths $\mathrm{Dyck}(n)$ of semi-length n where C_n acts by changing peaks to valleys (and vice versa) from left to right whenever possible, or the set of triangulation of a regular $(n+2)$-gon where C_{n+2} acts by rotating the triangulation. In the latter case we instead lack a simple natural statistic (as opposed to a natural action) on the set of triangulations that generates $C_n(q)$.

3.1. **A new CSP with stretched Schur polynomials.** In this section we conjecture a new cyclic sieving phenomenon involving stretched Schur polynomials. We prove our conjecture in the case of certain rectangular shapes for which it is straightforward to explicitly compute the data needed to verify the positivity condition (2.2) in Theorem 2.7. We begin by recalling the basic definitions required to state the conjecture.

A *partition* $\lambda=(\lambda_1,\ldots,\lambda_r)$ is a finite weakly decreasing sequence of non-negative integers $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_r\geq 0$. The *parts* of λ are the positive entries and the number of positive

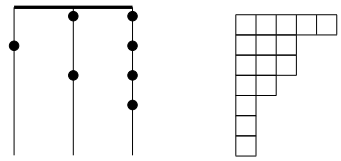

FIGURE 1. The abacus representation of $\lambda = (5, 3^2, 2, 1^3)$ with $m = 7$ beads and $d = 3$ runners, next to the Young diagram representation of λ.

parts is the *length* of λ, denoted $l(\lambda)$. The quantity $|\lambda| := \lambda_1 + \cdots + \lambda_r$ is called the *size* of λ. The *empty partition* $\emptyset$ is the partition with no parts. We use exponents to denote multiplicities *e.g.* $\lambda = (5, 3, 3, 2, 1, 1, 1) = (5, 3^2, 2, 1^3)$. Scalar multiplication on partitions is performed elementwise *e.g.* with $n \in \mathbb{N}$ and λ as above we have $n\lambda = (5n, (3n)^2, 2n, n^3)$. If $\mu = (\mu_1, \ldots, \mu_r)$ is a partition such that $\lambda_i \geq \mu_i$ for all $i = 1, \ldots, r$ then we say that $\mu \subseteq \lambda$. This is called the *inclusion order* on partitions.

Partitions are commonly visualized in at least two different ways. The first and most common way to represent a partition is via its Young diagram. A *skew Young diagram* of *shape* λ/μ is an arrangement of boxes in the plane with coordinates given by $\{(i, j) \in \mathbb{Z}^2 : \mu_i \leq j \leq \lambda_i\}$. The first coordinate represents the row and the second coordinate the column. If $\mu = \emptyset$, then we simply write λ instead of λ/μ and refer to the corresponding skew Young diagram as the *(regular) Young diagram* of λ. A *border strip* (or *rim hook*) of size d is a connected skew Young diagram consisting of d boxes and containing no 2×2 square. The *height* of a border strip is one less than its number of rows. A *border strip tableau* of shape λ/μ and type $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a sequence $\mu = \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^r = \lambda$ such that λ^i/λ^{i-1} is a border strip of size α_i.

A second way to visually represent a partition λ is via an *abacus* with $m \geq r$ beads: Let $d \in \mathbb{N}$. For $i = 1, \ldots, m$, write $\lambda_i + m - i = s + dt$, with $0 \leq s \leq d - 1$, and place a bead on the sth runner in the tth row. The operation of sliding a bead one row upwards on its runner into a vacant position corresponds to removing a border strip of size d from λ. Sliding all beads up as far as possible produces an abacus representation of the *d-core* partition of λ, a partition from which no further border strip tableaux of size d can be removed. It is worth mentioning that the d-core of λ is independent of the way in which border strip tableaux are removed. For $i = 0, 1, \ldots, d - 1$, let $\lambda_j^{(i)}$ be the number of unoccupied positions on the ith runner above the jth bead from the bottom. Then $\lambda^i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_d^{(i)})$ is a partition and the d-tuple $[\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}]$ is called the *d-quotient* of λ.

A *semi-standard Young tableau (SSYT)* is a Young diagram whose boxes are filled with non-negative integers, such that each row is weakly increasing and each column is strictly increasing. Denote the set of SSYT of shape λ with entries in $\{0, \ldots, m - 1\}$ by $\mathrm{SSYT}(\lambda, m)$. Given $T \in \mathrm{SSYT}(\lambda, m)$, the *type* of T is the vector $\alpha(T) = (\alpha_0(T), \alpha_1(T), \ldots, \alpha_{m-1}(T))$ where $\alpha_k(T)$ counts the number of boxes of T containing the number k.

The *Schur polynomial* is defined as

$$s_\lambda(x_0, \ldots, x_{m-1}) = \sum_{T \in \mathrm{SSYT}(\lambda, m)} x_0^{\alpha_0(T)} x_1^{\alpha_1(T)} \cdots x_{m-1}^{\alpha_{m-1}(T)}.$$

The polynomial $s_\lambda(x_0,\dots,x_{m-1})$ is symmetric and has several alternative definitions, see [Sta99]. The *principal specialization* of $s_\lambda(x_0,\dots,x_{m-1})$ is given by

$$s_\lambda(1,q,q^2,\dots,q^{m-1}) = \sum_{T\in \mathrm{SSYT}(\lambda,m)} q^{|T|},$$

where $|T|$ denotes the sum of all entries in T. The following explicit formula is referred to as the *q-hook-content formula* and is due to Stanley (see [Sta99, Thm 7.21.2]),

$$s_\lambda(1,q,q^2,\dots,q^{m-1}) = q^{b(\lambda)} \prod_{(i,j)\in\lambda} \frac{[m+c_{i,j}]_q}{[h_{i,j}]_q}, \tag{3.1}$$

where $b(\lambda)=\sum_{i=1}^r (i-1)\lambda_i$, $c_{i,j}=j-i$ (the *content*) and $h_{i,j}$ is defined as the number of boxes in λ to the right of (i,j) in row i plus the number of boxes below (i,j) in column j plus 1 (the *hook length*). In particular

$$|\mathrm{SSYT}(\lambda,m)| = s_\lambda(1^m) = \prod_{(i,j)\in\lambda} \frac{m+c_{i,j}}{h_{i,j}}. \tag{3.2}$$

If G is a group and V a (finite-dimensional) vector space over $\mathbb{C}$, then a *representation* of G is a group homomorphism $\rho: G\to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the group of invertible linear transformations of V. A representation $\rho: G\to\mathrm{GL}(V)$ is *irreducible* if it has no proper subrepresentation $\rho|_W: G\to \mathrm{GL}(W)$, $0<W<V$ closed under the action of $\{\rho(g): g\in G\}$. The *character* of G on V is a function $\chi: G\to\mathbb{C}$ defined by $\chi(g)=\mathrm{tr}(\rho(g))$. Note that characters are invariant under conjugation by G. A character χ is said to be *irreducible* if the underlying representation is irreducible. If $G=\mathfrak{S}_m$, then the irreducible characters χ^λ of $\mathfrak{S}_m$ are indexed by partitions λ of weight m and may be computed combinatorially (on each conjugacy class of type α in $\mathfrak{S}_m$) using the *Murnaghan–Nakayama rule* [Sta99, Thm 7.17.3]

$$\chi^\lambda_\alpha = \sum_{T\in\mathrm{BST}(\lambda,\alpha)} (-1)^{\mathrm{ht}(T)}, \tag{3.3}$$

where the sum runs over all border strip tableaux $\mathrm{BST}(\lambda,\alpha)$ of shape λ and type α and $\mathrm{ht}(T)$ is the sum of all heights of the border strips in T. In particular this implies χ^λ takes integer values.

The following theorem provides an expression for the root of unity evaluation of the principal specialization $s_\lambda(1,q,\dots,q^{m-1})$.

Theorem 3.2 (Reiner–Stanton–White [RSW04])**.** *Let $d|m$ and ω_d be a primitive d^{th} root of unity. Then $s_\lambda(1,\omega_d,\dots,\omega_d^{m-1})$ is zero unless the d-core of λ is empty, in which case*

$$\mathrm{s}_\lambda(1,\omega_d,\omega_d^2,\dots,\omega_d^{m-1}) = sgn(\chi^\lambda_{d^{|\lambda|/d}}) \prod_{i=0}^{d-1} \mathrm{s}_{\lambda^{(i)}}(1^{m/d}),$$

where χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_{|\lambda|}$ indexed by λ.

Lemma 3.3. *Suppose ω_d is a primitive d^{th} root of unity with $d|m,n$, then*

$$\mathrm{s}_{n\lambda}(1,\omega_d,\omega_d^2,\dots,\omega_d^{m-1}) = \prod_{i=0}^{d-1} \mathrm{s}_{(n\lambda)^{(i)}}(1^{m/d}) \in \mathbb{N}.$$

Proof. By Theorem 3.2 we only need to verify that $\chi^{n\lambda}_{d^{n|\lambda|/d}} \geq 0$. A result by White [Whi83, Cor. 10] (see also [Pak00, Thm. 3.3]), implies that the Murnaghan–Nakayama rule (3.3) is cancellation-free in this instance. Furthermore, it is clear that there is a border-strip tableau of shape $n\lambda$ with border-strips of size d with positive sign. For example, take all strips to be horizontal — this is possible since $d|n$. □

We are now ready to state our conjecture.

Conjecture 3.4. *Let $n, m \in \mathbb{N}$ and let λ be a partition. Then the triple*

$$(\mathrm{SSYT}(n\lambda, m), C_n, \mathrm{s}_{n\lambda}(1, q, q^2, \ldots, q^{m-1}))$$

exhibits a CSP for C_n acting on $\mathrm{SSYT}(n\lambda, m)$ in some fashion.

We believe that a natural action is realized by some type of promotion on semi-standard Young tableaux similar to [Rho10]. In the case $\lambda = (1)$ we have

$$\mathrm{s}_{n\lambda}(1, q, q^2, \ldots, q^m) = \begin{bmatrix} n+m-1 \\ n \end{bmatrix}_q$$

and this polynomial exhibits a cyclic sieving phenomenon under C_n, see [RSW04].

We have verified Conjecture 3.4 using Theorem 2.7 for all partitions λ such that $|\lambda| \leq 6$, all $m \leq 6$ and all $n \leq 12$.

Below we prove the conjecture for certain rectangular shapes λ.

Lemma 3.5. *The n-quotient of the rectangular shape $(na)^{nb+r}$ with $0 \leq r < n$ is given by*

$$[\underbrace{a^b, a^b, \ldots, a^b}_{n-r \text{ times}}, \underbrace{a^{b+1}, a^{b+1}, \ldots, a^{b+1}}_{r \text{ times}}].$$

Proof. The abacus representation of $\lambda = (na)^{nb+r}$ with $m = nb + r$ beads and $d = n$ runners is given via

$$na + (nb + r) - i = s + nt,$$

for $i = 1, \ldots, nb + r$ where $0 \leq s \leq n - 1$, see Figure 2. Thus we see that each of the n runners have no bead in the first a rows. Since all parts of λ are the same, we also note that the $nb + r$ beads are distributed evenly from right to left on the n runners with no vacant positions in between the beads on each runner. Thus there are b beads on the first $n - r$ runners and $b + 1$ beads on the last r runners. Moreover each bead have exactly a vacant positions above it on its runner, so the n-quotient is given as in the lemma. □

Lemma 3.6. *We have*

$$\mathrm{s}_{(a^b)}(1^m) = \prod_{j=0}^{a-1} \binom{m+j}{b} \binom{b+j}{b}^{-1}$$

Proof. By the hook-content formula (3.2) we have

$$s_{(a^b)}(1^m) = \prod_{(i,j)\in(a^b)} \frac{m+j-i}{(a-j)+(b-i)+1},$$

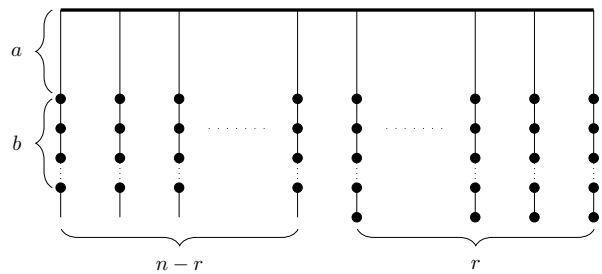

FIGURE 2. The abacus representation of $\lambda = (na)^{nb+r}$ with $m = nb + r$ beads and $d = n$ runners.

which after rearrangement equals

$$\prod_{j=0}^{a-1}\prod_{i=0}^{b-1}\frac{m+j-i}{b+j-i} = \prod_{j=0}^{a-1}\frac{(m+j)!}{(m-b+j)!}\frac{j!}{(b+j)!} = \prod_{j=0}^{a-1}\binom{m+j}{b}\binom{b+j}{b}^{-1}.$$

□

Theorem 3.7. *Let $n, m, a, b \in \mathbb{N}$ with $b < m$ and $n|b, m$. If $\lambda = (a^b)$, then the triple*

$$(\mathrm{SSYT}(n\lambda, m), C_n, s_{n\lambda}(1, q, q^2, \ldots, q^{m-1}))$$

exhibits a CSP for some ad-hoc action of C_n on $\mathrm{SSYT}(\lambda, m)$.

Proof. By Lemma 3.3 it follows that $\mathrm{s}_{n\lambda}(1, \omega_n^j, \omega_n^{2j}, \ldots, \omega_n^{(m-1)j}) \in \mathbb{N}$ for all $j = 1, \ldots, n$. By Theorem 2.7 it therefore remains to show that for all $k|n$,

$$\sum_{j|k}\mu(k/j)\mathrm{s}_{n\lambda}(1, \omega_n^j, \omega_n^{2j}, \ldots, \omega_n^{(m-1)j}) \geq 0. \tag{3.4}$$

Note that ω_n^j is a $(n/j)^{th}$ root of unity. By Lemma 3.3 and Lemma 3.5 the left hand side of (3.4) rewrites as

$$\sum_{j|k}\mu(k/j)\prod_{i=0}^{n/j-1}\underbrace{\mathrm{s}_{(ja)^{bj/n}}(1^{mj/n})}_{\text{independent of } i} = \sum_{j|k}\mu(k/j)\left(\mathrm{s}_{(ja)^{bj/n}}(1^{mj/n})\right)^{n/j}. \tag{3.5}$$

Using Lemma 3.6, this equals

$$\sum_{j|k}\mu(k/j)\left(\prod_{i=0}^{ja-1}\binom{mj/n+i}{bj/n}\binom{bj/n+i}{bj/n}^{-1}\right)^{n/j}, \tag{3.6}$$

which is greater or equal to

$$\left(\prod_{i=0}^{ka-1}\binom{mk/n+i}{bk/n}\binom{bk/n+i}{bk/n}^{-1}\right)^{\frac{n}{k}} - \sum_{\substack{j|k\\ j<k}}\left(\prod_{i=0}^{ja-1}\binom{mj/n+i}{bj/n}\binom{bj/n+i}{bj/n}^{-1}\right)^{\frac{n}{j}}. \tag{3.7}$$

By Lemma 8.4 and the fact that the number of divisors of k, excluding k, is bounded above by $2\sqrt{k}-1$ we get that (3.7) is greater than or equal to

$$\left(\prod_{i=k'a}^{ka-1}\frac{\binom{mk/n+i}{bk/n}^{n/k}}{\binom{bk/n+i}{bk/n}^{n/k}}-(2\sqrt{k}-1)\right)\left(\prod_{i=0}^{k'a-1}\frac{\binom{mk/n+i}{bk/n}^{n/k}}{\binom{bk/n+i}{bk/n}^{n/k}}-\prod_{i=0}^{k'a-1}\frac{\binom{mk'/n+i}{bk'/n}^{n/k'}}{\binom{bk'/n+i}{bk'/n}^{n/k'}}\right), \tag{3.8}$$

where $k'=\lfloor k/2\rfloor$. The remaining steps needed are given in the appendix Section 8, where it is shown that the left factor in (3.8) is non-negative by Lemma 8.6 and the right factor is non-negative by Lemma 8.4 for all $k|n$. This concludes the proof of the theorem. □

4. The CSP cone

In the following sections we offer a geometric perspective on the cyclic sieving phenomenon by associating a polyhedral cone that captures joint information about the cyclic action and statistics on the object X. The cone has the property that all cyclic sieving phenomena with a polynomial generated by a choice of statistic (modulo n) on the set X corresponds to a lattice point in the cone.

As presented in the introduction, the polynomial $f(q)$ is often given by some natural statistic $\tau : X \to \mathbb{N}$ on X. Define

$$f_\tau(q) := \sum_{x\in X} q^{\tau(x)}.$$

Moreover for each $n \in \mathbb{N}$, define $\tau_n : X \to \mathbb{Z}_n$ by

$$\tau_n(x) := \tau(x) \pmod{n}.$$

More than understanding the individual components of the CSP triple $(X, C_n, f_\tau(q))$, one is also interested in the behaviour and distribution of the statistic τ with respect to the cyclic action. Given an action of C_n on X and a statistic $\tau : X \to \mathbb{N}$, we can associate a $n \times n$ matrix $A_{(X,C_n,\tau)} = (a_{ij})$ which keeps track of the coefficients of the generating function

$$\sum_{x\in X} q^{\tau_n(x)}t^{o(x)} := \sum_{i=0}^{n-1}\sum_{j=1}^{n} a_{ij}q^i t^j,$$

where $o(x) := \min\{j \in [n] : \sigma_n^j \cdot x = x\}$ denotes the order of $x \in X$ under C_n. We remark that the rows of $A_{(X,C_n,\tau)}$ are indexed from 0 to $n-1$.

We can now restate CSP as follows:

Proposition 4.1. *Suppose X is a finite set on which C_n acts and let $\tau : X \to \mathbb{N}$ be a statistic. Then the triple $(X, C_n, f_\tau(q))$ exhibits CSP if and only if $A_{(X,C_n,\tau)} = (a_{ij})$ satisfies the condition that for each $1 \le k \le n$,*

$$\sum_{\substack{0\le i<n\\ 1\le j\le n}} a_{ij}\omega_n^{ki} = \sum_{0\le i<n}\sum_{j|k} a_{ij}. \tag{4.1}$$

where ω_n is a primitive nth root of unity.

Proof. For each $1 \leq k \leq n$ we have that

$$\begin{aligned} X^{\sigma_n^k} &= \bigcup_{i=0}^{n-1} \{x \in X : \tau_n(x) = i,\ \sigma_n^k \cdot x = x\} \\ &= \bigcup_{i=0}^{n-1} \bigcup_{j|k} \{x \in X : \tau_n(x) = i,\ o(x) = j\}. \end{aligned}$$

Hence $(X, C_n, f_\tau(q))$ exhibits CSP if and only if for each $1 \leq k \leq n$,

$$\sum_{\substack{0 \leq i < n \\ 1 \leq j \leq n}} a_{ij} \omega_n^{ki} = f_{\tau_n}(\omega_n^k) = |X^{\sigma_n^k}| = \sum_{0 \leq i < n} \sum_{j|k} a_{ij}. \tag{4.2}$$

□

This motivates the following definition.

Definition 4.2. A $n \times n$-matrix $A = (a_{ij}) \in \mathbb{R}_{\geq 0}^{n \times n}$ is called a *CSP-matrix* if it fulfills the conditions in Equation (4.1). Let $\mathrm{CSP}(n)$ denote the set of all $n \times n$ CSP-matrices and $\mathrm{CSP}_{\mathbb{Z}}(n) := \mathrm{CSP}(n) \cap \mathbb{Z}^{n \times n}$ the set of integer CSP-matrices.

Example 4.3. Consider all binary words of length 6, with group action being shift by 1 and τ being the the major index statistic. Then

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 11 \\ 0 & 0 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 1 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 0 & 2 & 0 & 0 & 7 \end{pmatrix}$$

is the corresponding CSP matrix. The entry in the upper left hand corner correspond to the two binary words 000000 and 111111. These have major index 0 and are fixed under a single shift. The words corresponding to the second column are 010101 and 101010. These have major index $6 \equiv 0 \pmod 6$ and $9 \equiv 3 \pmod 6$ respectively and are fixed under two consecutive shifts etc.

By linearity of the CSP-condition (4.1), it follows that for all $A, B \in \mathrm{CSP}(n)$ we have $sA + tB \in \mathrm{CSP}(n)$ for any $s, t \geq 0$. Hence $\mathrm{CSP}(n)$ forms a real convex cone. In fact by Theorem 7.1 in Section 7 we have the following corollary.

Corollary 4.4. *The set* $\mathrm{CSP}(n)$ *forms a real convex rational polyhedral cone.*

5. General properties of the CSP cone

Since $\mathrm{CSP}(n)$ is a rational cone by Corollary 4.4, its extreme rays are spanned by integer matrices. Every element in $\mathrm{CSP}(n)$ is therefore a conic combination of elements in $\mathrm{CSP}_{\mathbb{Z}}(n)$. In particular, properties of $\mathrm{CSP}_{\mathbb{Z}}(n)$ closed under conic combinations can be lifted to $\mathrm{CSP}(n)$.

A priori an integer lattice point $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$ need not be realizable by a cyclic sieving phenomenon with CSP-matrix A. However thanks to Lemma 5.1 we shall see that this property does indeed hold.

Lemma 5.1. *Let $A = (a_{ij}) \in \mathrm{CSP}_{\mathbb{Z}}(n)$. Then there exists a CSP-triple (X, C_n, τ) with $A_{(X,C_n,\tau)} = A$.*

Proof. According to (4.1), the polynomial $f(q) = \sum_{i=0}^{n-1} r_i q^i$ where $r_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, \dots, n-1$ defines a polynomial such that $f(\omega_n^k) = \sum_{j|k} S_j \in \mathbb{N}$ for $k = 1, \dots, n$, where $S_j = \sum_{i=0}^{n-1} a_{ij}$ for $j = 1, \dots, n$. By Möbius inversion as in Theorem 2.7 we have $S_k = \sum_{j|k} \mu(k/j) f(\omega_n^j)$. Hence by Lemma 2.5, $k|S_k$. Therefore a CSP-instance having CSP-matrix A can be realized through any triple (X, C_n, τ) with C_n acting in an ad-hoc manner on a set X with $\sum_{i,j} a_{ij}$ elements divided into S_k/k orbits of size k for each $k|n$ where $\tau : X \to \mathbb{N}$ is any statistic distributed according to A. □

Let $\{E_{ij} : 0 \le i < n, 1 \le j \le n\}$ denote the standard basis of $\mathbb{R}^{n\times n}$.

Definition 5.2. Call a matrix $\delta_a(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n\times n}$ a *swap* if

$$\delta_a(\mathbf{u}, \mathbf{v}) := a(E_{u_1u_2} + E_{v_1v_2} - E_{v_1u_2} - E_{u_1v_2}),$$

where $a \in \mathbb{R}$.

Lemma 5.3. *Let $A \in \mathrm{CSP}(n)$ and suppose $\delta_a(\mathbf{u}, \mathbf{v}) + A \in \mathbb{R}^{n\times n}_{\ge 0}$. Then $\delta_a(\mathbf{u}, \mathbf{v}) + A \in \mathrm{CSP}(n)$.*

Proof. Since adding $\delta_a(\mathbf{u}, \mathbf{v})$ does not alter column nor row-sums we have that the CSP-condition (4.1) remains intact. Hence $\delta_a(\mathbf{u}, \mathbf{v}) + A \in \mathrm{CSP}(n)$. □

The next lemma follows by repeated applications of Lemma 5.3.

Lemma 5.4. *Let $A = (a_{ij}) \in \mathrm{CSP}(n)$. Suppose i and i' are two row indices such that $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{i'j}$. If A' is the matrix obtained from A by interchanging rows i and i', then $A' \in \mathrm{CSP}(n)$.*

Remark 5.5. The corresponding statement of Lemma 5.4 also holds for the column indices instead of row indices.

Proposition 5.6. *Let $n \in \mathbb{N}$ and suppose i and i' are row indices such that $\gcd(n, i) = \gcd(n, i')$. If $A \in \mathrm{CSP}(n)$, then $A' \in \mathrm{CSP}(n)$ where A' is obtained from A by interchanging rows i and i'.*

Proof. Let $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$. Then the polynomial $f(q) = \sum_{i=0}^{n-1} c_i q^i \in \mathbb{N}[q]$, where $c_i = \sum_{j=1}^{n} a_{ij}$, satisfies $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \dots, n$. By Lemma 2.2 it follows that $c_{i \pmod n} = c_{\gcd(n,i)}$ for all $i = 1, \dots, n$. Hence $A' \in \mathrm{CSP}_{\mathbb{Z}}(n)$ by Lemma 5.4. Moreover from above, row i and i' clearly have the same row sum in $sA + tB$ for any $A, B \in \mathrm{CSP}_{\mathbb{Z}}(n)$ and $s, t \ge 0$. Hence the property can be lifted to all matrices in $\mathrm{CSP}(n)$. □

6. The universal CSP cone

Let W_α be the set of words with content α, that is, α_i is the number of occurrences of the letter i in the words, and let n be the length of the words. Then C_n acts on such words by cyclic shift. In [AS17], the authors construct a statistic, $\mathrm{flex}(\cdot)$, which is equidistributed modulo n with major index on W_α. Furthermore, flex has the property that for every orbit

$\mathcal{O}$, the triple $(\mathcal{O}, C_n, \text{flex})$ exhibits the cyclic sieving phenomenon. They show that flex is universal in the following sense:

Definition 6.1. A cyclic sieving phenomena (X, C_n, τ) is called *universal* if $(\mathcal{O}, C_n, \tau)$ exhibits the cyclic sieving phenomenon for every orbit C_n-orbit $\mathcal{O}$ of X. This is shown in [AS17] to be equivalent with the property that for every C_n-orbit $\mathcal{O} \subseteq X$ with length k, the sets

$$\{\tau_n(x) : x \in \mathcal{O}\} \text{ and } \left\{0, \frac{n}{k}, \frac{2n}{k}, \ldots, \frac{(k-1)n}{k}\right\}$$

coincide. In other words, the statistic τ is "evenly distributed" on each C_n-orbit modulo n. We also refer to τ as being a universal statistic (with respect to X and C_n).

Clearly a universal statistic is uniquely determined modulo n by the orbit structure of X under C_n (up to a choice of total order on the orbits). We remark that most cyclic sieving phenomena in the literature are *not* universal. We shall see below how a non-universal statistic can be turned into a universal one without changing the generating polynomial.

Definition 6.2. A matrix $A = (a_{ij}) \in \mathrm{CSP}(n)$ is called *universal* if there are constants $K_1, \ldots, K_n \in \mathbb{R}_{\geq 0}$ such that

$$a_{ij} = \begin{cases} K_j, & \text{if } i \equiv 0 \pmod{\frac{n}{j}}, \\ 0, & \text{otherwise.} \end{cases}$$

for all $1 \leq i, j \leq n$. Let $\widetilde{\mathrm{CSP}}(n)$ denote the subset of all universal CSP-matrices. Moreover if $\mathbf{s} = (S_1, \ldots, S_n) \in \mathbb{N}^n$ is a sequence such that $j | S_j$ for $j = 1, \ldots, n$ and $S_j = 0$ for $j \nmid n$, then we let $U(\mathbf{s}) \in \widetilde{\mathrm{CSP}}(n)$ denote the unique universal CSP-matrix with column sums given by $S_1, \ldots, S_n$.

Remark 6.3. Note that $\widetilde{\mathrm{CSP}}(n)$ forms a subcone of $\mathrm{CSP}(n)$ and that the lattice points $\widetilde{\mathrm{CSP}}_{\mathbb{Z}}(n)$ are realized by universal cyclic sieving phenomena.

Every CSP-matrix can be linearly projected onto a universal CSP-matrix. Indeed the map

$$\begin{aligned} P : \mathrm{CSP}(n) &\to \widetilde{\mathrm{CSP}}(n) \\ a_{ij} &\mapsto \begin{cases} \frac{1}{j} \sum_{i=1}^{n} a_{ij}, & \text{if } i \equiv 0 \pmod{\frac{n}{j}}, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

is clearly linear in each entry with $P^2 = P$. By Proposition 5.1 the projection P restricts to a map $P : \mathrm{CSP}_{\mathbb{Z}}(n) \to \widetilde{\mathrm{CSP}}_{\mathbb{Z}}(n)$.

If $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$, then $\delta_1(\mathbf{u}, \mathbf{v}) + A$ corresponds to swapping statistic between two elements belonging to orbits of different size.

Next we show that every CSP matrix $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$ can be obtained from a universal CSP-matrix with the same column sums via a sequence of such swaps while keeping inside $\mathrm{CSP}_{\mathbb{Z}}(n)$. We prove this fact by showing a slightly more general result over the class of non-negative integer matrices with matching row and column sums.

Proposition 6.4. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be integer $n \times n$ matrices with non-negative entries having matching row and column sums i.e. $\sum_{i=1}^n a_{ij_0} = \sum_{i=1}^n b_{ij_0}$ and $\sum_{j=1}^n a_{i_0 j} = \sum_{j=1}^n b_{i_0 j}$ for $1 \leq i_0, j_0 \leq n$. Then there exists swaps $\delta_1(\mathbf{u}_r, \mathbf{v}_r)$ for $r = 1, \ldots, t$ such that*

$$A = B + \sum_{r=1}^{t} \delta_1(\mathbf{u}_r, \mathbf{v}_r). \tag{6.1}$$

Moreover the swaps $\delta_1(\mathbf{u}_r, \mathbf{v}_r)$ can be chosen such that $B + \sum_{r=1}^{t_0} \delta_1(\mathbf{u}_r, \mathbf{v}_r)$ has non-negative entries for all $1 \leq t_0 \leq t$.

Proof. Define $\Delta(A)$ to be the quantity

$$\Delta(A) := ||A - B||$$

where $||A|| = \sum_{i,j} |a_{ij}|$. We say that an entry a_{ij} is in *deficit* if $a_{ij} < b_{ij}$ and in *surplus* if $a_{ij} > b_{ij}$. We argue by induction on $\Delta(A)$. If $\Delta(A) = 0$, then clearly $A = B$ since A and B both have non-negative entries. Suppose $\Delta(A) > 0$. Then there exists indices i and j such that $a_{ij} - b_{ij} \neq 0$. If a_{ij} is in surplus, then there must exists some row index i' such that $a_{i'j}$ is in deficit, otherwise the sum of column j in A is strictly greater than sum of column j in B which leads to a contradiction. Therefore we may assume a_{ij} is in deficit. Since a_{ij} is in deficit there exists $j' \neq j$ such that $a_{ij'}$ is in surplus, otherwise the sum of row i in B is strictly greater than the sum of row i in A. Similarly, there exists a row index $i' \neq i$ such that $a_{i'j}$ is in surplus. It follows that

$$A' := A - \delta_1((i, j'), (i', j))$$

has non-negative entries by construction with row and column sums matching that of A (and hence that of B). Moreover

$$\Delta(A') = \begin{cases} \Delta(A) - 4, & \text{if } a_{i'j'} \text{ is in deficit,} \\ \Delta(A) - 2, & \text{otherwise.} \end{cases}.$$

Hence by induction

$$\begin{aligned} A &= A' + \delta_1((i, j'), (i', j)) \\ &= B + \sum_{r=1}^{t} \delta(\mathbf{u}_r, \mathbf{v}_r) + \delta_1((i, j'), (i', j)). \end{aligned}$$

□

Corollary 6.5. *Let $A = (a_{ij}) \in \mathrm{CSP}_{\mathbb{Z}}(n)$. Write $S_j = \sum_{i=0}^{n-1} a_{ij}$ for the column sums of A for $j = 1, \ldots, n$ and set $\mathbf{s} = (S_1, \ldots, S_n)$. Then there exists swaps $\delta_1(\mathbf{u}_r, \mathbf{v}_r)$ for $r = 1, \ldots, t$ such that*

$$A = U(\mathbf{s}) + \sum_{r=1}^{t} \delta_1(\mathbf{u}_r, \mathbf{v}_r). \tag{6.2}$$

Moreover $U(\mathbf{s}) + \sum_{r=1}^{t_0} \delta_1(\mathbf{u}_r, \mathbf{v}_r) \in \mathrm{CSP}_{\mathbb{Z}}(n)$ for all $1 \leq t_0 \leq t$.

Proof. Let $R_i = \sum_{j=1}^n a_{ij}$ denote the row sums of A for $i = 0, 1, \ldots, n-1$. Note that the row sums of A are determined uniquely by the column sums of A via

$$R_i = \sum_{j : \frac{n}{j} | i} \frac{1}{j} S_j, \quad \text{for } i = 0, \ldots, n-1,$$

since both sides count the number of orbits whose stabilizer-order divides i in the corresponding CSP-instance, according to (1.2) and Remark 2.8. Since A and $U(\mathbf{s})$ have the same column sums they must therefore have the same row sums. The corollary now follows from Proposition 6.4 and Lemma 5.3. □

Remark 6.6. Proposition 6.4 shows that every $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$ can be uniquely expressed as $U(\mathbf{s})+B$ where $B = (b_{ij}) \in \mathbb{Z}^{n\times n}$ is a matrix with zero row and column-sums and non-negative values in all entries $b_{k\ell}$ unless $(k,\ell) = (\frac{ni}{j}, j)$ where $0 \le i < j$ and $j|n$.

Construction 6.7. If $(C_m, X, f(q))$ and $(C_n, Y, g(q))$ are two CSP-triples, then we can construct a new CSP-triple of the form $(C_{mn}, X \times Y, h(q))$ where $h(q)$ is a polynomial of degree less than mn which may be expressed as certain convolution of f and g.

Let $(x, y) \in X \times Y$ and suppose $o(x) = i$, $o(y) = j$ with respect to the actions of C_m on X and C_n on Y respectively. Let C_{mn} act on (x, y) via

$$\sigma_{mn}^{is+t} \cdot (x, y) \coloneqq (\sigma_m^t \cdot x, \sigma_n^s \cdot y)$$

where $0 \le t < i$ and $s \in \mathbb{Z}$. Note that (x, y) has order ij under the above action. By Remark 2.8, the number of elements of order i and j with respect to the actions of C_m on X and C_n on Y are given respectively by

$$S_i = \sum_{\ell|i} \mu(\ell/i) f(\omega_m^\ell), \;\; T_j = \sum_{\ell|j} \mu(\ell/j) g(\omega_n^\ell).$$

Therefore the action of C_{mn} on $X \times Y$ has

$$\sum_{ij=k} S_i T_j,$$

elements of order k. By (1.2) the coefficients c_r of the unique polynomial $h(q) = \sum_{r=0}^{mn-1} c_r q^r \pmod{q^{mn}-1}$ complementing the action of C_{mn} on $X \times Y$ to a CSP is given by the number of orbits whose stabilizer-order divides r, that is,

$$c_r = \sum_{k:\frac{mn}{k}|r} \sum_{ij=k} \frac{1}{k} S_i T_j.$$

The above construction gives rise to a natural product on universal CSP-matrices. Given a vector $\mathbf{s} = (S_d)$, we define its *number-theoretical series* as the formal power-series

$$NS(\mathbf{s}) \coloneqq \sum_{1\le d} S_d x_{p_1}^{e_1} \dots x_{p_\ell}^{e_\ell} \tag{6.3}$$

where $d = p_1^{e_1} \dots p_\ell^{e_\ell}$ is the prime factorization of d.

Given two vectors $\mathbf{s}$ and $\mathbf{t}$ of length m and n, respectively, define the vector $\mathbf{s} \boxtimes \mathbf{t}$ of length mn via the identity

$$NS(\mathbf{s} \boxtimes \mathbf{t}) = NS(\mathbf{s}) \cdot NS(\mathbf{t}).$$

In other words, coordinate k in $\mathbf{s} \boxtimes \mathbf{t}$ is given by $\sum S_i T_j$, where the sum ranges over all natural numbers i, j such that $ij = k$. Note that $\boxtimes$ is symmetric and transitive, and $|\mathbf{s} \boxtimes \mathbf{t}| = |\mathbf{s}| \cdot |\mathbf{t}|$ where $|\cdot|$ denotes the sum of the entries.

Proposition 6.8. *Let $U(\mathbf{s}) \in \widetilde{\mathrm{CSP}}(m)$ and $U(\mathbf{t}) \in \widetilde{\mathrm{CSP}}(n)$. Then*

$$U(\mathbf{s}) \boxtimes U(\mathbf{t}) \coloneqq U(\mathbf{s} \boxtimes \mathbf{t}) \in \widetilde{\mathrm{CSP}}(mn).$$

Proof. We have that $i|S_i$ and $j|T_j$ for $i = 1, \dots, m$, $j = 1, \dots, n$ and $S_i, T_j = 0$ for $i \not| m$, $j \not| n$. It follows that

$$(\mathbf{s} \boxtimes \mathbf{t})_k = \sum_{\substack{ij=k \\ i|m, j|n}} S_i T_j,$$

with $k|(\mathbf{s} \boxtimes \mathbf{t})_k$ for $k = 1, \dots, mn$ and $(\mathbf{s} \boxtimes \mathbf{t})_k = 0$ if $k \not| mn$. □

7. Geometry of the CSP cone

The below theorem provides the half-space description of CSP(n), showing that it is indeed a rational convex polyhedral cone.

Theorem 7.1. *Let $n \in \mathbb{N} \setminus \{0\}$ and $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. Let the divisors of n be given by*

$$1 = c_1 < c_2 < \cdots < c_d = n.$$

Let

$$H_k(\mathbf{x}) \coloneqq \sum_{i=0}^{n-1} \sum_{j=2}^{d} \alpha_{ijk} x_{ij} \in \mathbb{Z}[\mathbf{x}],$$

where

$$\alpha_{ijk} \coloneqq \begin{cases} -n + \frac{n}{c_j}, & \text{if } i = k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ -n & \text{if } i = k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}, \\ \frac{n}{c_j}, & \text{if } i \neq k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ 0 & \text{if } i \neq k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}. \end{cases}$$

Then A is a CSP matrix if and only if

$$A = (\mathbf{a}_1 | \mathbf{a}_2 | \cdots | \mathbf{a}_n),$$

where

$$\mathbf{a}_1 = (x_{01}, H_1(\mathbf{x}), \dots, H_{n-1}(\mathbf{x}))^t,$$

$$\mathbf{a}_c = \begin{cases} (nx_{0c}, nx_{1c}, \dots, nx_{(n-1)c})^t, & \text{if } c|n, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

for $c = 2, \dots, n$ with $H_k(\mathbf{x}) \geq 0$ and $x_{ij} \geq 0$ for all i, j, k.

Proof. For $\mathbf{z} \in \mathbb{C}^{n-1}$, let

$$V(\mathbf{z}) \coloneqq \begin{pmatrix} z_1 & z_1^2 & \dots & z_1^{n-1} \\ z_2 & z_2^2 & \dots & z_2^{n-1} \\ \vdots & \vdots & & \vdots \\ z_{n-1} & z_{n-1}^2 & \dots & z_{n-1}^{n-1} \end{pmatrix}.$$

Let $\boldsymbol{\omega} \coloneqq (\omega_n, \omega_n^2, \dots, \omega_n^{n-1})$ and set

$$B_j \coloneqq (\mathbf{1}^t | V(\boldsymbol{\omega})) - J_{c_j}$$

for $j = 1, \dots, d$ where $\mathbf{1} \coloneqq (1, \dots, 1) \in \mathbb{R}^{n-1}$ and

$$J_{c_j}(k, \ell) \coloneqq \begin{cases} 1 & \text{if } c_j | k, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq k \leq n-1$ and $1 \leq \ell \leq n$. Consider the matrix

$$B := [B_1|B_2|\cdots|B_d]\,.$$

Then $A = (a_{ij}) \in \mathbb{R}^{n\times n}_{\geq 0}$ satisfies (4.1) if and only if

$$B\mathbf{a} = \mathbf{0}, \tag{7.1}$$

where $\mathbf{a} = (\mathbf{a}_1|\cdots|\mathbf{a}_d)^t$ and $\mathbf{a}_j = (a_{1c_j}, \ldots, a_{nc_j})$ for $j = 1, \ldots, d$. Note that the defining CSP-equations (4.1) immediately give that $a_{ij} = 0$ for all $1 \leq i \leq n$ and $j \nmid n$. We claim that the real solutions to (7.1) are of the form

$$\mathbf{a}_1 = \begin{pmatrix} x_{01} \\ H_1(\mathbf{x}) \\ \vdots \\ H_{n-1}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{a}_j = \begin{pmatrix} nx_{0j} \\ nx_{1j} \\ \vdots \\ nx_{(n-1)j} \end{pmatrix} \tag{7.2}$$

where $x_{01}, x_{ij} \in \mathbb{R}$ for $0 \leq i \leq n-1$, $2 \leq j \leq d$ and

$$H_k(\mathbf{x}) = \sum_{i=0}^{n-1}\sum_{j=2}^{d} \alpha_{ijk} x_{ij}$$

for some $\alpha_{ijk} \in \mathbb{Z}$, $k = 1, \ldots, n-1$. Since B has full rank $n-1$, the solutions (7.2) make up the whole null space of B for dimensional reasons. Thus we only need to concern ourselves with the existence of solutions of the form (7.2).

Given (7.1) and supposing (7.2) we thus require

$$(V(\boldsymbol{\omega}) - J_1)\boldsymbol{\alpha}^{(ij)} = \mathbf{u}^{(ij)}, \tag{7.3}$$

for $i = 0, \ldots, n-1$ and $j = 2, \ldots, d$ where

$$\boldsymbol{\alpha}^{(ij)} := \begin{pmatrix} \alpha_{ij1} \\ \alpha_{ij2} \\ \vdots \\ \alpha_{ij(n-1)} \end{pmatrix}, \quad \mathbf{u}^{(ij)} := \begin{pmatrix} u_1^{(ij)} \\ u_2^{(ij)} \\ \vdots \\ u_{n-1}^{(ij)} \end{pmatrix}, \quad u_k^{(ij)} := \begin{cases} -n\omega_n^{ik} + n, & \text{if } c_j|k, \\ -n\omega_n^{ik}, & \text{otherwise} \end{cases}.$$

Note that

$$(V(\boldsymbol{\omega}) - J_1)^{-1} = \frac{1}{n}V(\overline{\boldsymbol{\omega}}).$$

Therefore

$$\boldsymbol{\alpha}^{(ij)} = \frac{1}{n}V(\overline{\boldsymbol{\omega}})\mathbf{u}^{(ij)},$$

which gives

$$\begin{aligned}\alpha_{ijk} &= \sum_{\ell=1}^{n-1} \frac{\overline{\omega}_n^{k\ell}}{n}(-n\omega_n^{i\ell}) + \sum_{\substack{\ell=1\\ c_j|\ell}}^{n-1} \frac{\overline{\omega}_n^{k\ell}}{n} n \\ &= -\sum_{\ell=0}^{n-1} (\omega_n^{(i-k)})^\ell + \sum_{s=0}^{\frac{n}{c_j}-1} ((\omega_n^{c_j})^k)^s \\ &= \begin{cases} -n+\frac{n}{c_j}, & \text{if } i=k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ -n & \text{if } i=k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}, \\ \frac{n}{c_j}, & \text{if } i \neq k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ 0 & \text{if } i \neq k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}. \end{cases}\end{aligned}$$

Hence the theorem follows. □

The following corollary follows immediately from Theorem 7.1.

Corollary 7.2. *Let $n \in \mathbb{N} \setminus \{0\}$ and $d := |\{c \in \mathbb{N} : c|n\}|$ denote the number of divisors of n. Then* $\mathrm{CSP}(n)$ *has dimension $n(d-1)+1$.*

Recall that a polyhedral cone is given by $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \geq \mathbf{b}\}$ for some $n \times n$ matrix A. A non-zero element $\mathbf{x}$ of a polyhedral cone P is called an *extreme ray* if there are $d-1$ linearly independent constraints that are active at $\mathbf{x}$ (*i.e.* hold with equality at $\mathbf{x}$). If $\mathbf{x}$ is an extreme ray, then $\lambda\mathbf{x}$ is also an extreme ray for $\lambda > 0$. Two extreme rays that are positive multiples of each other are called *equivalent*. Equivalent extreme rays correspond to the same $d-1$ active constraints. Extreme rays can also be defined as points in $\mathbf{x} \in P$ that cannot be expressed as a convex combination of two points in the interior of P.

Below we give an explicit description of a subset of the extreme rays of $\mathrm{CSP}(n)$. This subset includes all extreme rays of the universal CSP-cone $\widetilde{\mathrm{CSP}}(n)$ (see Corollary 7.4). When $n = p$ for some prime number p, then we get all the extreme rays (see Corollary 7.5).

Theorem 7.3. *Let $n \in \mathbb{N} \setminus \{0\}$ and suppose*

$$1 = c_1 < c_2 < \cdots < c_{d-1} < c_d = n$$

are the divisors of n. Let $\ell_0 \in [d]$. Then $\mathbf{r} = (r_{ij}) \in \mathbb{R}^{n\times n}$ is an extreme ray of $\mathrm{CSP}(n)$ *if*

$$r_{ij} = \begin{cases} 1, & \text{if } (i,j) = (0, c_{\ell_0}), \\ \frac{1}{c_{\ell_0}-|I|}, & \text{if } i \in I \text{ and } j = c_{\ell_0}, \\ 0, & \text{otherwise} \end{cases}$$

for $0 \leq i \leq n-1$ and $1 \leq j \leq n$ where $I \subseteq \{t\frac{n}{c_{\ell_0}} \in \mathbb{N} : 1 \leq t < c_{\ell_0}\}$. In particular the number of extreme rays of $\mathrm{CSP}(n)$ *is at least*

$$\frac{1}{2}\sum_{\ell=1}^{d} 2^{c_\ell}.$$

Proof. By Theorem 7.1, $\mathrm{CSP}(n)$ is isomorphic to the polyhedral cone

$$\{\mathbf{x} \in \mathbb{R}^{n(r-1)+1} : n\mathbf{x} \geq 0 \text{ and } H_k(\mathbf{x}) \geq 0 \text{ for all } k = 1, \ldots, n-1\}.$$

Let $\mathbf{r} = (r_{ij}) \in \mathbb{R}^{n\times n}$ be an extremal ray of CSP(n) such that $r_{ij} = 0$ if $j \neq c_{\ell_0}$. Note that the defining inequalities of CSP(n) imply in particular that $r_{ij} \geq 0$ for all $0 \leq i < n$ and $1 \leq j \leq d$.

Suppose first that $r_{0c_{\ell_0}} = 0$. Let $k \in [n-1]$ be such that $r_{kc_{\ell_0}} \geq r_{ic_{\ell_0}}$ for all $i \in [n-1]$. Suppose for a contradiction that $r_{kc_{\ell_0}} > 0$. The maximality of $r_{kc_{\ell_0}}$ implies $\frac{r_{ic_{\ell_0}}}{r_{kc_{\ell_0}}} \leq 1$. The defining inequalities of the polyhedral cone CSP(n) gives $-nr_{ic_{\ell_0}} \geq 0$ for $i \not\equiv 0 \pmod{\frac{n}{c_{\ell_0}}}$, which implies that $r_{ic_{\ell_0}} = 0$ for $i \not\equiv 0 \pmod{\frac{n}{c_{\ell_0}}}$. Thus we may assume $k \equiv 0 \pmod{\frac{n}{c_{\ell_0}}}$. Now, $H_k(\mathbf{x}) \geq 0$ gives

$$0 \leq -n + \frac{n}{c_{\ell_0}} + \sum_{\substack{i\in[n-1]\setminus k \\ \frac{n}{c_{\ell_0}}|i}} \frac{r_{ic_{\ell_0}}}{r_{kc_{\ell_0}}} \leq -n + \frac{n}{c_{\ell_0}} + c_{\ell_0} - 2,$$

which holds if and only if $c_{\ell_0} \leq \frac{n+2}{2} - \Delta$ or $c_{\ell_0} \geq \frac{n+2}{2} + \Delta$ where $\Delta = \left(\left(\frac{n+2}{2}\right)^2 - n\right)^{1/2}$. Since $\frac{n+2}{2} - \Delta < 1$ and $\frac{n+2}{2} + \Delta > n$ for $n > 0$ whereas $1 \leq c_{j_0} \leq n$ this gives a contradiction.

Hence we may assume $r_{0c_{\ell_0}} > 0$. Let

$$M_\ell := \left\{ t\frac{n}{c_\ell} \in \mathbb{N} : 1 \leq t < c_\ell \right\}.$$

Suppose $I \subseteq M_{c_{\ell_0}}$ such that $r_{ic_{\ell_0}} > 0$ for $i \in I$ and $r_{ic_{\ell_0}} = 0$ for $i \in M_{c_{\ell_0}} \setminus I$. Since $\mathbf{r}$ is an extreme ray there are by definition $n(d-1)$ linearly independent constraints active at $\mathbf{r}$. Since $r_{ij} = 0$ for $j \neq c_{\ell_0}$ and $r_{ic_{\ell_0}} = 0$ for $i \in [n-1] \setminus I$ there are $n(d-2) + 1 + (n-1) - |I|$ active constraints covered. Note that we have $(-n + \frac{n}{c_{\ell_0}})r_{kc_{\ell_0}} + \sum_{i\neq k} \frac{n}{c_{\ell_0}} r_{ic_{\ell_0}} > 0$ for $k \notin I$ and $nr_{kc_{\ell_0}} > 0$ for $k \in M_{c_{\ell_0}} \setminus I$. Hence the remaining $|I|$ inequalities must be active at $\mathbf{r}$ which gives

$$\left(-n + \frac{n}{c_{\ell_0}}\right) r_{kc_{\ell_0}} + \sum_{i\neq k} \frac{n}{c_{\ell_0}} r_{ic_{\ell_0}} = 0 \tag{7.4}$$

for $k \in I$. If $I = \emptyset$, then the only non-zero entry of $\mathbf{r}$ is $r_{0c_{\ell_0}}$. Suppose $I \neq \emptyset$. Summing the equations (7.4) and dividing by $\frac{n}{c_{\ell_0}} r_{0c_{\ell_0}}$, we get

$$0 = \frac{c_{\ell_0}}{nr_{0c_{\ell_0}}} \sum_{i\in I} \left(\left(-n + \frac{n}{c_{\ell_0}}\right) r_{ic_{\ell_0}} + \sum_{k\neq i} \frac{n}{c_{\ell_0}} r_{kc_{\ell_0}} \right) = (-c_{\ell_0} + |I|) \sum_{i\in I} \frac{r_{ic_{\ell_0}}}{r_{0c_{\ell_0}}} + |I|.$$

Hence we get the average ratio

$$\frac{1}{|I|} \sum_{i\in I} \frac{r_{ic_{\ell_0}}}{r_{0c_{\ell_0}}} = \frac{1}{c_{\ell_0} - |I|} \tag{7.5}$$

Suppose

$$\frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}} > \frac{1}{c_{\ell_0} - |I|}$$

for some $k \in I$. Then by dividing (7.4) with $\frac{n}{c_{\ell_0}} r_{0c_{\ell_0}}$ and using (7.5) we have

$$\begin{aligned} 0 &= (-c_{\ell_0}+1)\frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}} + 1 + \sum_{i\in I}\frac{r_{ic_{\ell_0}}}{r_{0c_{\ell_0}}} - \frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}} \\ &= -c_{\ell_0}\frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}} + 1 + \frac{|I|}{c_{\ell_0}-|I|} \\ &< \frac{-c_{\ell_0}}{c_{\ell_0}-|I|} + 1 + \frac{|I|}{c_{\ell_0}-|I|} = 0, \end{aligned}$$

which gives a contradiction. Hence by (7.5) we have that

$$r_{ic_{\ell_0}} = \frac{r_{0c_{\ell_0}}}{c_{\ell_0}-|I|}$$

for all $i \in I$ proving the theorem.

□

Corollary 7.4. *Let $n \in \mathbb{N}\setminus\{0\}$ and suppose $1 = c_1 < c_2 < \cdots < c_{d-1} < c_d = n$ are the divisors of n. Let $M_\ell = \{t\frac{n}{c_\ell} : 0 \le t < c_\ell\}$ and define $\mathbf{r}^{(\ell)} = (r_{ij}^{(\ell)}) \in \mathbb{R}^{n\times n}$ by*

$$r_{ij}^{(\ell)} = \begin{cases} 1, & \text{if } i \in M_\ell \text{ and } j = c_\ell, \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \le \ell \le d$. Then the extreme rays of $\widetilde{\mathrm{CSP}}(n)$ are given by $\{\mathbf{r}^{(\ell)} : 1 \le \ell \le d\}$.

Proof. By Theorem 7.3 the set $\{\mathbf{r}^{(\ell)} : 1 \le \ell \le d\}$ are indeed extreme rays and they clearly generate all universal CSP matrices (cf. Definition 6.2). □

Corollary 7.5. *Let $p \in \mathbb{N}$ be a prime number. Then the extreme rays of $\mathrm{CSP}(p)$ are given by $E_{01} \in \mathbb{R}^{p\times p}$ and $\mathbf{r} = (r_{ij}) \in \mathbb{R}^{p\times p}$ such that*

$$r_{ij} = \begin{cases} 1, & \text{if } (i,j) = (0,p), \\ \frac{1}{p-|I|}, & \text{if } i \in I \text{ and } j = p, \\ 0, & \text{otherwise,} \end{cases}$$

where $I \subseteq \{1,\dots,p-1\}$. In particular the number of extreme rays of $\mathrm{CSP}(p)$ is given by $2^{p-1}+1$.

By adding a size restriction on the set X we can also talk about a natural family of polytopes associated with cyclic sieving phenomena.

Definition 7.6. Let $m \in \mathbb{N}$. The mth *CSP-polytope* is the convex rational polytope defined by

$$\mathrm{CSP}(n,m) := \{A \in \mathrm{CSP}(n) : ||A|| = m\}.$$

Let $\mathrm{CSP}_{\mathbb{Z}}(n,m) := \mathrm{CSP}(n,m) \cap \mathbb{Z}^{n\times n}$ denote the set of integer lattice points in $\mathrm{CSP}(n,m)$.

Once again, in the case where $n = p$ for some prime number $p \in \mathbb{N}$ we are able to make explicit computations. In the following two propositions we compute the vertices and the number of integer lattice points of $\mathrm{CSP}(n,m)$.

Proposition 7.7. *Let $p \in \mathbb{N}$ be a prime number and $m \in \mathbb{N}$. Then the vertices of* $\mathrm{CSP}(p,m)$ *are given by $mE_{01} \in \mathbb{R}^{p\times p}$ and $\mathbf{v} = (v_{ij}) \in \mathbb{R}^{p\times p}$ such that*

$$v_{ij} = \begin{cases} C, & \text{if } (i,j) = (0,p), \\ \frac{C}{p-|I|}, & \text{if } i \in I \text{ and } j = p, \\ 0, & \text{otherwise,} \end{cases}$$

where $I \subseteq \{2,\dots,p\}$ and

$$C = \frac{m}{2p-1+\frac{(p-1)|I|}{p-|I|}}.$$

In particular the number of vertices of $\mathrm{CSP}(p,m)$ *is given by $2^{p-1}+1$.*

Proof. Suppose $\mathbf{v} = (v_{ij}) \in \mathbb{R}^{p\times p}$ is a vertex of $\mathrm{CSP}(p,m)$.

If $v_{0p} = 0$, then arguing as in the first part of the proof of Theorem 7.3 gives that $v_{0p} = v_{1p} = \cdots = v_{p-1p} = 0$. Therefore $v_{ij} = 0$, unless $j = 1$ by Lemma 5.1. The additional constraint $||\mathbf{v}|| = m$ thus gives

$$m = \sum_{\substack{0\le i<p \\ 1\le j\le p}} v_{ij} = \sum_{j=1}^{p} v_{0j} = x_{01} + \sum_{k=1}^{p-1} H_k(\mathbf{x}), \tag{7.6}$$

which is the same as

$$x_{01} + (2p-1)x_{0p} + (p-1)x_{1p} + \cdots + (p-1)x_{p-1p} = m. \tag{7.7}$$

Since $x_{ip} = v_{ip}$ for $i = 0, 1\dots, p-1$ we get that $v_{01} = x_{01} = m$, so that $\mathbf{v} = mE_{01}$.

Therefore suppose $v_{0p} > 0$. Moreover suppose $I \subseteq \{1,\dots,p-1\}$ such that $v_{ip} > 0$ for $i \in I$ and $v_{ip} = 0$ for $i \in \{1,\dots,p-1\} \setminus I$. Since $\mathbf{v}$ is a vertex, there are by definition $p+1$ linearly independent constraints active at $\mathbf{v}$. Since p of these constraints arise from the polyhedral description of $\mathrm{CSP}(p)$ in Theorem 7.1 it follows, as in Corollary 7.5, that

$$v_{ip} = \begin{cases} C, & \text{if } (i,j) = (0,p), \\ \frac{C}{p-|I|}, & \text{if } i \in I, \\ 0, & \text{if } i \in \{1,\dots,p-1\} \setminus I, \end{cases}$$

for some $C > 0$. The remaining active constraint is Equation (7.7). Inserting the above into Equation (7.7) and solving for C yields

$$C = \frac{m}{2p-1+\frac{(p-1)|I|}{p-|I|}},$$

from which the proposition follows. □

Proposition 7.8. *Let $p, m \in \mathbb{N}$ where p is a prime number. The number of lattice points in* $\mathrm{CSP}(p,m)$ *is given by*

$$|\mathrm{CSP}_{\mathbb{Z}}(p,m)| = \sum_{j=0}^{m} \sum_{r\in\left[\frac{2j}{2p-1},\frac{j}{p-1}\right]\cap\mathbb{Z}} C\left(r(2p-1)-2j, p-1, \lfloor r - j/p\rfloor\right),$$

where

$$C(n,k,w) = \sum_{j=0}^{k} (-1)^j \binom{k}{j}\binom{n-jw-1}{k-1}.$$

Proof. Let $x = x_{0p}$, $y = x_{1p} + \cdots + x_{p-1p}$ and $z = x_{01}$. According to the constraint $||A|| = m$ we seek non-negative integer solutions to

$$(2p-1)x + (p-1)y + z = m,$$

(cf. Equation (7.7)) satisfying $H_k(\mathbf{x}) \geq 0$. We therefore consider the Diophantine equations

$$(2p-1)x + (p-1)y = j,$$

for $j = 0, \ldots, m$ which have the non-negative integer solutions

$$x = j - r(p-1) \text{ and } y = -2j + r(2p-1),$$

for $r \in \left[\frac{2j}{2p-1}, \frac{j}{p-1}\right] \cap \mathbb{Z}$. The constraints $H_k(\mathbf{x}) \geq 0$ for $k = 1, \ldots, p-1$ give

$$x - (p-1)x_{kp} + (y - x_{kp}) \geq 0,$$

which implies

$$x_{kp} \leq r - \frac{j}{p}$$

for $k = 1, \ldots, p-1$. Hence the lattice points in $\mathrm{CSP}(p, m)$ are in one-to-one correspondence with weak compositions of $y = -2j + r(2p-1)$ into $p-1$ parts of size at most $\lfloor r - j/p \rfloor$. By [Abr76] the number of such compositions are given by $C(r(2p-1) - 2j, p-1, \lfloor r - j/p \rfloor)$. □

8. Appendix

In this appendix we prove inequalities needed for the estimations in Theorem 3.7. The first inequality below gives a sufficient condition for a Riemann sum to be monotonically increasing. A slightly weaker result appears in [BJ00, Theorem 3A].

Proposition 8.1. *Let $f(x)$ be a* decreasing *convex*[1] *function on $\mathbb{R}_{\geq 0}$, let p be a positive integer and $r \geq 0$. Then*

$$\frac{1}{p}\sum_{\ell=1}^{p} f\left(\frac{\ell + r}{p}\right) \leq \frac{1}{p+1}\sum_{\ell=1}^{p+1} f\left(\frac{\ell + r}{p+1}\right). \tag{8.1}$$

Proof. Let $x_i \coloneqq (i+r)/p$ and $y_i \coloneqq (i+r)/(p+1)$ and note that

$$x_i = \left(1 - \frac{i}{p}\right) y_i + \frac{i}{p} y_{i+1} + \frac{r}{p(p+1)}. \tag{8.2}$$

Since f is decreasing and convex, we have that

$$f(x_i) \leq f\left[\left(1 - \frac{i}{p}\right) y_i + \frac{i}{p} y_{i+1}\right] \leq \left(1 - \frac{i}{p}\right) f(y_i) + \frac{i}{p} f(y_{i+1})$$

Now let $a_i \coloneqq f(x_i)$ and $b_i \coloneqq f(y_i)$ and note that the decreasing property implies

$$a_i \leq \left(1 - \frac{i}{p}\right) b_i + \frac{i}{p} b_{i+1} \leq \left(1 - \frac{i}{p+1}\right) b_i + \frac{i}{p+1} b_{i+1} \text{ for } i = 1, \ldots, p.$$

We add all these inequalities and obtain

$$\sum_{i=1}^{p} a_i \leq \frac{1}{p+1}\sum_{i=1}^{p}(p+1-i) b_i + \frac{1}{p+1}\sum_{i=1}^{p} i b_{i+1}.$$

[1]For all a, b we have $f(\frac{a+b}{2}) \leq \frac{f(a)+f(b)}{2}$, or equivalently for twice differentiable functions, $f'' \geq 0$.

We then have

$$\begin{aligned}(p+1)(a_1+\cdots+a_p) &\le \sum_{i=1}^{p}(p+1-i)b_i + \sum_{i=2}^{p+1}(i-1)b_i \\ &\le p\sum_{i=1}^{p} b_i + \sum_{i=1}^{p}(1-i)b_i + pb_{p+1} + \sum_{i=2}^{p}(i-1)b_i \\ &\le p(b_1+\cdots+b_{p+1}).\end{aligned}$$

This implies (8.1). □

Corollary 8.2. *Let $r, s \ge 0$ and $p \in \mathbb{N}$. Then the expression*

$$g(p) = \frac{1}{p}\sum_{\ell=1}^{p}\frac{1}{s+(r+\ell)/p} \tag{8.3}$$

is increasing with p.

Proof. Choosing the decreasing convex function $f(x) = 1/(s+x)$ in Proposition 8.1 together with the given r yields

$$\frac{1}{p}\sum_{\ell=1}^{p}\frac{1}{s+(r+\ell)/p} \le \frac{1}{p+1}\sum_{\ell=1}^{p+1}\frac{1}{s+(r+\ell)/(p+1)}.$$

□

Corollary 8.3. *If a, t, i, j and k are non-negative integers such that $a \le t$ and $j \le k$, then*

$$\sum_{\ell=0}^{ka-1}\frac{1}{kt+i-\ell} \ge \sum_{\ell=0}^{ja-1}\frac{1}{jt+i-\ell}. \tag{8.4}$$

Proof. Choosing $p = ak$, $s = (t-a)/a$ and $r = i$ in (8.3) gives that

$$f(ak) = \frac{1}{ka}\sum_{\ell=1}^{ak}\frac{1}{s+(i+\ell)/(ak)} = \sum_{\ell=1}^{ka}\frac{1}{kt-ka+i+\ell} = \sum_{\ell=0}^{ka-1}\frac{1}{kt+i-\ell}.$$

The fact that $f(ka) \ge f(kj)$ if $k \ge j$ now gives the desired inequality. □

Lemma 8.4. *If a, b, i, j and k are non-negative integers such that $a \le b$ and $j \le k$, then*

$$\frac{\binom{kb+i}{ka}^{1/k}}{\binom{ka+i}{ka}^{1/k}} \ge \frac{\binom{jb+i}{ja}^{1/j}}{\binom{ja+i}{ja}^{1/j}}. \tag{8.5}$$

Proof. The inequality can be rewritten as $f(b) \ge f(a)$, where

$$f(t) := \frac{\binom{kt+i}{ka}^{j}}{\binom{jt+i}{ja}^{k}}. \tag{8.6}$$

Thus, it suffices to show that $f(t)$ is increasing. Computing the derivative and factoring out positive terms reduces to Equation (8.4). □

Remark 8.5. In the case where $j|k$, the binomial inequality (8.5) admits the following combinatorial interpretation. A certain organization wants ka members to sit on its executive committee and ja members to sit on the committee for each of its k/j factions. Then the number of possible committee constellations with $kb+i$ candidates for the executive committee and $ja+i$ candidates for each of the factions, is greater than the number of committee constellations with $ka+i$ candidates for the executive committee and $jb+i$ candidates for each faction.

Lemma 8.6. *If a, b and k are non-negative integers such that $b > a$, then for each $0 \leq i \leq ka$ we have*

$$\binom{kb+i}{ka}\binom{ka+i}{ka}^{-1} \geq \left(\frac{b+a}{2a}\right)^{ka}$$

Proof. If $B > A$, then the function $f(x) = \frac{B+x}{A+x}$ is decreasing as x increases. Thus for $0 \leq i \leq ka$ we have

$$\begin{aligned}\binom{kb+i}{ka}\binom{ka+i}{ka}^{-1} &= \frac{(kb+i)(kb+i-1)\cdots(kb+i-ka+1)}{(ka+i)(ka+i-1)\cdots(i+1)} \\ &\geq \left(\frac{kb+i}{ka+i}\right)^{ka} \\ &\geq \left(\frac{b+a}{2a}\right)^{ka}\end{aligned}$$

□

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