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Nicolae Mihalache Ciurdea

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LA CONDITION DE COLLET-ECKMANN POUR LES ORBITES CRITIQUES RÉCURRENTES

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M. David Guy
M. Graczyk Jacek (directeur de thèse)
M. Sands Duncan
M. Smirnov Stanislav (rapporteur)
Mme Tan Lei
M. Yoccoz Jean-Christophe (rapporteur)

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Table des matières

Chapitre 1

Introduction

L'itération des applications en dimension un a une longue histoire. La méthode de Newton-Raphson pour déterminer les zéros d'une fonction en est un exemple vieux de plus de trois siècles. En essayant de trouver des méthodes effectives de calcul des itérées des fonctions rationnelles, Schröder introduit la notion de conjugaison conforme. Toujours vers la fin du XIXème, siècle Poincaré étudie les propriétés des homéomorphismes du cercle. Au début du XXème siècle, en utilisant le théorème de Montel, Julia et Fatou étudient de façon plus systématique l'itération des applications rationnelles. Ils partitionnent la sphère de Riemann en deux ensembles d'après le comportement des itérées de l'application. Ainsi l'ensemble de Fatou est l'ensemble des points qui possèdent un voisinage sur lequel les itérées de l'application forment une famille normale. On appelle son complémentaire l'ensemble de Julia. L'ensemble de Fatou est ouvert et invariant. L'ensemble de Julia est soit toute la sphère, soit d'intérieur vide et il est lui aussi invariant. Son caractère fractal a été mis en évidence dès cette époque.

Les images obtenues à l'aide de l'ordinateur par Mandelbrot de l'ensemble des paramètres c pour lesquels l'ensemble de Julia du polynôme $z^2 + c$ est connexe a attiré l'attention de la communauté mathématique par leur complexité et par leur beauté. L'étude de la dynamique complexe et spécialement de la famille quadratique s'est intensifiée depuis le début des années 80, en commençant par les travaux de Douady et Hubbard. Toujours dans les années 80, Sullivan montre que l'ensemble de Fatou n'a pas de composantes errantes et par la suite on obtient une classification de ses composantes connexes. En même temps la dynamiques des applications de l'intervalle est le sujet de nombreux travaux. Milnor et Thurston élaborent une théorie combinatoire des application unimodales. Collet et Eckmann introduisent une condition sur la croissance de la dérivée sur l'orbite critique pour ces applications. À partir des années 80 la dynamique en dimension un connaît un développement impressionnant. Par la suite on se focalise sur les propriétés topologiques et analytiques des orbites critiques et leur conséquences.

Les orbites critiques

Soit f une application rationnelle de degré $d \geq 2$ et J son ensemble de Julia. L'ensemble de Julia est l'adhérence des orbites périodiques répulsives et l'ensemble de Fatou contient toutes les orbites périodiques attractives et leurs bassins d'attraction. Tout bassin d'attraction d'une orbite périodique attractive contient un point critique de f. Si f est un polynôme, J est connexe si et seulement si toutes les orbites critiques sont bornées. On dit que f est hyperbolique s'il existe des constantes $C > 0$ et $\lambda > 1$ telles que

$$\left|(f^n)'(z)\right| > C\lambda^n \text{ pour tous } z \in J \text{ et } n \geq 1.$$

Les dynamiques hyperboliques sont totalement comprises et la conjecture de Fatou affirme que pour un degré d fixé, l'ensemble des applications rationnelles hyperboliques est dense dans l'ensemble des applications rationnelles de degré d. C'est toujours un problème ouvert sauf pour la famille quadratique réelle. On sait que f est hyperbolique si et seulement si la fermeture de ses orbites critiques est disjointe de J. On peut aussi montrer que dans ce cas J est de mesure de Lebesgue nulle. Il a été conjecturé que l'ensemble de Julia est soit toute la sphère soit il est de mesure nulle. On dispose aujourd'hui d'un contre-exemple quadratique construit récemment par Buff et Chéritat, voir [1].

Dans le cas où l'ensemble de Julia contient des points critiques on peut se demander si on peut obtenir une expansion uniforme sur des sous ensembles compacts invariants de l'ensemble de Julia. Cela est vrai par exemple pour l'ensemble des points d'accumulation $\omega(c)$ des orbites critiques $O(c)$ tels que $\omega(c)$ est disjoint de l'ensemble des points critiques Crit en l'absence d'orbite périodique parabolique (orbite périodique indifférente de multiplicateur rationnel). C'est la condition de Misiurewicz qui a été ensuite généralisée en demandant seulement $c \notin \omega(c)$ pour tout $c \in \mathrm{Crit} \cap J$ en l'absence d'orbite périodique parabolique. On appelle cette condition *semi-hyperbolicité*. En dynamique unimodale ou quadratique la condition de Collet-Eckmann est impliquée par la semi-hyperbolicité. On dit qu'un point critique $c \in \mathrm{Crit}$ satisfait à la condition de Collet-Eckmann s'il existe des constantes $C > 0$ et $\lambda > 1$ telles que

$$\left|(f^n)'(f(c))\right| > C\lambda^n \text{ pour tout } n \geq 1.$$

On dit que f satisfait à la condition de Collet-Eckmann si tous ses points critiques dans J sont Collet-Eckmann en l'absence d'orbite périodique parabolique et on dénote cette condition par CE. En dynamique réelle, au début des années 80, van Strien, Guckenheimer et Misiurewicz posent le problème de l'invariance topologique de la condition de Collet-Eckmann pour les application unimodales avec dérivée Schwarzienne négative (S-unimodales). Nowicki et Sands ([10]) démontrent vers la fin des années 90 que la condition CE pour les applications S-unimodales est équivalente à *la deuxième condition de Collet-Eckmann* (CE_2) mais aussi à *l'hyperbolicité uniforme sur les orbites périodiques.* Soient g une application S-unimodale et c son point critique. On dit qu'elle satisfait à la condition $CE_2(c)$ s'il existe des constantes $C > 0$ et $\lambda > 1$ telles que pour toute préimage $y \in g^{-n}(c)$ avec $n > 0$ du point critique on a

$$\left|(g^n)'(y)\right| > C\lambda^n.$$

On dit que g est *uniformément hyperbolique sur les orbites périodiques* (UHP) s'il existe $\lambda > 1$ tel que pour tout point périodique x si $g^m(x) = x$ et $m > 0$ alors

$$\left|(g^m)'(x)\right| > \lambda^n.$$

On généralise cette définition pour les applications rationnelles en considérant seulement les orbites périodiques dans l'ensemble de Julia. En introduisant une condition formulée exclusivement en termes topologiques, la condition de *Collet-Eckmann topologique* (TCE), et en démontrant qu'elle est aussi équivalente à $CE_2(c)$ et à UHP, Nowicki et Przytycki montrent l'invariance topologique de CE dans le cas S-unimodal, voir [9]. On obtient des contre-exemples S-multimodales (applications avec dérivée Schwarzienne négative sur l'intervalle avec plusieurs points critiques) pour l'invariance topologique de CE. Tous ces contre-exemples sont semi-hyperboliques et la question de l'invariance de CE pour les points critiques récurrents persiste en dynamique S-multimodale à la fin des années 90, voir

[15]. L'étude de la condition CE pour les applications rationnelles a commencé seulement dans les années 90 dans l'article [12] de Przytycki.

L'étude de la régularité des composantes de l'ensemble de Fatou a été initiée par Carleson, Jones et Yoccoz dans l'article [2]. Ils démontrent qu'un polynôme est semi-hyperbolique si et seulement si le bassin d'attraction de l'infini est un domaine de John. Graczyk et Smirnov montrent plus tard dans [6] que les composantes de l'ensemble de Fatou sont des domaines de Hölder pour les applications rationnelles CE, voir aussi [7] pour une généralisation de la condition de Collet-Eckmann. Przytycki, Rivera-Letelier et Smirnov établissent en [13] l'équivalence entre TCE, UHP, $CE_2(z_0)$ pour un $z_0 \in \overline{\mathbb{C}}$ et la *décroissance exponentielle du diamètre* des composantes (*ExpShrink*). On dit que l'application rationnelle f satisfait à *ExpShrink* s'il existe $r > 0$ et $\lambda > 1$ tels que pour tout $z \in J$ et $n > 0$

$$\operatorname{diam} \operatorname{Comp} f^{-n} \left(B(z, r) \right) < \lambda^{-n}.$$

En utilisant aussi le résultat de Graczyk et Smirnov, en présence des cycles attractifs, la regularité Hölder des domaines de l'ensemble de Fatou devient équivalente a toutes ces conditions. Carleson, Jones et Yoccoz ([2]) montrent aussi que les polynômes semi-hyperboliques satisfont à *ExpShrink* et donc à toutes ces conditions équivalentes. La réciproque n'est pas vraie, voir [13]. Également, la condition CE pour les applications rationnelles n'est pas impliquée par ces conditions, sauf pour le cas où l'application a un seul point critique, voir [11]. Une application rationnelle satisfaisant à ces conditions équivalentes a une dynamique presque hyperbolique, par exemple la dimension de Hausdorff de l'ensemble de Julia est strictement inférieure à 2 dans le cas polynomial, voir [6]. De plus, de telles dynamiques sont abondantes dans l'espace des paramètres, voir [16], [17], [14] et [8].

La condition de Collet-Eckmann pour les orbites critiques récurrentes

Cette thèse etudie une condition plus générale que la semi-hyperbolicité et que la condition de Collet-Eckmann. On l'appelle *Collet-Eckmann pour les orbites critiques récurrentes* (RCE) et son étude a été inspirée par les résultats de [2] et [6]. Une application rationnelle f satisfait à cette condition si elle ne possède pas d'orbite périodique parabolique et tout point critique récurrent dans l'ensemble de Julia est Collet-Eckmann. On démontre qu'elle a comme conséquence la régularité Hölder des composantes de l'ensemble de Fatou. On construit aussi un contre-exemple pour la réciproque.

La condition CE pour les orbites critiques récurrentes a été déjà formulée dans le cas S-multimodal, voir [15]. Disposant seulement de contre-exemples semi-hyperboliques pour l'invariance topologique de CE pour ces applications, Świątek conjecture l'invariance topologique de RCE pour les applications S-multimodales. Les techniques développées dans le troisième chapitre pour construire un polynôme *ExpShrink* qui ne satisfait pas à RCE produisent aussi un contre-exemple pour cette conjecture.

Cette thèse comporte trois chapitres. Dans les sections 1.1, 1.2 et 1.3 ci-dessous nous décrivons les résultats de chacun de ces trois chapitres et l'essentiel des méthodes utilisées.

1.1 La condition de Collet-Eckmann pour les orbites critiques récurrentes implique l'hyperbolicité uniforme sur les orbites périodiques répulsives

Le deuxième chapitre est dédié exclusivement à la preuve du théorème suivant.

Théorème 1. *Les composantes de l'ensemble de Fatou de toute application rationnelle qui satisfait à la condition de Collet-Eckmann pour les orbites critiques récurrentes sont des domaines de Hölder.*

Soient f une application rationnelle RCE, J son ensemble de Julia et $C > 0$, $\lambda > 1$ tels que tout point critique récurrent $c \in J$ satisfait

$$\left|(f^n)'(f(c))\right| > C\lambda^n \text{ pour tout } n > 0.$$

Nous démontrons que f satisfait à la condition de décroissance exponentielle du diamètre des composantes.

Une étape importante avant de démontrer la décroissance exponentielle est la *stabilité en arrière* (BS, backward stability).

Definition 2.1.4. *On dit que f est* stable en arrière *si pour tout $\varepsilon > 0$ il existe $\delta > 0$ tel que pour tout $z \in J$ et $n \geq 0$*

$$\operatorname{diam} \operatorname{Comp} f^{-n}(B(z,\delta)) < \varepsilon.$$

Dans le cas semi-hyperbolique, la stabilité en arrière garantit que le degré sur les préimages des petits disques reste borné. Dans l'article [2] Carleson, Jones et Yoccoz montrent que dans ce cas la distorsion en termes de diamètres est bornée, ce qui implique une décroissance uniforme des diamètres des préimages. Grâce à la même borne de la distorsion on peut itérer cette décroissance uniforme pour obtenir la décroissance exponentielle du diamètre.

Graczyk et Smirnov ([6]) utilisent aussi une construction du type télescope mais ils calculent la dérivée sur une orbite en arrière au lieu de considérer le diamètre des disques. Ils obtiennent une borne explicite de la distorsion en utilisant la méthode des voisinages emboîtés. Ils récupèrent ainsi la croissance exponentielle de la dérivée sur une orbite en arrière en enchaînant trois types de tubes du télescope. À chaque étape ils considèrent des préimages univalentes.

Un outil employé pour développer la technique de voisinages emboîtés mais aussi fort utile dans la preuve de la proposition 2.2.3 est le lemme de Koebe. C'est essentiellement une borne de la variation de la dérivée d'une application holomorphe loin des points critiques qui engendre aussi une borne pour la déformation des disques.

Lemme de Koebe. *Soit $g : B \to \mathbb{C}$ une application holomorphe univalente du disque unité dans le plan complexe. L'image $g(B)$ contient le disque $B\left(g(0), \frac{1}{4}|g'(0)|\right)$ et pour tout $z \in B$ on a*

$$\frac{(1-|z|)}{(1+|z|)^3} \leq \frac{|g'(z)|}{|g'(0)|} \leq \frac{(1+|z|)}{(1-|z|)^3}$$

et

$$|g(z) - g(0)| \leq |g'(z)| \frac{|z|(1+|z|)}{1-|z|}.$$

Pour démontrer le théorème 1 on construit un télescope avec trois types de tubes. Nous avons choisi de considérer les diamètres des préimages au lieu de la dérivée sur une orbite en arrière à cause des orbites critiques qui ne satisfont pas la condition de Collet-Eckmann. Après avoir démontré la stabilité en arrière, en l'absence des orbites Collet-Eckmann on obtient une majoration du degré. On obtient une estimation explicite de la distorsion en termes de diamètre dans ce cas. Toutes les distances sont considérées dans la métrique sphérique.

Lemme 2.2.1. *Soient g une application rationnelle, $z \in \overline{\mathbb{C}}$ et $0 < r < R < 1$. Soient W et W' deux composantes connexes de $g^{-1}(B(z,R))$ et de $g^{-1}(B(z,r))$ respectivement, avec $W' \subseteq W$ et $\operatorname{diam} W < 1$. Si $\deg_W g \leq \mu$ alors*

$$\frac{\operatorname{diam} W'}{\operatorname{diam} W} < 64 \left(\frac{r}{R}\right)^{\frac{1}{\mu}}.$$

On peut ainsi utiliser les techniques de [2] et [6] ensemble, *a priori* de natures très différentes. Il faut remarquer que dans le cas rationnel, les préimages des disques ne sont plus nécessairement simplement connexes. Grâce à la stabilité en arrière on peut quand même choisir une échelle où les préimages de composantes simplement connexes sont simplement connexes. Toujours grâce à BS le télescope peut admettre des tubes avec un degré arbitraire. C'est le cas des tubes qui contiennent des orbites critiques Collet-Eckmann. La proposition suivante montre que leur diamètre décroît exponentiellement. On définit un voisinage Ω de J stable par préimage et qui ne rencontre pas d'orbite critique dans l'ensemble de Fatou.

Proposition 2.2.3. *Pour tout $1 < \lambda_0 < \lambda$ et $\theta < 1$ il existe $\delta > 0$ avec la propriété suivante. Soient $N > 0$ et W une composante connexe de $f^{-N}(B(z,R))$ avec $B(z,R) \subseteq \Omega$ et $\operatorname{diam} f^n(W) \leq \delta$ pour tout $n = 0, \ldots, N$, si $\overline{f^{-N-1}(W)}$ contient un point critique Collet-Eckmann et $\cup_{i=0}^{N-1} \overline{f^i(W)}$ contient aussi un point critique, alors*

$$\operatorname{diam} W < \theta R \lambda_0^{-N}.$$

Comme dans [2] on obtient une décroissance uniforme du diamètre quand le degré est borné. On l'utilise en l'absence de points critiques Collet-Eckmann dans une suite donnée de préimages.

Proposition 2.2.2. *Pour tous $\beta > 1$, $\mu \geq 1$ il existe $\delta > 0$ tel que pour tous $0 < r < R < \frac{\delta}{\beta}$ il existe $N > 0$ avec la propriété suivante. Pour tout $z \in J$ avec $B(z, \beta R) \subseteq \Omega$ et $n \geq N$ si W' et W sont deux composantes connexes de $f^{-n}(B(z,R))$ et de $f^{-n}(B(z,\beta R))$ respectivement telles que $W' \subseteq W$ et $\deg_W(f^n) \leq \mu$ alors*

$$\operatorname{diam} W' < r.$$

Une première difficulté rencontrée est le fait que sans disposer de la stabilité en arrière on doit travailler avec des préimages qui ne sont pas nécessairement simplement connexes. Par conséquent on considère des anneaux et on les découpe pour éviter les valeurs critiques. La preuve du lemme 2.2.1 utilise les propriétés de module des anneaux parmi lesquelles le problème extrémal de Teichmüller.

Une deuxième difficulté majeure de la preuve est la construction du télescope qui doit comprendre des tubes de degré arbitraire. En utilisant le diamètre au lieu de la dérivée sur une orbite en arrière, on ne peut pas *multiplier* les estimées trouvées. Cela se fait en utilisant le lemme 2.2.1 pour *lire* la contraction au bout du télescope. Par contre le lemme s'applique seulement quand le degré est borné. La proposition 2.2.3 fonctionne dès que les diamètres des préimages ne dépassent pas une borne fixée, ce qui permet de multiplier les estimées sur des tubes consécutifs qui contiennent des orbites critiques CE. Ces deux méthodes permettent de démontrer la décroissance exponentielle du diamètre des composantes.

1.2 Exemples et contre-exemples

On se pose le problème de la réciproque du théorème 1. En utilisant le lemme de Koebe ou l'inegalité (3.27) on observe que si W est ouvert et connexe et W^{-1} est une composante connexe de $f^{-1}(W)$ tels que $\frac{\operatorname{dist}\left(W^{-1},\operatorname{Crit}\right)}{\operatorname{diam} W^{-1}}$ est grand alors $\frac{\operatorname{diam} W}{\operatorname{diam} W^{-1}}$ est comparable avec $|f'(z)|$ pour tout $z \in W^{-1}$, voir aussi le lemme 3.4.3. Cela implique une certaine équivalence entre des conditions sur la dérivée et des conditions exprimées en termes de diamètre de composantes connexes. Par contre, dans le cas contraire, $|f'(z)|$ peut être beaucoup plus petit que $\frac{\operatorname{diam} W}{\operatorname{diam} W^{-1}}$. L'inégalité (3.28) affirme qu'il existe $M > 0$ tel que si le diamètre de W est suffisamment petit alors pour tout $z \in W^{-1}$ on a

$$\operatorname{diam} W^{-1} \leq M \left|f'(z)\right|^{-1} \operatorname{diam} W. \tag{2.17}$$

Le terme de droite peut être beaucoup plus grand que celui de gauche et cela est précisément le motif pour lequel il existe une application rationnelle *ExpShrink* qui ne satisfait pas à RCE.

On développe une technique basée sur les propriétés combinatoires des applications multimodales pour construire un contre-exemple pour la réciproque du théorème 1. C'est un polynôme de degré 3 très proche du deuxième polynôme de Chebyshev, on le dénote par g. Le premier point critique c_1 est envoyé en 1 qui est un point fixe répulsif. Przytycki montre en [11] que tout point critique d'une application rationnelle TCE qui ne s'accumule pas sur d'autres points critiques est CE. On doit alors rendre la deuxième orbite critique récurrente mais il faut aussi qu'elle s'accumule sur c_1. Ce sont exactement aux temps p_i où cette orbite se rapproche de c_1 que sa dérivée $\left|(g^{p_i})'(g(c_2))\right|$ peut être rendue plus petite que 1, pour que c_2 ne soit pas CE. On construit g de telle façon que sur les segments d'orbites qui ne comportent pas de tels moments p_i la croissance de la dérivée soit exponentielle. En utilisant ensuite les outils développés pour démontrer le théorème 1 et une analyse plus fine lorsqu'une préimage est très proche de c_1, on montre que g satisfait à *ExpShrink*.

Pour pouvoir construire l'application g en suivant cette schéma de preuve on doit disposer déjà de plusieurs constantes - notamment les échelles - qui *a priori* dépendent de g. Pour résoudre ce problème on démontre des versions uniformes des résultats de contraction utilisés dans la preuve du théorème 1. Comme g est une limite d'une suite décroissante de familles de polynômes, on peut utiliser ces résultats - corollaire 3.4.2 et proposition 3.4.4 - pour montrer le théorème suivant.

Théorème 2. *Il existe un polynôme ExpShrink qui ne satisfait pas à RCE.*

Les techniques développées pour construire ce contre-exemple produisent aussi une paire d'applications polynomiales 2-modales h et $\tilde{h}$ conjuguées et avec dérivées Schwarzienne négatives telles qu'une seule des deux satisfait à RCE. C'est un contre-exemple pour la conjecture de Świątek, [15].

Théorème 3. *La propriété RCE n'est pas topologiquement invariante dans la classe des applications S-multimodales.*

En changeant le degré du point critique c_1 on obtient des phénomènes différents pour h et $\tilde{h}$ aux moments où la deuxième orbite critique approche c_1. On peut remarquer que cette stratégie ne peut pas être employée pour infirmer l'invariance topologique de RCE complexe, où le degré des points critiques est preservé par conjugaison topologique.

Un exemple de polynône RCE

En utilisant le même type de construction on peut obtenir un polynôme 2-modal semi-hyperbolique tel que c_2 ne soit pas CE. On choisit aussi une dynamique quadratique réelle avec l'orbite critique récurrente et CE et on *colle* les deux dynamiques pour produire un exemple de polynôme RCE qui n'est pas semi-hyperbolique ni CE. On peut réaliser cela grâce à la théorie générale des application multimodales. Elle garantit l'existence d'un polynôme 3-modal de degré 4 qui réalise le triplet des itinéraires critiques (kneading sequences) de l'application 3-modale continue qu'on vient de décrire. Toutes les racines de sa dérivée sont réelles, par conséquent le polynôme obtenu a dérivée Schwarzienne négative. Comme il n'a pas d'orbite attractive, on peut conjuguer ses restrictions aux dynamiques initiales et démontrer les propriétés annoncées.

1.3 Sur la dimension de Hausdorff des attracteurs fractals des applications unimodales

Cette annexe présente un travail qui n'est pas lié à la condition RCE. On y étudie les applications unimodales infiniment renormalisables. Le motivation principale est le résultat suivant obtenu par Graczyk et Kozlovski dans l'article [4].

Théorème. *Il existe une constante universelle $\sigma < 1$ telle que tout attracteur d'une application C^4 unimodale dont le point critique est non-dégénéré a dimension de Hausdorff plus petite que σ ou est une réunion finie d'intervalles fermés et non-dégénérés.*

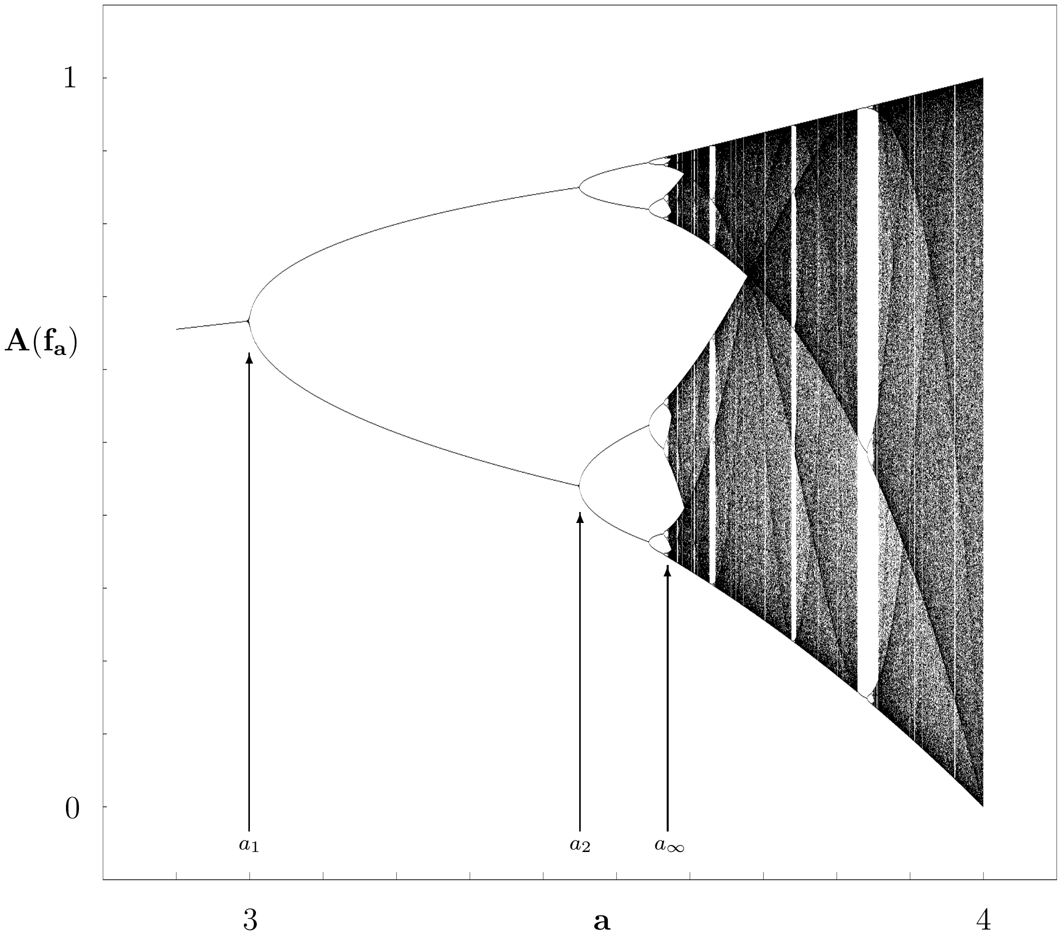

FIG. 1.1 – Les attracteurs dans la famille quadratique

Ce théorème a été généralisé par Li et Shen ([3]) pour les applications multimodales dont les points critiques sont non-plats.

Des phénomènes universels ont été observés au début des années 80 dans plusieurs familles d'applications parmi lesquelles la famille quadratique $f_a(x) = ax(1-x)$ pour $0 < a \leq 4$. Un de ces phénomènes est la convergence exponentielle des bifurcations. Plus précisément, soit a_n le plus petit paramètre pour lequel f_{a_n} a une orbite périodique d'ordre 2^n. On obtient une convergence $a_n \to a_\infty$ mais aussi

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{a_{n+2}-a_{n+1}} = 4,6692\ldots$$

qui est universelle. La figure 1.1 représente les attracteurs dans la famille quadratique.

Les orbites périodiques attractives constituent l'exemple le plus simple d'attracteur. Un *attracteur* est un ensemble invariant sur lequel s'accumulent une partie importante des orbites de la dynamique, minimal pour cette propriété. On peut alors considérer une définition topologique ou métrique de l'attracteur, voir la section A.1. Graczyk, Sands et Świątek montrent que pour les applications C^3 unimodales dont le point critique est non-dégénéré les deux notions d'attracteurs coïncident, voir [5].

L'application f_{a_∞} est appelée *application de Feigenbaum* et c'est un premier exemple d'application infiniment renormalisable. De façon générale, on dit que f est renormalisable si elle possède un intervalle restrictif J sur lequel une itérée f^n avec $n > 1$ est unimodale, voir aussi la définition A.3.1. Une application infiniment renormalisable possède une infinité d'intervalles restrictifs. Les applications S-unimodales ont exactement un attracteur. Si f_a n'est pas infiniment renormalisable son attracteur est soit une orbite périodique soit une réunion finie d'intervalles fermés. Par conséquent, du point de vue de la dimension de Hausdorff de l'attracteur, seules les applications infiniment renormalisables sont intéressantes. On appelle ces attracteurs fractals.

On démontre le théorème suivant qui caractérise les applications quadratiques infiniment renormalisables et leur type de renormalisation en termes d'itinéraires critiques (kneading sequences). On dénote par $\underline{I}_f(J)$ l'itinéraire fini de l'intervalle restrictif J de f et par $\mathcal{R}$ l'opérateur de renormalisation.

Theorem 4. *Si f est une application quadratique et $\underline{K}_f$ son itinéraire critique alors f est infiniment renormalisable si et seulement si $\underline{K}_f$ est une composition infinie d'itinéraires finis maximaux non-triviaux.*

Pour toute suite $(\underline{K}_n)_{n\geq 1}$ d'itinéraires maximaux finis non-décomposables non-trivials il existe une unique application quadratique g telle que

$$\underline{K}_g = \underline{K}_1 * \underline{K}_2 * \dots$$

et

$$\underline{I}_{\mathcal{R}^{i-1}(g)}(J_i) = \underline{K}_i \text{ pour tout } i \geq 1$$

où J_i est l'intervalle restrictif maximal associé à la i-ème renormalisation de g.

La preuve est basée sur l'identification des permutations unimodales non-renormalisables à des itinéraires maximaux finis non-décomposables.

En utilisant ce théorème et la théorie de Milnor et Thurston sur la monotonie de l'itinéraire critique dans la famille quadratique on obtient un algorithme qui pour tout type de renormalisation $(\sigma_1, \sigma_2, \dots)$ produit une suite convergente à $a \in (0, 4]$ tel que f_a est infiniment renormalisable du type $(\sigma_1, \sigma_2, \dots)$. Jusqu'à présent, les seules applications infiniment renormalisables accessibles numériquement étaient celles qui possèdent une renormalisation conjuguée à l'application de Feigenbaum. Ces applications se trouvent aux limites supérieures des *fenêtres* de bifurcations, voir la figure 1.1. On estime numériquement la dimension de Hausdorff de plusieurs types d'attracteurs. La plus grande valeur obtenue est la dimension de l'attracteur de Feigenbaum.

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Chapitre 2

Collet-Eckmann condition for recurrent critical orbits implies uniform hyperbolicity on periodic orbits

Abstract. We prove that the Collet-Eckmann condition for recurrent critical orbits inside the Julia set of a rational map with no parabolic periodic orbits implies uniform hyperbolicity on periodic orbits.

2.1 Introduction

Let f be a rational map, J its Julia set and Crit the set of critical points. We know that J is hyperbolic if and only if the closure of the postcritical set $O(\text{Crit})$ is disjoint from J. If we let some critical points with finite orbit be in the Julia set it becomes sub-hyperbolic. The next step is to allow infinite critical orbits in J as long as they do not accumulate on any critical point and rule out parabolic periodic orbits, the Misiurewicz condition. Semi-Hyperbolicity is even weaker, it requires that critical orbits should not be recurrent, in the absence of parabolic periodic points. Under this assumption Carleson, Jones and Yoccoz show that the Fatou components are John domains for polynomials (see [4]). Every John domain is a Hölder domain. The property that all Fatou components are Hölder is equivalent to uniform hyperbolicity on periodic orbits (see [5] and [10]). Another advance in this direction was done by Graczyk and Smirnov (initiated in [5] and developed in [6]) by allowing recurrent critical points in the Julia set. If all the critical points in J are *Collet-Eckmann* then all Fatou components are Hölder. We propose a new sufficient condition for Uniform Hyperbolicity on Periodic Orbits. It allows for both non-recurrent and Collet-Eckmann critical points in the Julia set, in the absence of parabolic periodic orbits.

It is known that a semi-hyperbolic rational map is not necessarily Collet-Eckmann and vice-versa, see Section 6.1.1 in [10] and Section 1.2 in [7]. Therefore Uniform Hyperbolicity on Periodic Orbits does not imply Collet-Eckmann nor Semi-Hyperbolicity. For unicritical polynomials however, the Collet-Eckmann condition is equivalent to Uniform Hyperbolicity on Periodic Orbits, see [5] and [9]. To our knowledge there is no counterexample to the converse of our main theorem.

A related problem is the invariance of such regularity or growth conditions under topological conjugacy. Semi-Hyperbolicity (by definition) and Uniform Hyperbolicity on Periodic Orbits (see [10]) are topologically invariant but the Collet-Eckmann condition is not topological, except for unicritical polynomials, see Appendix C in [10]. In [2] the expansion along every orbit in the Julia set is estimated with respect to the distance to critical points and to the growth of the derivative along the critical orbits. Therefore the recurrence of critical points and the growth of the derivative along their orbits play an important role in the understanding of the dynamics.

Definition 2.1.1. *We say that* $c \in \text{Crit}$ *satisfies the* Collet-Eckmann condition ($c \in CE$) *if* $|(f^n)'(f(c))| > C\lambda^n$ *for all* $n > 0$ *and some constants* $C > 0, \lambda > 1$*. We say that* f *has the* Collet-Eckmann *property if all critical points in* J *are* CE*.*

Given $c \in \text{Crit}$ *we say that it is* non-recurrent ($c \in NR$) *if* $c \notin \omega(c)$*, where* $\omega(c)$ *is the* ω-limit set*, the set of accumulation points of the orbit* $(f^n(c))_{n>0}$*. We call* f semi-hyperbolic *if all critical points in* J *are* NR *and* f *has no parabolic periodic orbits.*

Our new condition on critical orbits is weaker than Collet-Eckmann and Semi-Hyperbolicity.

Definition 2.1.2. *We say that f satisfies the* Recurrent Collet-Eckmann (RCE) *condition if every critical point in the Julia set is either CE or NR and f has no parabolic periodic orbits.*

Let us remark that a RCE rational map may have critical points in J that are Collet-Eckmann and non-recurrent in the same time. Moreover any critical orbit may accumulate on the other critical points.

Several standard conditions are equivalent to *Uniform Hyperbolicity on Periodic Orbits* (UHP), see [10]. Therefore any of them may be considered as a definition. Among these conditions we recall *Topological Collet-Eckmann condition* (TCE), *Exponential Shrinking of components* ($ExpShrink$) and *Backward Collet-Eckmann condition at some* $z_0 \in \overline{\mathbb{C}}$ ($CE2(z_0)$).

Definition 2.1.3. *We say that f satisfies the Exponential Shrinking of components condition if there are $\lambda > 1,\ r > 0$ such that for all $z \in J,\ n > 0$ and every connected component W of $f^{-n}(B(z,r))$*

$$\operatorname{diam} W < \lambda^{-n}.$$

If not stated explicitly all the distances and derivatives are considered with respect to the spherical metric. We denote by $B_{\mathrm{e}}(z,R)$, $\operatorname{dist}_{\mathrm{e}}(z,W)$ and $\operatorname{diam}_{\mathrm{e}} W$ the Euclidean balls, distances and diameters respectively.

Theorem 1. *The Recurrent Collet-Eckmann condition implies Uniform Hyperbolicity on Periodic orbits for rational maps.*

We show that the Recurrent Collet-Eckmann implies the equivalent condition Exponential Shrinking of components. An intermediary step to *ExpShrink* is to show that arbitrary pullbacks of small balls stay small. This property is called *Backward Stability* in [8].

Definition 2.1.4. *We say that a rational map f has* Backward Stability (BS) *if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in J,\ n \geq 0$ and every connected component W of $f^{-n}(B(z,\delta))$*

$$\operatorname{diam} W < \varepsilon.$$

Inevitably, we borrow some ideas from [4], [5] and [10] to prove our theorem.

In [4], Carleson, Jones and Yoccoz prove the Backward Stability property in the Semi-Hyperbolic case. Then there is $r > 0$ such that the degree of any pullback of a ball $B(x,r)$, with $x \in J$, is bounded, as the critical points are non-recurrent. They use a telescope construction which we sketch to prove the Exponential Shrinking of components condition. There is $n_0 > 0$ such that any pullback of length n_0 of some $B(x,r)$ with $x \in J$ is contracting. So it can be nested inside some $B(y,r)$ with $y \in J$ and inductively build the telescope. A bounded degree distortion argument yields an exponential contraction of pullbacks of $B(x, r)$ of arbitrary length. We should remark that this is done for polynomials, a fact that guarantees that pullbacks of balls are simply connected.

We refine the distortion argument and obtain some specific bound for the distortion in Lemma 2.2.1. Proposition 2.2.2 proves the contraction of long bounded degree pullbacks.

In [5], Graczyk and Smirnov prove the *Backward Collet-Eckmann condition* for some $z_0 \in \overline{\mathbb{C}}$. They also pull back balls around the backward orbit of z_0, considering only univalent pullbacks. Depending on the presence of critical points close to the backward orbit of z_0, there are three types of pullback. Using distortion arguments (the method of shrinking neighborhoods) and the Collet-Eckmann property, they obtain exponential growth of the derivative on the backward orbit.

Among our new methods we may count a precise bound for the distortion in a bounded degree setting (Lemma 2.2.1) and a way to deal with non-simply connected pullbacks, using rings. We build a telescope and show the Exponential Shrinking of components condition. Although the idea of a telescope is not new, its originality consists in combining bounded degree and unbounded degree segments. For a precise description of its construction one may refer to Section 2.4. The general picture is that we modified the techniques of [4], [5] and [10] to make them work together. As in [5], we distinguish three types of pullback. A pullback of the first type does not have a bounded degree so Proposition 2.2.3, which deals with this case, is crucial. Note that the Backward Stability property is needed to apply Proposition 2.2.3 and it is proved in Section 2.3. In the absence of Collet-Eckmann critical points, the Backward Stability property gives a bound for the degree of a pullback, as in the semi-hyperbolic case. This case defines the second type of pullback. The third type has a bound for the degree in the presence of Collet-Eckmann critical points. We obtain exponential contraction along every block of the telescope, with the eventual exception of the last one. Lemma 2.2.1 helps assemble all these estimates to obtain the Exponential Shrinking of components condition.

2.2 Preliminaries

Without loss of generality we may assume that critical orbits $(f^n(c))_{n\geq 1}$ with $c \in \mathrm{Crit} \cap J$ do not contain critical points, needed in the proof of Proposition 2.2.3. Indeed, suppose some critical orbit contains a critical point inside the Julia set. Then we consider the iterate of f that connects the critical points as one iterate of the dynamics. The critical points that are on the same orbit collapse into a critical "block" for the new local dynamics. As there are only finitely many such situations, our procedure does not affect global compactness properties of the dynamics. The multiplicity of the critical block is the product of multiplicities of critical points involved. This is a standard construction, see for example [5] or [10].

Notation. For $B \subseteq \overline{\mathbb{C}}$ connected we write B^{-n} or $f^{-n}(B)$ for some connected component of $f^{-n}(B)$, $0 \leq n$. When $z \in B$ and some backward orbit $z_n \in f^{-n}(z)$ are fixed, B^{-n} is the connected component of $f^{-n}(B)$ that contains z_n.

Let us recall that we denote by $B_e(z, R)$, $\mathrm{dist}_e(z, W)$ and $\mathrm{diam}_e W$ the Euclidean balls, distances and diameters respectively. We also recall some classical properties of the spherical metric and of the modulus of an annulus (or *ring* or *doubly connected region*). The spherical metric $d\sigma$ satisfies

$$d\sigma = \frac{2|dz|}{1+|z|^2}$$

so on $\overline{B_e(0,1)}$ we have

$$|dz| \leq d\sigma \leq 2|dz|.$$

Thus for every $W \subseteq B_e(0,1)$

$$\operatorname{diam_e} W \leq \operatorname{diam} W \leq 2 \operatorname{diam_e} W. \tag{2.1}$$

Moreover for $0 \neq z \in B_e(0,1)$ and $0 < \alpha < 1$ we have

$$\frac{\operatorname{dist}(0,z)}{\operatorname{dist}(0,\alpha z)} < \frac{1}{\alpha} = \frac{\operatorname{dist_e}(0,z)}{\operatorname{dist_e}(0,\alpha z)}.$$

Let $A(r,R) = B(0,R) \setminus \overline{B(0,r)}$ and $A_e(r,R) = B_e(0,R) \setminus \overline{B_e(0,r)}$ for $0 < r < R$. If $A(r,R) = A_e(r',R')$ with $R' < 1$, by the previous inequality we obtain

$$\frac{R}{r} < \frac{R'}{r'}.$$

As $\operatorname{mod} A_e(r',R') = \frac{\log(R'/r')}{2\pi}$ (modulus of $A_e(r',R')$),

$$\operatorname{mod} A(r,R) > \frac{\log(R/r)}{2\pi}, \; 0 < r < R < 1. \tag{2.2}$$

Let $g : A \to A'$ be a conformal proper map of degree d and A, A' two doubly connected regions. Then

$$\operatorname{mod} A = \frac{1}{d} \operatorname{mod} A'. \tag{2.3}$$

In particular, the modulus is a conformal invariant.

Let A be an annulus and B_1, B_2 the two connected components of $\overline{\mathbb{C}} \setminus A$. If $A_1, \ldots, A_k \subseteq A$ are disjoint annuli that separate B_1 from B_2 then

$$\operatorname{mod} A \geq \sum_{i=1}^{n} \operatorname{mod} A_i. \tag{2.4}$$

For any connected open $U \subseteq \overline{\mathbb{C}}$, every connected component of $\overline{\mathbb{C}} \setminus \overline{U}$ is simply connected. If $\operatorname{diam} U \leq 1$ then there is only one component of $\overline{\mathbb{C}} \setminus \overline{U}$ with diameter greater than 1. Denote it by $\operatorname{ext}(U)$. Let $\operatorname{fill}(U) = \overline{\mathbb{C}} \setminus \overline{\operatorname{ext}(U)}$. It is a simply connected open with $\operatorname{diam} U = \operatorname{diam}(\operatorname{fill}(U))$ and $\operatorname{diam_e} U = \operatorname{diam_e}(\operatorname{fill}(U))$.

Let us also recall the Teichmüller extremal problem - Theorem 4-7 and relation (4-21) in [1].

Proposition 2.2.1. *Let $T > 0$,*

$$\Lambda(T) = \operatorname{mod}\left(\overline{\mathbb{C}} \setminus ([-1,0] \cup [T,\infty])\right),$$

and A some annulus that separates $\{-1,0\}$ from $\{\omega_0,\infty\}$ with $|\omega_0| = T$. Then $\operatorname{mod} A \leq \Lambda(T)$*,*

$$16T \leq \exp\left(2\pi\Lambda(T)\right) \leq 16\,(T+1)\,. \tag{2.5}$$

and

$$\Lambda(T)\Lambda(T^{-1}) = \frac{1}{4}.$$

Therefore

$$\lim_{T\to\infty} \Lambda(T) = \infty \text{ and } \lim_{T\to 0} \Lambda(T) = 0. \tag{2.6}$$

We are ready to prove our first lemma. It provides a way to control distortion in terms of diameters of a bounded degree pullback.

Lemma 2.2.1. *Let g be a rational map, $z \in \overline{\mathbb{C}}$ and $0 < r < R < 1$. Let $W = B(z,R)^{-1}$ and $W' = B(z,r)^{-1}$ with $W' \subseteq W$ and* $\operatorname{diam} W < 1$*. If* $\deg_W(g) \leq \mu$ *then*

$$\frac{\operatorname{diam} W'}{\operatorname{diam} W} < 64\left(\frac{r}{R}\right)^{\frac{1}{\mu}}.$$

Proof. Let $A_1,\ldots,A_m$ be disjoint concentric annuli inside $A(z,r,R)$ that avoid critical values of $g|_W$ and such that $\cup_{i=1}^m \overline{A_i} = \overline{A(z,r,R)}$. In this setting

$$\sum_{i=1}^{m} \operatorname{mod} A_i = \operatorname{mod} dA(z,r,R). \tag{2.7}$$

It is easy to check that $g : A_i^{-1} \to A_i$ is a finite proper cover for all $i = 1,\ldots,m$ and $A_i^{-1} \subseteq W$ a connected component of $g^{-1}(A_i)$. Then A_i^{-1} is a doubly connected region and by equality (2.3)

$$\operatorname{mod} A_i^{-1} = \frac{1}{\deg_{A_i^{-1}}(g)} \operatorname{mod} A_i \geq \frac{1}{\mu} \operatorname{mod} A_i. \tag{2.8}$$

For every $w \in \partial W$ and every A_i there exists some $A_i^{-1} \subseteq W$ that separates W' from w. Suppose there is not. Then W' could be joined to w by a path $\gamma \subseteq W \setminus g^{-1}(A_i)$, as W connected. Then $g(\gamma)$ joins $B(z,r)$ to $\partial B(z,R)$ and $g(\gamma) \cap A_i = \emptyset$, a contradiction.

We may suppose $0 \in W$ so $W \subseteq B(0,1)$. Let $U = \operatorname{fill}(W)$ and $U' = \operatorname{fill}(W')$. Let $a \in \partial U'$ and $w \in \partial U$ with $|a-w| = \operatorname{dist}_{\mathrm{e}}(\partial U', \partial U)$. Let also $b \in \partial U'$ be such that $|b-a| = \sup_{x\in\partial U'} |x-a|$. The linear map $h : z \to \frac{z-a}{a-b}$ sends a to 0, b to -1 and w to $\omega_0 = \frac{w-a}{a-b}$.

Let A' be the annulus $U \setminus \overline{U'}$. For all $1 \leq i \leq m$ there exists an A_i^{-1} which separates W' from w. Denote it A_i'. As $\overline{W'} \cap A_i' = \emptyset$, A_i' separates U' from w, inside W. Therefore $A_i' \subseteq A'$. Using inequalities (2.4), (2.8), (2.7) and (2.2), we may immediately compute

$$\operatorname{mod} A' \geq \sum_{i=1}^{m} \operatorname{mod} A_i' \geq \frac{1}{\mu} \operatorname{mod} A(z, r, R) > \frac{\log(R/r)}{2\pi\mu}.$$

Let us remark that $h(A')$ satisfies the hypothesis of the Teichmüller extremal problem. Combining the previous inequality with inequality (2.5), we obtain

$$\left(\frac{R}{r}\right)^{\frac{1}{\mu}} < \exp(2\pi \operatorname{mod} A') \leq \exp(2\pi\Lambda(|\omega_0|)) \leq 16(|\omega_0| + 1),$$

as $\operatorname{mod} A' = \operatorname{mod} h(A')$. Therefore

$$\left(\frac{R}{r}\right)^{\frac{1}{\mu}} < 32 \frac{\operatorname{diam}_e U}{\operatorname{diam}_e U'} = 32 \frac{\operatorname{diam}_e W}{\operatorname{diam}_e W'}, \tag{2.9}$$

as

$$|\omega_0| + 1 = \frac{|w-a| + |a-b|}{|a-b|} \leq 2 \frac{\operatorname{diam}_e W}{2|a-b|} \leq 2 \frac{\operatorname{diam}_e W}{\operatorname{diam}_e W'}.$$

Combining inequalities (2.1) and (2.9) we may conclude that

$$\left(\frac{R}{r}\right)^{\frac{1}{\mu}} < 64 \frac{\operatorname{diam} W}{\operatorname{diam} W'}.$$

□

Lemma 2.2.2. *Let g be a rational map, $z \in \overline{\mathbb{C}}$ and $R > 0$. Let $W = B(z, R)^{-1}$ and $\mu = \deg_W(g)$. If g has no critical points on ∂W then the number of components of $\overline{\mathbb{C}} \setminus \overline{W}$ satisfies*

$$\# \operatorname{Comp}\left(\overline{\mathbb{C}} \setminus \overline{W}\right) \leq \mu.$$

Proof. It is easy to check that ∂W is a disjoint union of smooth closed paths. Moreover, if γ is such a path then

$$g(\gamma) = \partial B(z, R). \tag{2.10}$$

Let $D_1, \ldots, D_s$ be the connected components of $\overline{\mathbb{C}} \setminus \overline{W}$. As $\cup_{i=1}^{s} \partial D_i = \partial W$ and $(\partial D_i)_{1 \leq i \leq s}$ are disjoint

$$s \leq \# \operatorname{Comp} \partial W.$$

For some $x \in \partial B(z, R)$ consider $\{x_1, \ldots, x_k\} = g^{-1}(x) \cap \partial W$. On a neighborhood of x on which g^{-1} can be defined we see that $k = \mu$. By equality (2.10), any component of ∂W contains at least one x_i with $1 \leq i \leq \mu$. We conclude that

$$\#\operatorname{Comp}\left(\overline{\mathbb{C}} \setminus \overline{W}\right) = s \leq \#\operatorname{Comp} \partial W \leq \mu.$$

□

If A is an annulus and C_1, C_2 are the connected components of $\overline{\mathbb{C}} \setminus \overline{A}$ then we denote

$$\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A\right) = \operatorname{dist}\left(C_1, C_2\right).$$

Lemma 2.2.3. *Let $A \subseteq \overline{\mathbb{C}}$ be an annulus and C_1, C_2 the components of $\overline{\mathbb{C}} \setminus \overline{A}$, with $\operatorname{diam} C_1 \leq \operatorname{diam} C_2$. For each $\alpha > 0$ there exists $\delta_\alpha > 0$ that depends only on α such that if $\operatorname{mod} A \geq \alpha$ then*

$$\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A\right) \geq \delta_\alpha \operatorname{diam} C_1.$$

Proof. Let $a \in \partial C_1$ and $w \in \partial C_2$ with $\operatorname{dist}(a, w) = \operatorname{dist}\left(\overline{\mathbb{C}} \setminus A\right) < 1$. By rotation we may assume that $a = 0$. If $C_1 \subseteq B_e(0,1)$ let $b \in \partial C_1$ with $\operatorname{dist}(0, b) = \max_{x \in \partial C_1} \operatorname{dist}(0, x)$. If $C_1 \nsubseteq B_e(0,1)$ let $b \in C_1 \cap \partial B_e(0,1)$. In either case we may assume $b \in [-1, 0)$, again by rotation. The linear map $h : z \to -\frac{z}{b}$ sends b to -1 and w to ω_0. The Teichmüller extremal problem, combined with equality (2.6), gives

$$\alpha \leq \Lambda(|\omega_0|) \text{ and } \lim_{\varepsilon \to 0} \Lambda(\varepsilon) = 0.$$

Thus, there exists $\delta'_\alpha > 0$ a lower bound for $|\omega_0|$. So, by inequality (2.1)

$$\delta'_\alpha \leq |\omega_0| = \frac{|w|}{|b|} \leq \frac{\operatorname{dist}(0, w)}{|b|}.$$

If $|b| < 1$ then $C_1 \subseteq B_e(0,1)$ and $|b| \geq \operatorname{diam}_e(C_1)/2$. Moreover, by inequality (2.1) we have $|b| \geq \operatorname{diam}(C_1)/4$. If $|b| = 1$ then $|b| \geq \operatorname{diam}(C_1)/\pi$ as $\operatorname{diam} \overline{\mathbb{C}} = \pi$. In either case

$$|b| \geq \frac{\operatorname{diam}(C_1)}{4}.$$

We choose $\delta_\alpha = \delta'_\alpha/4$. As $\operatorname{dist}(0, w) = \operatorname{dist}\left(\overline{\mathbb{C}} \setminus A\right)$, we conclude that

$$\delta_\alpha \operatorname{diam}(C_1) = \frac{\delta'_\alpha}{4} \operatorname{diam}(C_1) \leq \operatorname{dist}\left(\overline{\mathbb{C}} \setminus A\right).$$

□

Let $I \subseteq \overline{\mathbb{C}}$ be an infinite set. Then for every $m > 0$, there is $d'_m > 0$ such that I cannot be covered by m balls of radius d'_m. Moreover, if $B_1, \ldots, B_m \subseteq \overline{\mathbb{C}}$ are balls of radius $d_m(I) = d'_m/2$ there is $x \in I$ such that

$$\left(\bigcup_{i=1}^{m} B_i\right) \cap B\left(x, d_m(I)\right) = \emptyset. \tag{2.11}$$

We use the construction developed in the proof of Lemma 2.2.1 to obtain the next corollary.

Corollary 2.2.1. *Let f a rational map, $\beta > 1$ and $\mu \geq 1$ be fixed. For all $z \in \overline{\mathbb{C}}$, $n \geq 0$ and $0 < R < \min(1, \operatorname{diam} J_f/2)$, let $W' = B(z,R)^{-n}$ and $W = B(z, \beta R)^{-n}$ with $W' \subseteq W$. There is $\Delta_{\beta,\mu} > 0$ that depends only on β, μ and f such that if $\deg_W(f^n) \leq \mu$ then at least one of the following conditions is satisfied*

1. *There exists an annulus $A \subseteq W \setminus \overline{W'}$ with $\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A\right) \geq \Delta_{\beta,\mu} \operatorname{diam} W'$. Moreover, $f^n(A) \subseteq B(z, \beta R)$ separates z from $\partial B(z, \beta R)$.*
2. *There is $z_0 \in J_f \cap W$ such that $B(z_0, \Delta_{\beta,\mu}) \subseteq W$.*

In particular, if $\operatorname{diam} W < 1$ then the first condition is satisfied.

Proof. By inequality (2.2)

$$\operatorname{mod} A(z, R, \beta R) > \frac{\log \beta}{2\pi}$$

and this is the only way β enters in the following estimates. So we may decrease β to some β' and still have

$$\operatorname{mod} A(z, R, \beta' R) > \frac{\log \beta}{2\pi}.$$

So, without loss of generality, we may assume that f^n has no critical values on $\partial B(z, \beta R)$. By Lemma 2.2.2, the number of components of $\overline{\mathbb{C}} \setminus \overline{W}$ is bounded by μ. As $\deg_W(f^n) \leq \mu$ there are at most $\mu - 1$ critical points of f^n in W. So we may decompose, as in Lemma 2.2.1, the annulus $A(z, R, \beta R)$ into $A_1, \ldots, A_m$ concentric and disjoint annuli, with $m \leq \mu$. By equality (2.7) there is A_{i_0} with

$$\operatorname{mod} A_{i_0} > \frac{\log \beta}{2\pi\mu}.$$

Moreover, using inequality (2.8), for all $A'_{i_0} \in f^{-n}(A_{i_0})$ with $A'_{i_0} \subseteq W$

$$\operatorname{mod} A'_{i_0} > \frac{\log \beta}{2\pi\mu^2}. \tag{2.12}$$

Let $d = d_\mu(J_f)$ be the positive number defined by the equality (2.11).

Suppose that

$$\operatorname{diam} D < d, \ \forall D \in \operatorname{Comp}\left(\overline{\mathbb{C}} \setminus \overline{W}\right).$$

Then every such component D is contained in a ball of radius d. There exists $z_0 \in J_f$ such that $B(z_0, d) \subseteq W$. It is enough to choose $\Delta_{\beta,\mu} \leq d$ to satisfy the second condition of the corollary.

Suppose now that there is $D_0 \in \operatorname{Comp}\left(\overline{\mathbb{C}} \setminus \overline{W}\right)$ with

$$\operatorname{diam} D_0 \geq d.$$

Note that this is true if $\operatorname{diam} W < 1$. Proceeding as in Lemma 2.2.1 we find an annulus $A'_{i_0} \in f^{-n}(A_{i_0})$ with $A'_{i_0} \subseteq W$ that separates W' and D_0. We may apply Lemma 2.2.3 and obtain $\delta = \delta_{\beta,\mu}$ that depends only on $\operatorname{mod} A'_{i_0} > \frac{\log \beta}{2\pi\mu^2}$ with the following property

$$\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A'_{i_0}\right) \geq \delta_{\beta,\mu} \min(\operatorname{diam} W', \operatorname{diam} D_0).$$

If $\operatorname{diam} W' \leq \operatorname{diam} D_0$, choosing $\Delta_{\beta,\mu} \leq \delta_{\beta,\mu}$ satisfies the first condition of the Corollary. If $\operatorname{diam} W' > \operatorname{diam} D_0 \geq d$, we may choose any $\Delta_{\beta,\mu} \leq \delta_{\beta,\mu} \frac{d}{\pi} < \delta_{\beta,\mu}$ as the previous inequality becomes

$$\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A'_{i_0}\right) \geq \delta_{\beta,\mu} d \geq \Delta_{\beta,\mu} \pi \geq \Delta_{\beta,\mu} \operatorname{diam} W'.$$

Finally, independently of the existence of D_0, if we choose

$$\Delta_{\beta,\mu} = \delta_{\beta,\mu} \frac{d}{\pi} < d,$$

then at least one of the two conditions of the conclusion holds. $\square$

Our first goal is to prove contraction of a long, bounded degree pullback. This is true only in a neighborhood Ω of the Julia set J. We define Ω with the following properties

1. $f^{-1}(\Omega) \subseteq \Omega$,
2. $(\Omega \cap \operatorname{Crit}) \setminus J = \emptyset$ and
3. $\overline{\Omega}$ does not intersect attracting periodic orbits.

Let us fix a *RCE* rational map f. All the following statements apply to f. In the absence of parabolic periodic orbits, the critical orbits in the Fatou set F do not accumulate on J. Indeed, any critical point $c \in \operatorname{Crit} \cap F$ is sent to a periodic Fatou component which is not parabolic. By Sullivan's classification of Fatou components, the orbit of c stays away from J. Let $O(\operatorname{Crit} \cap F) = \{f^n(c) | n \geq 0, c \in \operatorname{Crit} \cap F\}$. Then

$$d_1 = \operatorname{dist}\left(J, O(\operatorname{Crit} \cap F)\right) > 0.$$

There is an open neighborhood V of the attractive periodic orbits such that $\overline{V} \subseteq F$ and $\overline{f(V)} \subseteq V$. So

$$d_2 = \operatorname{dist}\left(J, V\right) > 0.$$

Let $0 < \eta < \min(d_1, d_2)$ and U the η-neighborhood of J. Then

$$U \cap \left(O(\operatorname{Crit} \cap F) \cup V\right) = \emptyset.$$

We define

$$\Omega = \bigcup_{0 \leq n} f^{-n}(U).$$

Then $\Omega \cap O(\operatorname{Crit}) = O(\operatorname{Crit} \cap J)$, $\Omega \cap V = \emptyset$ and $f^{-1}(\Omega) \subseteq \Omega$. We prove a variant of Mañé's lemma. Loosely speaking, the diameter of bounded degree pullbacks of small balls in Ω stays small. It is stated for a fixed *RCE* rational map but it applies to all rational maps with no parabolic orbits.

Lemma 2.2.4. *Let $\varepsilon \in (0,1), \beta > 1$ and $\mu \geq 1$ be fixed. For all $z \in \Omega$, $R > 0$ such that $B(z, \beta R) \subseteq \Omega$ and $n \geq 0$, let $W' = B(z, R)^{-n}$ and $W = B(z, \beta R)^{-n}$ with $W' \subseteq W$. There is $\delta_{\varepsilon,\beta,\mu} > 0$ that depends only on ε, β and μ such that if $\beta R \leq \delta_{\varepsilon,\beta,\mu}$ and $\deg_W(f^n) \leq \mu$ then*

$$\operatorname{diam} W' < \varepsilon.$$

Proof. Suppose the statement does not hold. Then there exist sequences $(z_i)_{0<i} \subseteq \Omega$, $(R_i)_{0<i}$ decreasing to 0 and $(n_i)_{0<i} \subseteq \mathbb{N}$ increasing such that $\deg_{W_{n_i}}(f^{n_i}) \leq \mu$ and

$$\operatorname{diam} W'_{n_i} \geq \varepsilon.$$

We apply Corollary 2.2.1 and we get $\Delta = \Delta_{\beta,\mu}/4 > 0$ such that for all $i > 0$ there is $a_i \in W_{n_i}$ with

$$B(a_i, \varepsilon\Delta) \subseteq W_{n_i}. \tag{2.13}$$

Indeed, suppose that only the first condition of the conclusion of Corollary 2.2.1 is satisfied. As $\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A_i\right) > \Delta_{\beta,\mu} \operatorname{diam} W'_{n_i}$, we may choose $a_i \in A_i \subseteq W_{n_i}$ such that

$$\operatorname{dist}\left(a_i, \overline{\mathbb{C}} \setminus A_i\right) > \frac{\Delta_{\beta,\mu}}{4} \operatorname{diam} W'_{n_i} \geq \varepsilon\Delta. \tag{2.14}$$

Let a be an accumulation point of $(a_i)_{0<i}$. By (2.13) there exists a subsequence $(n_j)_{0<j}$ of $(n_i)_{0<i}$ such that

$$B\left(a, \frac{\varepsilon}{2}\Delta\right) \subseteq W_{n_j}, \forall j > 0.$$

So $f^n\left(B\left(a, \frac{\varepsilon}{2}\Delta\right)\right) \subseteq \Omega$ for any $n \geq 0$ and

$$\operatorname{diam} f^{n_j}\left(B\left(a, \frac{\varepsilon}{2}\Delta\right)\right) \to 0 \text{ as } j \to \infty. \tag{2.15}$$

$B\left(a, \frac{\varepsilon}{2}\Delta\right)$ cannot intersect J because of the "eventually onto" property of the Julia set. It cannot be in the basin of attraction of a periodic orbit as all of its images stay in Ω. As there are no parabolic components, some image has to land inside a Siegel disk or inside a Herman ring. But this contradicts (2.15). $\square$

Let us state inequality (2.14) in a more general form.

Lemma 2.2.5. *Let A be an annulus with $\operatorname{diam} A < 1$ and $\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A\right) \geq 4\alpha$. Then there is an annulus $A' \subseteq A$ with $\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A'\right) \geq 2\alpha$ and*

$$\operatorname{dist}\left(A', \overline{\mathbb{C}} \setminus A\right) \geq \alpha.$$

Moreover, A' separates the two connected components of $\overline{\mathbb{C}} \setminus A$.

Proof. For any set $E \subseteq \overline{\mathbb{C}}$, let us define the α-neighborhood of E by

$$E_{+\alpha} = B\,(E, \alpha) = \{x \in \overline{\mathbb{C}} |\, \mathrm{dist}\,(x, E) < \alpha\}.$$

Analogously, we define the α-cut of E by

$$E_{-\alpha} = E \setminus \overline{(\overline{\mathbb{C}} \setminus E)_{+\alpha}} = \{x \in \overline{\mathbb{C}} |\, \mathrm{dist}\left(x, \overline{\mathbb{C}} \setminus E\right) > \alpha\}.$$

The α-neighborhood of E and the α-cut of E are open sets. If E is connected then $E_{+\alpha}$ is connected. Moreover, if E is simply connected then every connected component of $E_{-\alpha}$ is simply connected.

Let $U = \mathrm{fill}\,(A)$ and $U' = U \setminus \overline{A}$. Then $U' \subseteq U$ and U, U' are simply connected open sets. Let $V' = \mathrm{fill}\left(U'_{+\alpha}\right)$ which is a simply connected open set. Moreover, as $\mathrm{dist}\left(U', \overline{\mathbb{C}} \setminus U\right) \geq 4\alpha$

$$\mathrm{dist}\left(V', \overline{\mathbb{C}} \setminus U\right) \geq 3\alpha.$$

Let also $V = U_{-\alpha}$. Then there is at most one connected component V of $U_{-\alpha}$ that intersects V'. This open V is simply connected. It is also easy to check that $\overline{V'} \subseteq V$ and

$$\mathrm{dist}\left(V', \overline{\mathbb{C}} \setminus V\right) \geq 2\alpha.$$

Finally, we may set $A' = V \setminus \overline{V'}$. Then $\mathrm{dist}\left(\overline{\mathbb{C}} \setminus A'\right) \geq 2\alpha$ and

$$\mathrm{dist}\left(A', \overline{\mathbb{C}} \setminus A\right) \geq \alpha.$$

There could be no path disjoint from A' that connects ∂A. $\square$

Now we have the tools needed to prove our first contraction result.

Proposition 2.2.2. *Let $\beta > 1$, $\mu \geq 1$. There exists $\delta = \delta_{\beta,\mu} > 0$ such that for all $0 < r < R < \frac{\delta}{\beta}$ there exists $N = N_{\beta,\mu,r,R} > 0$ such that the following statement holds. For all $z \in J$ with $B(z, \beta R) \subseteq \Omega$ and for all $n \geq N$, let $W' = B(z, R)^{-n}$ and $W = B(z, \beta R)^{-n}$ with $W' \subseteq W$. If $\deg_W(f^n) \leq \mu$ then*

$$\mathrm{diam}\, W' < r.$$

Proof. There are finitely many Herman rings in the Fatou set. Let us denote them by $H_1, ..., H_m$. Let $h = \min\left\{\mathrm{diam}\,\mathrm{Comp}\left(\overline{\mathbb{C}} \setminus H_i\right) | i = 1, \ldots, m\right\}$. Let also

$$0 < \delta < \min\left\{\delta_{1,\sqrt{\beta},\mu}, \frac{\mathrm{diam}\, J}{2}, \frac{h}{2}, \frac{1}{2}\right\},$$

where $\delta_{1,\sqrt{\beta},\mu}$ is provided by Lemma 2.2.4.

Let $W'' = B(z, \sqrt{\beta}R)^{-n}$ with $W' \subseteq W'' \subseteq W$. Thus as $\beta R < \delta$ and $\deg_W(f^n) \leq \mu$

$$\mathrm{diam}\, W' \leq \mathrm{diam}\, W'' < 1.$$

Now suppose that the conclusion of the proposition does not hold for the chosen δ. Then there are sequences $(n_i)_{0<i}$ increasing and $(z_i)_{0<i} \subseteq J$ such that for all $i > 0$, W'_{n_i} and W_{n_i} satisfy the hypothesis but

$$\operatorname{diam} W'_{n_i} \geq r.$$

Let $\Delta = \Delta_{\sqrt{\beta},\mu} > 0$ and the annulus $A_i \subseteq W''_{n_i} \setminus \overline{W'_{n_i}}$ be provided by Corollary 2.2.1. Then $\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A_i\right) \geq \Delta r$ and we may apply Lemma 2.2.5 for $\alpha = \Delta \frac{r}{4}$ and obtain $A'_i \subseteq A_i$ with $\operatorname{dist}\left(\overline{\mathbb{C}} \setminus A'_i\right) \geq 2\alpha$ and

$$B(x, \alpha) \subseteq A_i, \forall x \in A'_i.$$

At least one of the following conditions holds for infinitely many $i > 0$

1. $A'_i \cap J \neq \emptyset$,
2. $A'_i \cap J = \emptyset$.

So, taking a subsequence, we may assume at least one condition holds for all $i > 0$. That could not be condition 1 as the compactness argument used in Lemma 2.2.4 would yield $a \in J$ such that for i sufficiently big

$$\operatorname{diam} f^{n_i}\left(B\left(a, \frac{\alpha}{2}\right)\right) \leq 2\beta R < 2\delta < \operatorname{diam} J.$$

Thus no image of $B(a, \alpha)$ could contain J. This contradicts the "eventually onto" property of the Julia set.

The only possibility is that the second condition holds for all i, so $A'_i \subseteq F$. We apply Lemma 2.2.5 to check that A'_i contains some ball $B(a_i, \alpha/2)$. If a is an accumulation point of $(a_i)_{0<i}$ then, taking a subsequence,

$$B\left(a, \frac{\alpha}{4}\right) \subseteq A'_i \subseteq \Omega \text{ so } f^{n_i}(a) \in \Omega.$$

Thus a cannot be contained in the basin of attraction of some periodic orbit. There are no parabolic components. So a is sent to a rotation domain P, a Siegel disk or a Hermann ring. We fix some big i and omit it from notations.

Recall that we assumed the first condition of the conclusion of Corollary 2.2.1, as $\operatorname{diam} W' < \operatorname{diam} W'' < 1$. Thus $f^n(A)$ separates z from $\partial B(z, \beta R)$. We show that $f^n(A')$ has the same property. Suppose this is false. Then there is a path γ that joins the two components of $\partial f^n(A)$ which does not intersect $f^n(A')$. Then there is some pullback of γ that connects ∂A and does not intersect A'. This contradicts Lemma 2.2.5. We may conclude that

$$z \in \operatorname{fill}\left(f^n(A')\right).$$

Let us also recall that $z \in J$, $\operatorname{diam} f^n(A') < 2\delta < \operatorname{diam} J$ and $f^n(A') \subseteq f^{n-k_0}(P)$ for some $0 \leq k_0 < n$, where P is a rotation domain. Thus $f^n(A')$ separates the Julia set and $H = f^{n-k_0}(P)$ is a Hermann ring. But this contradicts

$$\operatorname{diam} f^n(A') < 2\delta < h.$$

$\square$

Let us recall some general distortion properties of rational maps.

Distortion. This is a reformulation of the classical Koebe distortion lemma in the complex case, see for example Lemma 2.5 in [2]. For all $D > 1$ there exists $\varepsilon > 0$ such that if the connected open W satisfies

$$\operatorname{diam} W \leq \varepsilon \operatorname{dist}(W, \mathrm{Crit}) \tag{2.16}$$

then the distortion of f in $\overline{W}$ is bounded by D, that is

$$\sup_{x,y\in\overline{W}} \left|\frac{f'(x)}{f'(y)}\right| \leq D.$$

Pullback. Once a small $r > 0$ is fixed, there exists $M \geq 1$ such that for any connected open W with $\operatorname{diam} W \leq r$ and for all $z \in \overline{W^{-1}}$

$$\operatorname{diam} W^{-1} \leq M|f'(z)|^{-1} \operatorname{diam} W. \tag{2.17}$$

We shall use this estimate for W^{-1} close to Crit.

The second goal of this section is to obtain contraction when there is no bound for the degree of the pullback. This can be done only in the presence of Collet-Eckmann critical points. In the next section we show that if the pullback does not meet CE points then the degree is bounded.

Proposition 2.2.3. *For any $1 < \lambda_0 < \lambda$ and $\theta < 1$ there exists $\delta = \delta_{\lambda_0,\theta} > 0$ such that for all $N > 0$ and for any ball $B = B(z, R) \subseteq \Omega$ with* $\operatorname{diam} B^{-n} \leq \delta$ *for all $0 \leq n \leq N$, if $\overline{B^{-N-1}} \cap CE \neq \emptyset$ and $\cup_{i=1}^{N} \overline{B^{-i}} \cap \mathrm{Crit} \neq \emptyset$ then*

$$\operatorname{diam} B^{-N} < \theta R \lambda_0^{-N}. \tag{2.18}$$

Note that the hypothesis does not involve any condition on the length N of the orbit. Instead, there is additional information on critical points. This situation occurs naturally in our construction.

Proof. Let us fix $z \in \overline{\mathbb{C}}$ and $D \in (1, \lambda/\lambda_0)$. Let $\varepsilon > 0$ be provided by inequality (2.16). Let also $r > 0$ be small and $M \geq 1$ defined by the inequality (2.17). Let $l \geq 1$ such that

$$2M^{j/l} C^{-1} D^j \lambda^{-j} \leq \theta \lambda_0^{-j} \text{ for all } j \geq l. \tag{2.19}$$

Let us recall that no critical point is sent to another critical point. Then there is $r_1 < r$ such that for any $c \in \mathrm{Crit}$, $B(c, 2r_1)^{-k}$ satisfies the inequality (2.16) for all $0 < k \leq l$. Let us define

$$\delta = \varepsilon r_1.$$

By hypothesis, $\operatorname{diam} B(z,R)^{-n} \leq \varepsilon r_1 < r$ for all $0 \leq n \leq N$.

Let $c_0 \in \overline{B(z,R)^{-N-1}} \cap CE$. Denote $x_k = f^{N+1-k}(c_0) \in W_k = B^{-k}$. By hypothesis, there exists $0 < k' \leq N$ with $\overline{W_{k'}} \cap \mathrm{Crit} \neq \emptyset$. Let $0 < k_0 < k_1 < \ldots < k_t \leq N$ be all the integers $0 < k \leq N$ such that W_k does not satisfy the inequality (2.16). As $\varepsilon r_1 \geq \operatorname{diam} W_{k_i}$, we have $r_1 > \operatorname{dist}(W_{k_i}, \mathrm{Crit})$ for all $0 \leq i \leq t$. Then for all $0 \leq i \leq t$ there is $c_i \in \mathrm{Crit}$ such that $W_{k_i} \subseteq B(c_i, 2r_1)$. By the definition of r_1

$$k_{i+1} - k_i > l, \ \forall\, 0 \leq i \leq t, \tag{2.20}$$

where $k_{t+1} = N+1$. In fact W_{N+1} cannot satisfy inequality (2.16), as $c_0 \in \overline{W_{N+1}}$.

We may begin estimates. For all $0 < j \leq N$ with $j \neq k_i$ for all $0 \leq i \leq t$, W_j satisfies the inequality (2.16) so the distortion on W_j is bounded by D. Thus

$$\operatorname{diam} W_j \leq D|f'(x_j)|^{-1} \operatorname{diam} W_{j-1}. \tag{2.21}$$

If $j = k_i$ for some $0 \leq i \leq t$ we use inequality (2.17) to obtain

$$\operatorname{diam} W_j \leq M|f'(x_j)|^{-1} \operatorname{diam} W_{j-1}. \tag{2.22}$$

Let us recall that $x_N = f(c_0)$ with $c_0 \in CE$ and that $W_0 = B$ so $\operatorname{diam} W_0 = 2R$. Inequality (2.20) yields $t+1 < N/l$. Multiplying all the relations (2.21) and (2.22) for all $0 < j \leq N$ we obtain

$$\begin{aligned} \operatorname{diam} W_N &\leq M^{t+1} D^{N-t-1} |(f^N)'(x_N)|^{-1} \operatorname{diam} W_0 \\ &< 2M^{N/l} D^N C^{-1} \lambda^{-N} R \\ &\leq \theta R \lambda_0^{-N}. \end{aligned}$$

The last inequality is inequality (2.19). □

2.3 Backward Stability

As we have already announced, an important intermediary step to NUH is BS, see page 19. It is a generalization of Lemma 2.2.4. Basically, the diameter of any pullback of a small ball is small. The first condition in the hypothesis of Proposition 2.2.3 will be satisfied automatically, thanks to BS. So the only hypothesis of Proposition 2.2.3 will be the presence of critical points in some pullbacks.

All the following constructions take place inside Ω, the neighborhood of J constructed in the previous section. Therefore critical point in the Fatou set do not play any role in the sequel. For transparency we introduce additional notation $\mathrm{Crit}_J = \mathrm{Crit} \cap J$, $NR_J = NR \cap J$ and $CE_J = \mathrm{Crit} \setminus NR$. So $NR_J, CE_J \subseteq J$ form a partition of Crit_J. For any $c \in \mathrm{Crit}$ let μ_c be the multiplicity of c, that is the degree of f at c. Let $\mu_{max} = \max\{\mu_c | c \in \mathrm{Crit}_J\}$, $\mu_f = \prod_{c \in \mathrm{Crit}_J} \mu_c$, $\mu_0 = \mu_f^2$ and $\mu_1 = \left(\prod_{c \in NR_J} \mu_c\right) \cdot \max\{\mu_c | c \in CE_J\}$. Let us observe that

$$\mu_{max} < \mu_1 \leq \mu_f < \mu_0.$$

Proposition 2.3.1. *RCE implies BS.*

Proof. Let $0 < \varepsilon_1 = min\{\operatorname{dist}(c, c') \, | c \neq c'; \; c, c' \in \operatorname{Crit}_J\}$ be the smallest distance between two critical points.

Let us remark that there exists $\varepsilon_2 > 0$ such that every connected component U of $f^{-1}(V)$ is simply connected provided V is simply connected and $\operatorname{diam} V < \varepsilon_2$. We may assume that ε_2 is so small that $\operatorname{diam} B^{-1}(z, \varepsilon_2) < \varepsilon_1$ for all $z \in \overline{\mathbb{C}}$. By the choice of ε_1, U contains at most 1 critical point. Hence, $f : U \to V$ is univalent if $\operatorname{Crit} \cap U = \emptyset$ or its degree is equal to μ_c if $c \in \operatorname{Crit} \cap U$.

Let $0 < \varepsilon_3 = \min\{\operatorname{dist}(c, O(c)) \, | c \in NR_J)\}$ be the smallest distance of a non-recurrent critical point to its orbit.

Fix some $\lambda_0 \in (1, \lambda)$ and consider $\delta_{\lambda_0,1/2} > 0$ supplied by Proposition 2.2.3. Choose $\varepsilon > 0$ such that

$$100\mu_0\varepsilon \leq \varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_{\lambda_0,1/2}). \tag{2.23}$$

Let $\delta_{\varepsilon,2,\mu_0} > 0$ be supplied by Lemma 2.2.4 and $\delta = \delta_{\varepsilon,2,\mu_0}/2$. We call $B\,(z, r)$ *admissible* if $B\,(z, 4r) \subseteq \Omega$. By diminishing δ, we may suppose that any ball with radius at most δ that intersects J is admissible.

Suppose BS is not satisfied. Consider n_0 the smallest n with the property that there is an admissible ball $B\,(z, r)$ with $r \leq \delta$ such that $\operatorname{diam} B\,(z, r)^{-n} \geq \varepsilon$. Let us denote $B\,(z, r)^{-n_0} = W' \subseteq W = B\,(z, 2r)^{-n_0}$. By Lemma 2.2.4, this choice of constants implies that $\deg_W(f^{n_0}) > \mu_0$.

Remark 2.3.1. *This is the only exception to our construction of blocks of critical points, developed in the first part of Section 2.2, in order to ensure that critical orbits avoid critical points. Here n_0 and m (to be defined) are the "original" lengths of the orbits. Note that a CE_J critical point cannot be sent to* Crit_J. *Thus Proposition 2.2.3 still applies as it does not assume anything on N, the length of the orbit.*

We may cover $\partial B\,(z, 2r)$ with less than 100 balls $B_i = B\,(z_i', r/2)$ centered on $\partial B\,(z, 2r)$. They are admissible as $B\,(z, r)$ is admissible. Therefore $\operatorname{diam} B_i^{-n} < \varepsilon$ for all $n < n_0$. Thus for all $n < n_0$

$$\operatorname{diam} B\,(z, 2r)^{-n} < 100\varepsilon \deg_{B(z,2r)^{-n}}(f^n). \tag{2.24}$$

Let us denote $W_k = f^{n_0-k}(W)$ for all $0 \leq k \leq n_0$. In particular $W_0 = B\,(z, 2r)$ and $W_{n_0} = W$. Let also $d_k = \deg_{W_k}(f^k)$ for all $0 \leq k \leq n_0$. Thus $d_{n_0} > \mu_0$ and $d_{k+1} = d_k \deg_{W_{k+1}}(f)$ for all $0 \leq k < n_0$. Let $m = \max\{0 \leq k < n_0 \mid d_k \leq \mu_0\}$. Inequalities (2.23) and (2.24) show that

$$\operatorname{diam} W_k < \varepsilon_0, \text{ for all } k \leq m. \tag{2.25}$$

Recall that $\varepsilon_0 \leq \varepsilon_1$, by its definition (2.23). Thus W_k contains at most one critical point for all $k \leq m$. Equally by (2.23), $\varepsilon_0 \leq \varepsilon_2$. Therefore $\operatorname{diam} W_{m+1} < \varepsilon_1$ so W_{m+1} contains at most one critical point also. For $k \leq m+1$, $\deg_{W_k}(f) = \mu_c$ if $c \in W_k \cap \operatorname{Crit}_J$

and $\deg_{W_k}(f) = 1$ if otherwise. Thus for all $n \leq m+1$

$$d_n = \prod_{\substack{c \in W_k \cap \mathrm{Crit} \\ 0<k\leq n}} \mu_{c,k} \tag{2.26}$$

counted with multiplicity if some $c \in \mathrm{Crit}_J \cap W_{k_1} \cap W_{k_2}$, with $0 < k_1 < k_2 \leq m$. By the definition of m, $d_m \leq \mu_0 < d_{m+1}$. Thus W_{m+1} contains exactly one critical point so $1 < \deg_{W_{m+1}}(f) \leq \mu_{max}$. Therefore $\mu_0/\mu_{max} < d_m$. By definition $\mu_0 = \mu_f^2$ and $\mu_{max} < \mu_f$ thus

$$\mu_1 \leq \mu_f < \mu_0/\mu_{max} < d_m. \tag{2.27}$$

Moreover, $\varepsilon_0 \leq \varepsilon_3$ thus

$$NR_J \cap W_{k_1} \cap W_{k_2} = \emptyset, \text{ for all } 0 < k_1 < k_2 \leq m.$$

Otherwise there is $c \in NR_J$ with $c, f^{k_2-k_1}(c) \in W_{k_1}$ and $\operatorname{diam} W_{k_1} < \varepsilon_3$, a contradiction. Thus, in the product (2.26) that defines d_m, non-recurrent critical points are counted at most once. Since $\mu_1 < d_m$ by inequality (2.27), there are at least two integers $0 < m_0 < m_1 \leq m$ such that each W_{m_0} and W_{m_1} contains exactly one Collet-Eckmann critical point. As $\varepsilon_0 \leq \delta_{\lambda_0,1/2}$ by inequality (2.23) and for all $k \leq m$ $\operatorname{diam} W_k < \varepsilon_0$ by inequality (2.25), we are in position to apply Proposition 2.2.3 for $N = m_1 - 1$. Thus

$$\operatorname{diam} W_N < \frac{2r}{2}\lambda_0^{-N} < r \leq \delta.$$

As $W_{N+1} = W_{m_1}$ contains a critical point, W_N contains a critical value v inside Ω. As Ω does not intersect critical orbits in the Fatou set, $v \in J$. So $W_N \subseteq B(v,r) \subseteq B(v,\delta)$, thus $B(v,r)$ is admissible. Therefore

$$\varepsilon \leq \operatorname{diam} W' \leq \operatorname{diam} W_{n_0} \leq \operatorname{diam}\left(B(v,r)^{-(n_0-N)}\right),$$

which contradicts the minimality of n_0 and hence proves the proposition. □

In the previous section we fixed f a RCE rational map. We also defined Ω, a neighborhood of $J = J_f$. Let us also define some constants using Proposition 2.3.1. They will be used in the next section, in the proof of the main Theorem. They are also used to state the following corollary.

Let $\beta = 2$, $\mu = \mu_1$, $\lambda_0 \in (1, \lambda)$ and $\theta = \frac{1}{2}64^{-\mu}$. Proposition 2.2.2 provides $\delta_{\beta,\mu} > 0$ and Proposition 2.2.3 provides $\delta_{\lambda_0,\theta} > 0$ that depend only on β, μ, λ_0 and θ. Let

$$\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_{\beta,\mu}, \delta_{\lambda_0,\theta}).$$

Proposition 2.3.1 provides δ such that the diameter of any pullback of a ball of radius at most δ centered on J is smaller than ε. We may assume that ε and δ are small $\delta \leq \varepsilon < \operatorname{diam} J/10 < 1/2$ and that any ball of radius δ that intersects J is contained in Ω. We set

$$R = \delta/4.$$

Corollary 2.2.1 and Propositions 2.2.2, 2.2.3 and 2.3.1 apply for pullbacks of balls centered on J of radius $R' \leq 2R$. Moreover, if we set $r_0 < \theta R$, Proposition 2.2.2 yields $N_0 = N_{\beta,\mu,r_0,R} \geq 1$, the minimum length of the orbits on which it applies.

If W is an open set and $\overline{f^k(W)}$ contains at most one critical point for all $0 \leq k < n$, let us define

$$\deg_{\overline{W}}(f^n) = \prod_{\substack{c \in \overline{f^k(W)} \cap \mathrm{Crit} \\ 0 \leq k < n}} \mu_{c,k}. \tag{2.28}$$

Corollary 2.3.1. *For all $z \in J$, $0 < r \leq 2R$ and $(W_k)_{k \geq 0}$ a backward orbit of $B(z,r)$, if $\overline{d}_n > \mu$, where $\overline{d}_k = \deg_{\overline{W_k}} f^k$ for all $k \geq 0$, then there is $0 < n_{CE} \leq n$ such that $W_{n_{CE}} \cap CE \neq \emptyset$ and $\overline{d}_{n_{CE}-1} > 1$.*

Proof. This is a reformulation of the definition of m_0 and m_1 in the end of the proof of the previous proposition. □

2.4 RCE implies UHP

We have discussed in the introduction some telescope-like constructions in the literature and we also have announced that our proof uses one of its own. We do not give any general definition of a telescope, instead we provide a self-contained description of the one we use. We consider a pullback of an arbitrary ball $B(z,R)$ with $z \in J$, of length N. We prove the Exponential Shrinking of components condition. We show that there are constants $C_1 > 0$ and $\lambda_1 > 1$ that do not depend on z nor on N such that

$$\operatorname{diam} B(z,R)^{-N} \leq C_1 \lambda_1^{-n}.$$

It is easy to check that the previous inequality for all $z \in J$ and $N > 0$ implies the *ExpShrink* condition.

Let us describe the construction. We nest $B(z,R)$ inside a ball $B(z,R'_0)$ with $R'_0 \leq 2R$ and consider its preimages up to time N. We show that there is some moment N'_0 when the pullback observes a strong contraction. Then $B(z,R'_0)^{-N'_0}$ can be nested inside some ball $B(z_{N'_0}, R'_1)$ where $z_{N'_0} \in f^{-N'_0}$ and $R'_1 \leq 2R$. This new ball is pulled back and the construction is achieved inductively. The pullbacks $B(z,R'_0)$, $B(z,R'_0)^{-1} \ldots B(z,R'_0)^{-N'_0}$ form the first block of the telescope. The pullbacks $B(z_{N'_0},R'_1)$, $B(z_{N'_0},R'_1)^{-1} \ldots B(z_{N'_0},R'_1)^{-N'_1}$ form the second block and so on. Lemma 2.2.1 is essential to manage the passage between two such consecutive telescope blocks. We show contraction for every block using either Proposition 2.2.2 or Proposition 2.2.3. This leads to a classification of blocks depending on the presence and on the type of critical points inside them.

Let us recall that $\beta, \mu, \lambda_0, \theta, \varepsilon, R, r_0$ and N_0 were defined at the end of the previous section. Let R' be the radius of the initial ball of some block and N' be its length. Let $r' \leq R$ be the diameter of the last pullback of the previous block. It is a lower bound for R' thus consecutive blocks are nested. Recall that $(z_n)_{1 \leq n \leq N}$ is a fixed backward orbit of

z, contained in the pullback. A block that starts at time n is defined by the choice of R' with $r' \leq R' \leq 2R$ and of N' with $1 \leq N' \leq N-n$. It is the pullback of length N' of $B(z_n, R')$. For all $n, t \geq 0$ and $r > 0$ we denote

$$\begin{aligned} d(n,r,t) &= \deg_{B(z_n,r)^{-t}}(f^t) \text{ and} \\ \overline{d}(n,r,t) &= \deg_{\overline{B(z_n,r)^{-t}}}(f^t). \end{aligned}$$

Fix $n \geq 0$ and $t \geq 1$ and consider the maps d and $\overline{d}$ defined on $[r', 2R]$. They are increasing and $d \leq \overline{d}$. The set $\{r \in [r', 2R] \mid d(n,r,t) < \overline{d}(n,r,t)\}$ is the common set of discontinuities of d and $\overline{d}$. Note also that d is lower semi-continuous and $\overline{d}$ is upper semi-continuous.

For transparency, we also denote

$$W_k = B(z_n, R')^{-k}.$$

Let us define three types of block before we make any further considerations.

Type 1 Blocks with $R' = r'$ and N' such that $\overline{d}(n, R', N') > 1$ and $\overline{W_{N'+1}} \cap CE \neq \emptyset$.

Type 2 Blocks with $R' = R$, $N' = \min(N_0, N-n)$ and $d(n, 2R, N-n) \leq \mu$.

Type 3 Blocks with $\overline{d}(n, R', N') > 1$, $\overline{W_{N'+1}} \cap CE \neq \emptyset$ and $d(n, R', N-n) \leq \mu$.

The proof of the theorem has two parts. The first part is the construction of the telescope. That is, every pullback of some $B(z, R)$ with $z \in J$ of length N, can be nested inside a telescope built of blocks of the three types. We show that $\operatorname{diam} B(z_{n_i}, R_i')^{-N_i'} < R$, that is, the contraction along the i-th block is strong enough and that there is at least one type of block to continue with. An upper bound $C_1 \lambda_1^{-N}$ for the diameter of the pullback of length N of $B(z, R_0')$ completes the proof of the theorem.

Theorem 1. *A rational map that satisfies the Recurrent Collet-Eckmann condition is Uniformly Hyperbolic on Periodic orbits.*

Proof. We shall reuse the notations $z_n, r', R', N', d, \overline{d}$ and W_k described before the definition of the three types of pullback.

For the first block of the telescope we set $r' = R$. If $\overline{d}(0, r', N+1) > \mu$ then by Corollary 2.3.1 there is $1 \leq N' \leq N$ that defines a type 1 pullback for $R' = r'$. If $\overline{d}(0, 2R, N) \leq \mu$ then we define a type 2 block, as $d \leq \overline{d}$. If $\overline{d}(0, 2R, N+1) > \mu$ there is a smallest R', with $r' < R' \leq 2R$, such that $\overline{d}(0, R', N+1) > \mu$. Thus R' is a point of discontinuity of $\overline{d}$ so $d(0, R', N+1) < \overline{d}(0, R', N+1)$, so $d(0, R', N) \leq d(0, R', N+1) \leq \mu$. Then by Corollary 2.3.1 there is $1 \leq N' \leq N$ that defines a type 3 pullback.

In the general case we replace 0 by n and N by $N-n$ in the previous construction. Let us be more precise with our notations. We denote by n_i', N_i', r_i' and R_i' the parameters n, N', r' and R' of the i-th block. Let also $W_{i,k}$ be W_k in the context of the i-th block with $i \in \{0, \ldots, b\}$, where $b+1$ is the number of blocks of the telescope. So $n_0' = 0$, $r_0' = R$, $n_1' = N_0'$ and $r_1' = \operatorname{diam} W_{0,N_0'}$. In the general case $i > 0$, we have

$$\begin{aligned} n_i' &= n_{i-1}' + N_{i-1}' \text{ and} \\ r_i' &= \operatorname{diam} W_{i-1, N_{i-1}'}. \end{aligned}$$

Let us also denote by $T_i \in \{1, 2, 2', 3\}$ the type of the i-th block. The type $2'$ is a particular case of the second type, when $N' < N_0$. This could only happen for the last block, when $N - n'_b < N_0$. So $T_i \in \{1, 2, 3\}$ for all $0 \le i < b$. We may code our telescope by the type of its blocks, from right to left

$$T_b \dots T_2 T_1 T_0.$$

Our construction shows that

$$r'_{i+1} < R \text{ for all } 0 \le i < b \tag{2.29}$$

is a sufficient condition for the existence of the telescope that contains the pullback of $B(z, R)$ of length N. If $T_i \in \{1, 3\}$ we apply Proposition 2.2.3 and find that

$$\operatorname{diam} W_{i,N'_i} < \theta R'_i \lambda_0^{-N'_i} < \frac{2R}{2}, \tag{2.30}$$

as $\theta < \frac{1}{2}$, $R'_i \le 2R$ and $\lambda_0^{-N'_i} < 1$. If $T_i = 2$ we apply Proposition 2.2.2 so

$$\operatorname{diam} W_{i,N'_i} < r_0 < \theta R < R. \tag{2.31}$$

In either case, $r'_{i+1} = \operatorname{diam} W_{i,N'_i}$ satisfies inequality (2.29). Thus the telescope is well defined.

We may start estimates. First note that if $T_i = 1$ then, using Proposition 2.2.3

$$r'_{i+1} < \theta r'_i \lambda_0^{-N'_i} < r'_i \lambda_0^{-N'_i}. \tag{2.32}$$

Recall also that if there are $\lambda_1 > 1$ and $C_1 > 0$ such that

$$\operatorname{diam} B(z, R)^{-N} \le \operatorname{diam} W_{0,N} < C_1 \lambda_1^{-N} \tag{2.33}$$

then the theorem holds. We may already set

$$\lambda_1 = \min\left(2^{\frac{1}{\mu N_0}}, \lambda_0^{\frac{1}{\mu}}\right). \tag{2.34}$$

As inequality (2.32) provides an easy way to deal with the first type of block, we compute estimates only for sequences of blocks of types $1 \dots 1$, $1 \dots 12$ and $1 \dots 13$, as the sequence $T_b \dots T_2 T_1 T_0$ can be decomposed in such sequences. Sequences with only one block of type 2 or 3 are allowed as long as the following block is not of type 1. For a sequence of blocks $T_{i+p} \dots T_i$, let

$$N'_{i,p} = N'_{i+p} + \dots + N'_i$$

be its length.

A sequence $1 \dots 1$ may only occur as the first sequence of blocks, thus $i = 0$. As $r'_0 = R$, iterating inequality (2.32) for such a sequence

$$\begin{aligned} r'_{p+1} &< \theta^{p+1} R \lambda_0^{-N'_{0,p}} \\ &< 2\theta R'_0 \lambda_1^{-\mu N'_{0,p}}. \end{aligned} \tag{2.35}$$

Combining inequalities (2.32), (2.31) and the definition (2.34) of λ_1, for a sequence $1 \dots 12$

$$\begin{aligned} r'_{i+p+1} &< r'_{i+1} \lambda_0^{-N'_{i+1,p-1}} \\ &< 2\theta R 2^{-1} \lambda_0^{-N'_{i+1,p-1}} \\ &\leq 2\theta R'_i \lambda_1^{-\mu N'_{i,p}}, \end{aligned} \tag{2.36}$$

as $N'_i = N_0$, $N'_{i,p} = N'_i + N'_{i+1,p-1}$ and $R'_i = R$.

For a sequence $1 \dots 13$, inequalities (2.32) and (2.30) yield

$$\begin{aligned} r'_{i+p+1} &< r'_{i+1} \lambda_0^{-N'_{i+1,p-1}} \\ &< \theta R'_i \lambda_0^{-N'_{i,p}} \\ &< 2\theta R'_i \lambda_1^{-\mu N'_{i,p}}. \end{aligned} \tag{2.37}$$

We also find a bound for r'_{b+1} in the case $T_b = 2'$. Let us remark that $R'_b = R$ therefore

$$\begin{aligned} r'_{b+1} &< \varepsilon = \varepsilon R^{-1} R'_b \\ &= (\varepsilon R^{-1} \lambda_1^{N'_b}) R'_b \lambda_1^{-N'_b}. \end{aligned} \tag{2.38}$$

We decompose the telescope into $m+1$ sequences $1 \dots 1$, $1 \dots 12$, $1 \dots 13$ and eventually $2'$ on the leftmost position

$$S_m \dots S_2 S_1 S_0.$$

Consider a sequence of blocks

$$S_j = T_{i+p} \dots T_i.$$

Denote $n''_j = n'_i$, $N''_j = N'_{i,p}$, $r''_j = r'_i$ and $R''_j = R'_i$. Let also

$$\Delta_j = \operatorname{diam} W_{i,N-n'_i}$$

be the diameter of the pullback of the first block of the sequence up to time $-N$.

With the eventual exception of S_m, inequalities (2.35), (2.36) and (2.37) provide good contraction estimates for each sequence S_j

$$r''_{j+1} < 2\theta R''_j \lambda_1^{-\mu N''_j}.$$

If $T_b = 2'$ then inequality (2.38) yields a constant $\varepsilon R^{-1} \lambda_1^{N'_b} < C_1 = \varepsilon R^{-1} \lambda_1^{N'_0}$ such that

$$r''_{m+1} < C_1 R''_m \lambda_1^{-N''_m},$$

as $R''_m = R'_b$ and $N''_m = N'_b$. Note that the previous inequality also holds if $T_b \in \{1, 2, 3\}$. We cannot simply multiply these inequalities as $R''_j > r''_j$ for all $0 < j \leq m$.

By the definitions of types 2 and 3, the degree $d(n''_j, R''_j, N - n''_j)$ is bounded by μ in all cases. So Lemma 2.2.1 provides a bound for the distortion of pullbacks up to time $-N$

$$\begin{aligned} \frac{\Delta_{j-1}}{\Delta_j} &< 64\left(\frac{r''_j}{R''_j}\right)^{\frac{1}{\mu}} \\ &< 64\left(2\theta\lambda_1^{-\mu N''_{j-1}}\frac{R''_{j-1}}{R''_j}\right)^{\frac{1}{\mu}} \\ &= \lambda_1^{-N''_{j-1}}\left(\frac{R''_{j-1}}{R''_j}\right)^{\frac{1}{\mu}}, \end{aligned}$$

for all $0 < j \leq m$. Therefore

$$\frac{\Delta_0}{\Delta_m} < \lambda_1^{-N+N''_m}\left(\frac{R''_0}{R''_m}\right)^{\frac{1}{\mu}}.$$

Recall that $R''_j \leq 2R < 1$ for all $0 \leq j \leq m$ and $\Delta_m = r''_{m+1}$, so

$$\begin{aligned} \Delta_0 &< \lambda_1^{-N+N''_m} C_1 R''_m \lambda_1^{-N''_m}\left(\frac{2R}{R''_m}\right)^{\frac{1}{\mu}} \\ &< \lambda_1^{-N} C_1 (R''_m)^{1-\frac{1}{\mu}} \\ &< C_1\lambda_1^{-N}. \end{aligned}$$

By definition $\Delta_0 = \operatorname{diam} W_{0,N}$, therefore the previous inequality combined with inequality (2.33) completes the proof of our theorem. $\square$

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Chapitre 3

Counterexamples

Abstract. We find a counterexample to the converse of Theorem 1, that is a polynomial with Uniform Hyperbolicity on repulsive Periodic orbits (UHP) that is not Recurrent Collet-Eckmann (RCE). Using the same techniques we also show that the Collet-Eckmann property of recurrent critical orbits is not topological for real polynomials with negative Schwarzian derivative.

3.1 Introduction

We studied the Collet-Eckmann condition for the Recurrent critical orbits RCE in the complex setting in an attempt to characterize the Topological Collet-Eckmann condition TCE in terms of properties of critical orbits. The results obtained by Carleson, Jones and Yoccoz in [2], by Graczyk and Smirnov in [4] and by Przytycki, Rivera-Letelier and Smirnov in [9] inspired the proof of Theorem 1 which states that RCE implies the equivalent conditions TCE and UHP. Finally we found that TCE does not imply RCE as recurrent critical orbits may approach other critical points and lose the expansion of the derivative, see also [8]. The diameters of pullbacks of small components may still decay exponentially in a similar fashion to semi-hyperbolic dynamics which are not CE. This property that we call Exponential Shrinking of components is again equivalent to TCE, see [9].

The first part of this chapter describes a technique of building real polynomials with prescribed topological and analytical properties by specifying their combinatorial properties. By combinatorial properties we understand symbolic dynamics induced by means of discretization of the phase space $[0, 1]$. This is done using the partition induced by the critical points. It turns out that it is enough to consider only the dynamics of critical orbits. The sequences of symbols associated to the points of the critical orbits are called *kneading sequences* and they are central objects in our study. Using the monotonicity of the multimodal map on each interval of this partition we may define an order on the space of itinerary sequences. The theory of multimodal maps and kneading sequences is well understood but it is related mostly to topological properties of the dynamics. In Section 3.3 we develop a special theory of one parameter families of 2-modal polynomials which provides the tools to obtain a prescribed growth of the derivative on the second critical orbit.

The main idea of Theorem 2 is very simple but the construction of the counterexample and the proof that it has UHP are rather technical. In the vicinity of critical points the diameter of a small domain decreases at most in the power rate while the derivative can approach 0 as fast as one wants. This important difference in the behavior of derivative and diameter is used to produce an aforementioned counterexample.

Using careful estimates of the derivative on the critical orbits we construct two polynomials with negative Schwarzian derivative with the same combinatorics thus topologically conjugated such that only one is RCE. An important feature of our counterexample is that the corresponding critical points of this two polynomials are of different degree. We rely on the tools developed in Section 3.3.

All our examples of dynamics in this chapter are polynomial therefore all distances and diameters are considered with respect to the Euclidean metric.

3.2 Preliminaries

Let us define multimodal maps and state some classical theorems about their dynamics. This results will be used in the construction of a counterexample.

Definition 3.2.1. *Let I be the compact interval $[0,1]$ and $f : I \to I$ a* piecewise strictly monotone *continuous map. This means that f has a finite number of turning points $0 < c_1 < \ldots < c_l < 1$, points where f has a local extremum, and f is strictly monotone on each of the $l+1$ intervals $I_1 = [0, c_1), I_2 = (c_1, c_2), \ldots, I_{l+1} = (c_l, 1]$. Such a map is called* l-modal *if $f(\partial I) \subseteq \partial I$. If $l = 1$ then f is called* unimodal. *If f is $C^1 + r$ with $r \geq 0$ it is called a smooth l-modal map if f' has no zeros outside $\{c_1, \ldots, c_l\}$.*

If f is a l-modal map, let us denote by Crit_f the set of turning points - or critical points

$$\mathrm{Crit}_f = \{c_1, \ldots, c_l\}.$$

For all $x \in I$ we denote by $O(x)$ or $O^+(x)$ its forward orbit

$$O(x) = (f^n(x))_{n \geq 0}.$$

Analogously, let $O^-(x) = \{y \in f^{-n}(x) \mid n \geq 0\}$ and $O^\pm(x) = \{y \in f^n(x) \mid n \in \mathbb{Z}\}$. We also extend these notations to orbits of sets. For $S \subseteq I$ let $O^+(S) = \{f^n(x) \mid x \in S, n \geq 0\}$, $O^-(S) = \{y \in f^{-n}(x) \mid x \in S, n \geq 0\}$ and $O^\pm(S) = O^+(S) \cup O^-(S)$.

One of the most important questions in all areas of dynamics is when two systems have similar underlaying dynamics. A natural equivalence relation for multimodal maps is topological conjugacy.

Definition 3.2.2. *We say that two multimodal maps $f, g : I \to I$ are* topologically conjugate *or simply* conjugate *if there is a homeomorphism $h : I \to I$ such that*

$$h \circ f = g \circ h.$$

We may remark that if f and g are conjugate by h then $h(f^n(x)) = g^n(h(x))$ for all $x \in I$ and $n \geq 0$ so h maps orbits of f onto orbits of g. It is easy to check that h is a monotone bijection form the critical set of f to the critical set of g. We may also consider combinatorial properties of orbits and use the order of the points of critical orbits to define another equivalence relation between multimodal maps. Theorem II.3.1 in [5] tells us that it is enough to consider only the forward orbit of the critical set.

Theorem 3.2.1. *Let f, g be two l-modal maps with turning points $c_1 < \ldots < c_l$ respectively $\tilde{c}_1 < \ldots < \tilde{c}_l$. The following properties are equivalent.*

1. *There exists an order preserving bijection h from $O^+(\mathrm{Crit}_f)$ to $O^+(\mathrm{Crit}_g)$ such that*

$$h(f(x)) = g(h(x)) \text{ for all } x \in O^+(\mathrm{Crit}_f).$$

2. There exists an order preserving bijection $\tilde{h}$ from $O^{\pm}(\mathrm{Crit}_f)$ to $O^{\pm}(\mathrm{Crit}_g)$ such that

$$\tilde{h}(f(x)) = g(\tilde{h}(x)) \text{ for all } x \in O^{\pm}(\mathrm{Crit}_f).$$

If f and g satisfy the properties of the previous theorem we say that they are *combinatorially equivalent.* Note that if f and g are conjugate by an order preserving homeomorphism h then the restriction of h to $O^{+}(\mathrm{Crit}_f)$ is an order preserving bijection onto $O^{+}(\mathrm{Crit}_g)$ so f and g are combinatorially equivalent. The converse is true only in the absence of homtervals. It is the case of all the examples in this chapter. There is a very convenient way to describe the combinatorial type of a multimodal map using symbolic dynamics. We associate to every point $x \in I$ a sequence of symbols $\underline{i}(x)$ that we call the *itinerary* of x. The itineraries $\underline{k}_1, \ldots, \underline{k}_l$ of the critical values $f(c_1), \ldots, f(c_l)$ are called the kneading sequences of f and the ordered set of kneading sequences the kneading invariant. Combinatorially equivalent multimodal maps have the same kneading invariants but the converse is true only in the absence of homtervals. We use the kneading invariant to describe the dynamics of multimodal maps in one-dimensional families. We build sequences $(\mathcal{F}_n)_{n\geq 0}$ of compact families of C^1 multimodal maps with $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ for all $n \geq 0$ and obtain our examples as the intersection of such sequences.

When not specified otherwise, we assume f to be a multimodal map.

Definition 3.2.3. *Let $O(p)$ be a periodic orbit of f. This orbit is called* attracting *if its* basin

$$B(p) = \left\{x \in I \mid f^k(x) \to O(p) \text{ as } k \to \infty\right\}$$

contains an open set. The immediate basin $B_0(p)$ *of $O(p)$ is the union of connected components of $B(p)$ which contain points from $O(p)$. If $B_0(p)$ is a neighborhood of $O(p)$ then this orbit is called a* two-sided attractor *and otherwise a* one-sided attractor*. Suppose f is C^1 and let $m(p) = |(f^n)'(p)|$ where n is the period of p. If $m(p) < 1$ we say that $O(p)$ is attracting respectively super-attracting if $m(p) = 0$. We call $O(p)$* neutral *if $m(p) = 1$ and we say it is* repulsive *if $m(p) > 1$.*

Let us denote by $B(f)$ the union of the basins of periodic attracting orbits and by $B_0(f)$ the union of immediate basins of periodic attractors. The basins of attracting periodic contain intervals on which all iterates of f are monotone. Such intervals do not intersect $O^{-}(\mathrm{Crit}_f)$ and they do not carry too much combinatorial information.

Definition 3.2.4. *Let us define a* homterval *to be an interval on which f^n is monotone for all $n \geq 0$.*

Homtervals are related to *wandering intervals* and they play an important role in the study of the relation between conjugacy and combinatorial equivalence.

Definition 3.2.5. *An interval $J \subseteq I$ is* wandering *if all its iterates $J, f(J), f^2(J), \ldots$ are disjoint and if $(f^n(J))_{n\geq 0}$ does not tend to a periodic orbit.*

Homtervals have simple dynamics described by the following lemma, Lemma II.3.1 in [5].

Lemma 3.2.1. *Let J be a homterval of f. Then there are two possibilities:*

1. *J is a wandering interval;*
2. *$J \subseteq B(f)$ and some iterate of J is mapped into an interval L such that f^p maps L monotonically into itself for some $p \geq 0$.*

Multimodal maps satisfying some regularity conditions have no wandering intervals. Let us say that f is *non-flat* at a critical point c if there exists a C^2 diffeomorphism $\phi : \mathbb{R} \to I$ with $\phi(0) = c$ such that $f \circ \phi$ is a polynomial near the origin.

The following theorem is Theorem II.6.2 in [5].

Theorem 3.2.2. *Let f be a C^2 map that is non-flat at each critical point. Then f has no wandering intervals.*

Guckenheimer proved this theorem in 1979 for unimodal maps with *negative Schwarzian derivative* with *non-degenerate* critical point, that is with $|f''(c)| \neq 0$. The Schwarzian derivative was first used by Singer to study the dynamics of quadratic unimodal maps $x \to ax(1-x)$ with $a \in [0,4]$. He observed that this property is preserved under iteration and that is has important consequences in unimodal and multimodal dynamics.

Definition 3.2.6. *Let $f : I \to I$ be a C^3 l-modal map. The* Schwarzian derivative *of f at x is defined as*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f(x)}\right)^2,$$

for all $x \in I \setminus \{c_1, \ldots, c_l\}$.

We may compute the Schwarzian derivate of a composition

$$S(g \circ f)(x) = Sg(f(x)) \cdot |f'(x)|^2 + Sf(x), \tag{3.1}$$

therefore if $Sf < 0$ and $Sg < 0$ then $S(f \circ g) < 0$ so negative Schwarzian derivative is preserved under iteration. Let us state an important consequence of this property for C^3 maps of the interval proved by Singer (see Theorem II.6.1 in [5]).

Theorem 3.2.3 (Singer)**.** *If $f : I \to I$ is a C^3 map with negative Schwarzian derivative then*

1. *the immediate basin of any attracting periodic orbit contains either a critical point of f or a boundary point of the interval I;*
2. *each neutral periodic point is attracting;*
3. *there are no intervals of periodic points.*

Combining this result with Theorem 3.2.2 and Lemma 3.2.1 we obtain the following

Corollary 3.2.1. *If f is C^3 multimodal map with negative Schwarzian derivative that is non-flat at each critical point and which has no attracting periodic orbits then it has no homterval. Therefore $O^-(\mathrm{Crit}_f)$ is dense in I.*

The following corollary is a particular case of the corollary of Theorem II.3.1 in [5].

Corollary 3.2.2. *Let f, g and h be as in Theorem 3.2.1. If f and g have no homtervals then they are topologically conjugate.*

All our examples of multimodal maps in this chapter are polynomials with negative Schwarzian derivative and without attracting periodic orbits. We prefer however to use slightly more general classes of multimodal maps, as suggested by the previous two corollaries. As combinatorially equivalent multimodal maps have the same monotony type we only use maps that are increasing on the leftmost lap I_1, that is exactly the multimodal maps f with $f(0) = 0$. Let us define some classes of multimodal maps

$$\mathcal{S}_l = \{f : I \to I \mid f \text{ is a smooth } l\text{-modal map with } f(0) = 0\},$$

$$\mathcal{S}'_l = \left\{f \in \mathcal{S}_l \mid f \text{ is } C^3 \text{ and } Sf < 0\right\},$$

$$\mathcal{P}_l = \{f \in \mathcal{S}'_l \mid f \text{ non-flat at each critical point}\} \text{ and}$$

$$\mathcal{P}'_l = \{f \in \mathcal{P}_l \mid \text{all periodic points of } f \text{ are repulsive}\}.$$

We have seen that in the absence of homtervals combinatorially equivalent multimodal maps are topologically conjugate. Using symbolic dynamics we find a more convenient way to describe the combinatorial properties of forward critical orbits. Let $\mathcal{A}_I = \{I_1, \ldots, I_{l+1}\}$ and $\mathcal{A}_c = \{c_1, \ldots, c_l\}$ be two alphabets and $\mathcal{A} = \mathcal{A}_I \cup \mathcal{A}_c$. Let

$$\Sigma = \mathcal{A}_I^{\mathbb{N}} \cup \bigcup_{n \geq 0} (\mathcal{A}_I^n \times \mathcal{A}_c)$$

be the space of sequences of symbols of $\mathcal{A}$ with the following property. If $\underline{i} \in \Sigma$ and $m = |\underline{i}| \in \overline{\mathbb{N}}$ is its length then $m = \infty$ if and only if $\underline{i}$ consists only of symbols of $\mathcal{A}_I$. Moreover, if $m < \infty$ then $\underline{i}$ contains exactly one symbol of $\mathcal{A}_c$ on the rightmost position. Let $\Sigma' = \Sigma \setminus \mathcal{A}_c$ be the space of sequences $\underline{i} \in \Sigma$ with $|\underline{i}| > 1$. Let us define the shift transformation $\sigma : \Sigma' \to \Sigma$ by

$$\sigma(i_0 i_1 \ldots) = i_1 i_2 \ldots.$$

If $f \in \mathcal{S}_l$ let $\underline{i} : I \to \Sigma$ be defined by $\underline{i}(x) = i_0(x) i_1(x) \ldots$ where $i_n(x) = I_k$ if $f^n(x) \in I_k$ and $i_n(x) = c_k$ if $f^n(x) = c_k$ for all $n \geq 0$. The map $\underline{i}$ relates the dynamics of f on $I \setminus \{c_1, \ldots, c_l\}$ with the shift transformation σ on Σ'

$$\underline{i}(f(x)) = \sigma(\underline{i}(x)) \text{ for all } x \in I \setminus \{c_1, \ldots, c_l\}.$$

Moreover, we may define a *signed lexicographic ordering* on Σ that makes $\underline{i}$ increasing. It becomes strictly increasing in the absence of homtervals.

Definition 3.2.7. *A* signed lexicographic ordering $\prec$ *on* Σ *is defined as follows. Let us define a sign* $\epsilon : \mathcal{A} \to \{-1, 0, 1\}$ *where* $\epsilon(I_j) = (-1)^{j+1}$ *for all* $j = 1, \ldots, l+1$ *and* $\epsilon(c_j) = 0$ *for all* $j = 1, \ldots, l$. *Using the natural ordering on* $\mathcal{A}$ *we say that* $\underline{x} \prec \underline{y}$ *if there exists* $n \geq 0$ *such that* $x_i = y_i$ *for all* $i = 0, \ldots, n-1$ *and*

$$\left(\prod_{i=0}^{n-1} \epsilon(x_i)\right) x_n < \left(\prod_{i=0}^{n-1} \epsilon(y_i)\right) y_n.$$

Let us observe that $\prec$ is a complete ordering and that $\epsilon \cdot f' > 0$ on $I \setminus \{c_1, \ldots, c_l\}$, that is ϵ represents the monotony of f. The product $\prod_{i=0}^{n-1} \epsilon(x_i)$ represents therefore the monotony of f^n. This is the main reason for the monotony of $\underline{i}$ with respect to $\prec$.

Proposition 3.2.1. *Let* $f \in \mathcal{S}_l$ *for some* $l \geq 0$.

1. *If* $x < y$ *then* $\underline{i}(x) \preceq \underline{i}(y)$.
2. *If* $\underline{i}(x) \prec \underline{i}(y)$ *then* $x < y$.
3. *If* $f \in \mathcal{P}_l'$ *then* $x < y$ *if and only if* $\underline{i}(x) \prec \underline{i}(y)$.

Proof. The first two points are Lemma II.3.1 in [5]. If $f \in \mathcal{P}_l'$ then by Corollary 3.2.1 $O^-(\mathrm{Crit}_f)$ is dense in I. Let us note that

$$O^-(\mathrm{Crit}_f) = \{x \in I \mid |\underline{i}(x)| < \infty\}.$$

Moreover, $O^-(\mathrm{Crit}_f)$ is countable as $f^{-1}(x)$ is finite for all $x \in I$, therefore $\underline{i}$ is strictly increasing. $\square$

Let us define the kneading sequences of $f \in \mathcal{S}_l$ by $\underline{k_i} = \underline{i}(f(c_i))$ for $i = 1, \ldots, l$, the itineraries of the critical values. The kneading invariant of f is the sequence $\underline{K}(f) = (\underline{k}_1, \ldots, \underline{k}_l)$. The last point of the previous lemma shows that if $f, g \in \mathcal{P}_l'$ and $\underline{K}(f) = \underline{K}(g)$ then there is an order preserving bijection $h : O^+(\mathrm{Crit}_f) \to O^+(\mathrm{Crit}_g)$. Therefore, by Corollaries 3.2.1 and 3.2.2, f and g are topologically conjugate.

Let us define one-dimensional smooth families of multimodal maps. It is the main tool in our constructions of examples of multimodal maps.

Definition 3.2.8. *We say that* $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ *is a* family of l-modal maps *if* $\mathcal{F}$ *is continuous with respect to the* C^1 *topology of* $\mathcal{S}_l$.

Note that we do not assume the continuity of critical points in such a family - as in the general definition of a family of multimodal maps in [5] - as it is a direct consequence of the smoothness conditions we impose.

When not stated otherwise we suppose $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ is a family of l-modal maps and denote $f_\gamma = \mathcal{F}(\gamma)$.

Lemma 3.2.2. *The critical points* $c_i : [\alpha, \beta] \to I$ *of* f_γ *are continuous maps for all* $i = 1, \ldots, l$.

Proof. Fix $\gamma_0 \in [\alpha, \beta]$ and $\varepsilon > 0$. Let $A = \{x \in [0,1] \mid \min_i \operatorname{dist}(x, c_i(\gamma_0)) \geq \varepsilon\}$ a finite union of compact intervals and

$$\theta = \min_{x \in A} |f'_{\gamma_0}(x)| > 0$$

by Definition 3.2.1. Let $\delta > 0$ be such that $||f_\gamma - f_{\gamma_0}||_{C^1} < \frac{\theta}{2}$ for all $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$. Therefore the critical points $c_i(\gamma)$ satisfy

$$|c_i(\gamma) - c_i(\gamma_0)| < \varepsilon$$

for all $i = 1, \ldots, l$ and $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$ as $f'_\gamma(x) \cdot f'_{\gamma_0}(x) > 0$ for all $x \in A$. □

Let us show that the C^1 continuity of families of multimodal maps is preserved under iteration.

Lemma 3.2.3. *Let $G, H : [a,b] \to C^1(I, I)$ be continuous. Then the map*

$$c \to G(c) \circ H(c) \text{ is continuous on } [a,b].$$

Proof. Fix $c_0 \in [a,b]$ and $\varepsilon > 0$. We show that there is $\delta > 0$ such that

$$||G(c) \circ H(c) - G(c_0) \circ H(c_0)||_{C^1} < \varepsilon \text{ for all } c \in (c_0 - \delta, c_0 + \delta) \cap [a,b].$$

For transparency we denote $g_c = G(c)$ and $h_c = H(c)$ for all $c \in [a,b]$. Let

$$M = \max\left\{||g_c||_{C^1}, ||h_c||_{C^1}, 1 \mid c \in [a,b]\right\}.$$

By the compactness of I, g'_{c_0} is uniformly continuous therefore there is $\delta' > 0$ such that

$$|g'_{c_0}(x) - g'_{c_0}(y)| < \frac{\varepsilon}{4M} \text{ for all } x, y \in I \text{ with } |x - y| < \delta'.$$

Let $\delta > 0$ such that

$$\sup\left\{||g_c - g_{c_0}||_{C^1}, ||h_c - h_{c_0}||_{C^1} \mid c \in (c_0 - \delta, c_0 + \delta) \cap [a,b]\right\} < \min\left(\frac{\varepsilon}{4M}, \delta'\right).$$

We compute a bound for $||g_c \circ h_c - g_{c_0} \circ h_{c_0}||_{C^1}$ for all $c \in (c_0 - \delta, c_0 + \delta) \cap [a,b]$

$$\begin{array}{rcl} ||g_c \circ h_c - g_{c_0} \circ h_{c_0}||_\infty & \leq & ||g_c \circ h_c - g_c \circ h_{c_0}||_\infty + ||g_c \circ h_{c_0} - g_{c_0} \circ h_{c_0}||_\infty \\ & \leq & M\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} \\ & < & \varepsilon. \end{array}$$

Analogously

$$\begin{array}{rcl} ||g'_c \circ h_c \cdot h'_c - g'_{c_0} \circ h_{c_0} \cdot h'_{c_0}||_\infty & \leq & ||g'_c \circ h_c \cdot h'_c - g'_{c_0} \circ h_c \cdot h'_c||_\infty + \\ & & ||g'_{c_0} \circ h_c \cdot h'_c - g'_{c_0} \circ h_{c_0} \cdot h'_c||_\infty + \\ & & ||g'_{c_0} \circ h_{c_0} \cdot h'_c - g'_{c_0} \circ h_{c_0} \cdot h'_{c_0}||_\infty \\ & \leq & \frac{\varepsilon}{4M}M + \frac{\varepsilon}{4M}M + M\frac{\varepsilon}{4M} \\ & < & \varepsilon \end{array}$$

as $||h_c - h_{c_0}||_\infty < \delta'$. □

We may remark that by iteration $\gamma \to f_\gamma^n$ is continuous for all $n \geq 1$. This property is therefore assumed in the sequel for all families of multimodal maps.

The following proposition shows that pullbacks of given combinatorial type of continuous maps are continuous in a family of multimodal maps.

Proposition 3.2.2. *Let $y : [\alpha, \beta] \to I$ be continuous and $S \in \mathcal{A}_I^n$. The maximal domain of definition of the map $\gamma \to x_\gamma$ such that*

$$\begin{array}{l} f_\gamma^n(x_\gamma) = y(\gamma) \text{ and} \\ \underline{i}(x_\gamma) \in S \times \Sigma \end{array}$$

is open and $\gamma \to x_\gamma$ is unique and continuous.

Proof. Suppose that for some γ there are $x_1 < x_2 \in I$ with $f_\gamma^n(x_1) = f_\gamma^n(x_2) = y(\gamma)$ and such that $\underline{i}(x_1) = \underline{i}(x_2) = S\underline{i}(y(\gamma))$ for some $\gamma \in [\alpha, \beta]$. But $S \in \mathcal{A}_I^n$ so f^n is strictly monotone on $[x_1, x_2]$, which contradicts $f_\gamma^n(x_1) = f_\gamma^n(x_2)$ so $\gamma \to x_\gamma$ is unique.

Let x_{γ_0} be as in the hypothesis and $\varepsilon > 0$. We show that there exists $\delta > 0$ such that $\gamma \to x_\gamma$ is defined on $(\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$ and takes values in $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$. Let

$$\theta = (f_{\gamma_0}^n)'(x_{\gamma_0}) \neq 0$$

and by eventually diminishing ε we may suppose that

$$|(f_{\gamma_0}^n)'(x) - \theta| < \frac{\theta}{4} \text{ for all } x \in (x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon).$$

Let $1 > \delta_1 > 0$ be such that

$$||f_\gamma^n - f_{\gamma_0}^n||_{C^1} < \frac{\theta\varepsilon}{4} < \frac{\theta}{4} \text{ for all } \gamma \in (\gamma_0 - \delta_1, \gamma_0 + \delta_1) \cap [\alpha, \beta].$$

Let also $\delta_2 > 0$ be such that

$$|y(\gamma) - y(\gamma_0)| < \frac{\theta\varepsilon}{4} \text{ for all } \gamma \in (\gamma_0 - \delta_2, \gamma_0 + \delta_2) \cap [\alpha, \beta].$$

We choose $\delta = \min(\delta_1, \delta_2)$ and show that

$$y(\gamma) \in f_\gamma^n((x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon) \cap I) \text{ for all } \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta].$$

Indeed, f_γ^n is monotone on $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$ and

$$|f_\gamma^n(x_{\gamma_0} \pm \varepsilon) - y(\gamma_0)| > \frac{\theta\varepsilon}{4}$$

for all $\gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta]$ as $|f_\gamma^n(x_{\gamma_0} \pm \varepsilon) - y(\gamma_0)| = |f_\gamma^n(x_{\gamma_0} \pm \varepsilon) - f_{\gamma_0}^n(x_{\gamma_0} \pm \varepsilon) + f_{\gamma_0}^n(x_{\gamma_0} \pm \varepsilon) - f_{\gamma_0}^n(x_{\gamma_0})|$ and $|f_{\gamma_0}^n(x_{\gamma_0} \pm \varepsilon) - f_{\gamma_0}^n(x_{\gamma_0})| > \frac{3}{4}\theta\varepsilon$. $\square$

As an immediate consequence of the previous proposition and Lemma 3.2.2 we obtain the following corollary.

Corollary 3.2.3. *If $\mathcal{F}$ realizes a finite itinerary sequence $\underline{i}_0 \in \Sigma$, that is for all $\gamma \in [\alpha, \beta]$ there is $x(\underline{i}_0)(\gamma) \in I$ such that*

$$\underline{i}(x(\underline{i}_0)(\gamma)) = \underline{i}_0,$$

then $x(\underline{i}_0) : [\alpha, \beta] \to I$ is unique and continuous.

We may observe that if $x, y : [\alpha, \beta] \to I$ are continuous and for some $k \geq 0$

$$\left(f_\alpha^k(x(\alpha)) - y(\alpha)\right) \cdot \left(f_\beta^k(x(\beta)) - y(\beta)\right) < 0$$

then there exists $\gamma \in [\alpha, \beta]$ such that

$$f_\gamma^k(x(\gamma)) = y(\gamma). \tag{3.2}$$

Therefore if $\underline{i}(x(\alpha)) \neq \underline{i}(x(\beta))$ then there exists $\gamma \in [\alpha, \beta]$ such that $\underline{i}(x(\gamma))$ is finite. Let $m = \min\{k \geq 0 \mid \exists \gamma \in [\alpha, \beta] \text{ such that } \underline{i}(x(\alpha))(k) \neq \underline{i}(x(\gamma))(k)\}$ then the itinerary $\sigma^m \underline{i}(x(\gamma)) = \underline{i}(f_\gamma^m(x(\gamma)))$ changes the first symbol on $[\alpha, \beta]$. Without loss of generality we may assume that $\sigma^m \underline{i}(x(\alpha)) \prec \sigma^m \underline{i}(x(\beta))$. Therefore there exists $i \in \{1, \ldots, l\}$ such that $f_\gamma^m(x(\alpha)) \leq c_i(\alpha)$ and $f_\gamma^m(x(\beta)) \geq c_i(\alpha)$, which yields γ using the previous remark.

A simplified version of the proof of Proposition 3.2.2 shows that if $F : [\alpha, \beta] \to C^1(I)$ is continuous, $r_0 \in I$ is a root of $F(\gamma_0)$ and $(F(\gamma_0))'(r_0) \neq 0$ then there are $J \subseteq [\alpha, \beta]$ a neighborhood of γ_0 and $r : J \to I$ continuous such that $F(\gamma)(r(\gamma)) = 0$ for all $\gamma \in J$. For $F(\gamma)(x) = f_\gamma^n(x) - x$ we obtain the following corollary.

Corollary 3.2.4. *Let r_0 be a periodic point of f_{γ_0} of period $n \geq 1$ that is not neutral. There exists a connected neighborhood $J \subseteq [\alpha, \beta]$ of γ_0 and $r : J \to I$ continuous such that $r(\gamma)$ is a non-neutral periodic point of f_γ of period n. Moreover, if $r(\gamma)$ is not super-attracting for all $\gamma \in J$ then the itinerary $\underline{i}(r(\gamma))$ is constant.*

Proof. As a periodic point, $r(\gamma)$ exists and is continuous on a connected neighborhood J_0 of γ_0, using the previous remark. As $|(f_{\gamma_0}^n)'(r_0)| \neq 1$, there is a connected neighborhood J_1 of γ_0 such that

$$|(f_\gamma^n)'(r(\gamma))| \neq 1 \text{ for all } \gamma \in J_1.$$

Let $J = J_0 \cap J_1$ so $r(\gamma)$ is a non-neutral periodic point of period n for all $\gamma \in J$. Suppose that its itinerary $\underline{i}(r(\gamma))$ is not constant, then there is $\gamma_1 \in J$ such that $\underline{i}(r(\gamma_1))$ is finite so the orbit of $r(\gamma_1)$ contains a critical point thus it is super-attracting. $\square$

Let us define the *asymptotic kneading sequences* $\underline{k}_j^-(\gamma)$ and $\underline{k}_j^+(\gamma)$ for all $\gamma \in [\alpha, \beta]$ and $j = 1, \ldots, l$. When they exist, the asymptotic kneading sequences capture important information about the local variation of the kneading sequences.

Definition 3.2.9. *Let $j \in \{1, \ldots, l\}$ and $\gamma \in [\alpha, \beta]$. If $\gamma > \alpha$ and for all $n \geq 0$ there exists $\delta > 0$ such that $\underline{k}_j(\gamma - \theta) \in S_n \times \Sigma$ with $S_n \in \mathcal{A}_I^n$ for all $\theta \in (0, \delta)$ then we set $\underline{k}_j^-(\gamma)(k) = S_n(k)$ for all $0 \leq k < n$. Analogously, if $\gamma < \beta$ and for all $n \geq 0$ there exists $\delta > 0$ such that $\underline{k}_j(\gamma + \theta) \in S'_n \times \Sigma$ with $S'_n \in \mathcal{A}_I^n$ for all $\theta \in (0, \delta)$ then we set $\underline{k}_j^+(\gamma)(k) = S'_n(k)$ for all $0 \leq k < n$.*

Let us define a necessary and sufficient condition for the existence of the asymptotic kneading sequences for all $\gamma \in [\alpha, \beta]$.

Definition 3.2.10. *We call a family $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ of l-modal maps* natural *if the set*

$$\underline{k}_j^{-1}(\underline{i}) = \left\{\gamma \in [\alpha, \beta] \mid \underline{k}_j(\gamma) = \underline{i}\right\} \text{ is finite for all } \underline{i} \in \Sigma \text{ finite.}$$

This property does not hold in general for C^1 families of multimodal maps, even polynomial, as such a family could be reparametrized to have intervals of constance in the parameter space. It is however generally true for analytic families such as the quadratic family $a \to ax(1-x)$ with $a \in [0, 4]$.

One may easily check that the previous condition is necessary for the existence of all asymptotic kneading sequences, considering an accumulation point of some $\underline{k}_j^{-1}(\underline{i})$ with $\underline{i}$ finite. The following proposition shows that it is also sufficient.

Proposition 3.2.3. *Let $\mathcal{F} : [\alpha, \beta] \to \mathcal{S}_l$ be a natural family of l-modal maps and $j \in \{1, \ldots, l\}$. Then $\underline{k}_j^-(\gamma)$ exists for all $\gamma \in (\alpha, \beta]$ and $\underline{k}_j^+(\gamma)$ exists for all $\gamma \in [\alpha, \beta)$. Moreover, if $\underline{k}_j(\gamma) \in \mathcal{A}_I^\infty$ for some $\gamma \in (\alpha, \beta)$ then $\underline{k}_j^-(\gamma) = \underline{k}_j(\gamma) = \underline{k}_j^+(\gamma)$. If $\underline{k}_j(\gamma) = Sc_i$ with $S \in \mathcal{A}_I^n$ for some $n \geq 0$ and $i \in \{1, \ldots, l\}$ then $\underline{k}_j^-(\gamma) = Sl_1l_2\ldots$ and $\underline{k}_j^+(\gamma) = Sr_1r_2\ldots$ with $l_1, r_1 \in \{I_i, I_{i+1}\}$.*

Proof. If $\mathcal{F}$ is natural then the set of all $\gamma \in [\alpha, \beta]$ that have at least one kneading sequence of length at most n for some $0 < n$

$$K_n = \bigcup_{j=1}^{l} \left\{\gamma \in [\alpha, \beta] \mid |\underline{k}_j(\gamma)| \leq n\right\}$$

is finite. This is sufficient for the existence of all asymptotic kneading sequences.

If $\underline{k}_j(\gamma_0) \in S \times \Sigma$ with $S \in \mathcal{A}_I^n$ and $n \geq 0, j \in \{1, \ldots, l\}$ then by the continuity of $\gamma \to f_\gamma^m(c_j)$ and of $\gamma \to c_i$ for all $m = 0, \ldots, n-1$ and $i = 1, \ldots, l$ there exists $\delta > 0$ such that

$$\underline{k}_j(\gamma) \in S \times \Sigma \text{ for all } \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta) \cap [\alpha, \beta].$$

Therefore if $\underline{k}_j(\gamma) \in \mathcal{A}_I^\infty$ then $\underline{k}_j^-(\gamma) = \underline{k}_j(\gamma) = \underline{k}_j^+(\gamma)$. If $\underline{k}_j(\gamma) = Sc_i$ for some $i \in \{1, \ldots, l\}$ then $\underline{k}_j^-(\gamma) = Sl_1l_2\ldots$ and $\underline{k}_j^+(\gamma) = Sr_1r_2\ldots$. Again by the continuity of $\gamma \to f_\gamma^n(c_j)$ and of $\gamma \to c_k$ for all $k = 1, \ldots, l$

$$l_1, r_1 \in \{I_i, I_{i+1}\}.$$

□

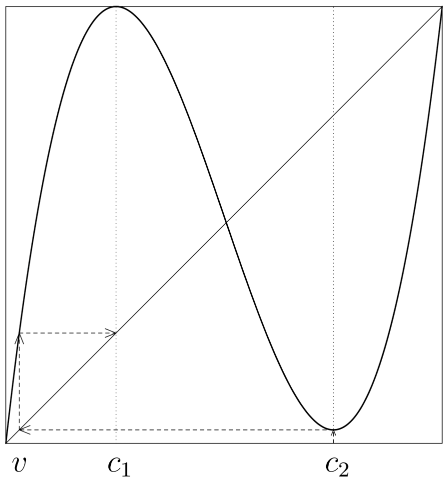

Figure 3.1: 2-modal map with $\underline{k}_2 = I_1 c_1$.

Note that we may omit the parameter γ whenever there is no danger of confusion but c_j, $\underline{i}$ and $\underline{k}_j$ for some $j \in \{1, \ldots, l\}$ should always be understood in the context of some f_γ. However, the symbols of the itineraries of Σ are $I_1, \ldots, I_{l+1}, c_1, \ldots, c_l$ and do not depend on γ.

3.3 One-parameter families of 2-modal maps

In this section we consider a natural family $\mathcal{G} : [\alpha, \beta] \to \mathcal{P}_2$ of 2-modal polynomials with negative Schwarzian derivative satisfying the following conditions

$$0, 1 \in \partial I \text{ are fixed and repulsive for } g_\alpha, \tag{3.3}$$

$$g_\gamma(c_1) = 1 \text{ for all } \gamma \in [\alpha, \beta], \tag{3.4}$$

$$g_\gamma(c_2) = 0 \text{ if an only if } \gamma = \alpha. \tag{3.5}$$

Let us denote by $v_n = g_\gamma^{n+1}(c_2)$ for $n \geq 0$ the points of the second critical orbit and let $\underline{k} = \underline{k}_2(\gamma) = k_0 k_1 \ldots$. If $S \in \mathcal{A}_I^k, k \geq 1$ and $n \geq 1$ we write S^n for $SS \ldots S \in \mathcal{A}_I^{kn}$ repeated n times and S^∞ for $SS \ldots \in \mathcal{A}_I^\infty$.

Proposition 3.2.3 shows the existence of $\underline{k}^+(\alpha) = \underline{k}(\alpha) = I_1^\infty$ therefore, there is $\delta_0 > 0$ such that

$$\underline{k} \in I_1^2 \times \Sigma \tag{3.6}$$

for all $\gamma \in [\alpha, \alpha + \delta_0]$. Figure 3.1 represents the graph of a 2-modal map with the second kneading sequence $I_1 c_1 \succ \underline{k}(\gamma)$ for all $\gamma \in [\alpha, \alpha + \delta_0]$.

Let us observe that $O^+(\mathrm{Crit}_{g_\alpha}) = \{0, c_1, c_2, 1\}$ and that by Singer's Theorem 3.2.3, g_α has no homtervals. Therefore by Corollary 3.2.2 if $\mathcal{H} : [\alpha', \beta'] \to \mathcal{P}_2$ is a natural family satisfying conditions (3.3) to (3.5) then g_α and $h_{\alpha'}$ are topologically conjugate. Therefore g_α is conjugate to the second Chebyshev polynomial (on $[-2,2]$) and topological properties of its dynamics are universal. Let us study this dynamics and extend by continuity some of its properties to some neighborhood of α in the parameter space.

We have seen that g_α has no homtervals and that all its periodic points are repulsive. Proposition 3.2.1 shows that the map

$$\underline{i}(g_\alpha) : I \to \Sigma \text{ is strictly increasing.}$$

Let us denote by $\sigma^-(\underline{i})$ the set of all preimages of $\underline{i}$ by some shift

$$\sigma^-(\underline{i}) = \left\{\underline{i}' \in \Sigma \mid \exists k \geq 0 \text{ such that } \sigma^k(\underline{i}') = \underline{i}\right\}.$$

As $(0,1) \subseteq g_\alpha(I_j)$ for all $j = 1,2,3$ and $\underline{i}(g_\alpha)(0) = I_1^\infty$, $\underline{i}(g_\alpha)(1) = I_3^\infty$

$$\underline{i}(g_\alpha)\left((0,1)\right) = \Sigma \backslash \left(\sigma^-(I_1^\infty) \cup \sigma^-(I_3^\infty)\right).$$

Let us denote by $\Sigma_0 = \underline{i}(g_\alpha)(I)$. Then

$$\underline{i}(g_\alpha) : I \to \Sigma_0 \text{ is an order preserving bijection.}$$

As g_α is decreasing on I_2, $g_\alpha(c_1) > c_1$ and $g_\alpha(c_2) < c_2$ it has exactly one fixed point $r \in I_2$ and it is repulsive. Moreover, g_α has no fixed points in I_1 or I_3 other than 0 and 1 as this would contradict the injectivity of $\underline{i}(g_\alpha)$. As 0 and 1 are repulsive fixed points $g_\alpha(x) > x$ for all $x \in (0, c_1)$ and $g_\alpha(x) < x$ for all $x \in (c_2, 1)$. Then by the C^1 continuity of $\mathcal{G}$ and Corollary 3.2.4 we obtain the following lemma.

Lemma 3.3.1. *There is $\delta_1 > 0$ such that g_γ has exactly one fixed point $r(\gamma)$ in $(0,1)$ and all its fixed points $0, 1$ and $r(\gamma)$ are repulsive for all $\gamma \in [\alpha, \alpha+\delta_1]$. Moreover, the map $\gamma \to r(\gamma)$ is continuous and $\underline{i}(r) = I_2^\infty$.*

Let p be a periodic point of period 2 of g_α. Then $\underline{i}(p)$ is periodic of period 2 and infinite as the critical points are not periodic. So $\underline{i}(p) \in \{(I_j I_k)^\infty \mid j,k = 1,2,3\}$. But $\underline{i}(g_\alpha)$ is injective, $\underline{i}(g_\alpha)(0) = I_1^\infty$, $\underline{i}(g_\alpha)(r) = I_2^\infty$ and $\underline{i}(g_\alpha)(1) = I_3^\infty$ so

$$\underline{i}(p) \in \{(I_j I_k)^\infty \mid j \neq k \text{ and } j,k = 1,2,3\} \subseteq \Sigma_0.$$

Therefore g_α has exactly 3 periodic orbits of period 2 with itinerary sequences $(I_1I_2)^\infty$, $(I_1I_3)^\infty$, $(I_2I_3)^\infty$ and their shifts. Figure 3.2 illustrates the periodic orbits of period 2 of g_α. By the C^1 continuity of $g \to g_\gamma^2$ and Corollary 3.2.4 we obtain the following lemma.

Lemma 3.3.2. *There is $\delta_2 > 0$ such that g_γ has exactly 3 periodic orbits of period 2 with itinerary sequences $(I_1I_2)^\infty$, $(I_1I_3)^\infty$, $(I_2I_3)^\infty$ and their shifts for all $\gamma \in [\alpha, \alpha+\delta_2]$. Moreover, the 6 periodic points of period 2 are repulsive and continuous with respect to γ on $[\alpha, \alpha+\delta_2]$.*

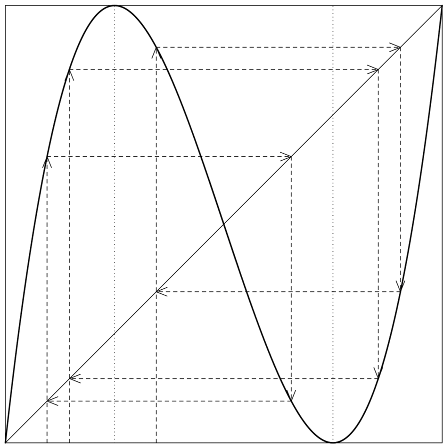

Figure 3.2: g_α and its periodic orbits of period 2, p_1 with $\underline{i}(p_1) = (I_1I_2)^\infty$, p_2 with $\underline{i}(p_2) = (I_1I_3)^\infty$ and p_3 with $\underline{i}(p_3) = (I_2I_3)^\infty$.

Let us define

$$\beta' = \alpha + \min\{\delta_0, \delta_1, \delta_2\} \tag{3.7}$$

so that $\mathcal{G}$ satisfies equality (3.6), Lemma 3.3.1 and the previous lemma for all $\gamma \in [\alpha, \beta']$.

Let us consider the dynamics of all maps g_γ with $\gamma \in [\alpha, \beta']$ from the combinatorial point of view. We observe that if $x \geq v = g_\gamma(c_2)$ then $g_\gamma^n(x) \geq v$ for all $n \geq 0$. This means that any itinerary of g_γ is of the form $\underline{i}_\gamma = I_1^k a \ldots \in \Sigma_0$ with $k \geq 0$, $a \neq I_1$ and such that $\sigma^{k+p}\underline{i}_\gamma \succeq \underline{k}$ for all $p \geq 0$. Let $\Sigma(\underline{k})$ denote the set of itineraries satisfying this condition. We observe that $(v,1)\subseteq g_\gamma(I_j)$ for $j = 1,2,3$ and $c_1, c_2 \in (v,1)$ for all $\gamma \in [\alpha, \beta']$ by relation (3.6) so we obtain the following lemma. The continuity is an immediate consequence of Proposition 3.2.2.

Lemma 3.3.3. *Let $\gamma_0 \in [\alpha, \beta']$ and $\underline{k} = \underline{k}_2(g_{\gamma_0})$. Then every finite itinerary*

$$\underline{i}_0 \in \{\underline{i} \in \Sigma(\underline{k}) \mid |\underline{i}| < \infty\}$$

is realized by a unique point $x(\underline{i}) \in I$ and $\gamma \to x(\underline{i})$ is continuous on a neighborhood of γ_0.

A kneading sequence $\underline{k} \in \Sigma(\underline{k})$ satisfies the following property.

Definition 3.3.1. *We call $\underline{m} \in \Sigma_0$* minimal *if*

$$\underline{m} \preceq \sigma^k \underline{m} \text{ for all } 0 \leq k < |m|.$$

The following proposition shows that the minimality an almost sufficient condition for an itinerary to be realized as the second kneading sequence in the family $\mathcal{G}$. This is very similar to the realization of maximal kneading sequences in unimodal families but the proof involves some particularities of our family $\mathcal{G}$. For the convenience of the reader, we include a complete proof.

Proposition 3.3.1. *Let $\alpha \leq \alpha_0 < \beta_0 \leq \beta'$ and $\underline{m}$ be a minimal itinerary such that*

$$\underline{k}(\alpha_0) \prec \underline{m} \prec \underline{k}(\beta_0).$$

Then there exists $\gamma \in (\alpha_0, \beta_0)$ such that

$$\underline{k}(\gamma) = \underline{m}.$$

Proof. Suppose that $\underline{k}(\gamma) \neq \underline{m}$ for all $\gamma \in (\alpha_0, \beta_0)$. Let $\gamma_0 = \sup\{\gamma \in [\alpha_0, \beta_0] \mid \underline{k}(\gamma) \preceq \underline{m}\}$ so $n = \min\{j \geq 0 \mid \underline{k}(\gamma_0)(j) \neq \underline{m}(j)\} < \infty$. Then, using the continuity of g_γ^n, c_1 and c_2 one may check that

$$k_n = \underline{k}(\gamma_0)(n) \in \mathcal{A}_c,$$

otherwise the maximality of γ_0 is contradicted as $\underline{k}(0), \ldots, \underline{k}(n-1)$ and $\underline{k}(n)$ would be constant on an open interval that contains γ_0. There are two possibilities

1. $k_n = c_1$ so $g_{\gamma_0}^n(c_2) = c_1$ therefore c_2 is preperiodic.

2. $k_n = c_2$ so $g_{\gamma_0}^n(c_2) = c_2$ therefore c_2 is super-attracting.

Therefore $\gamma_0 > \alpha$ and $\gamma_0 \leq \beta' < \beta$. Let us recall that $\mathcal{G}$ is a natural family so the asymptotic kneading sequences $\underline{k}^-(\gamma_0)$ and $\underline{k}^+(\gamma_0)$ do exist and are infinite. Then the definition of γ_0 shows that

$$\min(\underline{k}(\gamma_0), \underline{k}^-(\gamma_0)) \preceq \underline{m} \preceq \underline{k}^+(\gamma_0). \tag{3.8}$$

Let $\underline{m} = m_0 m_1 \ldots m_n \ldots$ and $S = m_0 \ldots m_{n-1} \in \mathcal{A}_I^n$ be the maximal common prefix of $\underline{k}(\gamma_0)$ and $\underline{m}$, so $\underline{k}(\gamma_0) = Sc_j$ with $j \in \{1, 2\}$. Therefore, using Proposition 3.2.3, $m_n \in I_j, I_{j+1}$.

Suppose $k_n = c_1$ so $g_{\gamma_0}^n = c_1$. Lemma 3.3.3 and property (3.6) show that the sequences $I_1 I_3^k c_2$ and $I_2 I_3^k c_2$ are realized as itineraries by all g_γ with $\gamma \in [\alpha, \beta']$ for all $k \geq 0$. Moreover $x(I_1 I_3^k c_2)$ is strictly increasing in k for all $\gamma \in [\alpha, \beta']$ and it is continuous in γ. Analogously, $x(I_2 I_3^k c_2)$ is strictly decreasing in k for all $\gamma \in [\alpha, \beta']$ and it is continuous in γ. Then by compactness and by the continuity of $\gamma \to g_\gamma^n$ and of $\gamma \to c_1$

$$\underline{k}^-(\gamma_0), \underline{k}^+(\gamma_0) \in S \times \{I_1, I_2\} \times I_3^\infty.$$

Therefore inequality (3.8) shows that

$$\min(c_1, I_1 I_3^\infty) = I_1 I_3^\infty \preceq \sigma^n \underline{m} \preceq I_2 I_3^\infty = \max(c_1, I_2 I_3^\infty).$$

But $\underline{m} \in \Sigma_0$ so

$$I_1 I_3^\infty \prec \sigma^n \underline{m} \prec I_2 I_3^\infty$$

therefore $m_n = c_1$ as $I_1 I_3^\infty = \max I_1 \times \Sigma$ and $I_2 I_3^\infty = \min I_2 \times \Sigma$, a contradiction.

Consequently $\underline{k}(\gamma_0) = Sc_2$ so $c_2(\gamma_0)$ is super-attracting. Then by Corollary 3.2.4 there is a neighborhood J of γ_0 such that $a(\gamma)$ is a periodic attracting point of period n for all $\gamma \in J$,

$\gamma \to a(\gamma)$ is continuous and $a(\gamma_0) = c_2(\gamma_0)$. By Singer's Theorem 3.2.3, c_2 is contained in the immediate basin of attraction $B_0(a(\gamma))$ for all $\gamma \in J$, which is disjoint from c_1. Therefore, considering the local dynamics of g_γ^n on a neighborhood of $a(\gamma)$, $\underline{k}(\gamma) = \underline{i}(g_\gamma(a))$ is also periodic of period n or finite of length n for all $\gamma \in J$. As the family $\mathcal{G}$ is natural, there exists $\varepsilon > 0$ such that c_2 is not periodic for all $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \backslash \{\gamma_0\}$. Again by Corollary 3.2.4, $\underline{k}(\gamma) = \underline{k}^-(\gamma_0)$ for all $\gamma \in (\gamma_0 - \varepsilon, \gamma_0)$ and $\underline{k}(\gamma) = \underline{k}^+(\gamma_0)$ for all $\gamma \in (\gamma_0, \gamma_0 + \varepsilon)$. Then Proposition 3.2.3 shows that

$$\underline{k}^-(\gamma_0), \underline{k}^+(\gamma_0) \in \{(SI_2)^\infty, (SI_3)^\infty\}.$$

Let $\underline{m}_1 = \min((SI_2)^\infty, (SI_3)^\infty)$ and $\underline{m}_2 = \max((SI_2)^\infty, (SI_3)^\infty)$ and

$$K = \{\underline{i} \in \Sigma \mid \underline{i} \text{ minimal and } \underline{m}_1 \prec \underline{i} \prec \underline{m}_2\}.$$

As the sequences Sc_2, $\underline{k}^-(\gamma_0)$ and $\underline{k}^+(\gamma_0)$ are all realized as a kneading sequence $\underline{k}(\gamma)$ with $\gamma \in [\alpha, \beta']$, using inequality (3.8) it is enough to show that

$$K = \{Sc_2\}.$$

Let $\underline{i} \in K \backslash \{Sc_2\}$ so

$$\underline{i} \in S \times \{I_2, I_3\} \times \Sigma.$$

Suppose $\epsilon(S) = 1$ so $\underline{m}_1 = (SI_2)^\infty$ and $\underline{m}_2 = (SI_3)^\infty$. Suppose $\underline{i}(n) = I_2$, then as $\epsilon(SI_2) = -1$ and $\underline{i}$ is minimal

$$\underline{i} \preceq \sigma^n(\underline{i}) \prec (SI_2)^\infty = \sigma^n(\underline{m}_1),$$

so $\underline{i} \in (SI_2)^2 \times \Sigma$. Therefore

$$\sigma^{2n}(\underline{m}_1) = (SI_2)^\infty \prec \sigma^{2n}(\underline{i}) \preceq \sigma^n(\underline{i}) \in SI_2 \times \Sigma,$$

so $\underline{i} \in (SI_2)^3 \times \Sigma$ and by induction $\underline{i} = \underline{m}_1 \notin K$. Suppose $\underline{i}(n) = I_3$, then

$$\underline{i} \preceq \sigma^n(\underline{i}) \prec (SI_3)^\infty = \sigma^n(\underline{m}_2),$$

as $\epsilon(SI_3) = 1$ and $\underline{i}$ minimal, so $\underline{i} \in (SI_3)^2 \times \Sigma$. For the same reason

$$\underline{i} \preceq \sigma^{2n}(\underline{i}) \prec (SI_3)^\infty = \sigma^{2n}(\underline{m}_2),$$

so $\underline{i} \in (SI_3)^3 \times \Sigma$ and by induction $\underline{i} = \underline{m}_2 \notin K$.

The case $\epsilon(S) = -1$ is symmetric so we may conclude that $K = \{Sc_2\}$ which contradicts our initial supposition. $\square$

Let us prove a complementary combinatorial property.

Lemma 3.3.4. *Let $S \in \mathcal{A}_I^n$ with $\underline{k}(\alpha) \preceq SI_2^\infty \preceq \underline{k}(\beta')$ and such that SI_2^∞ is minimal. If $i_1 i_2 \ldots \in \Sigma$ and $i_1, i_2, \ldots \in \mathcal{A} \backslash \{I_1\}$ then*

$$SI_2^k i_1 i_2 \ldots \in \Sigma \text{ is minimal for all } k \geq |S|.$$

Proof. Let $\underline{i} = SI_2^k i_1 i_2 \ldots \in \Sigma$, $n = |S|$ and $k \geq n$. Suppose there exists $j > 0$ such that

$$\sigma^j(\underline{i}) \prec \underline{i}.$$

As $SI_2^\infty \preceq \underline{k}(\beta') = I_1 \ldots$

$$\underline{i} \in I_1 \times \Sigma.$$

Then $j < n$ and we set $m = \min\{p \geq 0 \mid \sigma^j(\underline{i})(p) \neq \underline{i}(p)\}$. Therefore $m \leq n-1$ so

$$\sigma^j(SI_2^\infty) \prec SI_2^\infty$$

as $\underline{i}$ coincides with SI_2^∞ on the first $2n$ symbols, a contradiction. $\square$

Using relation (3.6), $\underline{k}(\gamma) = I_1 \ldots$ so $I_2^k c_j \in \Sigma(\underline{k}(\gamma))$ for all $k \geq 0$, $j = 1, 2$ and $\gamma \in [\alpha, \beta']$. Then by Lemma 3.3.3 the maps

$$\gamma \to p_k(\gamma) = x(I_2^k c_1)(\gamma) \text{ and } \gamma \to q_k(\gamma) = x(I_2^k c_2)(\gamma)$$

are uniquely defined and continuous on $[\alpha, \beta']$ for all $k \geq 0$. Let us recall that g_γ is decreasing on I_2 so

$$c_1 \prec I_2 c_2 \prec I_2^2 c_1 \prec I_2^3 c_2 \prec \ldots \prec I_2^\infty \prec \ldots \prec I_2^3 c_1 \prec I_2^2 c_2 \prec I_2 c_1 \prec c_2,$$

therefore

$$c_1 = p_0 < q_1 < p_2 < q_3 < \ldots < r < \ldots < p_3 < q_2 < p_1 < q_0 = c_2$$

for all $\gamma \in [\alpha, \beta']$.

Let us show that $p_k \to r$ and $q_k \to r$ as $k \to \infty$ for all $\gamma \in [\alpha, \beta']$. Let

$$\begin{aligned} r^- &= \lim_{k\to\infty} p_{2k} = \lim_{k\to\infty} q_{2k+1} \text{ and} \\ r^+ &= \lim_{k\to\infty} q_{2k} = \lim_{k\to\infty} p_{2k+1}. \end{aligned}$$

Suppose that $r^- < r^+$ then by continuity $g_\gamma(r^-) = r^+$ and $g_\gamma(r^+) = r^-$, as $g_\gamma(p_{k+1}) = p_k$ and $g_\gamma(q_{k+1}) = q_k$ for all $k \geq 0$. Then r^- and r^+ are periodic points of period 2 and with itinerary sequence I_2^∞, which contradicts Lemma 3.3.2. By compactness

$$p_k, q_k \to r \text{ uniformly as } k \to \infty. \tag{3.9}$$

The following proposition shows that these convergences have a counterpart in the parameter space.

Proposition 3.3.2. *Let $S \in \mathcal{A}_I^n$ for some $n \geq 0$ be such that SI_2^∞ is minimal and $\underline{k}^{-1}(SI_2^\infty)$ is finite. Let $\alpha \leq \alpha_0 < \beta_0 \leq \beta'$ be such that $\underline{k}(\alpha_0) \prec SI_2^\infty \prec \underline{k}(\beta_0)$ and $S' = SI_2^{k+1}$ with $k \geq 0$ and such that $\epsilon(S') = 1$. If $\underline{i}_1 = S'c_1$, $\underline{i}_2 = S'c_2$ and k is sufficiently big then we may define*

$$\begin{aligned} \gamma_1 &= \max\left(\underline{k}^{-1}(\underline{i}_1) \cap (\alpha_0, \beta_0)\right) \text{ and} \\ \gamma_2 &= \min\left(\underline{k}^{-1}(\underline{i}_2) \cap (\gamma_1, \beta_0)\right) \end{aligned} \tag{3.10}$$

and then

$$\lim_{k\to\infty} (\gamma_2 - \gamma_1) = 0.$$

Proof. First let us remark that the condition $\epsilon(S')=1$ guarantees that

$$\underline{i}_1 \prec SI_2^\infty \prec \underline{i}_2.$$

Using for example convergences (3.9) and the bijective map $\underline{i}(g_\alpha)$ there exists $N_0>0$ such that for all $k \geq N_0$, $\underline{k}(\alpha_0) \prec \underline{i}_1 \prec \underline{i}_2 \prec \underline{k}(\beta_0)$. Moreover, if $k \geq n$ then $\underline{i}_1$ and $\underline{i}_2$ are minimal, using Lemma 3.3.4.

Therefore for $k \geq \max(N_0, n)$ we may apply Proposition 3.3.1 to show that there exist $\gamma_1 \in \underline{k}^{-1}(\underline{i}_1) \cap (\alpha_0, \beta_0)$ and $\gamma_2 \in \underline{k}^{-1}(\underline{i}_2) \cap (\gamma_1, \beta_0)$. As $\underline{i}_1$ and $\underline{i}_2$ are finite and the family $\mathcal{G}$ is natural, $\underline{k}^{-1}(\underline{i}_1)$ and $\underline{k}^{-1}(\underline{i}_2)$ are finite.

We may apply again Proposition 3.3.1 to see that γ_1 is increasing to a limit γ^- as $k\to\infty$. Again by Proposition 3.3.1 and by the finiteness of $\underline{k}^{-1}(SI_2^\infty)$ there exists

$$\gamma_0 = \max\left(\underline{k}^{-1}(SI_2^\infty) \cap (\alpha_0, \beta_0)\right) < \beta_0 \text{ and } \gamma^- \leq \gamma_0.$$

For the same reasons there is $N>0$ such that $\gamma_2 > \gamma_0$ for all $k \geq N$, therefore γ_2 becomes decreasing and converges to some $\gamma^+ \geq \gamma_0$.

Suppose that the statement does not hold, that is

$$\gamma^- < \gamma^+.$$

The map $\underline{i}(g_\alpha): I \to \Sigma_0$ is bijective and order preserving and $p_i \to r$, $q_i \to r$ as $i\to\infty$ therefore

$$\{\underline{i} \in \Sigma_0 \mid \underline{i}_1 \preceq \underline{i} \preceq \underline{i}_2 \text{ for all } k>0\} = \{SI_2^\infty\}.$$

Then the definitions of γ^- and γ^+ imply that

$$\underline{k}(\gamma) = SI_2^\infty \text{ for all } \gamma \in [\gamma^-, \gamma^+]$$

which contradicts the hypothesis. □

From the previous proof we may also retain the following Corollary.

Corollary 3.3.1. *Assume the hypothesis of the previous proposition. Then*

$$\lim_{k\to\infty} \gamma_1 = \lim_{k\to\infty} \gamma_2 = \gamma_0$$

and $\underline{k}(\gamma_0) = SI_2^\infty$.

We may also control the growth of the derivative on the second critical orbit in the setting of the last proposition. In fact, letting $k\to\infty$, the second critical orbit spends most of its time very close to the fixed repulsing point r. Therefore the growth of the derivative along this orbit is exponential.

Let us also compute some bounds for the derivative along two types of orbits.

Lemma 3.3.5. *Let* $[\gamma_1, \gamma_2] \subseteq [\alpha, \beta']$*,* $n \geq 0$*,* $S \in \mathcal{A}_I^n$ *and* $\underline{i}_1, \underline{i}_2 \in S \times \Sigma$ *with* $\underline{i}_1 \prec \underline{i}_2$ *be finite or equal to* I_1^∞*,* I_2^∞ *or* I_3^∞*. If* $\underline{i}_1$*,* $\underline{i}_2$ *are realized on* $[\gamma_1, \gamma_2]$ *then there exists* $\theta > 0$ *such that*

$$\theta < \left| \left(g_\gamma^j\right)'(x) \right| < \theta^{-1}$$

for all $\gamma \in [\gamma_1, \gamma_2]$*,* $x \in [x(\underline{i}_1), x(\underline{i}_2)]$ *and* $j = 1, \ldots, n$*.*

Proof. Let us remark that $\underline{i}(x) \in S \times \Sigma$ therefore $\left(g_\gamma^j\right)'(x) \neq 0$ for all $\gamma \in [\gamma_1, \gamma_2]$, $x \in [x(\underline{i}_1), x(\underline{i}_2)]$ and $j = 1, \ldots, n$. As $x(\underline{i}_1)$ and $x(\underline{i}_2)$ are continuous by Lemmas 3.3.1 and 3.3.3, the set

$$\left\{(\gamma, x) \in \mathbb{R}^2 \mid \gamma \in [\gamma_1, \gamma_2], x \in [x(\underline{i}_1), x(\underline{i}_2)]\right\}$$

is compact. Therefore the continuity of $(\gamma, x) \rightarrow \left(g_\gamma^j\right)'(x)$ for all $j = 1, \ldots, n$ implies the existence of θ. $\square$

The previous lemma helps us estimate the derivative of $g_\gamma^n(x)$ on a compact interval of parameters if $\underline{i}(x) \in I_j^n \times \Sigma$ and n is sufficiently big. Let us denote

$$I_j(n)(\gamma) = \left\{x \in I_j \mid g_\gamma^k(x) \in I_j \text{ for all } k = 1, \ldots, n\right\}$$

for j=1,2,3, the interval of points of I_j that stay in I_j under n iterations. Let also s_j be the unique fixed point in I_j.

Lemma 3.3.6. *Let* $[\gamma_1, \gamma_2] \subseteq [\alpha, \beta']$*,* $j \in \{1, 2, 3\}$ *and* $\varepsilon > 0$*. Let also*

$$\lambda_1 = \min_{\gamma \in [\gamma_1, \gamma_2]} \left|g_\gamma'(s_j)\right|,$$

$$\lambda_2 = \max_{\gamma \in [\gamma_1, \gamma_2]} \left|g_\gamma'(s_j)\right|.$$

There exists $N > 0$ *such that for all* $k > 0$

$$\lambda_1^{k(1-\varepsilon)} < \left| \left(g_\gamma^k\right)'(x) \right| < \lambda_2^{k(1+\varepsilon)}$$

for all $\gamma \in [\gamma_1, \gamma_2]$ *and* $x \in I_j(m)$ *where* $m = \max(k, N)$*.*

Proof. Let us first observe that by the definition (3.7) of β'

$$1 < \lambda_1 < \lambda_2.$$

Lemma 3.3.3 shows that the itinerary sequences $I_j^n c_1$, $I_j^n c_2$ are realized on $[\alpha, \beta']$ for all $n \geq 0$. We may easily obtain analoguous convergences to (3.9) if $j \in \{1, 3\}$ therefore

$$x(I_j^n c_1), x(I_j^n c_2) \rightarrow s_j \text{ uniformly as } n \rightarrow \infty.$$

Moreover $\partial I_j(n) \subseteq \{x(I_j^n c_1), x(I_j^n c_2), s_j\}$ for all $\gamma \in [\gamma_1, \gamma_2]$ and $n \geq 0$. Using the continuity of s_j and of $(\gamma, x) \rightarrow g_\gamma'(x)$ there exists $N_0 > 0$ such that

$$\lambda_1^{1-\frac{\varepsilon}{2}} < \left|g_\gamma'(x)\right| < \lambda_2^{1+\frac{\varepsilon}{2}}$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in I_j(N_0)$.

Using Lemma 3.3.5 there exists $\theta > 0$ such that

$$\theta < \left| \left(g_\gamma^m\right)'(x) \right| < \theta^{-1}$$

for all $\gamma \in [\gamma_1, \gamma_2]$, $x \in I_j(N_0)$ and $1 \leq m \leq N_0$. Let $N_1 > 0$ be such that

$$\lambda_1^{N_1 \frac{\varepsilon}{2}} > \theta^{-1} \lambda_2^{N_0(1+\varepsilon)}$$

and set $N = N_0 + N_1$. Let $k > N_1$ and $n = \max(N_1, k - N_0)$ then

$$\theta \lambda_1^{n\left(1-\frac{\varepsilon}{2}\right)} < \left| \left(g_\gamma^k\right)'(x) \right| < \theta^{-1} \lambda_2^{n\left(1+\frac{\varepsilon}{2}\right)}$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in I_j(m)$. As $n \geq N_1$ and $1 < \lambda_1 < \lambda_2$

$$\lambda_1^{k(1-\varepsilon)} < \left| \left(g_\gamma^k\right)'(x) \right| < \lambda_2^{k(1+\varepsilon)}$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in I_j(m)$. If $k \leq N_1$ then $g_\gamma^n(x) \in I_j(N_0)$ for all $n = 0, \ldots, k-1$ so

$$\lambda_1^{k(1-\varepsilon)} < \lambda_1^{k\left(1-\frac{\varepsilon}{2}\right)} < \left| \left(g_\gamma^k\right)'(x) \right| < \lambda_2^{k\left(1+\frac{\varepsilon}{2}\right)} < \lambda_2^{k(1+\varepsilon)}$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in I_j(m)$. $\square$

We may remark that if we assume the hypothesis of the previous lemma then g_γ^k is monotone on $I_j(m)$ therefore

$$\lambda_2^{-k(1+\varepsilon)} < |I_j(m)| < \lambda_1^{-k(1-\varepsilon)}. \tag{3.11}$$

Let $d_n : [\alpha, \beta'] \to \mathbb{R}_+$ be defined by

$$d_n(\gamma) = \left| \left(g_\gamma^n\right)'(v) \right|,$$

where $v = g_\gamma(c_2)$ the second critical value. As $\gamma \to v$ is continuous and $\gamma \to g_\gamma^n$ is C^1 continuous, d_n is continuous. The family $\mathcal{G}$ is natural so d_n has finitely many zeros for all $n \geq 0$.

Corollary 3.3.2. *Assume the hypothesis of Proposition 3.3.2 and let* $\lambda_0 = |g'_{\gamma_0}(r)| > 1$. *For all* $0 < \varepsilon < 1$ *there exists* $N > 0$ *such that if* $k \geq N$ *then*

$$\lambda_0^{(n+k)(1-\varepsilon)} < d_{n+k}(\gamma) < \lambda_0^{(n+k)(1+\varepsilon)} \text{ for all } \gamma \in [\gamma_1, \gamma_2].$$

Proof. Let us remark that $|\underline{k}(\gamma)| > n$ for all $\gamma \in [\gamma_1, \gamma_2]$ therefore there exists $\theta > 0$ such that

$$\theta < d_n(\gamma) < \theta^{-1} \text{ for all } \gamma \in [\gamma_1, \gamma_2].$$

Using the previous lemma and Corollary 3.3.1 there exists $N_0 > 0$ such that if $k \geq N_0$ then

$$\lambda_0^{k\left(1-\frac{\varepsilon}{2}\right)} < \left| \left(g_\gamma^k\right)'(v_n) \right| < \lambda_0^{k\left(1+\frac{\varepsilon}{2}\right)} \text{ for all } \gamma \in [\gamma_1, \gamma_2].$$

Therefore it is enough to choose $N \geq N_0$ such that

$$\lambda_0^{N \frac{\varepsilon}{2}} > \theta^{-1} \lambda_0^{n(1-\varepsilon)}.$$

$\square$

3.4 UHP does not imply RCE

In this section we consider a family $\mathcal{G} : [\alpha, \beta] \to \mathcal{P}_2$ satisfying all properties (3.3) to (3.6) and Lemmas 3.3.1 and 3.3.2 for all $\gamma \in [\alpha, \beta]$. We build a decreasing sequence of families $\mathcal{G}_n : [\alpha_n, \beta_n] \to \mathcal{P}_2$ with $\mathcal{G}_0 = \mathcal{G}$, $\alpha_n \nearrow \gamma_0$ and $\beta_n \searrow \gamma_0$ as $n \to \infty$. This means that $\mathcal{G}_n(\gamma) = \mathcal{G}(\gamma)$ for all $n \geq 0$ and $\gamma \in [\alpha_n, \beta_n]$. We obtain our counterexample as a limit $g_{\gamma_0} = \mathcal{G}(\gamma_0) = \mathcal{G}_n(\gamma_0)$ for all $n \geq 0$. For all $n \geq 0$ we choose two finite minimal itinerary sequences $\underline{i}_1(n+1)$ and $\underline{i}_2(n+1)$ as in Proposition 3.3.2 such that

$$\underline{k}_2(\alpha_n) \prec \underline{i}_1(n+1) \prec \underline{i}_2(n+1) \prec \underline{k}_2(\beta_n).$$

We set $\alpha_{n+1} = \gamma_1$ and $\beta_{n+1} = \gamma_2$ and choosing sufficient long sequences $\underline{i}_1(n+1)$ and $\underline{i}_2(n+1)$ we obtain the convergences $\alpha_n \to \gamma_0$ and $\beta_n \to \gamma_0$ as $n \to \infty$.

Let $T_2(x) = x^3 - 3x$ be the second Chebyshev polynomial. Observe that -2, 0 and 2 are fixed and that the critical points $c_1 = -1$ and $c_2 = 1$ are sent to 2 respectively -2. Its Schwarzian derivative $S(T_2)(x) = -\frac{4x^2+1}{(x^2-1)^2}$ is negative on $\mathbb{R} \setminus \{c_1, c_2\}$. Let $h > 0$ small and for each $\gamma \in [0, h]$ two order preserving linear maps $P_\gamma(x) = x(4+\gamma) - 2 - \gamma$ and $Q_\gamma(y) = \frac{y - T_2(-2-\gamma)}{2 - T_2(-2-\gamma)}$ that map $[0, 1]$ onto $[-2-\gamma, 2]$ respectively $[T_2(-2-\gamma), T_2(2)]$ onto $[0, 1]$. Let then

$$g_\gamma = Q_\gamma \circ T_2 \circ P_\gamma \tag{3.12}$$

be a 2-modal degree 3 polynomial. As $S(P_\gamma) = S(Q_\gamma) = 0$ for all $\gamma \in [0, h]$, using equality (3.1), one may check that

$$S(g_\gamma) < 0 \text{ on } I \setminus \{c_1(\gamma), c_2(\gamma)\} \text{ for all } \gamma \in [0, h].$$

If we write

$$g_\gamma(x) = \sum_{k=0}^{3} a_k(\gamma) x^k \tag{3.13}$$

it is not hard to check that $\gamma \to a_k(\gamma)$ is continuous on $[0, h]$ for $k = 0, \ldots, 3$ therefore $\gamma \to g_\gamma$ is continuous with respect to the C^1 topology on I. By the definition of $\mathcal{P}_2$ (see page 46), as $g_\gamma(0) = 0$ for all $\gamma \in [0, h]$, $\mathcal{G} : [0, h] \to \mathcal{P}_2$ with $\mathcal{G}(\gamma) = g_\gamma$ for all $\gamma \in [0, h]$ is a family of 2-modal maps with negative Schwarzian derivative. Observe that 0 and 1 are fixed points for all $\gamma \in [0, h]$ and that they are repulsive for g_0, with $g_0'(0) = g_0'(1) = 9$, which is condition (3.3). Moreover, $g_\gamma(c_1) = 1$ for all $\gamma \in [0, h]$ thus $\mathcal{G}$ satisfies also (3.4). Observe that if $\gamma \in [0, h]$ then $Q_\gamma(-2) = 0$ if and only if $\gamma = 0$ so $\mathcal{G}$ satisfies also condition (3.5). We show that $\mathcal{G}$ is also natural and that any minimal sequence SI_2^∞ with $S \in \mathcal{A}_I^n$ and $n \geq 0$ equals the second kneading sequence $\underline{k}(\gamma)$ for at most finitely many $\gamma \in [0, h]$. This allows us to use all the results of the previous section for the family $\mathcal{G}$.

Let $G : [0, h] \times [0, 1] \to \mathbb{R}$ be defined by

$$G(\gamma, x) = g_\gamma(x) \text{ for all } \gamma \in [0, h] \text{ and } x \in [0, 1].$$

Then

$$G(\gamma, x) = \frac{P_1(\gamma, x)}{P_2(\gamma)},$$

where P_1 and P_2 are polynomials. Using definition (3.12), we may compute P_2 easily

$$P_2(\gamma) = 2 - T_2(-2-\gamma) = (\gamma+1)^2(\gamma+4).$$

We may therefore extend G analytically on a neighborhood $\Omega\subseteq\mathbb{R}^2$ of $[0,h]\times[0,1]$. The critical points c_1 and c_2 are continuously defined on $[0,h]$ by Lemma 3.3.3. They are also analytic in γ as a consequence of the Implicit Functions Theorem for real analytic maps applied to $\frac{\partial G}{\partial x}$. Therefore for all $n \geq 0$ the map $g_\gamma^n(c_2)$ is analytic on a neighborhood of $[0,h]$ so

$$c_j(\gamma) - g_\gamma^n(c_2) \text{ has finitely many zeros in } [0,h]$$

for all $j \in \{1,2\}$ and $n \geq 0$ as $g_0^n(c_2) = 0$ and $c_1(\gamma), c_2(\gamma) \in (0,1)$ for all $\gamma \in [0,h]$. The family $\mathcal{G}$ is therefore natural so by eventually shrinking h we may also suppose that $\mathcal{G}$ satisfies property (3.6) and Lemmas 3.3.1 and 3.3.2 for all $\gamma \in [0,h]$. Then the repulsive fixed point r is continuously defined on $[0,h]$ and again by the Implicit Functions Theorem applied to $G(\gamma,x)-x$, it is analytic on a neighborhood of $[0,h]$. Then

$$r(\gamma) - g_\gamma^n(c_2) \text{ has finitely many zeros in } [0,h]$$

for all $n \geq 0$ as $r(0) - g_0^n(c_2) = \frac{1}{2}$.

Let then $\mathcal{G}_0 = \mathcal{G}$ so $\alpha_0 = 0$ and $\beta_0 = h$. Our counterexample g_{γ_0} should be UHP but not RCE. Its first critical point is non-recurrent as $g_\gamma(c_1) = 1$ and 1 is fixed for all $\gamma \in [\alpha_0, \beta_0]$. Therefore the second critical point c_2 should be recurrent and not Collet-Eckmann. We let c_2 accumulate on c_1 also to control the expansion of the derivative along its orbit. In order to obtain UHP we build g_{γ_0} such that its second critical orbit spends most of the time near r or 1 so its derivative accumulates sufficient expansion.

3.4.1 A construction

The construction of the sequence $(\mathcal{G}_n)_{n\geq 0}$ is realized by imposing at the n-th step the behavior of the second critical orbit for a time span $t_{n-1}+1, t_{n-1}, \ldots, t_n$. This is achieved specifying the second kneading sequence and using Proposition 3.3.2. We set $t_0 = 0$.

We have seen that $\underline{k}^+(0) = I_1^\infty$ and that $g_\gamma(x) > x$ for all $x \in (0,c_1)$ and all $\gamma \in [0,h]$ as 0 is repulsive and g_γ has no fixed point in $(0,c_1)$. Therefore the backward orbit of c_1 in I_1 converges to 0 and by compactness the convergence is uniform. Then

$$\underline{k}^{-1}\left(I_1^k c_1\right) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

using Proposition 3.3.1 for their existence. Then for any $\varepsilon_0 > 0$ there is $k_0 > 0$ such that $I_1^{k_0} c_1 \prec \underline{k}(\beta_0)$ and $||g_0 - g_\gamma||_{C^1} < \varepsilon_0$ for all $\gamma \in [0, \underline{k}^{-1}(I_1^{k_0} c_1)]$. In particular, if

$$1 < \lambda < \lambda' < |g_0'(r)| = 3 < |g_0'(0)| = |g_0'(1)| = 9$$

then for ε_0 sufficiently small

$$\lambda' < \left|g_\gamma'(r)\right|, \lambda' < \left|g_\gamma'(0)\right| \text{ and } \lambda' < \left|g_\gamma'(1)\right| \tag{3.14}$$

for all $\gamma \in [0, \underline{k}^{-1}(I_1^{k_0}c_1)]$. Let $S_0 = I_1^{k_0+1} \in \mathcal{A}_I^{k_0+1}$ so $\underline{i} \prec I_1^{k_0}c_1$ for all $\underline{i} \in S_0 \times \Sigma$. Moreover, $S_0 I_2^\infty$ is minimal. Using Proposition 3.3.2 we find $\alpha_0 < \gamma_1 < \gamma_2 < \beta_0$ such that

$$\underline{k}(\alpha_0) \prec \underline{k}(\gamma_1) \prec S_0 I_2^\infty \prec \underline{k}(\gamma_2) \prec \underline{k}(\beta_0)$$

with $\underline{k}(\gamma_1), \underline{k}(\gamma_2) \in S_0 I_2 \times \Sigma$ and

$$|\gamma_2 - \gamma_1| < 2^{-1}.$$

We set $\alpha_1 = \gamma_1$ and $\beta_1 = \gamma_2$ and define $\mathcal{G}_1 : [\alpha_1, \beta_1] \to \mathcal{P}_2$ by $\mathcal{G}_1(\gamma) = \mathcal{G}(\gamma) = g_\gamma$ for all $\gamma \in [\alpha_1, \beta_1]$. Moreover, let $t_1 = k + |S_0|$ and $S_1 = S_0 I_2^k$, where k is specified by Proposition 3.3.2, then

$$\underline{k}(\gamma) \in S_1 I_2 \times \Sigma \tag{3.15}$$

for all $\gamma \in [\alpha_1, \beta_1]$, and using Corollary 3.3.2 we may also suppose that

$$d_m(\gamma) > \lambda^m \tag{3.16}$$

for all $\gamma \in [\alpha_1, \beta_1]$, where $m = t_1 = |S_1|$. Let us recall that $d_n(\gamma) = \left| \left(g_\gamma^n\right)'(v) \right|$.

Then we build inductively the decreasing sequence of families $(\mathcal{G}_n)_{n \geq 0}$ such that for all $n \geq 1$, $\mathcal{G}_n$ satisfies

$$|\underline{k}(\alpha_n)|, |\underline{k}(\beta_n)| < \infty, \tag{3.17}$$

$$|\beta_n - \alpha_n| < 2^{-n}, \tag{3.18}$$

$$\underline{k}(\alpha_n) \prec S_n I_2^\infty \prec \underline{k}(\beta_n) \tag{3.19}$$

and conditions (3.14) to (3.16) for all $\gamma \in [\alpha_n, \beta_n]$, for some $S_n \in \mathcal{A}_I^m$ with $S_n I_2^\infty$ minimal, where $m = t_n$. As the sequence $(\mathcal{G}_n)_{n \geq 0}$ is decreasing, inequality (3.14) is satisfied by all $\mathcal{G}_n$ with $n \geq 1$. For transparency we denote

$$d_{n,p}(\gamma) = \left| \left(g_\gamma^p\right)'(v_n) \right|$$

which also equals $d_{n+p}(\gamma) d_n^{-1}(\gamma)$ whenever $|\underline{k}(\gamma)| > n$ so $d_n(\gamma) \neq 0$.

Let us describe two types of steps, one that takes the second critical orbit near c_1 to control the growth of the derivative and the other that takes it near c_2 to make the second critical point c_2 recurrent. We alternate the two types of steps in the construction of the sequence $(\mathcal{G}_n)_{n \geq 0}$ to obtain our counterexample.

The following proposition describes the passage near c_1.

Proposition 3.4.1. *Let the family $\mathcal{G}_n$ with $n \geq 1$ satisfy conditions (3.14) to (3.19) and*

$$0 < \lambda_1 < \lambda_2 < \lambda.$$

Then there exists a subfamily $\mathcal{G}_{n+1}$ of $\mathcal{G}_n$ satisfying the same conditions and such that there exists $2t_n < p < t_{n+1}$ with the following properties

1. $\max\limits_{\gamma \in [\alpha_{n+1}, \beta_{n+1}]} \left| \log \left| g'_\gamma(r) \right| - \frac{1}{p-1} \log d_{p-1}(\gamma) \right| < \log \lambda_2 - \log \lambda_1$.
2. $\lambda_1^p < d_p(\gamma) < \lambda_2^p$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.
3. $d_{t_n,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \ldots, p-1-t_n$.
4. $d_{p,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \ldots, t_{n+1} - p$.
5. $d_{t_n, t_{n+1}-t_n}(\gamma) > \lambda^{t_{n+1}-t_n}$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.

Proof. This proof follows a very simple idea, to define the family $\mathcal{G}_{n+1}$ with

$$S_{n+1} = S_n I_2^{k_1} I_3^{k_2} I_2^{k_3},$$

as described by properties (3.15) and (3.19). For k_1 and k_3 sufficiently big there exist k_2 such that the conclusion is satisfied for $p = t_n + k_1$.

Let us apply Proposition 3.3.2 to S_n, α_n and β_n. Let $k_1 = k+1$, $\lambda_0 = \left| g'_{\gamma_0}(r) \right|$ and $\lambda_3 = \left| g'_{\gamma_0}(1) \right|$. By inequality (3.14)

$$0 < \lambda_1 < \lambda_2 < \lambda < \lambda_0$$

therefore there exists $\varepsilon_0 \in (0,1)$ such that

$$\frac{(1+\varepsilon_0)\log \lambda_0 - \log \lambda_2}{(1-\varepsilon_0)\log \lambda_3} < \frac{(1-\varepsilon_0)\log \lambda_0 - \log \lambda_1}{(1+\varepsilon_0)\log \lambda_3}.$$

We choose $0 < \varepsilon < \varepsilon_0$ such that

$$\varepsilon < \frac{\log \lambda_2 - \log \lambda_1}{8 \log \lambda_0}.$$

Let us recall that

$$\underline{k}(\gamma_1) = S_n I_2^{k_1} c_1 \prec S_{n+1} \times \Sigma \prec \underline{k}(\gamma_2) = S_n I_2^{k_1} c_2. \tag{3.20}$$

Using Lemma 3.3.6 and Corollaries 3.3.1 and 3.3.2 there exists N_0 such that if $k_1 > N_0$ then the first and the third conclusions are satisfied provided $[\alpha_{n+1}, \beta_{n+1}] \subseteq [\gamma_1, \gamma_2]$.

Let $y(\gamma) \in I$ with $\underline{i}(y) \in I_2 I_3^{k_2} I_2 \times \Sigma$ and $y' = g_\gamma(x)$. By Corollary 3.3.1, Lemma 3.3.6 and inequality (3.11) there exist $N_1, N'_0 > 0$ such that if $k_1 > N_1$ and $k_2 > N'_0$ then for all $\gamma \in [\gamma_1, \gamma_2]$

$$\lambda_3^{-k_2(1+\varepsilon)} < |1 - y'| < \lambda_3^{-(k_2-1)(1-\varepsilon)}, \tag{3.21}$$

as $y \in I_3(k_2 - 1) \backslash I_3(k_2)$.

Let us recall that $g_\gamma(x) = \sum_{k=0}^{3} a_k(\gamma) x^k$ with a_i continuous and $g'_\gamma(c_1) = 0$, $g''_\gamma(c_1) \neq 0$ for all $\gamma \in [\alpha, \beta']$ and c_1 is continuous. Therefore there exist constants $M > 1$, $\delta > 0$ and $N_2 > 0$ such that if $k_1 > N_2$ and $\gamma \in [\gamma_1, \gamma_2]$ then

$$\begin{array}{lclcl} M^{-1}(x-c_1)^2 & < & |1 - g_\gamma(x)| & < & M(x-c_1)^2 \text{ and} \\ M^{-1}(x-c_1) & < & \left| g'_\gamma(x) \right| & < & M(x-c_1) \end{array} \tag{3.22}$$

for all $x \in (c_1 - \delta, c_1 + \delta)$. Using inequality (3.21) there exists N_1' such that if $k_2 > N_1'$ then $|1 - y'| < M^{-1}\delta^2$ therefore

$$M^{-\frac{3}{2}}\lambda_3^{-\frac{k_2}{2}(1+\varepsilon)} < \left|g_\gamma'(y)\right| < M^{\frac{3}{2}}\lambda_3^{-\frac{k_2-1}{2}(1-\varepsilon)}.$$

Let $k_1 > \max(t_n, N_0, N_1, N_2)$ and $k_2 > \max(N_0', N_1')$. Lemma 3.3.4 shows that $S_{n+1}I_2^\infty$ is minimal. We may therefore apply Proposition 3.3.2 with $S = S_n I_2^{k_1+1} I_3^{k_2}$ using also inequality (3.20). Let $k_3 = k$ and α_{n+1} and β_{n+1} be the new bounds for γ provided by Proposition 3.3.2. Let us recall that $p = t_n + k_1$ and $v_n = g_\gamma^{n+1}(c_2)$ for all $n \geq 0$, therefore $\underline{i}(v_p) \in I_2 I_3^{k_2} \times \Sigma$ so we may set $y = v_p$ and $y' = v_{p+1}$. Let us remark that

$$d_p(\gamma) = d_{p-1}(\gamma) \cdot \left|g_\gamma'(y)\right| \text{ for all } \gamma \in [\alpha_{n+1}, \beta_{n+1}].$$

By the choice of k_1 and k_2, for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$

$$M^{-\frac{3}{2}}\lambda_0^{(p-1)(1-\varepsilon)}\lambda_3^{-\frac{k_2}{2}(1+\varepsilon)} < d_p(\gamma) < M^{\frac{3}{2}}\lambda_0^{(p-1)(1+\varepsilon)}\lambda_3^{-\frac{k_2-1}{2}(1-\varepsilon)}.$$

Therefore the second conclusion is satisfied if

$$p \log \lambda_1 < -\frac{3}{2}\log M + (p-1)(1-\varepsilon)\log \lambda_0 - \frac{k_2}{2}(1+\varepsilon)\log \lambda_3$$

and

$$p \log \lambda_2 > \frac{3}{2}\log M + (p-1)(1+\varepsilon)\log \lambda_0 - \frac{k_2-1}{2}(1-\varepsilon)\log \lambda_3.$$

We may let $p \to \infty$ and $\frac{k_2}{2p} \to \eta$ so it is enough to find $\eta > 0$ such that

$$\begin{array}{rcl} \log \lambda_1 & < & (1-\varepsilon)\log \lambda_0 - \eta(1+\varepsilon)\log \lambda_3 \text{ and} \\ \log \lambda_2 & > & (1+\varepsilon)\log \lambda_0 - \eta(1-\varepsilon)\log \lambda_3. \end{array}$$

The existence of η is guaranteed by the choice of $\varepsilon < \varepsilon_0$.

Again by inequality (3.14), Lemma 3.3.6 and Corollary 3.3.2, if k_2 and k_3 are sufficiently big then the last two conclusions are satisfied. If k_3 is sufficiently big then by Corollary 3.3.1 inequality (3.18) is also satisfied. $\square$

The following proposition describes the passage near c_2.

Proposition 3.4.2. *Let the family $\mathcal{G}_n$ with $n \geq 1$ satisfy conditions (3.14) to (3.19) and*

$$\Delta > 0.$$

Then there exists a subfamily $\mathcal{G}_{n+1}$ of $\mathcal{G}_n$ satisfying the same conditions and such that there exists $t_n < p < t_{n+1}$ with the following properties

1. $\left|g_\gamma^p(c_2) - c_2\right| < \Delta$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.

2. $d_{t_n,l}(\gamma) > \lambda^l$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ *and* $l = 1, \ldots, t_{n+1} - t_n$.

3. $d_{p,t_{n+1}-p}(\gamma) > \lambda^{t_{n+1}-p}$ *for all* $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.

Proof. Once again, we build the family $\mathcal{G}_{n+1}$ using the prefix of the kneading sequence

$$S_{n+1} = S_n I_2^{k_1} S_n I_2^{k_2+1} I_3 I_2^{k_3}$$

and show that we may choose k_2 such that if k_1 and k_3 are sufficiently big then the conclusion is satisfied for $p = t_n + k_1$.

We apply Proposition 3.3.2 to S_n, α_n and β_n. Let $k_1 = k+2$, $\lambda_0 = \left|g'_{\gamma_0}(r)\right| > \lambda'$ and

$$S' = S_n I_2^{k_2+1} I_3.$$

In the sequel k_2 is chosen such that $\epsilon(S') = 1$ therefore $\underline{k}(\gamma_0) = S_n I_2^{\infty} \prec S' \ldots$ so

$$S_{n+1} I_2^{\infty} \text{ is minimal if } k_1 - 1 > k_2 > t_n.$$

Indeed, suppose that there exists $j > 0$ such that $\sigma^j \left(S_{n+1} I_2^{\infty}\right) \prec S_{n+1} I_2^{\infty}$. Let us recall that $t_n = |S_n|$ and $S_n I_2^{\infty}$ is minimal, using property (3.19) of $\mathcal{G}_n$. A similar reason to the proof of Lemma 3.3.4 shows that j can only be equal to $t_n + k_1$ so

$$S' \ldots \prec S_n I_2^{k_1} \ldots$$

which contradicts $S_n I_2^{\infty} \prec S' \ldots$ as $k_1 \geq k_2 + 2$. Moreover

$$\underline{k}(\gamma_1) = S_n I_2^{k_1-1} c_1 \prec S_{n+1} I_2^{\infty} \prec \underline{k}(\gamma_2) = S_n I_2^{k_1-1} c_2$$

and $\underline{i}' = I_2^2 S' c_1 \prec I_2^2 S' c_2 = \underline{i}'' \prec c_2$ are realized for all $\gamma \in [\gamma_1, \gamma_2]$, using Lemma 3.3.3. Let us remark that g_{γ_0} has no homterval as $v_{t_n} = r$, using Singer's Theorem 3.2.3. Therefore

$$\lim_{k_2 \to \infty} g_{\gamma_0}(x(\underline{i}'')) = c_2$$

as $g_{\gamma_0}(x(\underline{i}'')) = x(\sigma \underline{i}'') < c_2$ is increasing with respect to k_2 and

$$\{\underline{i} \in \Sigma_0 \mid I_2 S' c_2 \prec \underline{i} \prec c_2 \text{ for all } k_2 > 0\} = \emptyset.$$

Let k_2 be such that $|c_2 - (x(\sigma \underline{i}''))| < \Delta$. Using Corollary 3.3.1 and the continuity of c_2 and of $x(\sigma \underline{i}'') < x(\sigma \underline{i}') < c_2$ there exists $N_0 > 0$ such that if $k_1 > N_0$ then

$$|c_2 - x| < \Delta$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $x \in [x(\sigma \underline{i}''), x(\sigma \underline{i}')]$. Lemma 3.3.5 applied to $\underline{i}'$ and $\underline{i}''$ yields $\theta > 0$ such that if $l = t_n + k_2 + 4$ then

$$\theta < \left| \left(g_{\gamma}^{j}\right)'(x) \right| < \theta^{-1} \tag{3.23}$$

for all $\gamma \in [\gamma_1, \gamma_2]$, $x \in [x(\underline{i}'), x(\underline{i}'')]$ and $j = 1, \ldots, l$. Lemma 3.3.6 provides $N_1 > 0$ such that if $k_1 > N_1$ then

$$(\lambda')^j < d_{t_n,j}(\gamma) \text{ for all } \gamma \in [\gamma_1, \gamma_2] \text{ for all } j = 1, \ldots, k_2 - 2. \tag{3.24}$$

As $\lambda' > \lambda$ there exists also $N_2 > 0$ such that

$$\theta^{-1}\lambda^{N_2-2+l} < (\lambda')^{N_2-2}.$$

Let $k_1 > \max(k_2+1, N_0, N_1, N_2)$ and $S'' = S_n I_2^{k_1} S'$. Let us remark that $S'' I_2^{\infty} = S_{n+1} I_2^{\infty}$ thus we may apply Proposition 3.3.2 to S'', γ_1 and γ_2. Let α_{n+1} and β_{n+1} be the new bounds for γ provided by Proposition 3.3.2 and $k_3 = k$.

As $g_\gamma^p(c_2) = v_{p-1}$ and $\sigma^{p-1}\left(S'' I_2^{k_3} \ldots\right) = I_2 S' I_2 \ldots$

$$g_\gamma^p(c_2) \in [x(\sigma \underline{i}''), x(\sigma \underline{i}')]$$

for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ thus the first conclusion is satisfied. Moreover, using inequalities (3.23) and (3.24)

$$\lambda^j < d_{t_n,j}(\gamma) \text{ for all } \gamma \in [\alpha_{n+1}, \beta_{n+1}] \text{ and } j = 1, \ldots, |S''| = k_1 - 2 + l.$$

Using Lemma 3.3.6 and Corollary 3.3.2, for k_3 sufficiently big the last two conclusions are satisfied. If k_3 is sufficiently big then by Corollary 3.3.1 inequality (3.18) is also satisfied. □

3.4.2 Some properties of polynomial dynamics

Let us recall some notation introduced in the previous chapter. For any set $E \subseteq \overline{\mathbb{C}}$, we defined the α-neighborhood of E by

$$E_{+\alpha} = B(E, \alpha) = \{x \in \overline{\mathbb{C}} | \operatorname{dist}(x, E) < \alpha\}.$$

One may easily check that if $f, g : \Omega \to \overline{\mathbb{C}}$ with $\Omega \subseteq \overline{\mathbb{C}}$ and $\delta > ||f - g||_\infty$ then for all $B \subseteq \overline{\mathbb{C}}$

$$g^{-1}(B) \subseteq f^{-1}(B_{+\delta}). \tag{3.25}$$

Using this simple observation we show that in a neighborhood of an *ExpShrink* polynomial (see Definition 2.1.3) some weaker version of Backward Stability (see Definition 2.1.4) is satisfied, see Proposition 3.4.3. Let us first show that the Julia set is *continuous* in the sense of Lemma 3.4.1. For transparency we introduce additional notations. We denote by $\mathbb{C}_d[z]$ the space of complex polynomials of degree d. If $f(z) = \sum_{i=0}^d a_i z^i \in \mathbb{C}_d[z]$ let us also denote

$$|f| = \max_{0 \le i \le d} |a_i|.$$

By convention, when $f \in \mathbb{C}_d[z]$ and we compare it to another polynomial g writing $|f - g|$ we also assume that $g \in \mathbb{C}_d[z]$.

Let us observe that the coefficients of $f^n = f \circ f \circ \ldots \circ f$, the n-th iterate of f, are continuous with respect to $(a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$ for all $n > 0$. Therefore given $f \in \mathbb{C}_d[z]$, $m > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then

$$\left|f^i - g^i\right| < \varepsilon \text{ for all } i = 1, \ldots, m.$$

Given a compact $K \subseteq \mathbb{C}$, the map $\mathbb{R}^{d+1} \ni (a_0, a_1, \ldots, a_d) \to f \in \mathbb{C}_d[z]$ is continuous with respect to the topology of $C(K, \mathbb{C})$. Therefore for all $f \in \mathbb{C}_d[z]$, $\varepsilon > 0$ and $m > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then

$$\left|\left|f^i - g^i\right|\right|_{\infty,K} < \varepsilon \text{ for all } i = 1, \ldots, m. \tag{3.26}$$

Lemma 3.4.1. *Let $f \in \mathbb{C}_d[z]$ with $d \geq 2$ and such that its Fatou set is connected and let J be its Julia set. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|f - g| < \delta$ then*

$$J_g \subseteq J_{+\varepsilon}.$$

Proof. The Fatou set of f is the basin of attraction of ∞ and J is compact and invariant. Let $|J| = \max\limits_{z \in J} |z|$, then for all $M \geq |J|$

$$J = \{z \in \mathbb{C} \mid |f^n(z)| \leq M \text{ for all } n \geq 0\}.$$

Let $f(z) = \sum_{i=0}^{d} a_i z^i \in \mathbb{C}_d[z]$. There exists $R > 1$ such that if $|f - g| < \frac{1}{2}|a_d|$ then

$$|J_g| \leq R.$$

Indeed, it is enough to choose

$$R > 4d + 2|a_d|^{-1} \left(1 + \sum_{i=0}^{d-1} |a_i|\right)$$

and check that if $|z| > R$ then $|g(z)| > |z| + 1$.

Let $T = \left\{z \in \overline{\mathbb{C}} \mid \operatorname{dist}(z, J) \geq \varepsilon\right\}$. As T is compact and contained in the basin of attraction of ∞, there is $m > 0$ such that

$$|f^m(z)| > R + 1 \text{ for all } z \in T.$$

Let $K = \overline{B(0, R+1)}$ a compact such that $J_{+\varepsilon}, J_g \subseteq K$ if $|f - g| < \frac{1}{2}|a_d|$. Inequality (3.26) yields $0 < \delta < \frac{1}{2}|a_d|$ such that if $|f - g| < \delta$ then

$$\left|\left|f^i - g^i\right|\right|_{\infty,K} < 1 \text{ for all } i = 1, \ldots, m.$$

Therefore by the definitions of R and m, if $|f - g| < \delta$ then

$$|g^m(z)| > R \text{ for all } z \in T$$

thus $J_g \cap T = \emptyset$. $\square$

Remark 3.4.1. *The hypothesis f polynomial and its Fatou set connected are somewhat artificial, introduced for the elegance of the proof. It may be easily generalized to rational maps with no parabolic periodic points and no rotation domains.*

Proposition 3.4.3. *Let f be an ExpShrink polynomial satisfying the hypothesis of Lemma 3.4.1. There exists $\delta > 0$ such that for all $0 < r < \delta$ there exist $N > 0$ and $d > 0$ such that for all g with $|f - g| < d$ and $z \in J_g$*

$$\operatorname{diam} \operatorname{Comp} g^{-N}(B(z, \delta)) < r.$$

Proof. Let us denote J the Julia set of f. Let $r_0 > 0$ and $\lambda_0 > 1$ be provided by Definition 2.1.3 such that for all $z \in J$

$$\operatorname{diam} \operatorname{Comp} f^{-n}\left(B(z, r_0)\right) < \lambda_0^{-n} \text{ for all } n \geq 0.$$

Let $\delta = \frac{r_0}{4}$ and choose $N \geq 1$ such that

$$\lambda_0^{-N} < r.$$

Inequality (3.26) provides d_0 such that if $|f - g| < d_0$ then

$$\left|f^N(z) - g^N(z)\right| < \delta \text{ for all } z \in \overline{J_{+r_0}}.$$

Lemma 3.4.1 yields $d_1 > 0$ such that if $|f - g| < d_1$ and $z \in J_g$ then there exists $z' \in J$ such that $|z - z'| < 2\delta$ therefore

$$B\left(z, 2\delta\right) \subseteq B\left(z', r_0\right).$$

We choose $d = \min(d_0, d_1)$ and $g \in \mathbb{C}_d[z]$ with $|f - g| < d$. Using inequality (3.25)

$$\operatorname{diam} \operatorname{Comp} g^{-N}\left(B(z, \delta)\right) < \lambda_0^{-N} < r \text{ for all } z \in J_g.$$

□

Corollary 3.4.1. *Let f satisfy the hypothesis of Proposition 3.4.3 and $\varepsilon > 0$. There exist $d, \delta > 0$ such that if $|f - g| < d$ then for all $z \in J_g$ and $n \geq 0$*

$$\operatorname{diam} \operatorname{Comp} g^{-n}(B(z, \delta)) < \varepsilon.$$

Proof. Let us use the notations defined by the proof of Proposition 3.4.3. It is straightforward to check that f has Backward Stability and that, by eventually decreasing r_0, we may also suppose

$$\operatorname{diam} \operatorname{Comp} f^{-n}\left(B(z, r_0)\right) < \varepsilon \text{ for all } z \in J \text{ and } n \geq 0.$$

Let $m \geq 1$ such that

$$\lambda_0^{-m} < \delta.$$

Inequality (3.26) provides d_0 such that if $|f - g| < d_0$ then

$$\left|f^i(z) - g^i(z)\right| < \delta \text{ for all } z \in \overline{J_{+r_0}} \text{ and } i = 1, \ldots, m.$$

Let d_1, d and g be as in the proof of Proposition 3.4.3. By inequality (3.25), for all $z \in J_g$

$$\operatorname{diam} \operatorname{Comp} g^{-m} \left(B(z, \delta)\right) < \delta$$

and

$$\operatorname{diam} \operatorname{Comp} g^{-i} \left(B(z, \delta)\right) < \varepsilon \text{ for all } i = 0, \ldots, m.$$

For some $z \in J_g$, let $W \in \operatorname{Comp} g^{-m} \left(B(z, \delta)\right)$ and $z_1 \in W \cap J_g$. Then

$$W \subseteq B(z_1, \delta)$$

and the proof is completed by induction. □

Let us show that the hypothesis of Lemma 3.4.1 is easy to check for polynomials in $\mathcal{G}_0$.

Lemma 3.4.2. *If $g_\gamma \in \mathcal{G}_0$ and its second critical orbit $(v_n)_{n \geq 0}$ accumulates on a repulsive periodic orbit then g_γ satisfies the hypothesis of Lemma 3.4.1. Moreover, if $(v_n)_{n \geq 0}$ is preperiodic then g_γ has ExpShrink.*

Proof. By Theorems III.2.2 and III.2.3 in [1] the immediate basin of attraction of an attracting or parabolic periodic point contains a critical point. But c_1 is preperiodic and $(v_n)_{n \geq 0}$ accumulates on a repulsive periodic orbit thus it cannot converge to some attracting or parabolic periodic point. Using Theorem V.1.1 in [1] we rule out Siegel disks and Herman rings as their boundary should be contained in the closure of the critical orbits which is contained in $[0, 1]$ for all $g_\gamma \in \mathcal{G}_0$. Using Sullivan's classification of Fatou components, Theorem IV.2.1 in [1], the Fatou set equals the basin of attraction of infinity which is connected for all polynomials by the maximum principle.

If $(v_n)_{n \geq 0}$ is preperiodic then g_γ is Semi-Hyperbolic therefore by Theorem 1 it has *ExpShrink*. □

3.4.3 A counterexample

Using Propositions 3.4.1 and 3.4.2 we build a sequence of families $(\mathcal{G}_n)_{n \geq 1}$ which converge to a 2-modal polynomial g that is Uniformly Hyperbolic on repulsive Periodic orbits (UHP). Its first critical point c_1 is non-recurrent as $g(c_1) = 1$ and 1 is a repulsive fixed point. The second critical point c_2 is recurrent and it does not satisfy the Collet-Eckmann condition. Therefore g does not satisfy the Recurrent Collet-Eckmann condition (RCE).

We obtain the following theorem which states that the converse of Theorem 1 does not hold.

Theorem 2. *There exists an UHP polynomial that is not RCE.*

The proof that g has UHP is analogous to that of Theorem 1. As g is not RCE we have to modify some of our tools like Propositions 2.2.2, 2.2.3 and 2.3.1. The polynomial g_0 is Collet-Eckmann and Semi-Hyperbolic thus RCE. By Theorem 1 g_0 has UHP and *ExpShrink*. Choosing the family $\mathcal{G}_1$ in a sufficiently small neighborhood of g_0 we show two contraction results similar to Propositions 2.2.2 and 2.2.3 that hold on $\mathcal{G}_1$, Corollary 3.4.2 and Proposition 3.4.4 below. As $g \in \mathcal{G}_1$ we may choose constants μ, θ, ε, R and N_0 - as described in the final part of Section 2.3 - that do not depend on g.

The main idea of the proof of Theorem 2 is that in inequality (2.17) the right term may be much bigger than the left term, see also Lemma 3.4.4. This means that when pulling back a ball B to B^{-1} near a second degree critical point, the diameter of B^{-1} is comparable to the square root of the radius of B but $|f'(z)|^{-1}$ may be as big as we want for some $z \in B^{-1}$. This is the main difference between growth conditions in terms of the derivative or in terms of the diameter of pullbacks.

An immediate consequence of Proposition 3.4.3 replaces Proposition 2.2.2 in the proof of Theorem 2.

Corollary 3.4.2. *There exists $\delta > 0$ such that for all $0 < r < R \leq \delta$ there exist $\beta > \alpha_0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta]$ and $z \in J$ the Julia set of g_γ*

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-N}(B(z,R)) < r.$$

Proof. Using Lemma 3.4.2, g_0 satisfies the hypothesis of Proposition 3.4.3. Using the continuity of coefficients of g_γ (3.13) there exists $\beta > \alpha_0$ such that

$$|g_0 - g_\gamma| < d \text{ for all } \gamma \in [\alpha_0, \beta].$$

□

The following consequence of Corollary 3.4.1 is a weaker version of uniform Backward Stability. We use it only twice therefore it can replace Proposition 2.3.1 in the proof of Theorem 2. The proof is analogous to the proof of the previous proposition.

Corollary 3.4.3. *For all $\varepsilon > 0$ there exist $\beta > \alpha_0$ and $\delta > 0$ such that for all $\gamma \in [\alpha_0, \beta]$ and $z \in J$ the Julia set of g_γ*

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-n}(B(z,\delta)) < \varepsilon \textit{ for all } n \geq 0.$$

Let us compute en estimate of the diameter of a pullback far from critical points.

Lemma 3.4.3. *Let $h : B(z, 2R) \to \mathbb{C}$ be an analytic univalent map and $U \ni z$ a connected open with* $\operatorname{diam} U \leq R$. *If*

$$\sup_{x,y \in B(z,2R)} \left| \frac{h'(x)}{h'(y)} \right| \leq D$$

then

$$\operatorname{diam} U \leq D \, |h'(z)|^{-1} \operatorname{diam} h(U).$$

Proof. Let $x, y \in \partial U$ such that $|x - y| = \operatorname{diam} U$. Let $a = h(x)$, $b = h(y)$ and consider the pullback of the line segment $[a, b]$ that starts at x. Then there exists $t_0 \in (0, 1]$ such that

$$[a, t_0 a + (1 - t_0) b] \subseteq h\left(B(z, 2R)\right)$$

and such that the length of $h^{-1}([a, t_0 a + (1-t_0)b])$ is at least $\operatorname{diam} U$. We also notice that

$$\left|\left(h^{-1}\right)'(ta + (1-t)b)\right| \leq D\,|h'(z)|^{-1} \text{ for all } t \in [0, t_0]$$

which completes the proof as $|(t_0-1)a + (1-t_0)b| \leq \operatorname{diam} h(U)$. □

Proposition 2.2.3 relies on inequalities (2.16) and (2.17). We remark that they are satisfied uniformly on a neighborhood of g_0. By the Koebe lemma, the definition (2.16) of ε does not depend on f. Let us prove the uniform version of inequality (2.17) in $\mathcal{G}$.

Lemma 3.4.4. *There exist $M > 1$, $\beta_M > \alpha_0$ and $r_M > 0$ such that for all $\gamma \in [\alpha_0, \beta_M]$ if W is a connected open with* $\operatorname{diam} W < r_M$*, W^{-1} a connected component of $g_\gamma^{-1}(W)$ and $x \in W^{-1}$ then*

$$\operatorname{diam} W^{-1} < M\left|g_\gamma'(x)\right|^{-1} \operatorname{diam} W.$$

Proof. Let $\gamma \in [\alpha_0, \beta_1]$, $x \in W^{-1}$ and suppose

$$3 \operatorname{diam} W^{-1} \leq \operatorname{dist}\left(W^{-1}, \operatorname{Crit}\right)$$

where we denote by Crit the set of critical points $\{c_1, c_2\}$. Then by the Koebe lemma the distortion is bounded by an universal constant $M_1 \geq 1$ on the ball $B(x, 2 \operatorname{diam} W^{-1})$. Using Lemma 3.4.3

$$\operatorname{diam} W^{-1} \leq M_1 \left|g_\gamma'(x)\right|^{-1} \operatorname{diam} W. \tag{3.27}$$

Let us remark some properties of the map $f_b : \mathbb{C} \to \mathbb{C}$ defined by $f_b(z) = bz^2$ for all $z \in \mathbb{C}$ and $b > 0$. Let U be a connected open and $V = f_b(U)$. If $3 \operatorname{diam} U > \operatorname{dist}(U, 0)$ then there exist universal constants $M_2, M_3 > 1$ such that

$$\begin{array}{rcccl} bM_2^{-1} \operatorname{diam} U & < & \sup_{z\in U} |f_b'(z)| & < & bM_2 \operatorname{diam} U, \\ bM_3^{-1} (\operatorname{diam} U)^2 & < & \operatorname{diam} V & < & bM_3 (\operatorname{diam} U)^2. \end{array}$$

Let us also remark that using equality (3.13) if $\gamma \in [\alpha_0, \beta_1]$ and $c \in \operatorname{Crit}$ then

$$g_\gamma(x) = g_\gamma(c) + \frac{g_\gamma''(c)}{2}(x-c)^2 + \frac{g_\gamma'''(c)}{6}(x-c)^6.$$

As $g_0''(c) \neq 0$ and $g_\gamma(c)$, $g_\gamma''(c)$ and $g_\gamma'''(c)$ are continuous there exist $r_M > 0$, $\beta_M > \alpha_0$ and $M_4 > 1$ such that if $\gamma \in [\alpha_0, \beta_M]$, $\operatorname{diam} W < r_M$ and

$$3 \operatorname{diam} W^{-1} > \operatorname{dist}\left(W^{-1}, \operatorname{Crit}\right)$$

then

$$\begin{array}{rcccl} M_4^{-1} \operatorname{diam} W^{-1} & < & \sup_{x\in W^{-1}} \left|g_\gamma'(x)\right| & < & M_4 \operatorname{diam} W^{-1}, \\ M_4^{-1} (\operatorname{diam} W^{-1})^2 & < & \operatorname{diam} W & < & M_4 (\operatorname{diam} W^{-1})^2. \end{array} \tag{3.28}$$

The previous inequality together with inequality (3.27) complete the proof. □

We may now prove a uniform contraction result on a neighborhood of g_0 in $\mathcal{G}$. It replaces Proposition 2.2.3 in the proof of theorem Theorem 2.

Proposition 3.4.4. *For any $1 < \lambda_0 < \lambda$ and $\theta < 1$ there exist $\beta > \alpha_0$, $\delta > 0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta]$, $0 < R \leq \delta$, $n \geq N$ and $z \in J_\gamma$ the Julia set of g_γ, if $W \in \operatorname{Comp} g_\gamma^{-n}(B(z,R))$ and there exists $x \in \overline{W}$ such that $\left|\left(g_\gamma^n\right)'(x)\right| > \lambda^n$ then*

$$\operatorname{diam} W < \theta R \lambda_0^{-n}. \tag{3.29}$$

Proof. Let us fix $z \in \mathbb{C}$ and $D \in (1, \lambda/\lambda_0)$. Let $\varepsilon \in (0,1)$ be provided by inequality (2.16). Let also $r_M > 0$ be small and $M > 1$ provided by the Lemma 3.4.4. Let $l \geq 1$ such that

$$2M^{j/l} D^j \lambda^{-j} \leq \theta \lambda_0^{-j} \text{ for all } j \geq l. \tag{3.30}$$

Let us define $N = 2l$. There exists $r_1 < r_M$ such that for all $i = 1, 2$, $k = 1, \ldots, N$ and any connected component W of $g_0^{-k}(B(c_i, 4r_1))$

$$\operatorname{diam} W \leq 2\varepsilon \operatorname{dist}(W, \operatorname{Crit}).$$

An argument similar to the proof of Proposition 3.4.3 and the continuity of the critical points and of the coefficients (3.13) of g_γ show that there exists $b_0 > \alpha_0$ such that for all $\gamma \in [\alpha_0, b_0]$, $i = 1, 2$ and $k = 1, \ldots, N$

$$g_\gamma^{-k}(B(c_i, 2r_1)) \subseteq g_0^{-k}(B(c_i, 4r_1)).$$

There are only a finite number of connected components of $g_0^{-k}(B(c_i, 4r_1))$ for all $i = 1, 2$ and $k = 1, \ldots, N$. Therefore by the continuity of the critical points there exists $b_1 > \alpha_0$ such that for all $\gamma \in [\alpha_0, b_1]$, $i = 1, 2$ and $k = 1, \ldots, N$ all connected components of $g_\gamma^{-k}(B(c_i, 2r_1))$ satisfy inequality (2.16).

Corollary 3.4.3 provides $b_2 > \alpha_0$ and $\delta > 0$ such that for all $\gamma \in [\alpha_0, b_2]$, $z \in J_\gamma$ and $k \geq 0$

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-k}(B(z, \delta)) < \varepsilon r_1.$$

Let us define $\beta = \min(\beta_M, b_0, b_1, b_2)$ and fix $\gamma \in [\alpha_0, \beta]$, $z \in J_\gamma$ and $n > N$. Then

$$\operatorname{diam} \operatorname{Comp} g_\gamma^{-k}\left(B(z,R)^{-k}\right) < \varepsilon r_1 < r \text{ for all } 0 \leq k \leq n.$$

Let us also fix W and x as in the hypothesis. Denote $x_k = g_\gamma^{n-k}(x) \in W_k = g_\gamma^{n-k}(W)$ for all $k = 0, \ldots, n$.

Let $0 < k_1 < \ldots < k_t \leq N$ be all the integers $0 \leq k \leq n$ such that W_k does not satisfy the inequality (2.16). As $\varepsilon r_1 \geq \operatorname{diam} W_{k_i}$

$$r_1 > \operatorname{dist}(W_{k_i}, \operatorname{Crit}) \text{ for all } 1 \leq i \leq t.$$

Then for all $1 \leq i \leq t$ there exists $c \in \{c_1, c_2\}$ such that $W_{k_i} \subseteq B(c, 2r_1)$. By the definition of r_1

$$k_{i+1} - k_i > N \text{ for all } 1 \leq i < t. \tag{3.31}$$

We may begin estimates. For all $0 < j \leq n$ with $j \neq k_i$ for all $1 \leq i \leq t$, W_j satisfies the inequality (2.16) so the distortion on W_j is bounded by D. Thus

$$\operatorname{diam} W_j \leq D|g'_\gamma(x_j)|^{-1} \operatorname{diam} W_{j-1}. \tag{3.32}$$

If $j = k_i$ for some $1 \leq i \leq t$ we use Proposition 3.2.2 to obtain

$$\operatorname{diam} W_j \leq M|g'_\gamma(x_j)|^{-1} \operatorname{diam} W_{j-1}. \tag{3.33}$$

Let us recall that $x_n = x$ with $\left|\left(g_\gamma^n\right)'(x)\right| > \lambda^n$ and that $W_0 = B(z, R)$ so $\operatorname{diam} W_0 = 2R$. If $t \geq 2$ inequality (3.31) yields $lt \leq 2l(t-1) = N(t-1) < n$. Consequently, as $n > 2l = N$,

$$t < \frac{n}{l}.$$

Multiplying all the relations (3.32) and (3.33) for all $0 < j \leq n$ we obtain

$$\begin{aligned} \operatorname{diam} W_n &\leq M^t D^{n-t} \left|\left(g_\gamma^n\right)'(x_n)\right|^{-1} \operatorname{diam} W_0 \\ &< 2M^{n/l} D^n \lambda^{-n} R \\ &\leq \theta R \lambda_0^{-n}. \end{aligned}$$

The last inequality is inequality (3.30). $\square$

As a direct consequence of inequality (3.31) we obtain the following corollary.

Corollary 3.4.4. *Assume the hypothesis of Proposition 3.4.4. If there exist*

$$-1 \leq k_1 < k_2 < n$$

such that $v \in \overline{g_\gamma^{k_1+1}(W)}$ *and* $\overline{g_\gamma^{k_2}(W)} \cap \{c_1, c_2\} \neq \emptyset$ *then* $k_2 - k_1 > N$ *therefore condition* $n \geq N$ *is superfluous.*

Let us compute a diameter estimate similar to (3.11).

Lemma 3.4.5. *There exist* $\delta > 0$ *and* $N > 0$ *such that for all* $\gamma \in [\alpha_0, \beta_1]$, $k \geq 1$ *and* $x \in I_3(N)$ *with* $\underline{i}(x) = I_3^k I_* \ldots$ *where* $I_* \in \{I_2, I_3\}$, *the following statement holds. If* $x \in W \subseteq \mathbb{C}$ *a connected open such that* $\operatorname{diam} g_\gamma^i(W) < \delta$ *for all* $i = 0, \ldots, k-1$ *then*

$$\operatorname{diam} W < \lambda^{-k} \operatorname{diam} g_\gamma^k(W).$$

Proof. Let us denote $x_i = g_\gamma^i(x)$ and $W_i = g_\gamma^i(W)$ for all $i = 0, \ldots, k$. Using Lemma 3.3.6, inequalities (3.14) and Lemma 3.3.5 for $\underline{i}_1 = I_3 c_1$, $\underline{i}_2 = I_3 c_2$ if $I_* = I_2$ and $\underline{i}_1 = I_3 c_2$, $\underline{i}_2 = I_3^\infty$ if $I_* = I_3$ there exists $N_0 > 0$ that does not depend on γ such that if $N \geq N_0$ then

$$\left|\left(g_\gamma^k\right)'(x)\right| > (\lambda')^k.$$

Let $D \in \left(1, \frac{\lambda'}{\lambda}\right)$ and $\varepsilon > 0$ given by inequality (2.16). Using Lemma 3.4.3 it is enough to show that $B(x_i, 2\delta)$ satisfies inequality (2.16) for all $i = 0, \ldots, k-1$.

Let us recall that $g_\gamma(y) < y$ for all $y \in (c_2, 1) = I_3 \setminus \{1\}$. Therefore for all $i = 0, \ldots, k-1$

$$\operatorname{dist}(x_i, \{c_1, c_2\}) \geq \operatorname{dist}(x_{k-1}, \{c_1, c_2\}) > \operatorname{dist}(x(I_3 c_1), \{c_1, c_2\}).$$

Let

$$d = \min_{\gamma \in [\alpha_0, \beta_1]} \operatorname{dist}(x(I_3 c_1), \{c_1, c_2\})$$

and recall that ε does not depend on γ. Therefore there exists

$$\delta = \frac{d}{2(1 + 2\varepsilon^{-1})} > 0$$

such that if $\operatorname{dist}(y, \{c_1, c_2\}) \geq d$ then $B(y, 2\delta)$ satisfies inequality (2.16). $\square$

The following corollary admits a very similar proof.

Corollary 3.4.5. *There exist $\delta > 0$ and $N > 0$ such that for all $\gamma \in [\alpha_0, \beta_1]$, $k \geq 1$ and $x \in I_3(\max(k, N))$ the following statement holds. If $x \in W \subseteq \mathbb{C}$ a connected open such that $\operatorname{diam} g_\gamma^i(W) < \delta$ for all $i = 0, \ldots, k-1$ then*

$$\operatorname{diam} W < \lambda^{-k} \operatorname{diam} g_\gamma^k(W).$$

Let us recall that all distances and diameters are considered with respect to the Euclidean metric, as we deal exclusively with polynomial dynamics. Let us state Lemma 2.2.1 in this setting.

Lemma 3.4.6. *Let f be a polynomial, $z \in \mathbb{C}$ and $0 < r < R$. Let $W \in \operatorname{Comp} f^{-1}(B(z, R))$ and $W' \in \operatorname{Comp} f^{-1}(B(z, r))$ with $W' \subseteq W$. If $\deg_W(f) \leq \mu$ then*

$$\frac{\operatorname{diam} W'}{\operatorname{diam} W} < 32 \left(\frac{r}{R}\right)^{\frac{1}{\mu}}.$$

Let us set some constants that define the telescope construction used in the proof of Theorem 2. Let $\mu = 2$ and $\theta = \frac{1}{2} 32^{-\mu}$. Let $\delta_0 > 0$ be provided by Corollary 3.4.2 and $\beta_0' > \alpha_0$, $\delta_1 > 0$, $N_1 > 0$ be provided by Proposition 3.4.4 applied to $\lambda^{\frac{1}{2}}$. Let $\delta' > 0$, $N_2 > 0$ be provided by Lemma 3.4.5, $\delta'' > 0$, $N_3 > 0$ be provided by Corollary 3.4.5 and $\beta_M > \alpha_0$, $r_M > 0$ and $M > 1$ defined by Lemma 3.4.4.

Let us observe that

$$I_1^\infty \prec I_1 c_2 \prec c_1 \prec I_2 c_2 \prec I_2^\infty \prec I_2 c_1 \prec c_2 \prec I_3 c_1 \prec I_3^\infty$$

and that all these sequences are continuously realized on $[\alpha_0, \beta_1]$. Let us define

$$\varepsilon_0 = \min_{\gamma \in [\alpha_0, \beta_1]} \left(|x(I_1 c_2) - c_1|, |x(I_2 c_2) - c_1|, |x(I_2 c_1) - c_2|, |x(I_3 c_1) - c_2| \right)$$

therefore $\varepsilon_0 > 0$ is smaller than $|c_1 - c_2|$, $|c_1|$ and $|1 - c_2|$ for all $\gamma \in [\alpha_0, \beta_1]$. We set

$$\varepsilon = \min\left(\varepsilon_0, \delta', \delta'', r_M\right). \tag{3.34}$$

Corollary 3.4.3 provides $\beta_1' > \alpha_0$ and $\delta_2 > 0$ such that for all $\gamma \in [\alpha_0, \beta_1']$ the diameter of any pullback of a ball of radius at most δ_2 centered on J_γ is smaller than ε. Let $\beta_2' = \min\left(\beta_1, \beta_0', \beta_1', \beta_M\right)$ and

$$R = \min\left(\delta_0, \delta_1, \delta_2\right)$$

such that Proposition 3.4.4 applies for balls centered on J_γ of radius at most R, for all $\gamma \in [\alpha_0, \beta_2']$. Moreover, Lemma 3.4.5 and Corollary 3.4.5 apply and inequalities (3.27) and (3.28) hold on all pullbacks of such balls.

Corollary 3.4.2 applied to $r = \theta R$ yields $\beta_3' > \alpha_0$ and $N_0 > 0$ the time span needed to contract the pullback of a ball of radius R into a component of diameter smaller than θR for all $\gamma \in [\alpha_0, \beta_3']$. We define

$$\beta = \min(\beta_2', \beta_3').$$

Let us also prove a version of Corollary 2.3.1 for all g_γ with $\gamma \in [\alpha_0, \beta]$ that works together with Corollary 3.4.4. Let us recall that $\deg_{\overline{W_k}} g_\gamma^k$ is defined by equality (2.28).

Corollary 3.4.6. *For all $\gamma \in [\alpha_0, \beta]$, $z \in J_\gamma$, $0 < r \leq R$ and $(W_k)_{k \geq 0}$ a backward orbit of $B(z, r) = W_0$, if $\overline{d}_n > \mu$, where $\overline{d}_k = \deg_{\overline{W_k}} g_\gamma^k$ for all $k \geq 0$, then there exist $0 < k_1 < k_2 \leq n$ such that $\overline{W_{k_1}} \cap \{c_1, c_2\} \neq \emptyset$ and $c_2 \in \overline{W_{k_2}}$.*

Proof. By the definition of R, $\operatorname{diam} W_k < \varepsilon \leq \varepsilon_0 < |c_1 - c_2|$ therefore $\overline{W_k}$ contains at most one critical point for all $k \geq 0$. As $\mu = \mu_{c_1} = \mu_{c_2}$ there exist $0 < k_1 < k_2 \leq n$ such that $\overline{W_{k_1}}$ and $\overline{W_{k_2}}$ contain exactly one critical point each. Suppose $c_1 \in \overline{W_{k_2}}$ therefore $1 \in \overline{W_k}$ for all $0 \leq k < k_2$ which contradicts $\operatorname{diam} W_{k_1} < \varepsilon \leq \varepsilon_0$. ☐

Let us prove the main result of this section.

Proof of Theorem 2. This proof has two parts. The first part describes the construction of a convergent sequence of families $\left(\mathcal{G}_n\right)_{n \geq 0}$ of 2-modal polynomials with negative Schwarzian derivative. Its limit g does not satisfy the RCE condition. The second part shows that g has $ExpShrink$ and it is very similar to the proof of Theorem 1.

Let us recall the construction of the family $\mathcal{G}_1$. It is described by the common prefix S_1 of its kneading sequences $\underline{k}(\gamma)$ for all $\gamma \in [\alpha_1, \beta_1]$. We defined $S_1 = I_1^{k_0+1} I_2^{k_1}$ so $\beta_1 < \underline{k}^{-1}\left(I_1^{k_0} c_1\right)$ which converges to $\alpha_0 = 0$ as $k_0 \to \infty$. Using this convergence, inequalities (3.14), Lemma 3.3.6 applied to v and Lemma 3.3.5 applied to $\underline{i}_1 = I_1 c_1$, $\underline{i}_2 = I_1 c_2$ to bound $\left|g_\gamma'(v_{k_0})\right|$ there exists $k_0 > 0$ such that the following inequalities hold

$$\beta_1 < \max \underline{k}^{-1}\left(I_1^{k_0} c_1\right) < \beta,$$

$$d_\gamma(k) > \lambda^k \text{ for all } \gamma \in [\alpha_0, \beta_1] \text{ and } k = 1, \ldots, k_0 + 1.$$

Again by Lemma 3.3.6, property (3.15) and inequalities (3.14), if k_1 is sufficiently big then

$$d_\gamma(k) > \lambda^k \text{ for all } \gamma \in [\alpha_1, \beta_1] \text{ and } k = 1, \ldots, t_1 \tag{3.35}$$

where $t_1 = k_0 + 1 + k_1 = |S_1|$. Let us choose k_1 such that the previous inequality holds and such that $t_1 > N_1$ and

$$\max\left(\varepsilon R^{-1}, 2M_4 (\theta R)^{-1}, \varepsilon^2 (\theta R)^{-2}, 2M_1\right) < \lambda^{t_1 - 1} \tag{3.36}$$

where M_1 and M_4 are defined by inequalities (3.27) respectively (3.28). This achieves the construction of the family $\mathcal{G}_1$.

For all $k \geq 1$ we construct $\mathcal{G}_{2k}$ using Proposition 3.4.1 with

$$\lambda^{-1} < \lambda_1 < \lambda_2 < 1$$

and $\mathcal{G}_{2k+1}$ using Proposition 3.4.2 with

$$\Delta_k = 2^{-k}.$$

Using inequality (3.18) the sequence $(\mathcal{G}_n)_{n\geq 1}$ converges to a map $g = g_{\gamma_0}$. Let us denote $d(n) = d_n(\gamma_0) = \left|(g^n)'(v)\right|$ and $d(n,p) = d_{n,p}(\gamma_0) = \left|(g^p)'(v_n)\right|$ for all $n, p \geq 0$, where v is the second critical value and $v_n = g^n(v)$. For all $n \geq 2$ let $p_n = p$ be provided by Proposition 3.4.1 or Proposition 3.4.2 used to construct $\mathcal{G}_n$. Therefore for all $n \geq 1$

$$t_n < p_{n+1} < t_{n+1}$$

where $t_n = |S_n|$ the length of the common prefix S_n of kneading sequences in $\mathcal{G}_n$. Let us set $t_0 = 1$. As $\gamma_0 \in [\alpha_n, \beta_n]$ for all $n \geq 1$

$$\underline{k} = \underline{k}(\gamma_0) \in S_n \times \Sigma \text{ for all } n \geq 1.$$

Let us also recall that for all $k \geq 1$

$$S_{2k} = S_{2k-1} I_2^{k_1+1} I_3^{k_2} I_2^{k_3}$$

and that we may choose k_1, k_2 and k_3 as big as we need. We impose therefore for all $k \geq 1$

$$k_1 > N_3, k_2 > N_2 \text{ and } k_3 > N_3. \tag{3.37}$$

Let us remark that $g(c_1) = 1$, $g(1) = 1$ and $|g'(1)| > 1$ therefore $c_1 \in J$ the Julia set of g and c_1 is non-recurrent and Collet-Eckmann. Let us remark that $\Delta_k \to 0$ as $k \to \infty$ and $\gamma_0 \in [\alpha_{2k+1}, \beta_{2k+1}]$ for all $k \geq 1$ therefore the second critical orbit is recurrent. By construction and inequality (3.11) the second critical orbit accumulates on r and on 1. Therefore $c_2 \in J$ using for example a similar argument to the proof of Lemma 3.4.2. Let us show that c_2 is not Collet-Eckmann. Indeed, by Proposition 3.4.1 for all $k \geq 1$

$$d(p_{2k}) < \lambda_2^{p_{2k}} < 1$$

and $p_{2k} \to \infty$ as $k \to \infty$. Therefore by Definition 2.1.2

$$g \text{ is not } RCE.$$

Combining inequalities (3.35) and (3.16), the third claim of Proposition 3.4.1 and the second claim of Proposition 3.4.2

$$d(n) > \lambda^n \text{ for all } n \in \bigcup_{k\geq 0} \{t_{2k}, \ldots, p_{2k+2}-1\}. \tag{3.38}$$

Let us check that for all $m > 0$ such that $|g^m(c_2) - c_2| < \varepsilon$

$$d(m) > \lambda^m. \tag{3.39}$$

Let us recall that $\varepsilon \leq \varepsilon_0$ by its definition (3.34) so $|g^m(c_2) - c_2| < \varepsilon$ implies that $v_m = g^{m+1}(c_2) \in I_1$ therefore $\underline{k}(m) = I_1$ so there exists $k \geq 1$ such that

$$t_{2k} < m < t_{2k+1}$$

as Proposition 3.4.1 extends S_{2n-1} to S_{2n} using only the symbols I_2 and I_3 for all $n \geq 1$. Therefore $m \in \{t_{2k}, \ldots, p_{2k+2}-1\}$ thus inequality (3.39) is a direct consequence of inequality (3.38).

Let us show that g has *ExpShrink*. We use a telescope that is very similar to the one used in the proof of Theorem 1. We make a minor change to the definition of type 2 as Corollary 3.4.2 does not need to consider a ball of radius $2R$. Let us reuse all notations defined in Section 2.4 and redefine the three type of blocks.

Type 1 Blocks with $R' = r'$ and N' such that $\overline{d}(n, R', N') > 1$ and $c_2 \in \overline{W_{N'+1}}$.

Type 2 Blocks with $R' = R$, $N' = \min(N_0, N-n)$ and $d(n, R, N-n) \leq \mu$.

Type 3 Blocks with $\overline{d}(n, R', N') > 1$, $c_2 \in \overline{W_{N'+1}}$ and $d(n, R', N-n) \leq \mu$.

Another minor modification of the telescope is that if $i > 0$ and $T_{i-1} = 2$ then we set

$$r_i' = \theta R$$

instead of the eventually smaller diameter of $W_{i,N_i'}$. This is harmless for the construction and estimates. We use Corollary 3.4.2 instead of Proposition 2.2.2 and Corollary 3.4.6 instead of Corollary 2.3.1 to construct the telescope. We cannot however replace Proposition 2.2.3 by Proposition 3.4.4 as c_2 is not Collet-Eckmann. We find λ_0 such that inequality (2.30) holds for all blocks of the first and the third type. This also implies inequality (2.32) thus it completes the proof as all estimates remain unchanged.

Let us fix $i \geq 0$ such that $T_i \in \{1,3\}$. Suppose that $i > 0$ and $T_{i-1} \in \{1,3\}$ also, therefore

$$c_2 \in \overline{W_{i-1,N_{i-1}'+1}} \subseteq \overline{W_{i,1}} = \overline{B(z_{n_i}, R_i')^{-1}} = \overline{g^{N_i'}\left(W_{i,N_i'+1}\right)} \subseteq \overline{B(z_{n_i}, R)^{-1}}.$$

But $c_2 \in \overline{W_{i,N_i'+1}}$ also and $\operatorname{diam} B(z_{n_i}, R)^{-1} < \varepsilon$ by the definition of R. Therefore by inequality (3.39)

$$d(N_i') > \lambda^{N_i'}$$

so by Corollary 3.4.4 we may apply Proposition 3.4.4 to obtain

$$\operatorname{diam} W_{i,N_i'} < \theta R_i' \lambda^{-\frac{N_i'}{2}}.$$

We have proved that for all $i > 0$ with $T_{i-1} \in \{1,3\}$ inequality (2.30) holds for all $\lambda_0 \leq \lambda^{\frac{1}{2}}$. If $i = 0$ or $T_{i-1} = 2$ then $R_i' \in [\theta R, R]$. Therefore it is enough to show that there exist $\lambda_0 > 1$ such that for all $z \in J$, $r \in [\theta R, R]$, $n > 0$ and W a connected component of $g^{-n}(B(z,r))$ the following statement holds. If $v \in \overline{W}$ and there exist $0 \leq m < n$ such that $g^m(W) \cap \mathrm{Crit} \neq \emptyset$ then

$$\operatorname{diam} W < \theta r \lambda_0^{-n}. \tag{3.40}$$

Again, if $d(n) > \lambda^n$ using Corollary 3.4.4 and Proposition 3.4.4 the previous inequality is satisfied for all $1 < \lambda_0 \leq \lambda^{\frac{1}{2}}$. Therefore using inequality (3.38) we may suppose that there exist $k' \geq 1$ such that

$$p_{2k'} \leq n < t_{2k'}.$$

Let us denote $p = p_{2k'}$, $t = t_{2k'-1}$ and $W_k = g^k(W)$ for all $k = 0, \ldots, n$. By the definition of p in Proposition 3.4.1

$$2t < p. \tag{3.41}$$

Using Corollary 3.4.5, inequalities (3.37) and (3.36)

$$\operatorname{diam} W_t < \lambda^{-(p-1-t)} \operatorname{diam} W_{p-1} < \lambda^{-(p-1-t)} \varepsilon < R.$$

As $t_1 > N_1$, inequality (3.16) lets us apply Proposition 3.4.4 to $B(v_t, \operatorname{diam} W_t)$ which combined to the previous inequality shows that

$$\operatorname{diam} W < \theta \lambda^{-\left(p-1-\frac{t}{2}\right)} \operatorname{diam} W_{p-1}. \tag{3.42}$$

Using Lemma 3.4.5 and eventually Corollary 3.4.5 if $v_n \in I_2$ and inequalities (3.37)

$$\operatorname{diam} W_p < \lambda^{-(n-p)} \operatorname{diam} W_n = 2\lambda^{-(n-p)} r. \tag{3.43}$$

Therefore the only missing link is an estimate of $\operatorname{diam} W_{p-1}$ with respect to $\operatorname{diam} W_p$. We distinguish the following two cases.

1. $\operatorname{dist}(W_{p-1}, c_1) < 3 \operatorname{diam} W_{p-1}$.
2. $\operatorname{dist}(W_{p-1}, c_1) \geq 3 \operatorname{diam} W_{p-1}$.

Suppose the first case. The by the definition (3.34) of ε we may use inequality (3.28) therefore

$$\begin{aligned} \operatorname{diam} W_{p-1} &< (M_4 \operatorname{diam} W_p)^{\frac{1}{2}} \\ &< (2M_4 r)^{\frac{1}{2}} \lambda^{-\frac{n-p}{2}} \end{aligned}$$

using inequality (3.43). Using also inequalities (3.42), (3.41) and (3.36) we obtain

$$\begin{aligned} \operatorname{diam} W &< \theta \lambda^{-\frac{n}{2}} r \left(2\lambda M_4 r^{-1}\right)^{\frac{1}{2}} \lambda^{-\frac{t}{2}} \\ &< \theta \lambda^{-\frac{n}{2}} r. \end{aligned}$$

Therefore in the first case it is enough to choose $\lambda_0 \leq \lambda^{\frac{1}{2}}$.

Suppose the second case. Using inequalities (3.42), (3.41) and (3.36) we may compute

$$\begin{aligned} \operatorname{diam} W &< \theta\lambda^{-\left(p-1-\frac{t}{2}\right)}\varepsilon \\ &< \theta\lambda^{-\frac{p}{2}}\theta R \leq \theta\lambda^{-\frac{p}{2}} r \\ &= \theta\lambda^{-n\left(\frac{p}{2n}\right)} r. \end{aligned} \tag{3.44}$$

This is not enough as λ_0 should depend only on g. We may remark that we are in position to use inequality (3.27) for W_p therefore

$$\operatorname{diam} W_{p-1} < M_1 \left|g'(v_{p-1})\right|^{-1} \operatorname{diam} W_p.$$

Let us compute an upper bound for $\left|g'(v_{p-1})\right|^{-1} = d(p-1,1)^{-1}$. Using the first two claims of Proposition 3.4.1

$$d(p)^{-1} = d(p-1)^{-1} d(p-1,1)^{-1} < \lambda_1^{-p} < \lambda^p$$

and

$$d(p-1) < \lambda_r^{p-1}\lambda^{p-1}$$

where we denote $\lambda_r = |g'(r(\gamma_0))|$ and $\nu = \frac{\log \lambda_r}{\log \lambda}$. Combining the previous inequalities

$$d(p-1,1)^{-1} < \lambda^{p(\nu+2)}$$

therefore using also inequalities (3.42), (3.43), (3.41) and (3.36)

$$\begin{aligned} \operatorname{diam} W &< 2M_1\theta\lambda^{-\left(p-1-\frac{t}{2}\right)}\lambda^{p(\nu+2)}\lambda^{-(n-p)} r \\ &< \theta(2M_1)\lambda^{\nu p+2p+1+\frac{t}{2}}\lambda^{-n} r \\ &< \theta\lambda^{-n+p(\nu+3)} r. \end{aligned}$$

If $n > 2p(\nu+3)$ then inequality (3.40) is satisfied for all $\lambda_0 \leq \lambda^{\frac{1}{2}}$. If $n \leq 2p(\nu+3)$ then using inequality (3.44), inequality (3.40) is satisfied for all

$$\lambda_0 \leq \lambda^{\frac{1}{4(\nu+3)}} \leq \lambda^{\frac{p}{2n}}$$

which completes the proof. ☐

3.5 RCE is not topological for real polynomials with negative Schwarzian derivative

Let $\mathcal{H} : [0,h] \to \mathcal{P}_2$ be equal to the family $\mathcal{G}$ defined in the previous section. Let us define another family of 2-modal maps $\tilde{\mathcal{H}} : [0,h'] \to \mathcal{P}_2$ in an analogous fashion. Let $T \in \mathbb{R}_7[x]$ be a degree 7 polynomial such that $T(0) = 0$ and such that $T'(x) = (x+1)^3(x-1)^3$. Therefore T has two critical points -1 and 1 of degree 4 and $T(-x) = -T(x)$ for all $x \in \mathbb{R}$. Let

$y_0 = T(-1)$ and $x_0 > 1$ such that $T(x_0) = y_0$. Let $h' > 0$ be small and for each $\gamma \in [0, h']$ two order preserving linear maps $R_{\gamma'}(x) = x(2x_0 + \gamma') - x_0 - \gamma'$ and $S_{\gamma'}(y) = \frac{y - T(-x_0-\gamma')}{y_0 - T(-x_0-\gamma')}$ that map $[0,1]$ onto $[-x_0 - \gamma', x_0]$ respectively $[T(-x_0-\gamma'), T(x_0)]$ onto $[0,1]$. One may show by direct computation that if a real polynomial P is such that all the roots of P' are real then P has negative Schwarzian derivative. Therefore

$$\tilde{h}_{\gamma'} = S_{\gamma'} \circ T \circ R_{\gamma'} \in \mathcal{P}_2 \text{ for all } \gamma' \in [0, h'].$$

We define $\tilde{\mathcal{H}}(\gamma') = \tilde{h}_{\gamma'}$ for all $\gamma' \in [0, h']$. Let us remark that $x_0 \in \left(\frac{3}{2}, 2\right)$ therefore all three fixed points of $\tilde{h}_0$ are repulsive. Let $\tilde{r}(\gamma')$ be the only fixed point of $\tilde{h}_{\gamma'}$ in $(0,1)$ and $\tilde{c}_1 < \tilde{c}_2$ its critical points. The proofs that for $h' > 0$ sufficiently small $\tilde{\mathcal{H}}$ satisfies properties (3.3) to (3.6), Lemmas 3.3.1 and 3.3.2, that it is natural, that $\tilde{r}$, $\tilde{c}_1$ and $\tilde{c}_2$ are continuous and that for all $n > 1$

$$\tilde{r}(\gamma') - \tilde{h}^n_{\gamma'}(\tilde{c}_2) \text{ has finitely many zeros in } [0, h']$$

go exactly the same way as for $\mathcal{G}$. As $h_0'(r(0)) = -3$, $h_0'(1) = 9$, $y_0 = \frac{16}{35}$ and $\frac{3}{2} < x_0 < 2$ one may compute that

$$\frac{1}{2}\frac{\log|h_0'(1)|}{\log|h_0'(r(0))|} = 1 < \frac{3}{4}\frac{\log\left|\tilde{h}_0'(1)\right|}{\log\left|\tilde{h}_0'(\tilde{r}(0))\right|}.$$

We may also suppose $h > 0$ and $h' > 0$ sufficiently small such that there exist $1 < \lambda < \lambda'$, $1 < \tilde{\lambda} < \tilde{\lambda}'$ and $\theta_1 < \theta_2$ such that for all $\gamma \in [0, h]$ and $\gamma' \in [0, h']$

$$\begin{aligned} \lambda' &< \min\left(\left|h_\gamma'(0)\right|, \left|h_\gamma'(r)\right|, \left|h_\gamma'(1)\right|\right) \text{ and} \\ \tilde{\lambda}' &< \min\left(\left|\tilde{h}_{\gamma'}'(0)\right|, \left|\tilde{h}_{\gamma'}'(\tilde{r})\right|, \left|\tilde{h}_{\gamma'}'(1)\right|\right) \end{aligned}$$

and

$$\frac{1}{2}\frac{\log\left|h_\gamma'(1)\right|}{\log\left|h_\gamma'(r(\gamma))\right|} < \theta_1 < \theta_2 < \frac{3}{4}\frac{\log\left|\tilde{h}_{\gamma'}'(1)\right|}{\log\left|\tilde{h}_{\gamma'}'(\tilde{r}(\gamma'))\right|}. \tag{3.45}$$

Let us denote $\underline{k}(\gamma)$ the second kneading sequence of h_γ and $\underline{\tilde{k}}(\gamma')$ the second kneading sequence of $\tilde{h}_{\gamma'}$. We construct two decreasing sequences of families of 2-modal maps $(\mathcal{H}_n)_{n\geq 1}$ and $(\tilde{\mathcal{H}})_{n\geq 1}$. Let $\mathcal{H}_n : [\alpha_n, \beta_n] \to \mathcal{P}_2$ with $\mathcal{H}_n(\gamma) = \mathcal{H}(\gamma)$ for all $\gamma \in [\alpha_n, \beta_n]$ and $\tilde{\mathcal{H}}_n : [\alpha_n', \beta_n'] \to \mathcal{P}_2$ with $\tilde{\mathcal{H}}_n(\gamma) = \tilde{\mathcal{H}}(\gamma)$ for all $\gamma' \in [\alpha_n', \beta_n']$. By construction we choose that for all $n \geq 1$

$$\underline{k}(\alpha_n) = \underline{\tilde{k}}(\alpha_n') \text{ and } \underline{k}(\beta_n) = \underline{\tilde{k}}(\beta_n').$$

Let us denote $v = h_\gamma(c_2)$, $\tilde{v} = \tilde{h}_{\gamma'}(\tilde{c}_2)$ and $v_n = h^n_\gamma(v)$, $\tilde{v}_n = \tilde{h}^n_{\gamma'}(\tilde{v})$ for all $n \geq 0$, $\gamma \in [\alpha_n, \beta_n]$ and $\gamma' \in [\alpha_n', \beta_n']$. Let also $d_n(\gamma) = \left|\left(h^n_\gamma\right)'(v)\right|$, $\tilde{d}_n(\gamma') = \left|\left(\tilde{h}^n_{\gamma'}\right)'(\tilde{v})\right|$, $d_{n,p}(\gamma) = \left|\left(h^p_\gamma\right)'(v_n)\right|$

and $\tilde{d}_{n,p}(\gamma') = \left|\left(\tilde{h}^p_{\gamma'}\right)'(\tilde{v}_n)\right|$ for all $n, p \geq 0$, $\gamma \in [\alpha_n, \beta_n]$ and $\gamma' \in [\alpha'_n, \beta'_n]$. The basic construction tool is again Proposition 3.3.2 and we build the sequences $(\mathcal{H}_n)_{n\geq 1}$ and $(\tilde{\mathcal{H}})_{n\geq 1}$ by specifying the common prefix S_n of the kneading sequences in $\mathcal{H}_n$ and $\tilde{\mathcal{H}}_n$ for all $n \geq 1$. We also reuse the notation $t_n = |S_n|$ for all $n \geq 1$. In an analogous way to the construction of the family $\mathcal{G}_1$, see inequality (3.35), we choose

$$S_1 = I_1^{k_0+1} I_2^{k_1}$$

such that

$$d_k(\gamma) > \lambda^k \text{ and } \tilde{d}_k(\gamma') > \tilde{\lambda}^k \tag{3.46}$$

for all $\gamma \in [\alpha_1, \beta_1]$, $\gamma' \in [\alpha'_1, \beta'_1]$ and $k = 1, \ldots, t_1$ and

$$\beta_1 < h \text{ and } \beta'_1 < h'.$$

Let us describe the construction of the sequences $(\mathcal{H}_n)_{n\geq 1}$ and $(\tilde{\mathcal{H}})_{n\geq 1}$ which satisfy properties (3.15) to (3.19) and

$$\tilde{d}_{t_n}(\gamma') > \tilde{\lambda}^{t_n} \tag{3.47}$$

for all $n \geq 1$.

Let us recall that Proposition 3.4.1 employs twice Proposition 3.3.2 to construct a subfamily $\mathcal{G}_{n+1}$ of $\mathcal{G}_n$ with

$$S_{n+1} = S_n I_2^{k_1} I_3^{k_2} I_2^{k_3}.$$

Let γ_0 and γ'_0 be provided by Proposition 3.3.2 such that $\underline{k}(\gamma_0) = \underline{\tilde{k}}(\gamma'_0) = S_n I_2^{\infty}$. We use the same strategy as in the proof of Proposition 3.4.1 to define both $\mathcal{H}_{n+1}$ and $\tilde{\mathcal{H}}_{n+1}$ with the same combinatorics. Taking k_1, k_2 and k_3 sufficiently big we may control the growth of $d_m(\gamma)$ and $\tilde{d}_m(\gamma')$ uniformly for all $t_n < m \leq t_{n+1}$. We let

$$\frac{k_1}{k_2} \to \eta > 0,$$

$p = t_n + k_1$ and compute some bounds for $d_p(\gamma)$ and $\tilde{d}_p(\gamma')$. For transparency, let us denote $\lambda_0 = \left|h'_{\gamma_0}(r)\right|$, $\tilde{\lambda}_0 = \left|\tilde{h}'_{\gamma'_0}(\tilde{r})\right|$, $\lambda_3 = \left|h'_{\gamma_0}(1)\right|$ and $\tilde{\lambda}_3 = \left|\tilde{h}'_{\gamma'_0}(1)\right|$. As in the proof of Proposition 3.4.1 we obtain

$$\lim_{k_1\to\infty} \frac{1}{k_1} \log d_p(\gamma) = \log \lambda_0 - \frac{1}{2\eta} \log \lambda_3 \text{ for all } \gamma \in [\alpha_{n+1}, \beta_{n+1}]. \tag{3.48}$$

We may observe that inequalities (3.22) hold exactly when c_1 is a second degree critical point. We may however write similar bounds for $\tilde{\mathcal{H}}_{n+1}$. By the same arguments there exist constants $\tilde{M} > 1$, $\tilde{\delta} > 0$ and $\tilde{N}_2 > 0$ such that if $k_1 > \tilde{N}_2$ and $\gamma' \in [\gamma'_1, \gamma'_2]$ then

$$\begin{array}{rcccl} \tilde{M}^{-1}(x - \tilde{c}_1)^4 & < & |1 - \tilde{h}_{\gamma'}(x)| & < & \tilde{M}(x - \tilde{c}_1)^4 \text{ and} \\ \tilde{M}^{-1}(x - \tilde{c}_1)^3 & < & \left|\tilde{h}'_{\gamma'}(x)\right| & < & \tilde{M}(x - \tilde{c}_1)^3 \end{array}$$

for all $x \in (\tilde{c}_1 - \tilde{\delta}, \tilde{c}_1 + \tilde{\delta})$, where γ_1', γ_2' are the bounds for γ' provided by Proposition 3.3.2 applied to S_n and $\tilde{\mathcal{H}}_n$. Therefore we obtain

$$\lim_{k_1 \to \infty} \frac{1}{k_1} \log \tilde{d}_p(\gamma') = \log \tilde{\lambda}_0 - \frac{3}{4\eta} \log \tilde{\lambda}_3 \text{ for all } \gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]. \tag{3.49}$$

Using inequalities (3.45) and the limits (3.48) and (3.49) it is enough to choose

$$\theta_1 < \eta < \theta_2$$

to obtain the following corollary of Proposition 3.4.1.

Corollary 3.5.1. *There exist*

$$0 < \lambda_1 < 1 < \lambda_2 < \min\left(\lambda, \tilde{\lambda}\right)$$

that depend only on $\mathcal{H}_1$ and $\tilde{\mathcal{H}}_1$ such that if $\mathcal{H}_n$ is a subfamily of $\mathcal{H}_1$ and $\tilde{\mathcal{H}}_n$ is a subfamily of $\tilde{\mathcal{H}}_1$ both satisfying conditions (3.15) to (3.19) and (3.47) then there exist $\mathcal{H}_{n+1}$ a subfamily of $\mathcal{H}_n$ and $\tilde{\mathcal{H}}_{n+1}$ a subfamily of $\tilde{\mathcal{H}}_n$ satisfying the same condition and $2t_n < p < t_{n+1}$ with the following properties

1. *$d_p(\gamma) > \lambda_2^p$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.*
2. *$\tilde{d}_p(\gamma') < \lambda_1^p$ for all $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$.*
3. *$d_{t_n,l}(\gamma) > \lambda^l$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ and $l = 1, \ldots, p-1-t_n$.*
4. *$\tilde{d}_{t_n,l}(\gamma') > \tilde{\lambda}^l$ for all $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$ and $l = 1, \ldots, p-1-t_n$.*
5. *$d_{p,l}(\gamma) > \lambda^l$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ and $l = 1, \ldots, t_{n+1} - p$.*
6. *$\tilde{d}_{p,l}(\gamma') > \tilde{\lambda}^l$ for all $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$ and $l = 1, \ldots, t_{n+1} - p$.*

Proposition 3.4.2 has an immediate corollary for the families $\mathcal{H}$ and $\tilde{\mathcal{H}}$.

Corollary 3.5.2. *Let the subfamilies $\mathcal{H}_n$ and $\tilde{\mathcal{H}}_n$ of $\mathcal{H}_1$ respectively $\tilde{\mathcal{H}}_1$ with $n \geq 1$ satisfy conditions (3.15) to (3.19) and (3.47) and*

$$\Delta > 0.$$

Then there exist subfamilies $\mathcal{H}_{n+1}$ of $\mathcal{H}_n$ and $\tilde{\mathcal{H}}_{n+1}$ of $\tilde{\mathcal{H}}_n$ satisfying the same conditions and such that there exists $t_n < p < t_{n+1}$ with the following properties

1. *$\left|h_\gamma^p(c_2) - c_2\right| < \Delta$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$.*
2. *$\left|\tilde{h}_{\gamma'}^p(\tilde{c}_2) - \tilde{c}_2\right| < \Delta$ for all $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$.*
3. *$d_{t_n,l}(\gamma) > \lambda^l$ for all $\gamma \in [\alpha_{n+1}, \beta_{n+1}]$ and $l = 1, \ldots, t_{n+1} - t_n$.*

4. $\tilde{d}_{t_n,l}(\gamma') > \tilde{\lambda}^l$ *for all* $\gamma' \in [\alpha'_{n+1}, \beta'_{n+1}]$ *and* $l = 1, \ldots, t_{n+1} - t_n$.

For all $k \geq 1$ we define $\mathcal{H}_{2k}$ and $\tilde{\mathcal{H}}_{2k}$ using Corollary 3.5.1 and $\mathcal{H}_{2k+1}$ and $\tilde{\mathcal{H}}_{2k+1}$ using Corollary 3.5.2 with $\Delta = 2^{-k}$. Let h be the limit of $(\mathcal{H}_n)_{n\geq 1}$ and $\tilde{h}$ be the limit of $(\tilde{\mathcal{H}}_n)_{n\geq 1}$. Then h is CE therefore RCE and the second critical point $\tilde{c}_2$ of $\tilde{h}$ is recurrent but not CE therefore $\tilde{h}$ is not RCE. Both h and $\tilde{h}$ have negative Schwarzian derivative and their second critical orbits accumulate on r and 1 respectively on $\tilde{r}$ and 1. Moreover, using Lemma 3.4.2, h and $\tilde{h}$ do not have attracting or neutral periodic points. We may therefore apply Corollaries 3.2.1 and 3.2.2 to obtain the following theorem that contradicts Conjecture 1 in [10].

Theorem 3. *The* RCE *condition for S-multimodal maps is not topologically invariant.*

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Annexe A

On the Hausdorff dimension of fractal attractors of unimodal maps

Abstract. We consider C^4 infinitely renormalizable unimodal maps of the interval with non-degenerate critical point. A recent result of Graczyk and Kozlovski (see [4]) shows that there is $\sigma < 1$ such that every attractor of such a map has Hausdorff dimension less than or equal to σ.

We find a correspondence between the renormalization type and the kneading sequence. This yields an algorithm that finds the quadratic map $x \to ax(1-x)$ with a given renormalization type. For periodic and preperiodic renormalization types we estimate the Hausdorff dimension of the fractal attractor. The results suggest that the attractor of the Feigenbaum map has the highest dimension.

A.1 Introduction

One-dimensional dynamics have been the subject of intense research during the last three decades. Despite their apparent simplicity these models present an interesting mathematical structure going far beyond the simple equilibrium solutions. They may arise as time-discretizations of higher dimension problems. They are computationally accessible and provide examples and counter-examples for a large spectrum of phenomena.

The simplest examples of one-dimensional dynamical systems are maps of the interval and maps of the circle. On the interval unimodal and multimodal maps are considered while on the circle homeomorphisms are usually studied.

Definition A.1.1. *We say that a map $f : I \to I$, where $I = [a, b]$ is a compact interval, is* unimodal *if*

1. *f is continuous,*

2. $f(a) = f(b) \in \partial I$,

3. *$\exists c \in (a, b)$ such that f is strictly monotone on $[a, c]$ and on $[c, b]$.*

We say that f is C^r-unimodal for $r \geq 1$ if f is C^r-continuous and

4. *$f'(x) \neq 0$ if $x \neq c$.*

If f is a C^2-unimodal map we say that its critical point is *non-degenerate* if

$$f''(c) \neq 0.$$

In the sequel, when not explicitly stated, we only consider unimodal maps on $[0, 1]$ that are increasing on the left lap $[0, c)$ and decreasing on the right lap $(c, 1]$.

Definition A.1.2. *We define a* family of unimodal maps *as a path $\mathcal{F} : [\alpha, \beta] \to \mathcal{U}^1$, where $\mathcal{U}^1$ is the topological space of C^1-unimodal maps.*

Note that $\mathcal{F}$ should be continuous with respect to the C^1 topology of $\mathcal{U}^1$ and that in such a family, the critical point $c(a)$ of $F(a)$ is continuous on $[\alpha, \beta]$. One may check [9] for the theory of families of unimodal and multimodal maps in its full generality - not necessarily smooth maps or families, for example.

A computationally accessible, *full family* of unimodal maps is the *quadratic family.*

Definition A.1.3. *We define the* quadratic family $\mathcal{Q}$ *of unimodal maps by*

$$\mathcal{Q} = \{f_a : [0,1] \to [0,1], a \in [1,4] | f_a(x) = ax(1-x)\}.$$

Indeed $a \to f_a$ is continuous in the C^1 topology and by Definition A.2.3 it is easy to check that $\mathcal{Q}$ is a full family ($f_l = f_1$ and $f_r = f_4$).

The understanding of high iterates of maps is a central problem in dynamics. The balance between expansion and contraction features plays an important role. In the quadratic case, the high degree of iterates (as polynomials) induces expansion and the presence of the critical point induces contraction. When the critical orbit accumulates on an attracting periodic orbit, the dynamics is well understood (hyperbolic). Attracting periodic orbits are the simplest example of *attractors*.

An attractor is an invariant set where a large part of the phase space accumulates. If f is an unimodal map, a forward invariant compact set A is called a (minimal) *metric attractor* of f if its basin of attraction $B(A) = \{x \in [0,1] \mid \omega(x) \subseteq A)\}$ has positive Lebesgue measure and A has no proper subset with the same property. A *topological attractor* is a minimal forward invariant compact set A with $B(A)$ of second Baire category.

The metric and topological attractors of a C^3 unimodal map with non-degenerate critical point coincide, see [5]. Unimodal maps with negative Schwarzian derivative are known to have exactly one metric attractor. In particular, quadratic maps have exactly one attractor - metric and topological. It is either an attracting periodic orbit, a transitive cycle of intervals or a Cantor set of solenoid type, see [5]. The interesting case from the point of view of the Hausdorff dimension is when the attractor is a Cantor set - we call it a *fractal attractor*. This happens exactly when the quadratic map is *infinitely renormalizable* (see Definition A.3.1). In [4], Graczyk and Kozlovski show that the Hausdorff dimension of such an attractor is bounded by an universal constant $\sigma < 1$. The Feigenbaum map is the limit of the period doubling cascade in the quadratic family. Its renormalization type is the simplest at any scale - period two renormalization. This work and [4] both suggest that its attractor maximizes the Hausdorff dimension of fractal attractors in the quadratic family.

A.2 The Kneading Sequence

Symbolic dynamics arose as an attempt to study dynamics by means of discretizing the phase space. One of its simplest forms is illustrated by the itinerary and kneading sequences of unimodal maps of the interval. Let f be a unimodal map and c its critical point. Let $\{[0,c), \{c\}, (c,1]\}$ be a partition of the interval $[0,1]$ corresponding to the monotonicity of f. We associate the symbols of an alphabet $\mathcal{A} = \{L, C, R\}$ to the elements of the partition, with respect to their order. We may assign to any orbit of the dynamics a sequence of symbols of $\mathcal{A}$. The dynamics of f on the orbit is represented by the left shift $\mathcal{S}$.

Definition A.2.1. *For f unimodal and $x \in [0,1]$ we define the* itinerary $\underline{I}_f(x)$ *of x, the sequence $(I_n)_{0 \leq n \leq \overline{n}}$ of symbols of $\mathcal{A}$ such that*

1. $\overline{n} = \min\{n \geq 0 | f^n(x) = c\} \in \overline{\mathbb{N}}$,

2. $I_n = C$ *if and only if* $f^n(x) = c$,

3. $I_n = L$ *if* $f^n(x) < c$ *and* $I_n = R$ *if* $f^n(x) > c$.

We define the kneading sequence $\underline{K}_f$ *of* f *by*

$$\underline{K}_f = \underline{I}_f(f(c)).$$

The map $\underline{I}_f$ conjugates the dynamics of f on $[0,1] \setminus \{c\}$ to the left shift $\mathcal{S}$, that is

$$\underline{I}_f(f(x)) = \mathcal{S}(\underline{I}_f(x)), \forall x \neq c, \tag{A.1}$$

where $\mathcal{S}(I_0 I_1 I_2 \ldots) = I_1 I_2 I_3 \ldots$. The shifts of $\underline{K}_f$ are the itineraries of the elements of the post-critical orbit.

Let

$$\underline{\mathcal{I}} = \{\underline{I}_f(x) | f \text{ unimodal}, x \in [0,1]\}$$

be the space of itinerary sequences of unimodal maps. We define a total order on $\underline{I}$ - a *signed lexicographic order* - that makes $\underline{I}_f$ increasing for each f unimodal (see Proposition II.3.1 in [9]).

Let us first define a sign function $\epsilon : \mathcal{A} \to \{-1, 0, 1\}$ - that corresponds to the sign of the derivative of a quadratic map - by $\epsilon(L) = 1, \epsilon(C) = 0$ and $\epsilon(R) = -1$. We extend ϵ to finite sequences $I_0 I_1 \ldots I_n$ of symbols of $\mathcal{A}$ by

$$\epsilon(I_0 I_1 \ldots I_n) = \prod_{0 \leq i \leq n} \epsilon(I_i).$$

If $f \in \mathcal{Q}$ and $\underline{I}_f(x) = I_0 I_1 \ldots I_n \ldots$ then

$$\epsilon(I_0 I_1 \ldots I_n) = \operatorname{sgn}(f^n)'(x).$$

Observe that for all $\underline{I} \neq \underline{I}' \in \underline{\mathcal{I}}$, $\underline{I}$ cannot be a prefix of $\underline{I}'$ as $\underline{I}'$ contains at most one symbol C on the rightmost position, if finite.

Definition A.2.2. *A* signed lexicographic order $\prec$ *on* $\underline{\mathcal{I}}$ *is defined as follows. We say that* $\underline{I} \prec \underline{I}'$ *if there exists* $n \geq 0$ *such that* $\underline{I}_i = \underline{I}'_i$ *for* $i = 0, 1, \ldots, n-1$ *and*

$$\epsilon(I_0 I_1 \ldots I_n) > \epsilon(I'_0 I'_1 \ldots I'_n).$$

We call a sequence $\underline{I} \in \underline{\mathcal{I}}$ maximal *if*

$$\mathcal{S}^k \underline{I} \preceq \underline{I}, \forall k \geq 0 \textit{ such that } \mathcal{S}^k \underline{I} \in \underline{\mathcal{I}}.$$

For f unimodal $\underline{I}_f$ is increasing and $f(c)$ is its maximal value therefore, using equality (A.1), the kneading sequence $\underline{K}_f$ is maximal. Let

$$\underline{\mathcal{K}} = \{\underline{K} \in \underline{\mathcal{I}} \mid \underline{K} \text{ maximal}\}$$

be the space of maximal itinerary sequences of unimodal maps.

Let us consider families of unimodal maps from the point of view of the kneading sequence. We may observe that $\underline{L} = LLL\ldots$ is the minimal element of $\underline{\mathcal{I}}$, $\underline{R} = RLLL\ldots$ is its maximal element and $\underline{L}, \underline{R} \in \underline{\mathcal{K}}$. Let us state a classical result on the realization of the kneading sequence - Theorem III.1.1 in [2].

Theorem A.2.1. *Let $\mathcal{F}$ be a family of unimodal maps, $f, g \in \mathcal{F}$ and $\underline{K} \in \underline{\mathcal{K}}$ such that*

$$\underline{K}_f \prec \underline{K} \prec \underline{K}_g.$$

Then there exists $h \in \mathcal{F}$ such that

$$\underline{K}_h = \underline{K}.$$

This result motivates the following definition of *full families* of unimodal maps, such families that realize all maximal sequences as kneading sequences.

Definition A.2.3. *Let $\mathcal{F}$ be a family of unimodal maps. We say that $\mathcal{F}$ is a full family if there exist $f_l, f_r \in \mathcal{F}$ with $\underline{K}_{f_l} = \underline{L}$ and $\underline{K}_{f_r} = \underline{R}$.*

The kneading sequence extracts important features of the quadratic dynamics. We use the following theorem to prove our main result (see [2], page 69). We formulate it for quadratic maps but it applies for a larger class (S-unimodal maps) that is stable by renormalization.

Theorem A.2.2. *If $f \in \mathcal{Q}$ then $\underline{K}_f$ is periodic or finite if and only if f has a stable periodic orbit. If $\underline{K}_f$ is not periodic and $\underline{K}_f = \underline{K}_g$ for some $g \in \mathcal{Q}$ then f and g are topologically conjugate.*

Let us define the composition of itinerary sequences.

Definition A.2.4. *Let $\underline{A} = A_1 \ldots A_n C$ and $\underline{B}$ be itinerary sequences and $\underline{A}' = A_1 \ldots A_n$ the maximal prefix of $\underline{A}$. Let $\overline{L} = R$ and $\overline{R} = L$ if $\epsilon(\underline{A}') = -1$ and $\overline{L} = L$,$\overline{R} = R$ otherwise. We define*

$$\underline{A} * \underline{B} = \begin{cases} \underline{A}'\overline{B}_1\underline{A}' \ldots \underline{A}'\overline{B}_m\underline{A}'C & \text{if } \underline{B} \text{ is finite,} \\ \underline{A}'\overline{B}_1\underline{A}'\overline{B}_2\underline{A}' \ldots & \text{if otherwise.} \end{cases}$$

It is easy to check that this operation is associative. The next lemma describes the maximality properties of itinerary sequences.

Lemma A.2.1. *Let $\underline{A}$ be a finite maximal sequence. If $\underline{B}$ is maximal then $\underline{A} * \underline{B}$ is maximal. Conversely, if $\underline{A} * \underline{B}$ is maximal then $\underline{B}$ is maximal.*

Proof. The first implication is Corollary II.2.4 in [2]. Considering the shifts $\mathcal{S}^{nk}(\underline{A} * \underline{B})$ for all $k \geq 0$, where $n = |\underline{A}|$, one may check the second implication. $\square$

A.3 Renormalization

We say that a unimodal map f is renormalizable if it has a *restrictive interval* on which an iterate of f is unimodal. If this is true for infinitely many iterates we call f *infinitely renormalizable*. In the quadratic family those are exactly the maps with a fractal attractor. The existence of restrictive intervals simplifies the study of high iterates of f. We define the combinatorial type of the renormalization and state a classical theorem of existence of any type of renormalization in full families of unimodal maps.

Definition A.3.1. *Let $f : I \to I$ be a unimodal map. A closed proper subinterval J of I that contains the critical point c is called* restrictive *with period $n \geq 2$ for f if*

1. *the interiors of $J, \ldots, f^{n-1}(J)$ are disjoint,*
2. *the restriction of f^n to J is unimodal,*
3. *J is maximal with respect to these properties: if J' is a closed interval with $J \subseteq J'$ and such that the previous properties also hold for J' (for the same integer n) then $J = J'$.*

Let f be renormalizable and J a restrictive interval of period n. We define a unimodal map $\mathcal{R}(f, J) : [0,1] \to [0,1]$ that is an affine conjugate of $f^n : J \to J$ such that it is increasing on the left lap and decreasing on the right lap. We define the *renormalization operator* $\mathcal{R}(f) = \mathcal{R}(f, J_0)$ where J_0 is the maximal restrictive interval (with minimal period).

We use the following lemma on several occasions to prove the existence of restrictive intervals. For a proof one may check [9] (Lemma II.5.1).

Lemma A.3.1. *Let $f : I \to I$ be unimodal. If $n \geq 2$ and J is an interval that contains the critical point c such that $f^n(J) \subseteq J$ and the interiors of $J, \ldots, f^{n-1}(J)$ are disjoint then J is contained in a restrictive interval of period n.*

Let us fix f a renormalizable unimodal map on $[0,1]$ that is increasing on $[0,c)$ and decreasing on $(c,1]$. Let J be a restrictive interval for f and n its period. We switch our attention to the combinatorial features of the renormalization. We define the itinerary of the restrictive interval $\underline{I}_f(J) = K_0K_1 \ldots K_{n-1}C$ that has the same prefix of length $n-1$ as the kneading sequence $\underline{K}_f$. The correspondence $\mathcal{S}^i\underline{I}_f(J) \to f^{i+1}(J)$ for $i = 0, \ldots, n-1$ is order preserving with respect to the natural order on the interval, as $f(J), \ldots, f^n(J)$ are intervals with disjoint interiors.

We define a permutation $\sigma_f \in S_n$ that captures the dynamics of f on the orbit of the restrictive interval with respect to the order on the real line. Let $\tau \in S_n$ be a permutation such that

$$f^{\tau(1)}(J) < f^{\tau(2)}(J) < \ldots < f^{\tau(n)}(J).$$

We set $\sigma_f(i) = \tau^{-1}(\tau(i)+1)$ for $i \neq \tau^{-1}(n)$ and $\sigma_f(\tau^{-1}(n)) = \tau^{-1}(1) = n$. Then f, J and σ_f satisfy

$$f\left(f^{\tau(i)}(J)\right) \subseteq f^{\tau(\sigma_f(i))}(J) \text{ for } i = 1, \ldots, n,$$

as $f^{n+1}(J) \subseteq f(J)$.

One may check that σ_f is a cycle that is increasing on $\{1, \dots, \tau^{-1}(n)\}$ and decreasing on $\{\tau^{-1}(n), \dots, n\}$. This motivates the following definition of an *unimodal permutation.*

Definition A.3.2. *We call a cycle $\gamma \in S_n$* unimodal *if there is $k \in \{1, \dots, n-1\}$ such that γ is increasing on $\{1, \dots, k\}$ and decreasing on $\{k, \dots, n\}$. If γ is unimodal we also call it* renormalizable *if there are $1 < k, m < n$ with $km = n$ such that γ acts on m blocks $B_i = \{ik+1, \dots, ik+k\}$, that is*

$$\forall\ 0 \leq i < m\ \exists\ 0 \leq j < m \text{ such that } \gamma(B_i) = B_j.$$

For a better picture of a unimodal permutation γ let us define its *graph* $G(\gamma) : [0, n+1] \to [0, n+1]$, a piecewise affine continuous map. Let $G(0) = G(n+1) = 0$, $G(i) = \gamma(i)$ and G affine on $[i, i+1]$ for all $i = 0, \dots, n$. It is easy to check that γ is unimodal if and only if $G(\gamma)$ is unimodal. Using Lemma A.3.1 one may check that γ is renormalizable if and only if $G(\gamma)$ is renormalizable. Moreover, σ_f is non-renormalizable if and only if J is a maximal restrictive interval for f.

Let us state the existence theorem of infinitely renormalizable unimodal maps of arbitrary combinatorial type for full families of unimodal maps (see [9], Theorem II.5.3).

Theorem A.3.1. *Let $f_\mu, \mu \in \Delta$ be a full family of unimodal maps and let $(\sigma_i)_{i\geq 0}$ be a sequence of non-renormalizable unimodal permutations. Then for each $n \in \mathbb{N}$, the set*

$$\{\mu \in \Delta \mid f_\mu \text{ } n \text{ times renormalizable, } \sigma(\mathcal{R}^i(f_\mu)) = \sigma_i, i = 0, \dots, n\}$$

is closed, non-empty and contains an interval $\Delta_{\sigma_0,\sigma_1,\dots,\sigma_n}$ such that $\mathcal{R}^i(f_\mu)$, $\mu \in \Delta_{\sigma_0,\sigma_1,\dots,\sigma_n}$ is a full family of unimodal maps. Furthermore $\Delta_{\sigma_0,\sigma_1,\dots,\sigma_n} \subset \Delta_{\sigma_0,\sigma_1,\dots,\sigma_{n-1}}$. In particular, $\Delta_{\sigma_0,\sigma_1,\dots}$ is non-empty and $\Delta_\infty = \cup\Delta_{\sigma_0,\sigma_1,\dots}$ contains a Cantor set.

A.4 Duality

In this section we identify finite maximal sequences to unimodal permutations and prove our main result, an existence theorem for infinitely renormalizable quadratic maps, in terms of the kneading sequence.

Let $\underline{K}$ be a fixed maximal sequence of length $n > 1$. We define $\sigma_{\underline{K}}$, the permutation that captures the dynamics of the left shift on $\underline{K}$. Let $\tau \in S_n$ be the permutation for which

$$\mathcal{S}^{\tau(1)-1}\underline{K} \prec \mathcal{S}^{\tau(2)-1}\underline{K} \prec \dots \prec \mathcal{S}^{\tau(n)-1}\underline{K},$$

and $k = \tau^{-1}(n)$. We construct $\sigma_{\underline{K}}$ in a similar way to σ_f, that is $\sigma_{\underline{K}}(i) = \tau^{-1}(\tau(i)+1)$ for $i \neq k$ and $\sigma_{\underline{K}}(k) = \tau^{-1}(1) = n$. We have

$$\mathcal{S}\left(\mathcal{S}^{\tau(i)-1}\underline{K}\right) = \mathcal{S}^{\tau(\sigma_{\underline{K}}(i))-1}\underline{K} \text{ for } i \neq k,$$

and $\mathcal{S}^{\tau(k)-1}\underline{K} = C$. As $\tau(n) = 1$ the orbit of n under $\sigma_{\underline{K}}$ is $\{1, \dots, n\}$ therefore $\sigma_{\underline{K}}$ is a cycle. Moreover $\mathcal{S}^{\tau(i)-1}\underline{K} \prec C$ for $i = 1 \dots k-1$ and $\mathcal{S}^{\tau(i)-1}\underline{K} \succ C$ for $i = k+1 \dots n$. As the left shift is increasing on sequences $L\dots$ and decreasing on sequences $R\dots$, $\sigma_{\underline{K}}$ is a unimodal permutation.

Let us consider the kneading sequence $\underline{K}_\sigma$ of a unimodal permutation $\sigma \in S_n$, defined by $\underline{K}_\sigma = \underline{K}_{G(\sigma)}$ the kneading sequence of the graph of σ. It is a maximal sequence of length n as the critical point of $G(\sigma)$ is periodic with period n. As they capture the same dynamics it is not hard to see that $\sigma \to \underline{K}_\sigma$ is the inverse of $\underline{K} \to \sigma_{\underline{K}}$. Therefore the correspondence is one to one and onto.

This duality comes into play in the following proposition and in the main theorem.

Proposition A.4.1. *If f is a renormalizable unimodal map and J its maximal restrictive interval then*

$$\underline{K}_f = \underline{I}_f(J) * \underline{K}\left(\mathcal{R}(f)\right).$$

Proof. Let n be the period of J and $\underline{K}_f = K_1 K_2 \dots$. As $f^{kn+i}(c) \in f^i(J)$ for all $k, i \geq 0$ and $c \notin f^i(J)$ for $i = 1, \dots, n-1$ we already obtain

$$K_{kn+i} = K_i \text{ for } i = 1, \dots, n-1 \text{ and } k \geq 0.$$

But $\underline{I}_f(J) = K_1 \dots \underline{K}_{n-1} C$ so

$$\underline{K}_f = \underline{I}_f(J) * \underline{K}'$$

and $\underline{K}'$ is maximal by Lemma A.2.1. Let $J = [a, b]$ so $\mathcal{R}(f) = g \circ f^n|_{[a,b]} \circ g^{-1}$ with g an affine homeomorphism from $[a, b]$ to $[0, 1]$. If $\epsilon(K_1 \dots K_{n-1}) = 1$ then $f^n(c)$ is a local maximum and $f^n(a) = f^n(b) = a$ so $\underline{K}' = \underline{K}(\mathcal{R}(f))$ as g is increasing. If $\epsilon(K_1 \dots K_{n-1}) = -1$ then $f^n(c)$ is a local minimum, $f^n(a) = f^n(b) = b$ and g is decreasing. Therefore $\underline{K}(\mathcal{R}(f))$ is the *complement* of $\underline{K}'$, that is, the sequence with the positions of L and R exchanged in $\underline{K}'$. By the definition of the composition this proves the proposition.

□

We say that a maximal finite sequence $\underline{K} \neq C$ is *prime* if there are no non-trivial maximal sequences $\underline{K}_1, \underline{K}_2$ such that

$$\underline{K} = \underline{K}_1 * \underline{K}_2.$$

Let us recall that if a unimodal permutation σ is renormalizable so is $G(\sigma)$. Applying the previous proposition, if $\underline{K}$ is a finite maximal prime sequence then $\sigma_{\underline{K}}$ is non-renormalizable.

Theorem 4. *If f is a quadratic unimodal map and $\underline{K}_f$ its kneading sequence then f is infinitely renormalizable if and only if $\underline{K}_f$ is the composition of infinitely many prime sequences. There exists a unique quadratic map f with*

$$\underline{K}_f = \underline{K}_1 * \underline{K}_2 * \dots$$

if $\underline{K}_i$ are prime sequences and

$$\underline{I}_{\mathcal{R}^{i-1}(f)}(J_i) = \underline{K}_i \text{ for all } i \geq 1, \tag{A.2}$$

where J_i is the maximal restrictive interval of $\mathcal{R}^i(f)$.

Proof. The first implication is obtained directly from Proposition A.4.1. Suppose $\underline{K} = \underline{K}_1 * \underline{K}_2 * \ldots$ with $\underline{K}_i$ finite maximal non-trivial sequences for all $i \geq 0$. Without loss of generality we may suppose $\underline{K}_i$ prime for all $i \geq 0$, otherwise we write it as a product of prime sequences. Therefore the unimodal permutations $\sigma_i = \sigma_{\underline{K}_i}$ for all $i \geq 0$ are non-renormalizable. By the existence Theorem A.3.1 there is a quadratic map f that is infinitely renormalizable and

$$\sigma_i = \sigma(\mathcal{R}^i(f)) \text{ for all } i \geq 0.$$

This also implies equality (A.2). We show that f is the unique quadratic map with $\underline{K}_f = \underline{K}$ and this ends the proof.

Suppose there is a quadratic map $g \neq f$ such that $\underline{K}_g = \underline{K}_f = \underline{K}$. Infinitely renormalizable maps do not have stable periodic orbits so by Theorem A.2.2 the maps f and g are conjugate. Using the definition of restrictive intervals one may check that g is infinitely renormalizable. Then by the Milnor-Thurston theory - the kneading sequence is increasing in the quadratic family - there is an interval $I \subset [1, 4]$ such that $f_a, a \in I$ is infinitely renormalizable. But this contradicts the main result of [6] that hyperbolic dynamics is dense in the quadratic family, as infinitely renormalizable maps are not hyperbolic. $\square$

Remark A.4.1. *Except for the uniqueness of f, one may prove the second part of the theorem directly, that is without using the result of [6]. This can be done using some variant of Proposition A.5.1.*

A.5 Applications

Motivated by the result of [4] that the Hausdorff dimension of fractal attractors is bounded away from 1, the aim of this section is to estimate the Hausdorff dimension of several types of fractal attractors of quadratic maps. Our work was inspired by the work of Grassberger [7]; however we do not use any of its methods. Grassberger estimates by two methods the dimension of the Feigenbaum attractor - with kneading sequence $RC * RC * \ldots$, or simply, with renormalization periods $(2, 2, \ldots)$ - using the box dimension. Attractors with renormalization periods $(k, 2, 2, \ldots)$ for some $k > 2$ are considered in [7] but they all have the same dimension. In fact, it is not hard to see that after renormalization, the dimension of the attractor does not change. Moreover, for periodic and preperiodic renormalization types of period $k \geq 1$, the sequence of renormalizations $\left(\mathcal{R}^{ki}(f)\right)_{i\geq 0}$ converges to a universal analytic map (see [11]). Therefore the Hausdorff dimension of such a map depends only on the periodic part of the renormalization type. This also means that the numerical approach could not deal with non-preperiodic renormalization types.

A.5.1 The Algorithm

Feigenbaum-like maps are easy to trace in the quadratic family as they are the limit of cascades of bifurcations - using the graph of $\omega(c)$, a forward invariant compact contained in the attractor of $f_a \in \mathcal{Q}$ for all $a \in [1,4]$. To the best of our knowledge there is no known method to search for quadratic maps of other renormalization types. This is the main application of Theorem 4, an algorithm that finds - up to arbitrarily small error - the quadratic map of a given renormalization type. We are able to estimate this error - in terms of the renormalization class of the result - and the time requirements of the algorithm (the space requirements are $O(1)$).

Let σ_i be non-renormalizable unimodal permutations and $\underline{K}_i = \underline{K}_{\sigma_i}$ the corresponding prime maximal sequences for $i \geq 1$. Let f be the unique quadratic map of renormalization type $(\sigma_i)_{i\geq 1}$ so

$$\underline{K}_f = \underline{K}_1 * \underline{K}_2 * \dots$$

Let $\underline{K}(a) = \underline{K}_{f_a}$, for all $a \in [1,4]$ and $f_a \in \mathcal{Q}$. We know that $\underline{K}(a)$ is increasing in a and that all maximal sequences are realized by this application. Our algorithm computes $a_0 \in [1,4]$ with

$$\underline{K}(a_0) = \underline{K}_f.$$

More precisely it computes $I_1 \supset I_2 \supset \dots$ such that $a_0 = \bigcap_{n\geq 1} I_n$. Let $I_0 = [1,4]$ and let b_i be the center of the interval I_i, then for all $i \geq 0$

$$I_{i+1} = \begin{cases} I_i \cap [1, b_i] & \text{if } \underline{K}_f \preceq \underline{K}(b_i), \\ I_i \cap [b_i, 4] & \text{if otherwise.} \end{cases}$$

Therefore

$$|I_i| = 3 \cdot 2^{-i} \text{ for all } i \geq 0. \tag{A.3}$$

This equality is used to compute the time requirements of the algorithm. For the evaluation of the quality of the answer we prefer a lower bound for the number of *good* renormalizations of f_{b_n}

$$r_n \leq \max\left\{j \geq 1 \mid \mathcal{R}^k(f_{b_n}) = \sigma_k \text{ for } k = 1, \dots, j\right\}. \tag{A.4}$$

Our algorithm simply computes the critical orbit and the kneading sequence so we need a method to compute r_n using only combinatorial properties of the critical orbit. The following proposition is a first step in that direction.

Let g be a unimodal map and $x \in [0,1]$. We define $s(x)$ to be such that $g(x) = g(s(x))$ and $s(x) \neq x$ if $x \neq c$, well defined on $[0,1]$. If g is quadratic or some renormalization of a quadratic map, it is symmetric thus $s(x) = 1 - x$. Let us also denote by $]a,b[$ the closed interval I with $\partial I = \{a,b\}$.

Proposition A.5.1. *Let g be unimodal and $c_i = g^i(c)$ for all $i \geq 0$ its critical orbit. If for some $n > 1$*

$$c_{2n+1} > c_{n+1} > c_k \text{ for all } k \in \{2, \dots, n\} \cup \{n+2, \dots, 2n\} \text{ and} \tag{A.5}$$

$$c \notin]c_j, c_{n+j}[\text{ for all } j \in \{1, \dots, n-1\} \tag{A.6}$$

then g is renormalizable of period n.

Proof. Using Lemma A.3.1 we show that the interval

$$J =]c_n, s(c_n)[$$

is contained in a restrictive interval of period n. As $c \in J$, one may check that $g(J) =]c_1, c_{n+1}[$ and using (A.6)

$$g^j(J) =]c_j, c_{n+j}[\text{ for all } j \in \{1, \dots, n\}\,.$$

As $c_{2n+1} > c_{n+1}$ we may observe that

$$g^n(J) \subseteq J.$$

Inequality (A.5) shows that $g(J)$ is disjoint from $g^k(J)$ for all $k \in \{2, \dots, n\}$ and using also (A.6), J is disjoint from $g^k(J)$ for all $k \in \{1, \dots, n-1\}$.

Suppose that there are $1 \le i < j < n$ such that $g^i(J)$ and $g^j(J)$ are not disjoint. Then $g^{n-(j-i)}(J)$ and $g^n(J) \subseteq J$ have a common point, a contradiction. □

The following proposition shows that conditions (A.5) and (A.6) are pertinent, that they are satisfied by a good approximation of an infinitely renormalizable map.

Proposition A.5.2. *If g is renormalizable of period n such that*

$$\underline{K}(\mathcal{R}(g)) = RL\ldots \prec \underline{R}$$

then g satisfies conditions (A.5) and (A.6). This hypothesis is satisfied by all infinitely renormalizable maps.

Proof. Let $J = [a, b]$ be the restrictive interval of period n of g. Then $g^n(a) = g^n(b) \in \{a, b\}$ so $s(a) = b$ and $s(b) = a$. As $c_n \in J$,

$$]c_n, s(c_n)[\subseteq J,$$

therefore $]c_k, c_{n+k}[\subseteq g^k(J)$ for all $k = 1, \dots, n$. Thus condition (A.6) is satisfied by g.

Let c' be the critical point of $g_1 = \mathcal{R}(g)$ and $c'_k = g_1^k(c')$ for all $k \ge 1$. As $\underline{K}(g_1) = RL\ldots$

$$c'_2 < c' < c'_1.$$

By definition $g(s(c'_1)) = g(c'_1) = c'_2$ and g is increasing on $[0, c']$. Then $c'_2 > s(c'_1)$ otherwise $\mathcal{S}^2\underline{K}(g_1) \preceq \mathcal{S}\underline{K}(g_1) = L\ldots$ so $\underline{K}(g_1) = RLLL\ldots = \underline{R}$ which violates the hypothesis. Therefore c_{2n} lies in the interior of $]c_n, s(c_n)[$ so

$$c_{2n+1} > c_{n+1}.$$

Moreover $c \in]c_n, c_{2n}[$ and $s(c_n)$ is not a fixed point for g^n so $]c_n, s(c_n)[$ is contained in the interior of J. As $J, g(J), \dots, g^{n-1}(J)$ have disjoint interiors, $c_i \ne c_j$ for all $1 \le i \ne j < 2n$.

The interval $g(J)$ is the rightmost among $J, g(J), \ldots, g^{n-1}(J)$, $c_n, c_{2n} \in J$ and $c_i, c_{n+i} \in g^i(J)$ for all $i = 1 \ldots n-1$ so

$$c_{n+1} > c_k \text{ for all } k \in \{2, \ldots, n\} \cup \{n+2, \ldots, 2n\}.$$

If g is infinitely renormalizable so is $\mathcal{R}(g)$ therefore $\underline{K}(\mathcal{R}(g))$ cannot be periodic or finite by Theorem A.2.2. The only maximal sequences that do not start with RL are $\underline{L} = LLL\ldots$, C, $RRR\ldots$ and RC. Thus

$$\underline{K}(\mathcal{R}(g)) = RL\ldots \prec \underline{R}.$$

□

Propositions A.5.1 and A.5.2 provide a method to check in $2n+1$ steps if a given quadratic map is renormalizable of period n. Moreover, it can be applied to any renormalization $\mathcal{R}^n, n \geq 1$. Let

$$P_n = \prod_{1\leq i\leq n} |\underline{K}_i|$$

be the renormalization period of $\mathcal{R}^n$. Let $\Delta_n = \Delta_{\sigma_1,\ldots,\sigma_n}$ be the interval defined by Theorem A.3.1. Then the time requirements for the algorithm to find some $b \in \Delta_n$, using equality (A.3), is

$$O(-P_n \cdot \log_2 |\Delta_n|).$$

One may check that for $n \in \{2, 3, 4\}$ there is only one finite prime maximal sequence of length n. For $n = 5$ there are three such sequences $RLRRC$, $RLLRC$ and $RLLLC$. Let us denote those renormalization types by 5_1, 5_2 and 5_3 respectively. Table A.1 presents the values of some parameters of quadratic maps as a function of the preperiodic renormalization type.

A.5.2 Hausdorff Dimension

Let us briefly discuss Grassberger's numerical method employed in [7] to compute the Hausdorff dimension of Feigenbaum-like fractal attractors. It is in fact an algorithm that approximates the *box dimension.* Let $A \subseteq [0,1]$ be the attractor. We divide the interval $[0,1]$ in N equal intervals. Let $A(N)$ be the number of such intervals that intersect A. We define the box dimension of A

$$BD(A) = \lim_{N\to\infty} \frac{\log A(N)}{\log N},$$

when the limit exists. For a detailed discussion on the box dimension and Hausdorff dimension one may check [12]. We know that

$$HD(A) \leq BD(A),$$

Table A.1: Parameter b_n as a function of the renormalization type.

Renormalization Type	b_n	$\lvert I_n \rvert$	r_n
$(2, 2, 2, 2, \ldots)$	3.5699456719	$8.5 \cdot 10^{-22}$	32
$(3, 2, 2, 2, 2, \ldots)$	3.8494336812	$3.4 \cdot 10^{-21}$	26
$(5_1, 2, 2, 2, 2, \ldots)$	3.7430055309	$8.5 \cdot 10^{-22}$	28
$(5_2, 2, 2, 2, 2, \ldots)$	3.9064536326	$2.1 \cdot 10^{-22}$	28
$(5_3, 2, 2, 2, 2, \ldots)$	3.9903214465	$2.1 \cdot 10^{-22}$	28
$(2, 3, 2, 3, \ldots)$	3.6330072770	$2.0 \cdot 10^{-28}$	23
$(3, 2, 3, 2, \ldots)$	3.8504152723	$2.5 \cdot 10^{-29}$	23
$(2, 2, 3, 2, 2, 3, \ldots)$	3.5833031348	$8.5 \cdot 10^{-22}$	24
$(2, 2, 2, 3, 2, 2, 2, 3, \ldots)$	3.5728060660	$2.1 \cdot 10^{-24}$	26
$(3, 3, 3, 3, \ldots)$	3.8540779636	$1.9 \cdot 10^{-34}$	20
$(4, 4, 4, 4, \ldots)$	3.9615565872	$8.8 \cdot 10^{-47}$	13

where $HD(A)$ is the Hausdorff dimension of A. The inequality is strict for $\mathbb{Q} \cap [0, 1]$ and $\left\{\frac{1}{n} \mid n \geq 1\right\}$. That is because the box dimension behaves rather badly under topological and set-theoretical operations. For example

$$BD(S) = BD(\overline{S}) \text{ for all } S \subseteq [0, 1].$$

Moreover the box dimension of a countable union of sets cannot be computed as a function of the dimensions of those sets - in the case of the Hausdorff dimension, it is the supremum of their dimensions.

Figures A.1 and A.2 represent the graph of $A(N)$ in a logarithmic scale, for the Feigenbaum attractor and for the attractor of renormalization type $(3, 2, 3, 2, \ldots)$ respectively. As the scale is logarithmic, the convergence of computer estimates is weak.

We propose a new method, inspired by the definition of the Hausdorff dimension and by the definition of restrictive intervals. We observe faster convergence compared to the previous method.

Let f be infinitely renormalizable and J be some restrictive interval of period n. Its attractor A is the closure of the critical orbit $(c_i)_{i \geq 0}$, a Cantor set. We have seen in Section A.5.1 that $A \subseteq \bigcup_{i=0}^{n-1} f^i(J)$. Moreover,

$$A \subseteq \bigcup_{i=1}^{n}]c_i, c_{n+i}[\tag{A.7}$$

and this is a minimal cover of A with n intervals - from the point of view of the inclusion of covers. For $\alpha \in [0, 1]$ we define

$$S(\alpha, n) = \sum_{i=1}^{n} |c_i - c_{n+i}|^{\alpha}.$$

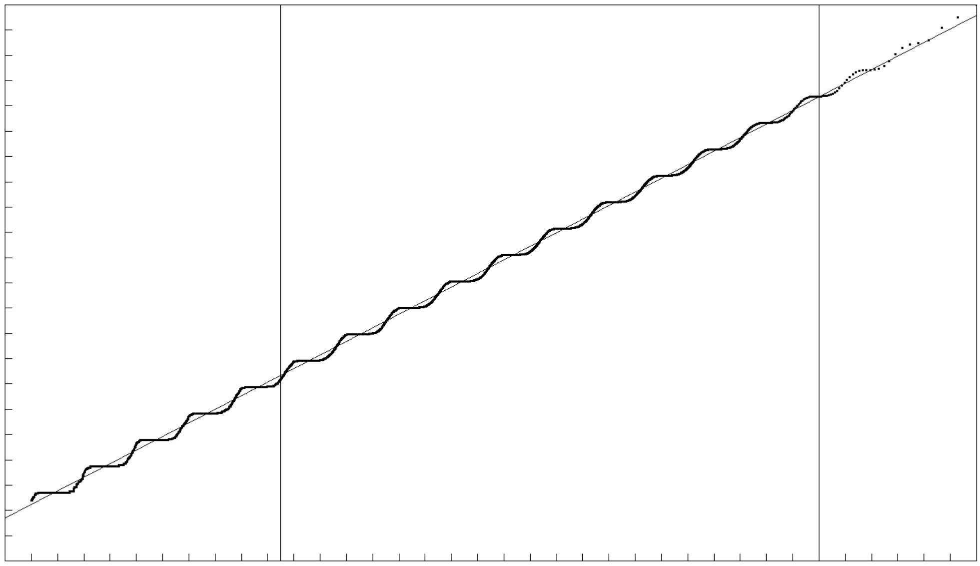

Figure A.1: Grassberger's method for the Feigenbaum attractor.

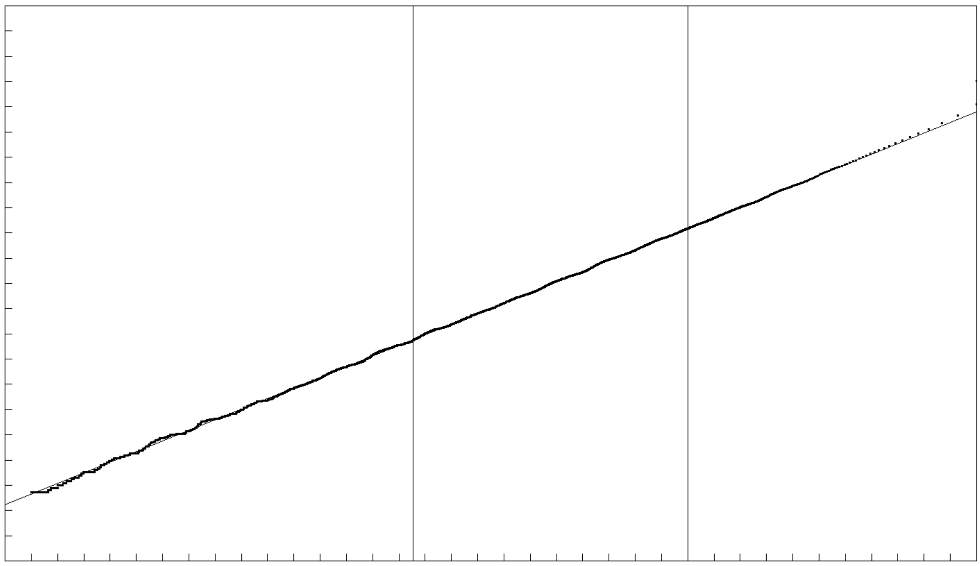

Figure A.2: Grassberger's method for the attractor of type $(3, 2, 3, 2, \ldots)$.

Table A.2: The Hausdorff dimension $HD(A)$ for several renormalization types.

Renormalization type	Hausdorff dimension
$(2, 2, 2, 2, \ldots)$	0.53804514358
$(3, 2, 2, 2, 2, \ldots)$	0.5380451436
$(5_1, 2, 2, 2, 2, \ldots)$	0.5380451436
$(5_2, 2, 2, 2, 2, \ldots)$	0.5380451436
$(5_3, 2, 2, 2, 2, \ldots)$	0.5380451436
$(2, 3, 2, 3, \ldots)$	0.420917432
$(3, 2, 3, 2, \ldots)$	0.420917432
$(2, 2, 3, 2, 2, 3, \ldots)$	0.4448735455
$(2, 2, 2, 3, 2, 2, 2, 3, \ldots)$	0.46275047
$(3, 3, 3, 3, \ldots)$	0.3502283975126
$(4, 4, 4, 4, \ldots)$	0.2689433270892

If (A.7) would be an *optimal* cover, then

$$\lim_{n\to\infty} S(\alpha, n) = \begin{cases} \infty & \text{if } \alpha < HD(A), \\ 0 & \text{if } \alpha > HD(A), \end{cases} \tag{A.8}$$

where $n \to \infty$ means for increasing n renormalization periods.

However, the Hausdorff dimension is constructed using countable covers of sets. Therefore, from the point of view of computer experiments, it is not computationally accessible. Let us recall that we consider only preperiodic renormalization types. Let k such a period, then $\left(\mathcal{R}^{ki}(f)\right)_{i\geq 0}$ converges uniformly to an analytic universal map - depending of the periodic renormalization type, see [11]. Therefore the attractor A is a *self-similar* set and [3] indicates that

$$HD(A) = BD(A).$$

Therefore we compute our estimates using our best finite cover (A.7). They are always an upper bound for $HD(A)$.

Let n_i be the renormalization period of $\mathcal{R}^{ki}(f)$. As one may expect, the experiments *show* that the following limit exists

$$c(\alpha) = \lim_{i\to\infty} \frac{S(\alpha, n_{i+1})}{S(\alpha, n_i)}$$

and it is decreasing. This means that the convergences (A.8) are exponential with base $c(\alpha)$. Therefore

$$HD(A) = c^{-1}(1).$$

Table A.2 presents the estimated Hausdorff dimension for several renormalization types.

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Résumé.

Cette thèse est consacrée à l'étude des relations entre les propriétés dynamiques des orbites critiques et la géométrie des ensembles de Fatou et de Julia des applications rationnelles. La régularité des composantes de l'ensemble de Fatou est équivalente à une version faible de l'hyperbolicité, conséquence des résultats de Graczyk et Smirnov et de Przytycki, Rivera-Letelier et Smirnov. Plus précisément, les composantes de l'ensemble de Fatou sont des domaines de Hölder si et seulement si le diamètre des préimages des petits disques centrés sur l'ensemble de Julia décroît exponentiellement. On s'intéresse désormais aux applications rationnelles sans orbite périodique parabolique. En dynamique polynomiale, Carleson, Jones et Yoccoz ont montré l'équivalence entre la semi-hyperbolicité (toute orbite critique dans l'ensemble de Julia est non-récurrente) et la régularité John (qui implique la régularité Hölder) des composantes de l'ensemble de Fatou. Graczyk et Smirnov ont montré plus tard que si tout point critique dans l'ensemble de Julia est Collet-Eckmann alors les composantes de l'ensemble de Fatou sont Hölder. On introduit la condition de Collet-Eckmann pour les orbites critiques récurrentes qui généralise ces deux dernières conditions et on montre qu'elle a comme conséquence la régularité Hölder. On construit un contre-exemple pour la réciproque. Un deuxième contre-exemple contredit la conjecture de Świątek qui affirme l'invariance topologique de la propriété Collet-Eckmann des points critiques récurrents dans la classe des applications S-multimodales. Le dernier chapitre présente une étude sur la dimension de Hausdorff des attracteurs des applications unimodales infiniment renormalisables.

Mots-Clés : dynamique rationnelle, orbites critiques, géométrie de l'ensemble de Fatou, hyperbolicité, semi-hyperbolicité, Collet-Eckmann, invariance topologique, attracteurs, dimension de Hausdorff.

Abstract.

This PhD thesis is devoted to the study of the relations between dynamical and geometric properties of the Julia set. The regularity of the components of the Fatou set is equivalent to a weaker version of hyperbolicity. This follows from results by Graczyk and Smirnov and by Przytycki, Rivera-Letelier and Smirnov. More precisely, the components of the Fatou set are Hölder domains if and only if the diameter of preimages of small balls centered on the Julia set decay exponentially. In the sequel we consider rational maps without parabolic orbits. In polynomial dynamics, Carleson, Jones and Yoccoz show that semi-hyperbolicity (every critical orbit in the Julia set is non-recurrent) and John regularity (which is stronger than Hölder) of the components of the Fatou set are equivalent. Graczyk and Smirnov show that if every critical point in the Julia set is Collet-Eckmann then the components of the Fatou set are Hölder domains. We introduce the recurrent Collet-Eckmann condition (every recurrent critical point in the Julia set is Collet-Eckmann) which is more general than semi-hyperbolicity and than Collet-Eckmann and show that it also implies Hölder regularity. We also provide a counter-example for the converse. A second counter-example shows that the Świąteks conjecture (topological invariance of the Collet-Eckmann property of recurrent critical orbits in the S-multimodal setting) does not hold. The last chapter presents a (numerical) study of the Hausdorff dimension of attractors of infinitely renormalizable unimodal maps.

Key-words : rational dynamics, critical orbits, geometry of the Fatou set, hyperbolicity, semi-hyperbolicity, Collet-Eckmann, topological invariance, attractors, Hausdorff dimension.

AMS Classification Codes (2000) : 37B10, 37B20, 37D25, 37E05, 37E15, 37F10, 37F15, 37F20, 37F25, 37F35, 37M05, 37M20.