PEGS, PEBBLES, PENNIES and PILES — a study of some combinatorial games

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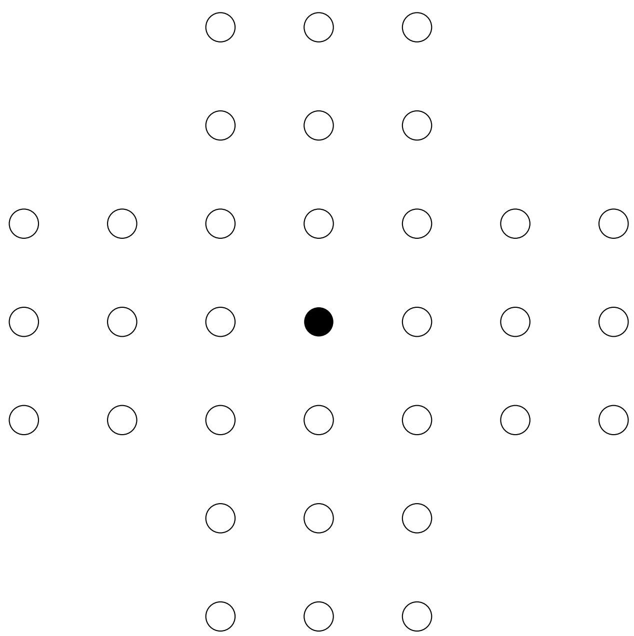


Master's thesis at KTH, 1999

Abstract

In this master's thesis, I will discuss some aspects of four combinatorial games. These are Pegs (also known as solitaire), Pebbles, Pennies and Piles, which are described in this report. The main contents can be summarised as follows.

- Already known results for Pegs, Pebbles and Piles, together with the standard techniques.
- A study of Pennies, which is an entirely new game. On most boards, Pennies is not solvable.
- The reachable area of Pegs in $\mathbb{Z}^d$, allowing diagonal moves, has been investigated. I have found that there still is a rather narrow limit on how far one can reach, but it is substantially larger than the limit obtained not allowing diagonal moves.
- The solvability of Pegs on some unconventional boards has been investigated. I have also obtained some results on the solvability of a game using a kind of sideways Pegs move.

Sammanfattning

Denna examensarbetesrapport behandlar fyra kombinatoriska spel. Dessa är Pegs (även känt som solitaire), Pebbles, Pennies och Piles, vilka beskrivs i rapporten. De viktigaste ämnena som behandlas är följande.

- En genomgång av redan kända resultat för Pegs, Pebbles och Piles, samt de standardtekniker som används.
- Det helt nya spelet Pennies presenteras. På de flesta bräden saknar Pennies lösning.
- Räckvidden för Pegs i $\mathbb{Z}^d$, om vi tillåter diagonala drag, har undersökts. Vi finner räckvidden är längre än i vanligt Pegs, d.v.s. utan diagonala drag, men ändå starkt begränsad.
- Lösbarheten för Pegs på en del ovanliga bräden har undersökts. Jag har även undersökt lösbarheten för det spel som fås, då man använder en sidledesvariant av dragen i Pegs.

Contents

Acknowledgements

A project like this could not be accomplished without the aid of others. I am deeply indebted to my supervisor Kimmo Eriksson and his father Henrik Eriksson, who have provided me with reading material, good ideas and much needed help in times of crisis. I would also like to thank Svante Linusson for teaching me combinatorics and answering tricky questions.

For valuable proofreading and comments on the text, I would like to thank Trygve Eriksen, Henrik Eriksson, Kimmo Eriksson, Mårten Larsson, Johan Wikberg and my ever-supporting fiancé Karin Lindberg.

1 Introduction

Solitaire (or PEGS) is a game which is played by (yes, you guessed it) one person on a board with 33 holes, 32 of which are occupied by marbles or pegs. The player picks any marble and jumps it over an adjacent marble, allowing him to remove the latter. The game ends when there are no two adjacent marbles, and the player has succeeded if there is single marble left in the middle of the board.

I played this game when I was a child, and I never managed to solve it. This may seem only natural, since I made my moves half randomly. It neither occurred to me that there could be a simple solution, nor that the game might be impossible to solve. To me, it was just a game, which remained interesting due to the fruitless search for a solution.

A true mathematician, however, would not be content with this. This mathematician would demand answers to a lot of questions. Is there a simple solution? Which is the simplest solution? Is there a solution at all? Are there some similar games with the same properties, or similar games with different properties? In this report, the approach to solitaire and the other games studied is the mathematician's. If you still wish to enjoy playing solitaire, you should stop reading now. But, if you want to know how to find a solution, or if you want to know an infinite number of other, related, games that you can play and enjoy, then you should keep on reading.

I will present a survey of four different games. PEGS is similar to solitaire, but in this report I will consider a wider variety of boards than just the ordinary solitaire board. I will present some new results concerning PEGS on different boards and some alternative PEGS rules. PENNIES is an entirely new game, and there will be some, obviously, new results on it as well. PEBBLES and PILES have been investigated quite thoroughly before, and the results presented are already well known.

Game theory may seem a rather popular, though not very serious, part of mathematics, and indeed it is. However, even though its results often have no important applications, game theory is important in two aspects. First, the techniques used and theorems obtained can occasionally be used in other parts of mathematics. Second, there are, in fact, a lot of people that want to know the answers to the questions found in game theory.

A word of advice to the reader: Don't read this report as you would read an ordinary scientific paper. Whenever you encounter a game in the text, take a break from your reading and play the game for a while. This will make it easier for you to relate to the games and their difficulties.

This report begins with a short description of the games (section 2). We will then proceed in section 3 with an introduction to some standard techniques used to obtain a solution, and section 4 presents a couple of ways to assign a value to a position. Finally, sections 5 and 6 contain results for finite and infinite boards, respectively.

Let the games begin!

2 The games

Four different kinds of games are studied in this report. They all deal with some objects (pegs, pennies, etc) on a board with a grid of squares or holes. The objects are moved according to rules, specific to each game. The boards, however, as well as the start and end configurations, may vary a lot.

2.1 Pegs

The game Pegs is, if played on one of the boards in Figure 1, also known as solitaire, a game most people are probably familiar with. We will call the objects *pegs* and they may be moved according to definition 2.1.

Definition 2.1 *Playing* Pegs*: Assume that we have two adjacent pegs next to an empty hole, all in the same row or column. To make a* Pegs *move, take the outer peg and jump it over the middle peg into the empty hole. The middle peg is then removed. Using white pegs and black holes, we get* $\bigcirc\bigcirc\bullet \longrightarrow \bullet\bullet\bigcirc$

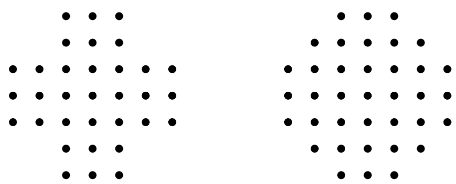


Figure 1: English (left) and continental board

The standard Pegs game, central solitaire, starts as in the front page figure, its *start configuration*, with the entire board filled save the central hole. The aim is to reduce the situation to a single peg in the central hole, the game's *end configuration*. This is an example of a *reversal game*, in which the status (filled/not filled) of each hole should be reversed. The central solitaire concept may be varied a lot, and a thorough investigation of these variations and some standard techniques can be found in the second volume of Winning Ways by Berlekamp, Conway and Guy [2]. We will examine most of these techniques in the following sections and we will also apply them to other finite boards, as well as some infinite boards.

2.2 Pebbles

The game Pebbles is usually played in $\mathbb{N}^2$[*], beginning with a single pebble at the origin and moving away from there. In each move, more pebbles are being constructed, contrary

[*] $\mathbb{N}^2$ is the set of all pairs, (a, b), where $a, b \in \mathbb{N}$, i. e., a and b are integers larger than, or equal to, zero. This means that $\mathbb{N}^2$ consists of the grid points in the first quadrant.

to Pegs, in which the objects are annihilated as the game proceeds. The idea for this game came from M. Kontsevich, who in 1981 in a mathematics competition asked for proof that a specified subset of the board could not be emptied, i. e. that it will always contain at least one pebble. This has been examined in a 1995 paper by Chung et al. [3] and an investigation of the problem in higher dimensions can be found in H. Eriksson, 1994 [6].

Definition 2.2 *Playing* Pebbles*: Assume that we have a pebble at square (x, y). If $(x+1, y)$ and $(x, y+1)$ are both empty, we can remove the pebble at (x, y) and put pebbles on the other two squares. In d dimensions, having a pebble at $(x_1, x_2, \ldots, x_d)$, we can replace it with pebbles at $(x_1+1, x_2, \ldots, x_d), (x_1, x_2+1, \ldots, x_d), \ldots, (x_1, x_2, \ldots, x_d+1)$. Using white pebbles and black squares in two dimensions, we get*

•
○ • ⟶ ○
• ○

2.3 Pennies

The game Pennies is quite similar to Pegs and can be played on the same boards. However, as we will see, the somewhat different moves make the game much harder to solve. Pennies is an invention of Henrik Eriksson (personal communication) and has, to my knowledge, not been studied before.

Definition 2.3 *Playing* Pennies*: Assume that we, in a row or column, have an empty square with a penny on each side. To make a* Pennies *move, we remove the two pennies and put one on the middle square. Using white pennies and black squares, we get*
○ • ○ ⟶ • ○ •

2.4 Piles

The game Piles is different from those above in that the number of objects, called chips, on the board is constant and that you are allowed to pile the objects. It is played on a subset of $\mathbb{Z}^2$[*] or, more generally, on a planar graph with directed edges. A nice summary and some references can be found in K. Eriksson, 1994 [8].

Definition 2.4 *Playing* Piles*: Assume that we have at least as many chips on one node as the number of outgoing edges, e. Then we can fire that node, removing e chips from it and giving one to each of the neighbour nodes.*

3 How to find a solution

3.1 Packages

Having played a few games of solitaire, most people will try to develop some kind of strategy. Are there any structural features that can be used? Let's look at the configuration in Figure 2 and think of the left smaller ring as a peg and the right as a hole.

[*] $\mathbb{Z}$ is the set of all integers.

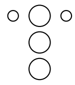


Figure 2: A Pegs configuration — one of the smaller circles is a peg, the other is an empty hole

If we jump to the right with this peg, jump upwards with the bottom one and then left with the first peg, the configuration will be similar to the original, except for the three pegs in the vertical column, which have been removed. We have found a way to remove three pegs in a row, without disturbing the rest of the board. This is a strong structural feature; instead of jumping back and forth randomly, we can now clinically clear parts of the board without affecting the surroundings.

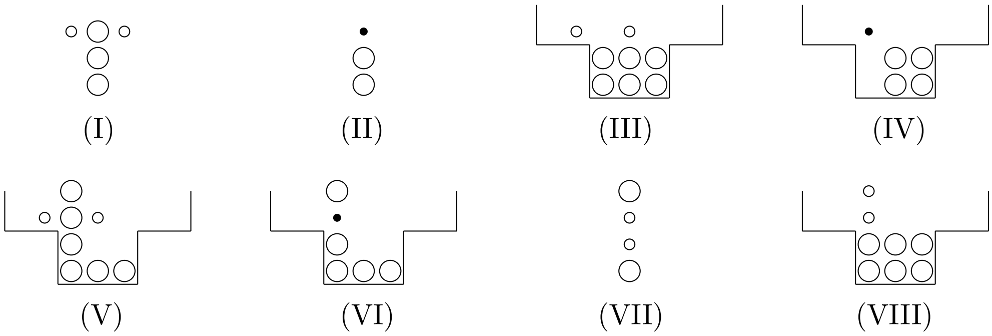


Figure 3: Our packages

Let's consider these three adjacent pegs as a unit, neatly packaged and ready to be removed. Are there any more of these packages? Yes, of course. Try to discover some yourself and then check Figure 3 for a (not complete) list of useful packages. Again, the smaller circles indicate pairs of positions, of which exactly one position should be occupied (it doesn't matter which one) and the dots now indicate holes to be filled. The second package, for example, is just a single move, leaving a peg a the top. Note that in the fourth package, we'll have to use holes outside the package. With these packages in mind, let's make an attempt to solve central solitaire.

The third (as well as the last) package looks very well suited to handle the areas near the border. However, in order to use them, we must first clear the area in front of them, which may be done with the first package. Finally, we must make sure that we have two pegs in position for the last move. All in all, one way to solve it is shown in Figure 4. The numbers indicate in which order we remove the packages. The order is important, since we have restrictions on the environment of the package we are removing.

The initial empty hole could of course be positioned differently. Berlekamp et al. [2]

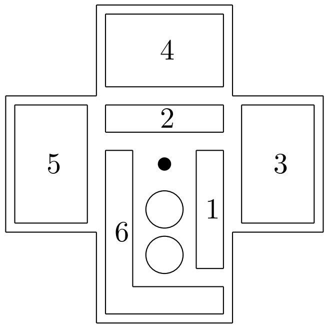


Figure 4: A solution to central solitaire

have solutions for all 6 alternatives (symmetry reduces the number of positions). Most of them are fairly simple and finding them forms an interesting exercise.

3.2 How many solutions are there?

Central solitaire on the English board have many solutions. Do the other games behave in the same way? PENNIES does, but PEBBLES and PILES do not. To explain why, we'll have to introduce a new concept (see [6] and [8]).

Definition 3.1 *A* ***strongly convergent*** *game has the following property: an instance of the game will either go on forever or always terminate in the same configuration after the same number of moves.*

An instance of the PEBBLES game is given by defining which squares should be emptied. If these are chosen badly, the game becomes unsolvable (Theorem 6.8 gives the smallest subset of $\mathbb{N}^2$ that can't be emptied). An attempt to empty such a subset will therefore go on forever. Let's assume that it is possible to empty the chosen squares. We will call a pebble *obstructing* if the square it occupies should be emptied or if it *covers* (stands in the way of) an obstructing pebble.

Theorem 3.1 (H. Eriksson, 1994 [6]) *If we restrict ourselves to moving only obstructing pebbles,* PEBBLES *in $\mathbb{N}^d$, where $d \in \mathbb{N}, d \geq 2$, is strongly convergent for all subsets to be emptied.*

The proof of this theorem, and a generalisation to posets, is given in [6].

This property means that PEBBLES is quite boring to play. Either the game goes on forever, or we can find the best solution by moving only obstructing pebbles.

4 Giving every configuration a value

When playing two-player games, an analysis is often facilitated by giving each configuration a value, which is positive for me and negative for my opponent. This makes it possible to play several games simultaneously (the player who is in turn has to choose which game to move in) and still predict the outcome. (The interested reader will certainly enjoy reading the first volume of Winning Ways [1], which gives a thorough introduction to the subject.) When playing one person games, like PEGS, we can use similar techniques, which are described below. Using *Quantum numbers* (section 4.1), we assign unchangeable values to each game configurations, thereby partitioning the configurations into equivalence classes, while applying *weight functions* (section 4.2) will assign a non-increasing value to every game configuration, producing an upper bound on the value of the end configuration.

4.1 Algebras and Quantum Numbers

We have previously discovered that the package technique enables us to find a solution to central solitaire on the English board. Let's now consider the continental board mentioned in section 2.1. It contains four more holes than the English board, but it shouldn't be too hard to find a solution for the continental board as well. Actually, it is impossible!

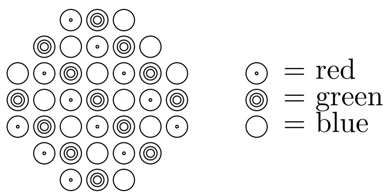


Figure 5: The continental board coloured

Theorem 4.1 (folklore) *Central solitaire can not be solved on the continental board.*

PROOF Let's bring out some paint tins and fill the holes of the board with paint as in Figure 5. Make sure you use enough paint to make all pegs turn into the colour of the hole you put them in. When making a move, you will take two pegs of different colour (e. g. red and green) and turn them into a peg of the remaining colour (in this case blue). Thus, a move will decrease the number of pegs of two of the colours by one and increase the number of pegs of the third colour by one. In short, a move will change the parity* of all the quantities r , g and b (the number of red, green and blue pegs, respectively), leaving the parity of $r + g$ and $r + b$ unchanged.

It is now easy to see that since $r = g = b = 12$ when we start playing (the central hole is empty and doesn't contribute), giving $r + g \equiv r + b \equiv 0$ (modulo 2), and the configuration sought for has $r = b = 0$ and $g = 1$, resulting in $r + g \equiv 1$ (modulo 2), we

*The parity of a quantity is either even or odd.

can't find a solution to central solitaire on the continental board. □

In the above proof, we found two entities ($r + g$ and $r + b$) that remained constant throughout the game. Such constants will be referred to as *quantum numbers* and are very useful when showing that a particular game can't be solved. These quantum numbers enables us to partition our games into equivalence classes. In this case, we have two possible values for both quantum numbers (there is no reason to consider $g + b$, since it isn't independent of the other quantities), which yields four equivalence classes.

Actually, this result can be improved. We can of course use a board colouring perpendicular to the one in Figure 5, which gives two new quantum numbers. Since these are independent of the first ones, we get 16 equivalence classes. Are there any more?

Let's name four holes in a row a, b, c, and d. Since we can replace two pegs at a and b with a peg at c, we may say that $ab = c$. We also have $bc = a$, and if we combine the two equations we get $abbc = ca \iff b^2 = 1$. We have reduced Pegs to pure algebra (see [2]). The interpretation is that if we allow more than one peg per hole, two pegs in the same hole can easily be erased, for example by jumping back and forth over the hole with another peg. Combining $a = bc$ with $bc = d$ gives $a = d$, so all pegs at distance three are equivalent in this sense. From these findings one easily obtains $ad = aa = 1$, which proves that the seventh package (see Figure 3) is removable. Similarly, we find the other packages to be removable, using $abc = aa = 1$.

This algebra may be handy when proving that our packages are correct, but it becomes really useful when used to reduce the size of the board. The fact that pegs at distance three are equal means that every hole on the board is equivalent to one of the nine middle holes (see Figure 6). The number of pegs in each of these equivalence classes is one or zero, since any two pegs cancel. But we can do better. It is easy to show that $b = ac$, $h = gi$, $f = ci$, $d = ag$ and $e = acgi$, which reduces the board to the four holes a, c, g and i, giving $2^4 = 16$ equivalence classes. Since this is an upper bound on the number of equivalence classes and the earlier result was a lower bound, we can conclude that the number of equivalence classes is precisely 16. This is valid for the English board as well.

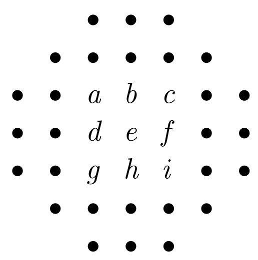


Figure 6: Reducing the size of the board

It should be noted that with this technique, we can only prove that a problem is unsolvable. If the start and end configurations belong to the same equivalence class, the game may be solvable, but it doesn't have to be, and the only way to find out is to find a solution.

Finally, a word about reversal games and colours. In a successful *reversal* game, the number of pegs in each hole will be changed by one, either removing one peg or adding one. Thus, the quantities $r + g$ and $r + b$, now counting the holes and not the pegs, must be even if we are to reach the end configuration using legal moves. This proves the following theorem.

Theorem 4.2 *A reversal game is not solvable if, counting the holes on the board, it does not satisfy* $r + g \equiv r + b \equiv 0$ *(modulo 2).*

On the English board, this theorem doesn't prove anything, since a colouring of the holes will satisfy the criterion above. On the continental board, however, we find that there is not a single reversal game that can be solved.

4.2 Weight Functions

Weight functions are usually applied when we have a multitude of objects to consider. The classic example arises when playing PEGS in $\mathbb{Z}^2$; it has been studied in Winning Ways [2] and, more generally, in a 1995 paper by Eriksson and Lindström [7]. We begin with the entire lower half plane filled and the upper half plane empty — a hole is filled if the y-coordinate is non-positive. We are looking for the lowest level (row) that can't be reached with the pegs, using ordinary moves and the result is surprisingly low. The proof will utilise a typical *weight function* (or *Pagoda function*).

Definition 4.1 *A **weight function** (or **Pagoda function**) P, is a function that assigns a value to every hole on the board, with the following restriction: Letting the value of a peg be the same as the value of the hole it is occupying, it should not be possible to increase the total value of the pegs on the board, using legal moves.*

The definition is similar if we play PEBBLES or PENNIES.

Theorem 4.3 (Berlekamp et al., 1982 [2]) *With the entire lower half plane filled with pegs, the highest reachable level is the fourth. Thus, no matter how many pegs we use, we cannot reach the fifth row.*

PROOF The Pagoda function is in this case taken to be

$$P(x, y) = \sigma^{|x|+|5-y|}$$

where $\sigma = \frac{\sqrt{5}-1}{2}$ (the golden ratio), and $(0, 5)$ is the hole we try to reach*. In other words, $P(x, y)$ is given by σ raised to the power of the axis parallel distances between

*Trying to reach $(k, 5)$ would be the same as trying to reach $(0, 5)$, since the plane is infinite (whichever k is used, there are a countable infinite number of holes to the left, as well as to the right).

(x, y) and $(0, 5)$. Since

$$\sigma^{a+2} + \sigma^{a+1} = \sigma^a(\sigma^2 + \sigma) = \sigma^a \cdot 1 = \sigma^a$$

the value of a configuration is maintained if a move is made towards $(0, 5)$, and decreased otherwise. Hence, $P(x, y)$ is a Pagoda function.

The value of the pegs with x-coordinate 0 is, using the geometric sum,

$$\sum_{k=5}^{\infty} \sigma^k = \sigma^5 \sum_{k=0}^{\infty} \sigma^k = \sigma^5 \frac{1}{1-\sigma} = \frac{\sigma^5}{\sigma^2} = \sigma^3$$

The pegs with $x = 1$ similarly have value σ^4, $x = 2$ gives σ^5 and so on. Not forgetting the negative x-values, the entire lower plane gives

$$\sigma^3 + 2\sum_{k=4}^{\infty} \sigma^k = \sigma^3 + 2\sigma^2 = \sigma^2 + \sigma = 1$$

Hence, the value of any configuration can't exceed 1. Since $P(0, 5) = \sigma^0 = 1$, we would have to use every peg on the board in order to reach $(0, 5)$. This is clearly not possible in a finite number of moves and $(0, 5)$ is thus unreachable. □

To reach the fourth row, only 20 pegs are needed. One sufficient set of pegs can be viewed in Figure 7.

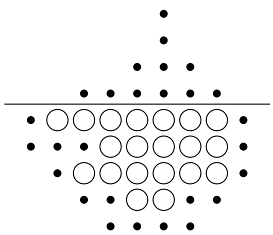


Figure 7: The 20 pegs needed to reach level four

On the finite solitaire board, Pagoda functions are useful as well ([2]). Let's consider the problem in Figure 8. We begin with the four empty holes, and, as usual, wish to reverse the status of every hole. Since it is a reversal game, the start and end configurations belong to the same equivalence class, so the game should perhaps be solvable (see theorem 4.2). Actually, it isn't. Consider the Pagoda function in Figure 9, where the numbers denote the value of each hole. The Pagoda function fulfills the Pagoda condition (i. e., the total value of the board doesn't increase when a move is made) and some simple arithmetic shows that the start value of the board (8) is lower than the end value (10). This shows the the game cannot be solved, although the algebra would allow it.

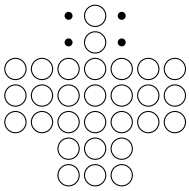


Figure 8: A reversal problem

		3	0	3		
		2	0	2		
0	1	1	0	1	1	0
0	1	1	0	1	1	0
0	0	0	0	0	0	0
		1	0	1		
		-1	0	-1		

Figure 9: The Pagoda function of the reversal problem

5 Playing some finite boards

This section considers only finite board, most of them two-dimensional and a few in three dimensions. We have already seen that small changes in the shape of the solitaire board have large effect on the solvability of the game and the following results are quite similar. Most of the results are new, and the remaining will include a reference.

5.1 PEGS on various boards

We have already discussed a couple of different games on the solitaire boards. Let's now expand outside these boards. Inspired by some of the exciting regular patterns of polygons in [5], some results concerning the solvability (or rather the unsolvability) of central PEGS on some boards have been obtained. To find an interesting board from a pattern, consider either the polygons or the nodes as holes. For instance, the pattern of triangles found on the Chinese checkers board produces an interesting board when the nodes are used as holes (as in Chinese checkers), but using the polygons wouldn't work, since a triangle does not have an even number of neighbours.

I have concentrated on showing that central PEGS is unsolvable on certain boards. On the remaining boards, I have not always tried to find a solution. The reader may find it amusing to search for solutions on the smaller boards, or even a general solution that works for boards of all sizes.

Theorem 5.1 *On a square board with side length n, central* PEGS *is unsolvable if $n \neq 6k+3, k > 0$.*

PROOF We will prove this by colouring the holes. Since the board is perfectly symmetric, it is enough to check one colouring. As this is a reversal game, it's enough to count the holes and check if $r \equiv g \equiv b$ (modulo 2), according to theorem 4.2.

If the side length is $3k$, it is easy to divide the board into square sections with side length 3. Each of the sections contains three holes of each colour, so for the entire board (counting the holes, not the pegs) we get $r = g = b$.

If the side length is $3k+1$ (with corner positions $(1,1)$ and $(3k+1, 3k+1)$), let's remove as many 3×3 sections as possible, leaving pegs at positions $(1,m)$ and $(m,1), m \in \{1,\ldots,3k+1\}$. We can also remove any section with size 1×3 or 3×1, leaving just a single red peg at $(3k+1, 3k+1)$. Thus, for the entire board we get $r = g+1 = b+1$.

If the side length is $3k+2$ (with corner positions $(1,1)$ and $(3k+2, 3k+2)$), let's remove any 3×3 sections that contain a hole. We will then have negative holes at positions $(3k+3, m)$ and $(m, 3k+3), m \in \{1,\ldots,3k+3\}$. Adding rectangular sections leaves us with a single blue hole at position $(3k+3, 3k+3)$. Thus we get $r+1 = g+1 = b$.

We have found that central PEGS is unsolvable with side lengths $n \neq 3k$. However, on a board with an even side length, we don't have a central element, so we need only consider boards with odd side lengths. Thus, PEGS is unsolvable for $n \neq 6k+3, k \geq 0$. Finally, it is easy to see that if $n = 3$, we can't make a single move, so central PEGS is unsolvable if $n \neq 6k+3, k > 0$. □

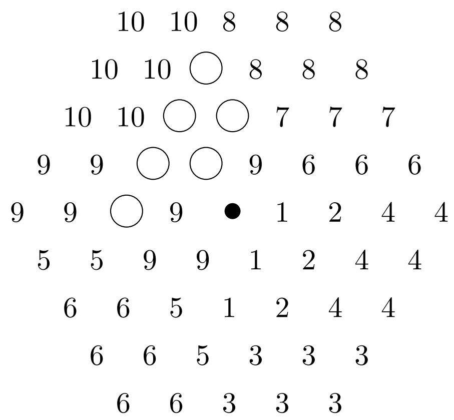


Figure 10: A solution to central PEGS on the Chinese checkers board

An interesting board to play is the hexagonal one. With three main directions, instead of the usual two, it is possible to invent some new packages. If you want to play it in real life, you may use the inner field of a Chinese checkers board. It has a side length of five and constitutes the smallest board on which central PEGS is solvable. A solution, using mostly familiar pacakages, is given in Figure 10. The pegs marked 9 may be tricky to remove. One way is to first clear pegs 3 to 5 on the fourth row and then use package IV in our collection to move pegs back to holes 3 and 4 on the fourth row.

Theorem 5.2 *On a hexagonal board with the side length n, central* PEGS *is unsolvable if $n \neq 3k+2, k > 0$.*

PROOF This will also be proven by colouring the holes, but we will count them differently. This way of counting could very well have been used in the last proof. Remember that we are looking for a solution with $r \equiv g \equiv b$ (modulo 2).

In Figure 11, a hexagonal board with side length 5 is coloured. Counting from the lower left corner, we see that we have 1 red peg, 2 green pegs, 3 blue pegs, 4 red pegs etc., with the pattern 12345454545454321. Since the pattern 545454 corresponds to an equal number of holes of each colour (in this case nine), it can be removed and we get 12345454321. This will be referred to as removing a middle segment. Let's now sum up each colour for boards of different sizes.

If the side length is $3k+2$, we get the pattern $1234 \ldots\ 3k+2, 3k+1, 3k+2 \ldots\ 4321$, after removing all possible middle segments. The first row will be red and the last one green, so symmetry gives $r = g$. Comparing blue with red, blue will be 2 higher than red (e. g., red 1 and blue 3) on k occasions when the pattern is increasing, once equal on the top and 2 lower than red on k occasions when the pattern is decreasing (compare with the pattern above, with side length 5). But there will be a last red row of two holes, resulting in $r = b + 2$. Thus, we have $r \equiv g \equiv b$ (modulo 2).

If the side length is $3k$, we get the pattern $1234 \ldots\ 3k-1 \ldots\ 4321$ after removing the middle segments. Symmetry gives $r = b$ and one easily finds $r + 1 = g$, since green will be 1 higher than red on k occasions and 1 lower on $k-1$ occasions. This means that we have no solutions.

The case is similar if the side length is $3k+1$. The pattern, is $1234 \ldots\ 3k+1 \ldots$ 4321 after reduction and symmetry gives $g = b$. It's easy to see that $r = g + 1$.

It can be proven that the algebra will divide the pegs into three equivalence classes, which are identical to the colour classes. This means that central PEGS on the Chinese checkers board is impossible to solve unless the side length is $3k+2, k > 0$ and that this result cannot be improved using the algebra technique. □

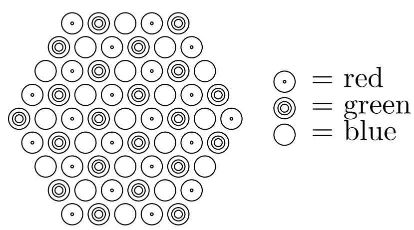


Figure 11: A coloured Chinese checkers board with side length 5

Let's see what we can do with a board like the one in Figure 12. One may consider it as built out of triangles and hexagons, or see it as the hexagonal board with some holes removed.

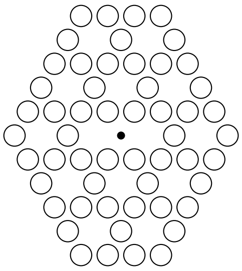


Figure 12: Board of triangles and hexagons, with length of top side length 4 and middle row length 5

Theorem 5.3 *On the board in Figure 12, with a top and bottom sides of equal length, not necessarily equal to the length of the slanting sides, central* PEGS *is not solvable when the number of holes in the middle row, n, satisfies* $n \neq 6k+3, k \geq 0$.

PROOF First of all, we conclude that to play central PEGS, the number of holes in the middle row must be odd. Now it's enough to prove that this number must be divisible by 3. We will show that it is only then that we have $r \equiv g \equiv b$ (modulo 2), using normal coulourings, which is required since this is a reversal game.

Take a look at Figure 13. The rows directly above and directly below the middle row are equal. Since any two holes of the same colour cancel, these rows cancel. The same goes for all other rows symmetrically placed around the middle row. This leaves us with the middle row, no matter how large the board is. It is easy to conclude that we need to have n divisible by three to have a chance to solve the game. □

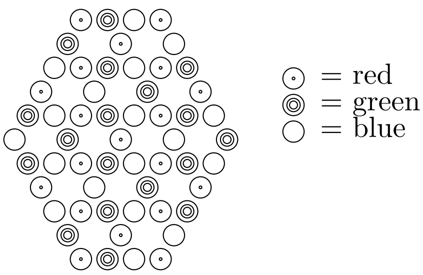


Figure 13: Board of triangles and hexagons, coloured

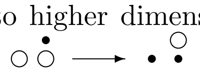
We saw that on the hexagonal board, the colouring was equivalent to the algebra. In this case, this is not valid; we actually have several kinds of green. Take a look at the numbering in Figure 14. Apart from the usual colour rule (the holes are coloured as in

Figure 13), we must also follow an adding rule. The adding rule says that two different numbers add to the third number (like the colours), but also that two equal numbers add to this number (for example 1 + 1 = 1). The second part of the adding rule makes it impossible to conclude that a game is not solvable using the numbers, but we can still use the numbers for orientation towards a solution. Not using the diagonals with equal numbers, we obey the first part of the adding rule, which is similar to the colour rule. Knowing how many times we need to violate this law helps us determine how many times we are to use the diagonal. (This technique of violating a law is also used in section 5.5, when playing Pegs on the torus.)

```
          3   1   2   3
        2       1       3
      1   2   3   1   2   3
    3       2       1       3
  2   3   1   2   3   1   2   3
1       3       2       1       3
  1   2   3   1   2   3   1   2
    1       3       2       1
      1   2   3   1   2   3
        1       3       2
          1   2   3   1
```

Figure 14: Board of triangles and hexagons, numbered

5.2 Pegs variations

In Eriksson and Lindström [7], the authors remark at the end that the result obtained (theorem 6.1, generalising theorem 4.3 to higher dimensions) can not be improved by allowing a kind of sideways Pegs move,  , together with the usual Pegs moves. The reason is that this move obeys the Pagoda function used in the proof. Still, one might think that some properties should be gained by relaxing the rules. In this case, this is true. Let's look at the algebra for sideways Pegs. Using the holes a, b, c, d placed clockwise in a square, we get $a = bc = d$. But since a and d are adjacent, all pegs are equal. We can't say that any sideways Pegs game, with or without the Pegs moves, is unsolvable!

In fact, most of them are really solvable. It may not come as a surprise to find that central sideways Pegs on the English board is solvable, but we can do better.

Theorem 5.4 *Central sideways* Pegs *is solvable on the continental board.*

Proof See Figure 15 □

Playing sideways Pegs on the solitaire board may seem somewhat boring, since virtually anything is possible. One way to complicate the game is to allow only those

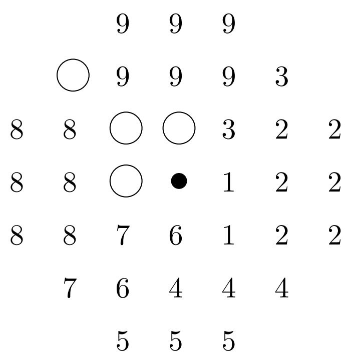
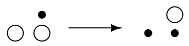
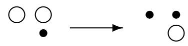


Figure 15: One way to solve central sideways PEGS on the continental board

moves that "turn left"; that is,  is allowed, but  isn't. How will this affect our algebra (remember that the moves showing that the algebra for sideways PEGS is trivial were both left moves and right moves)? It doesn't! Let's see why. Two adjacent, horizontal, pegs are equal to one above the right peg or one below the left peg. Thus, two holes are equal if a chess knight can jump from one to the other, jumping two steps forward and one to the *right*. If we mark the holes belonging to each equivalence class with different symbols, we see that we get five equivalence classes. A bit of arithmetic also shows that a hole raised to the power of four equals one.

But this is not the whole truth. Consider a row with holes a, b, c, d, e, each in a separate equivalence class. Since de equals the hole below d, which equals b, we get $de = b$, as well as $bc = e$, for instance. Combining these gives $dc = 1$. Correspondingly, we can obtain $bc = 1$, and suddenly we have $b = d$. Since two of the five equivalence classes are equal, they must all be equal.

Still, left sideways PEGS is trickier than ordinary sideways PEGS. I still haven't solved central left sideways PEGS on the English board, a task which is almost trivial playing ordinary sideways PEGS.

5.3 PENNIES

PENNIES is quite a tricky game. Even though it doesn't seem to be particularly different from PEGS, there is no solution to central PENNIES even on the English board.

Theorem 5.5 *Central* PENNIES *can't be solved on the English board.*

PROOF Playing PENNIES, we can use quantum numbers directly. If the weights of the holes in a row (or column) follow the repeated pattern `1 0 -1 -1 0 1`, as on the English board in Figure 16, the total weight will remain constant. There we see, that the start configuration has the weight -9 and the end configuration has the weight 1, making central PENNIESimpossible to solve. □

		0	-1	-1		
		-1	-1	0		
0	-1	-1	0	1	1	0
-1	-1	0	1	1	0	-1
-1	0	1	1	0	-1	-1
		1	0	-1		
		0	-1	-1		

Figure 16: Solving central PENNIES would alter the weight from -9 to 1

These quantum numbers could be applied in three different ways by translating the pattern, and another three by mirroring it. Having a zero in the middle, for instance, would make the game look solvable. For a game to be solvable, the quantum numbers of the start and end configurations must match whichever way we apply them. Therefore, there aren't many boards on which a PENNIES game is solvable.

Let's see what we get using the algebra. The quantum numbers clearly show that we don't get $b^k = 1$ for any penny b and $k \geq 2$. However, using the row $a\ b\ c\ d\ e\ f\ g$, we get $ac = b$ and $bd = c$, resulting in $ad = 1$. Similarly we get $dg = 1$, and together with $ad = 1$ we get $a = g$, equalling any pennies six holes apart. We can also see that pennies three diagonal holes apart are equal. Sadly, this is about all we can say and we're stuck with as many as 18 equivalence classes. Using these equivalences, we can reduce the board size to 6 by 3.

All in all, we can conclude that PENNIES may be solvable on some boards, but these boards are very few and simple calculations can in most cases reveal that a game is unsolvable.

What if we have a board that may be solvable? Even then, it may be hard to find a solution, which makes the game quite interesting. Earlier, we have seen that relaxing the rules by allowing multiple or negative pennies in holes, and reversed moves, may be a good technique. In this case, it won't help us find a solution to the ordinary game, but this new game becomes easier.

Theorem 5.6 *Allowing multiple pennies in each hole, as well as reversed moves, we can move a penny freely within its equivalence class, without disturbing the other pennies.*

PROOF If we can move a penny from the origin to the hole at $(3, 3)$, we can move it anywhere within the equivalence class. A sequence (though perhaps not the shortest) of moves that will do this is given in Table 1. H will be short for a horizontal move, V for a vertical and I will indicate a reversed move. Figure 17 shows how it works, taking a few steps at a time. It may seem that we are never having multiple pennies in a hole, but there may be any set of pennies on the board between (0, 0) and (3, 3) when we move the penny. □

Table 1: How to move a penny from (0, 0) to (3, 3)

(0,0)	H	I	(0,0)	H		(2,2)	V	
(1,0)	V	I	(1,0)	H		(2,2)	H	I
(1,1)	H	I	(3,3)	V	I	(1,1)	V	
(2,1)	V	I	(3,4)	H	I	(2,1)	V	
(2,2)	H	I	(3,2)	H	I	(2,2)	H	
(3,2)	V	I	(2,2)	V	I	(3,3)	V	
(1,2)	V	I	(1,1)	H		(3,4)	H	
(1,0)	V		(2,1)	H		(3,2)	H	

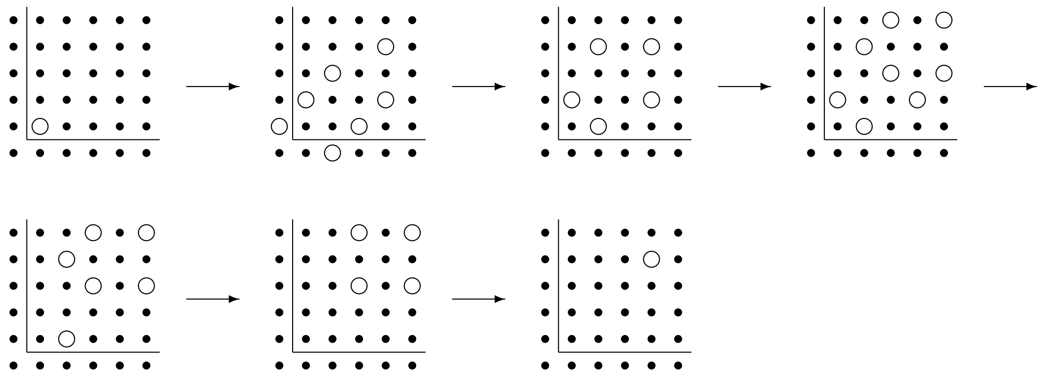


Figure 17: Moving a penny

Using this technique, we can gather all pennies in a 6-by-3 area. In this, we can concentrate the pennies to only four holes, making reversed moves to clear the middle row and using the rule $ad = 1$ to clear at least four squares in each of the other two. If the game is solvable, leaving only one penny on the board, it should be obvious how to clear the remaining, having gotten this far.

5.4 Piles

As described above, a Piles move is made by chosing one of the nodes with as many chips as the node's outgoing edges, and firing that node, giving one chip to each neighbour. Usually, there are plenty of nodes that can be fired, and one might think that the distribution of the chips when the game is finished — that is, when no node can be fired — might depend on which node we fire in each step, just as the outcome of an ordinary Pegs game depends on which moves are made. This is, however, not the case.

Theorem 5.7 (K. Eriksson, 1994 [8]) PILES *is strongly convergent.*

This has been proven by Kimmo Eriksson in his 1993 Ph. D. thesis, which is referred to in [8].

This property shows that there can be only two outcomes of a game. Either the game will go on forever, or the game will terminate in a specific number of moves. Let $\deg(v_k)$ denote the number of outgoing edges of a node v_k. If there are more chips than $\sum_k (\deg(v_k) - 1)$, the pidgeon hole principle says that there must always be a node that can be fired. Otherwise, we do not know the outcome. I Figure 18 there are two PILES games, played on a 2 by 2 grid. The left one terminates in two moves, but the right one will never terminate.

0	0
2	2

0	1
2	1

Figure 18: Two PILES games

One might be interested in the number of moves it takes to reach the final configuration (remember that this number is fixed, due to the strong convergence property). This is not known, generally, but the best upper bound known (see [8]) is $2nD^{n-1}$ moves, where $D = \max_k(v_k)$.

5.5 Nonplanar boards

So far, we have only played planar boards. But nonplanar boards could easily be imagined as well. First, let's consider the Platonic bodies, or more precisely, the cube and the octagon.

Theorem 5.8 *Of the Platonic bodies, the only ones playable are the cube (using the faces) and the octagon (using the vertices), resulting in the same board. Using the normal rules — beginning with one empty hole, the status of every hole should be reversed — we can solve both* PEGS *and* PENNIES.

PROOF It is easy to see that these are the only Platonic bodies playable (the Platonic bodies can be found in [9]). We will use the faces of the cube (or the vertices of the octagon) as holes and begin with the side facing upwards empty. Playing PEGS, we find that the jump sequence up, horizontally, down and up will clear the cube, save the top face.

Playing PENNIES may seem a little trickier, but it's not. In the first move, we put a penny on the top face and in the second move, we remove it. The third move is horizontal and the fourth is quite obvious, having gotten this far. □

What about other three-dimensional bodies? One interesting body is Rubik's cube, on which central PEGS is actually solvable, using the middle board of a side as the

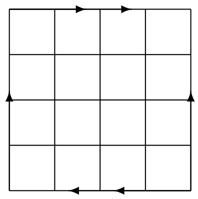


Figure 19: The torus with a 4 by 4 grid

rr	gg	bb	rr
gb	br	rg	gb
bg	rb	gr	bg
rr	gg	bb	rr

Figure 20: The torus with a 4 by 4 grid, coloured

central hole. The solution is easily obtainable by using the first package in our collection, removing three adjacent pegs over and over again. I haven't managed to find a solution to central PENNIES but since the quantum numbers technique doesn't work here (it gets messy around the corners), we can't prove it unsolvable either.

Another interesting body to play is the torus. If you cut it up, it will look as in Figure 19, in which we have applied a 4 by 4 grid to it. Let's first conclude that there is no real midpoint on this board, so when playing central PEGS, the empty square can be put anywhere. We also find that a solution is easily obtainable when the side length is a multiple of three, since the first package will clear a square board of this size, if we begin with the hole in a corner. But how do we find a solution when the side length isn't a multiple of three? If we don't use moves across the edges (when the torus has been cut up), the colouring will prove it impossible to find a solution. Using moves across the edges, the colour rule is invalid and does not give us any information. Or does it?

Actually, the colouring gives all the information we need. The idea is to apply a sufficient number of moves across the edges to restore the colouring to a configuration which may be solvable, and thereafter only use moves that do not go across the edges. This will usually work very well, espacially if we use both colourings. Let's take a look at Figure 20. Both colourings have $r = 6$, $g = 5$ and $b = 5$, if we consider the whole board. If we choose the moves across the edges cleverly, we can improve these numbers and hopefully find a solution. There exists one for side length 4, but I haven't tried larger boards.

6 Playing some infinite boards

6.1 Pegs in $\mathbb{Z}^d$

We have already seen, introducing the Pagoda functions, how limited the range of an infinite set of pegs is. Filling the lower half plane in $\mathbb{Z}^2$ with pegs, we couldn't even make these pegs reach the fifth row (theorem 4.3). Thanks to Henrik Eriksson and Bernt Lindström [7], we also know what happens in higher dimensions.

Theorem 6.1 (Eriksson and Lindström, 1995 [7]) *Suppose that a hole in $\mathbb{Z}^d$ with the coordinates $(x_1, x_2, \ldots, x_d)$ is filled with a peg if and only if $x_d \leq 0$. It is then impossible to put a peg in any hole with $x_d \geq 3d - 1$.*

Proof This proof is essentially the same as the one in theorem 4.3. We will use the Pagoda function

$$P(x) = \sigma^{\sum_{i=1}^{d} |x_i - y_i|}$$

where $\sigma = \frac{\sqrt{5}-1}{2}$ is the golden ratio, satisfying $\sigma + \sigma^2 = 1$, $x = (x_1, x_2, \ldots, x_d)$ is the hole we wish to assign a value to, and $y = (y_1, y_2, \ldots, y_d) = (0, 0, \ldots, 0, 3d - 1)$ is the hole we are trying to reach. Again, the value of a peg is the same as the value of the hole it is occupying and we find that a legal move will never increase the total value of the pegs on the board.

Let's first consider the pegs with $x_d = 0$. The number of pegs with $\sum_{i=1}^{d} |x_i - y_i| = 3d - 1 + k$ equals the number of integer solutions to $|x_1| + |x_2| + \ldots + |x_{d-1}| = k$. If we call this number q_k, we find that the generating function* for $q_0, q_1, \ldots$ is

$$\sum_{k=0}^{\infty} q_k x^k = (1 + x + x + x^2 + x^2 + x^3 + x^3 + \ldots)^{d-1}$$

How come? Well, the number of x^k on the left hand side is the number of solutions to $|x_1| + |x_2| + \ldots + |x_{d-1}| = k$. If a solution includes $|x_i| = j$, then this corresponds to taking an x^j term in the ith factor on the right hand side. This can be done in two ways (if $j > 0$), which is what we want, since we can obtain $|x_i| = j$ in two ways if $j > 0$.

Now, the geometric sum gives

$$(1 + x + x + x^2 + x^2 + x^3 + x^3 + \ldots)^{d-1} =$$

$$= \big((1 + x)(1 + x + x^2 + x^3 + \ldots)\big)^{d-1} = \left(\frac{(1 + x)}{(1 - x)}\right)^{d-1}$$

From this we see that the total value of the pegs with $x_d = 0$ is

$$\sum_{k=0}^{\infty} q_k \; x^{3d-1+k} = \sigma^{3d-1} \sum_{k=0}^{\infty} q_k \sigma^k = \sigma^{3d-1} \left(\frac{(1 + \sigma)}{(1 - \sigma)}\right)^{d-1}$$

*The generating function of a sequence of numbers is the function obtained when using the numbers as coefficients in an infinite series. This could be done as above, $\sum_{k=0}^{\infty} q_k x^k$, as an exponential generating function, $\sum_{k=0}^{\infty} \frac{q_k x^k}{k!}$, or in any other suitable way.

$$= \sigma^{3d-1}\left(\frac{\sigma^{-1}}{\sigma^2}\right)^{d-1} = \sigma^{3d-1}\sigma^{-3(d-1)} = \sigma^2$$

It is now easy to calculate the total value of all pegs. The value of the pegs with $x_d = m$ will decrease with a factor σ for each decrement of m. Thus, the value of the entire board is

$$\sigma^2(1+\sigma+\sigma^2+\dots) = \frac{\sigma^2}{1-\sigma} = 1$$

Consequently, we would have to use every peg on the board to reach $(0,0,\dots,0,3d-1)$, which is obviously impossible. □

Another interesting question is how many pegs, if any, we can put on the level immediately below the level mentioned in the last theorem. Eriksson and Lindström have the answer to this question as well.

Theorem 6.2 (Eriksson and Lindström, 1995 [7]) *Suppose that a hole in $\mathbb{Z}^d$ is filled with a peg if and only if $x_d \leq 0$. Then it is possible to put a peg in any hole with $x_d \geq 3d-2$, but any pegs on this level will have at least two empty holes between them.*

Proof Let's first show that two pegs on level $3d-2$ must be separated by at least one empty hole. Using the same Pagoda function as above, if one of the pegs has the coordinates $(0,0,\dots,0,3d-2)$ and the other is adjacent, their added value will be 1, which again is to high. However, to show that we can't have two pegs at distance one on level $3d-2$, we'll need another Pagoda function.

Let's try to put two pegs in the holes with coordinates $(\pm1,0,0,\dots,3d-2)$ and consider the Pagoda function obtained when using the Pagoda function above, but redefining $P(0,y_2,y_3,\dots,y_d) = 0$. The total value of the pegs will then be $1-\sigma^3$, where σ^3 is the value of the pegs with $x_1 = 0$ in the old Pagoda function. The value of the pegs with coordinates $(\pm1,0,0,\dots,3d-2)$ will be $2\sigma^2 = \sigma+\sigma^2-\sigma^3 = 1-\sigma^3$. Thus, the holes $(\pm1,0,0,\dots,3d-2)$ are simultaneously unreachable.

It remains to prove that level $3d-2$ can be reached by two pegs, only two holes apart. This will not be done here, and the interested reader is referred to [7]. □

6.2 Diagonal Pegs in $\mathbb{Z}^d$

In [7], the authors found that the use of sideways Pegs moves will not improve reachability (see section 5.2). But what about diagonal moves? It is obvious that the Pagoda function used in the proof of 6.1 will break down, if we allow diagonal Pegs (or Diapegs for short) moves. We'll simply have to consider another Pagoda function.

Lemma 6.3 *The function*

$$P(\boldsymbol{x}) = \sigma^{\max|x_i|}$$

is a Pagoda function, when playing diagonal Pegs *in $\mathbb{Z}^d$.*

PROOF Trivial. □

This Pagoda function works well with moves that are diagonal in all dimensions. Therefore, we will allow such moves. Clearly, the reachability ought to be much higher using these moves. But how much higher?

This time, we will try to find the total value of the holes with $x_n \leq -k$. When this value is less than one, we can't reach the origin (with value 1) with these pegs.

Definition 6.1 *The total value of all holes with $x_n \leq -k$ in $\mathbb{Z}^n$, using the Pagoda function from lemma 6.3, will be called $f(n, k)$.*

What can be said about these $f(n, k)$? Well, for starters, we can find a recursive formula.

Theorem 6.4 *The numbers $f(n, k)$ satisfy the recursion*

$$f(n, k) = 4 \sum_{l=k+1}^{\infty} f(n-1, l) + (2k+1) f(n-1, k)$$

using the initial values

$$f(1, k) = \sigma^{k-2}$$

PROOF Let's first verify the initial values. We first observe that in $\mathbb{Z}$, we get a single line of $\sigma, \sigma^2, \sigma^3, \ldots$. Simple calculations give

$$f(1, k) = \sum_{l=k}^{\infty} \sigma^l = \frac{\sigma^k}{1-\sigma} = \sigma^{k-2}$$

Now take a look at Figure 21, showing the σ-logarithm of the Pagoda function (the zero is, of course, at the origin). How would we calculate $f(2, 1)$? On the right side of the middle column, we have, diagonally, a $f(1, 1)$ running from (1, -1), two $f(1, 2)$s running from (1, -2) and (2, -1), two $f(1, 3)$s, etc. The same goes for the left side, and the middle column will produce a $f(1, 1)$, giving a total of $3f(1, 1) + 4f(1, 2) + 4f(1, 3) + \ldots$.

4	3	2	1	0	1	2	3	4
4	3	2	1	1	1	2	3	4
4	3	2	2	2	2	2	3	4
4	3	3	3	3	3	3	3	4
4	4	4	4	4	4	4	4	4

Figure 21: The σ-logarithm of the Pagoda function

Calculating $f(2,2)$ is not much harder. We get nice diagonals on the right and the sides if we first remove the three middle columns. The total weight will then be $5f(1,2) + 4f(1,3) + 4f(1,4) + \ldots$. An inductive reasoning will show that the recursion is valid for every $f(2,k)$, proving the formula correct for $n = 2$.

But what about higher dimensions? Actually, these can be dealt with in the same manner. Computing $f(3,1)$, use the vertical plane in the middle ($x_2 = 0$) as a divider and consider the right and left sides. On each, we have one $f(2,1)$, two $f(2,2)$, two $f(2,3)$ etc. Together with the dividing plane, which equals $f(2,1)$, we get $3f(1,1) + 4f(1,2) + 4f(1,3) + \ldots$. The rest of $f(3,k)$, as well as higher dimensions, will be similar. □

This may seem like an awkward way to calculate the value of a given $f(n,k)$. We would have to move up, recursively, through every dimension up to n, each time calculating, or at least approximating, an infinite sum. We can, however, do better. First, let's view a useful lemma and then we'll reduce the infinite sum to a finite sum.

Lemma 6.5 *The value of all holes with $x_n \leq -1$ in $\mathbb{Z}^n$, is given by*

$$f(n,1) = \sigma^2 \sum_{k=1}^{\infty} k\ (2k+1)^{n-1}\ \sigma^k = \sigma^{-2n} \sum_{m=0}^{n} b_{nm} \sigma^m$$

where the Eilerian numbers b_{nm} are given in Table 2. (There is also a recursive formula for the b_{nm}, which is treated in appendix A.)*

Table 2: The Eilerian numbers

1					
1	3				
1	14	9			
1	49	115	27		
1	156	918	764	81	
1	479	5994	11774	4549	243

Proof Let's take a look at $f(3,1)$. We wish to count the total value of the holes with $x_3 \leq -1$. This value is equal to the value of the holes with $x_3 \geq 1$. This means that we could just as well count the value of all holes in $\mathbb{Z}^3$ (we'll denote this value $g(3)$), retract the total value of the holes with $x_3 = 0$ and divide it by two. But the value of the holes with $x_3 = 0$ equals the value of all holes in $\mathbb{Z}^2$, $g(2)$. Since this reasoning is valid in any dimensions, we get

$$f(n,1) = \frac{g(n) - g(n-1)}{2} \tag{1}$$

*The author's full name is Niklas *Eiler* Eriksen

The holes with value $\sigma^k, k \geq 1$ in $\mathbb{Z}^n$ constitutes the outer shell of an n-dimensional hypercube* with a side length of $2k$ centered around the origin. We find that the number of such holes is given by $(2k+1)^n - (2k-1)^n$. This means that the number of holes with value σ^k in $f(n,1)$ is (using eq. 1)

$$\frac{((2k+1)^n - (2k-1)^n) - ((2k+1)^{n-1} - (2k-1)^{n-1})}{2}$$

$$= \frac{(2k+1-1)(2k+1)^{n-1} - (2k-1-1)^{n-1}}{2}$$

$$= k(2k+1)^{n-1} - (k-1)(2k-1)^{n-1}$$

and we get

$$f(n,1) = \sum_{k=1}^{\infty}(k(2k+1)^{n-1} - (k-1)(2k-1)^{n-1})\sigma^k$$

$$= \sum_{k=1}^{\infty} k(2k+1)^{n-1}\sigma^k - \sum_{k=1}^{\infty}(k-1)(2k-1)^{n-1}\sigma^k$$

$$= \sum_{k=1}^{\infty} k(2k+1)^{n-1}\sigma^k - \sum_{k=0}^{\infty} k(2k+1)^{n-1}\sigma^{k+1}$$

$$= (1-\sigma) \sum_{k=1}^{\infty} k \ (2k+1)^{n-1} \ \sigma^k$$

$$= \sigma^2 \sum_{k=1}^{\infty} k \ (2k+1)^{n-1} \ \sigma^k$$

This sum isn't to easy to calculate generally, but for each $n \geq 1$ we can use

$$\sum_{k=1}^{\infty} k^n x^k = (1-x)^{-(n+1)} \sum_{k=1}^{n} A(n,k) x^k$$

where $A(n,k)$ are the Eulerian numbers (see Comtet's Advanced Combinatorics [4] for an interesting treatment of the Eulerian numbers (and a lot of other combinatorics as well)), which can be viewed in table 3. Simple calculations now give the values in table 2. □

*An n-dimensional hypercube of sidelength k is a solid object, which fills out the space given by $a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, \ldots, a_n \leq x_n \leq b_n$, where $b_i - a_i = k, \forall i$. In 2 dimensions we get a square and in three dimensions an ordinary cube. Notice that a square with sidelength k contains $(k+1)^2$ holes, a cube contains $(k+1)^3$ holes, etc.

Table 3: The Eulerian numbers

					1					
				1		1				
			1		4		1			
		1		11		11		1		
	1		26		66		26		1	
1		57		302		302		57		1

Theorem 6.6 *The recursion formula for $f(n,k)$ can also be written*

$$f(n,k) = f(n,1) + f(n-1,1) - 4\sum_{l=1}^{k} f(n-1,l) + (2k+1)f(n-1,k)$$

$$= \sigma^{-2n}\sum_{m=0}^{n} b_{nm}\sigma^m + \sigma^{-2(n-1)}\sum_{m=0}^{n-1} b_{n-1,m}\sigma^m - 4\sum_{l=1}^{k} f(n-1,l) + (2k+1)f(n-1,k)$$

still using the initial values

$$f(1,k) = \sigma^{l-2}$$

Proof Using, from theorem 6.4,

$$f(n,k) = 4\sum_{l=k+1}^{\infty} f(n-1,l) + (2k+1)f(n-1,k)$$

and, consequently,

$$f(n,1) = 4\sum_{l=2}^{\infty} f(n-1,l) + 3f(n-1,1)$$

$$\Longleftrightarrow$$

$$4\sum_{l=2}^{\infty} f(n-1,l) = f(n,1) - 3f(n-1,1)$$

we get

$$f(n,k) = 4\sum_{l=k+1}^{\infty} f(n-1,l) + (2k+1)f(n-1,k)$$

$$= 4\sum_{l=1}^{\infty} f(n-1,l) - 4\sum_{l=1}^{k} f(n-1,l) + (2k+1)f(n-1,k)$$

$$= 4f(n-1,1) + (f(n,1) - 3f(n-1,1)) - 4\sum_{l=1}^{k} f(n-1,l) + (2k+1)f(n-1,k)$$

$$= f(n,1) + f(n-1,1) - 4\sum_{l=1}^{k} f(n-1,l) + (2k+1)f(n-1,k)$$

□

Now we have a finite recursive formula, with which we can compute, although usually not by hand, any reasonable $f(n,k)$. But we are still curious. Is there a direct formula? Yes, there is, but it isn't beautiful.

Theorem 6.7 *An expression for the numbers f(n, k) is given by*

$$f(n,k) = \sum_{m=2}^{n} f(m,1) - (2k-1)\sum_{m=2}^{n-1} f(m,1)$$

$$+ \sum_{l=2}^{k} \sigma^2 \left((1-2l)\sigma^{-1} - 2\sigma^{-2}\right) \sum_{m_1+m_2=n-2} (2l-1)^{m_1} (2l-3)^{m_2}$$

$$- (2k-1)\sum_{l=2}^{k} \sigma^2 \left((1-2l)\sigma^{-1} - 2\sigma^{-2}\right) \sum_{m_1+m_2=n-3} (2l-1)^{m_1} (2l-3)^{m_2}$$

Proof A generating function for $f(n,k)$ is

$$F_k(x) = \sum_{n=2}^{\infty} f(n,k)\ x^n$$

We will arrive at a recursive formula for $F_k(x)$. We have

$$F_k(x) = \sum_{n=2}^{\infty} f(n,k)\ x^n$$

$$= \sum_{n=2}^{\infty} \left(f(n,1) + f(n-1,1) - 4\sum_{l=1}^{k} f(n-1,l) + (2k+1)f(n-1,k) \right) x^n$$

$$= F_1(x) + x\sum_{n=2}^{\infty} f(n-1,1)x^{n-1} - 4x\sum_{l=1}^{k}\sum_{n=2}^{\infty} f(n-1,l)x^{n-1} + (2k+1)x\sum_{n=2}^{\infty} f(n-1,k)x^{n-1}$$

$$= F_1(x) + x\,F_1(x) + x^2\,f(1,1) - 4x\sum_{l=2}^{k} F_l(x) - 4x^2\sum_{l=2}^{k} f(1,l) + (2k+1)xF_k(x) + (2k+1)x^2\,f(1,k)$$

$$= F_1(x) + x\,F_1(x) - 4x\sum_{l=2}^{k} F_l(x) + (2k+1)xF_k(x) + x^2\left(f(1,1) + (2k+1)\,f(1,k) - 4\sum_{l=2}^{k} f(1,l)\right)$$

$$= F_1(x) + x\,F_1(x) - 4x\sum_{l=2}^{k} F_l(x) + (2k+1)xF_k(x) + x^2\left(\sigma^{-1} + (2k+1)\sigma^{k-2} - 4(\sigma^{-3} - \sigma^{k-3})\right)$$

This recursive formula for F_k can now be used to obtain a direct formula for F_k. From the difference

$$\begin{aligned} & F_k(x) - F_{k-1}(x) \\ = & \ -4x\ F_k(x) + (2k+1)\ x\ F_k(x) - (2k-1)\ x\ F_{k-1}(x) \\ & + x^2\ ((2k+1)\sigma^{k-2} - (2k-1)\sigma^{k-3} + 4\sigma^{k-3}(1-\sigma^{-1})) \\ = & \ (2k-3)\ x\ F_k(x) - (2k-1)\ x\ F_{k-1}(x) + x^2\ \sigma^k\ (-2k\ \sigma^{-1} - 3\sigma^{-2} + \sigma^{-}3) \end{aligned}$$

we can collect all F_k-terms and get

$$F_k(x) = \frac{1-x(2k-1)}{1-x(2k-3)}\ F_{k-1}(x) + \frac{x^2\ \sigma^k\ ((1-2k)\sigma^{-1} - 2\sigma^{-2})}{1-x(2k-3)}$$

If we calculate the first terms, we get

$$F_1(x) = \sum_{n=2}^{\infty} f(n,1)\ x^n$$

$$F_2(x) = \frac{1-3x}{1-x}\ F_1(x) - \frac{x^2\ \sigma^2\ (3\sigma^{-1} + 2\sigma^{-2})}{1-x}$$

$$F_3(x) = \frac{1-5x}{1-x}\ F_1(x) - \frac{x^2\ \sigma^2\ (3\sigma^{-1} + 2\sigma^{-2})\ (1-5x)}{(1-x)(1-3x)} - \frac{x^2\ \sigma^3\ (5\sigma^{-1} + 2\sigma^{-2})}{1-3x}$$

and the general formula may be written

$$F_k(x) = \frac{1-x(2k-1)}{1-x}\ F_1(x) - (1-x(2k-1))\ x^2 \sum_{l=2}^{k} \frac{\sigma^k\ ((2l-1)\sigma^{-1} + 2\sigma^{-2})}{(1-x(2l-3))(1-x(2l-1))}$$

To determine the $f(n,k)$, simply take the coefficient of x^n in $F_k(x)$. Using

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

to put all x's in the numerator, we can obtain the formula in the theorem. This will contain the terms $f(m,1)$, but actually, there is a direct formula for these as well. □

All these formulae are very fancy, but what is the result? A quick look in table 4 reveals that using DIAPEGS moves, we can probably get a lot further than using ordinary PEGS move. For each dimension, the reachability increases with about ten levels, compared to three for ordinary PEGS. Don't forget, though, that we have only proven that the levels k in Table 4 can not be reached. Whether the levels below can be reached is still an open question.

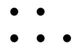
In $\mathbb{Z}^2$, Johan Karlsson and Richard Olofsson [10] have found that level seven can be reached using 49 pegs. Reaching level eight would require quite a lot of pegs, but there are no indications that it is impossible to reach. It is my belief that in the lowest dimensions, we can get really close to the upper limits in Table 4, but in higher dimensions, the distance will increase. Already in $\mathbb{Z}^5$, I would consider it really hard to find a way to reach level 39.

Table 4: Unreachable levels of DIAPEGS in $\mathbb{Z}^n$

n	k	$f(n,k)$	$f(n,k-1)$
1	2	1.000	1.618
2	9	0.877	1.308
3	18	0.872	1.286
4	28	0.967	1.424
5	40	0.689	1.017
6	51	0.932	1.376
7	64	0.703	1.041

6.3 PEBBLES

The original PEBBLES game of Kontsevich was played in $\mathbb{N}^2$, and the aim was to prove that a small subset of ten squares couldn't be emptied. We will say that the set is *unavoidable.* This result was soon improved and we now know that there are five squares that can't be emptied, using results from [3].

Theorem 6.8 (Chung et al., 1995 [3]) *This set of squares,*  *, where the lower left square is the origin, is unavoidable, i. e. it can not be emptied.*

This isn't altogether trivial, so before we prove this, we'll prove some lemmas.

Lemma 6.9 (Chung et al., 1995 [3]) *The set of six squares* $\{(x,y) : x+y \leq 2\}$ *is unavoidable.*

PROOF Let's first prove that the set of ten squares, $\{(x,y) : x+y \leq 3\}$ is unavoidable. We'll use the Pagoda function

$$P(x,y) = \frac{1}{2^{x+y}}$$

In a legal move, the total weight on the board will remain constant, and the weight of the start configuration is 1. Now, the total weight on the board is

$$\sum_{k=0}^{\infty} (k+1)\,\frac{1}{2^k} = \sum_{k=0}^{\infty} k\,\frac{1}{2^k} + \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1/2}{(1-1/2)^2} + \frac{1}{1-1/2} = 4$$

and the weight of the ten squares is

$$1 + 2\frac{1}{2} + 3\frac{1}{4} + 4\frac{1}{8} = 1 + 1 + \frac{3}{4} + \frac{1}{2} = \frac{13}{4}$$

If we are to clear these ten squares, the rest of the board would hold the weight 1, but we have seen that in can only hold the weight

$$4 - \frac{13}{4} = \frac{3}{4}$$

Proving that the set of six squares $\{(x, y) : x + y \leq 3\}$ is unavoidable isn't to hard either. It is easy to see that the squares with $y = 0$ can have only one pebble on them, since there is one at the beginning, and we can't add one without removing one. The same goes for $x = 0$. Thus, the largest weight holdable outside the six squares is less than

$$\frac{1}{8} + \frac{1}{8} + \sum_{k=3}^{\infty} (k-1) \, \frac{1}{2^k} \; = \frac{1}{4} + \frac{3}{4} = 1$$

which is the total weight on the board. □

To obtain the result in the theorem, we'll need to relax the rules somewhat. In this case, we will allow stacking of pebbles, that is, we'll allow more than one pebble per square. Since each pebble can only be played in one way, we can see that the set of moves used to reach a configuration is unique. We may, if possible, carry out the moves in a different order, but we can not vary them. Now, stacking will make it even easier to find an order to carry out the moves in. Having made all the moves, we should reach a configuration with at most one pebble per square. It is not to hard to see that the following lemma is valid.

Lemma 6.10 (Chung et al., 1995 [3]) *If we can reach a configuration with at most one pebble per square using the relaxed rules, we can reach it with the ordinary moves.*

We are now ready to prove the theorem above.

Proof (of theorem 6.8) Let's use the relaxed rules and empty the set of five squares, giving the situation in Figure 22, the numbers indicating the number of pebbles in each square. We have now done all necessary moves in the first 3 levels ($x + y < 3$), and we'll continue to play one level at a time. We will now prove that it is impossible to reduce the situation to one pebble per square if we have three pebbles in one square, which will prove that the set of five squares is unavoidable.

If we have three pebbles at the square (x, y), then either $(x-1, y+1)$ or $(x+1, y-1)$ must contain at least two pebbles. Let's assume that $(x - 1, y + 1)$ is this square. If we do all necessary moves on this level, reducing the pebbles to at most one per square (this is necessary if we are to reach a configuration with no more than one pebble per square on the entire board), we get three pebbles on square $(x, y + 1)$ (one from $(x - 1, y + 1)$ and two from (x, y)). Thus, the three-stack will propagate forever, and we'll never reach a configuration without stacks. □

1 2
• • 3
• • • 1

Figure 22: Trying to empty a set of five squares

The properties of PEBBLES in higher dimensions have been investigated by Henrik Eriksson in [6]. He has proven the following unexpected theorem.

Theorem 6.11 (H. Eriksson, 1995 [6]) *In $\mathbb{N}^d$, $d \geq 3$, a set of four squares including the origin and three of its neighbours, is unavoidable.*

PROOF Let's say that the squares to be emptied are, apart from the origin, $(1, 0, 0, \dots)$, $(0, 1, 0, \dots)$ and $(0, 0, 1, 0, \dots)$. Then, the squares $(1, 1, 0, \dots)$, $(1, 0, 1, \dots)$ and $(0, 1, 1, 0, \dots)$ will receive two pebbles and the square $(1, 1, 1, 0, \dots)$ will receive three pebbles. But we have already seen that if a square receives three pebbles, the game will go on forever. □

An interesting feature of PEBBLES is that it can be played on any poset* with $\hat{0}$, as well as in $\mathbb{N}^d$. A move is made by removing the pebble from an object x and adding one to every object that covers x. The property of strong convergence is still maintained.

6.4 PENNIES

We have already seen that PENNIES is hard to play. Another property of PENNIES, quite similar to the properties of PEBBLES, can be found when playing PENNIES using reversed moves.

Theorem 6.12 *Suppose we are playing reversed* PENNIES *in $\mathbb{Z}^d$, starting off with one penny at the origin and the rest of the board empty. Then a minimal unavoidable set is a unit hypercube in $\mathbb{Z}^d$ with a corner at the origin.*

PROOF Due to the symmetry, it is enough to consider the unit cube with corners (0, 0, ... , 0) and (1, 1, ... , 1). We will show that once a penny has entered the hypercube, there will always be at least one penny in the hypercube. Since we begin with a penny at the origin, this will prove that the hypercube is unavoidable.

The coordinates for the penny in the hypercube consists of a couple of zeros and a couple of ones. Not making a move won't empty the cube, so we make a move. Then one of the coordinates gets added and subtracted by one (we get two new pennies). If the coordinate was one, we now get the coordinate zero, giving a penny in the cube. The other penny is put outside the cube. The other case is similar. □

This pattern is not unique for the unit hypercubes around the origin. The theorem is easily generalised to the following.

Theorem 6.13 *When playing* PENNIES *in $\mathbb{Z}^d$, any unit hypercube in $\mathbb{Z}^d$ is unavoidable as soon as it contains at least one penny.*

Looking at it the other way, we find that with ordinary moves we can not fill a unit hypercube in $\mathbb{Z}^d$, once it has been partially emptied.

*A poset P is a partially ordered set. This means that for two objects $x, y \in P$, we may have either the relation $x < y$, the relation $x > y$ or no relation at all. If we have $x < y$ and $y < z$, then $x < z$. A $\hat{0}$ (zero) is an object with the property that $\forall x \in P, \hat{0} < x$. An object y covers x if $x < y$ and there is no $z \in P$ such that $x < z < y$.

A The Eilerian numbers

Just as there are nice recursive formulas for the numbers in Pascal's triangle and the Eulerian numbers, there is one for the Eilerian numbers.

Theorem A.1 *The Eilerian numbers b_{nm} satisfy the recurrence formula*

$$b_{nm} = (2(n-m)+1)\ b_{n-1,m-1} + (2m+1)\ b_{n-1,m}$$

This has been proven by Doron Zeilberger [11], as well as by Henrik and Kimmo Eriksson (personal communication).

The sum of the nth row in Pascals triangle is 2^n and the sum of the nth row in the triangle of Eulerian numbers is $n!$. What do we get for the Eilerian numbers?

Theorem A.2 *The sum of the nth row is*

$$\sum_{m=1}^{n} b_{nm} = 2^{n-1}\ n!$$

PROOF Let's prove this by using the principle of induction. It is easy to see that the relation is valid for $n = 1$. Let's assume that it is valid for $n = k-1$ and try to show it is valid for $n = k$. We've got

$$\sum_{m=1}^{k} b_{km} = (2(k-m)-1) \sum_{m=2}^{k} b_{k-1,m-1} + (2m+1) \sum_{m=1}^{k-1} b_{k-1,m} =$$

$$2k \sum_{m=2}^{k} b_{k-1,m-1} - (2m+1) \sum_{m=2}^{k} b_{k-1,m-1} + (2m+1) \sum_{m=1}^{k-1} b_{k-1,m} =$$

$$2k\ 2^{k-2}\ (k-1)! = 2^{k-1}\ k!$$

□

Finally, one might be interested in an explicit formula for the Eilerian numbers. This can be found by using the explicit formula for the Eulerian numbers (taken from Comtet [4]),

$$A(n,k) = \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} (k-j)^n$$

and applying a lot of calculations. Due to the length of the calculations, I will omit the proof.

Theorem A.3

$$b_{nm} = \sum_{s=0}^{m-1} \sum_{p=0}^{s} \sum_{r=m}^{n} \frac{(n-1)!\ (r-p+1)\ (r-p)}{p!\ (s-p)!\ (n-r)!\ (r-s+1)!} (-1)^s\ 2^{r-p-1}\ (m-s)^{r-p}$$

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