

## Lectures in analysis for Ph.d-students

Jan-Erik Björk

Stockholm University: e-mail@math.au.se

### Contents

- I: *Beurling's criterion for the Riemann hypothesis*
- II. *Product measures and a Schrödinger equation.*
- III.A *The Laplace operator and the Helmholtz equation.*
- III.B *Beurling's conformal mapping theorem.*
- IV. *Eigenvalues for the Laplace operator.*
- V. *Elliptic operators and their fundamental solutions*
- VI. *A Neumann problem with non-linear boundary values.*

### Introduction.

Lecture 1 is devoted to Beurling's criterion for the Riemann hypothesis which goes as follows: Let  $\rho(x)$  denote the 1-periodic function on the positive real  $x$ -line where  $\rho(x) = x$  if  $0 < x < 1$ . So if  $x > 1$  and  $\{x\}$  is the integral part of  $x$  then  $\rho(x) = x - \{x\}$ . If  $1 \leq j \leq M$  for a pair of integers we define the function on  $(0, 1)$  defined by

$$\rho_{j,M}(x) = \rho\left(\frac{j}{Mx}\right)$$

Let 1 be the identity function on  $(0, 1)$  and set

$$\beta(M) = \min_{c_1, \dots, c_M} \int_0^1 \left( \sum_{j=1}^{j=M} c_j \cdot \rho_{j,M}(x) - 1 \right)^2 dx \quad : \quad \sum_{j=1}^{j=M} j \cdot c_j = 0$$

Thus  $\beta(M)$  is Lagrange's multiplier to a minimum of a quadratic form in the  $c$ -variables with a linear constraint. Beurlings proved that the Riemann hypothesis is valid if and only if

$$\lim_{M \rightarrow \infty} \beta(M) = 0$$

The proof is exceedingly innovative and relies upon a closure theorem for Dirichlet series. Lecture 2 deals with a variational problem for product measures which is used to solve a non-linear Cauchy problem introduced by Schrödinger to describe phenomena in quantum physics. Lecture 3 gives an exposition about Beurling's studies of free boundary problems for the Laplace operator related to Helmholtz' equations. We have included § 3.B which contains the essential parts in Beurlings conformal mapping theorem from his article [Beurling:xx: Acta 1953]. Sections 4-6 expose results by Carleman in PDE-theory. The material in § 4 is classic but a notable point occurs after Theorem 1.2 in § 4 which in addition to standard asymptotic formulas for evaluations of eigenfunctions also give asymptotic formulas when one passes to higher order derivatives. In § 6 one studies the following non-linear boundary value problem:

A  $C^1$ -domain  $\Omega$  is given in  $\mathbf{R}^3$  and  $F(u, p)$  is a non-negative continuous function on  $\mathbf{R}^+ \times \partial\Omega$  is such that  $u \mapsto F(u, p)$  is strictly increasing for each fixed  $p \in \partial\Omega$  and tends uniformly to  $+\infty$  as  $u \rightarrow +\infty$ . Carleman proved that for every pair  $\Omega, F$  as above and each  $p_* \in \Omega$  there exists a unique function  $u$  which is harmonic except for the Newtonian singularity at  $p_*$  and the inner normal  $\frac{\partial u}{\partial n}$  satisfies

$$\frac{\partial u}{\partial n}(p) = F(u(p), p) \quad : \quad p \in \partial\Omega$$

The proof employs a homotopy method where non-linearity increases in a parametrized system of PDE-equations. § 5 describes to Carleman's construction of fundamental solutions to elliptic

(in general non-symmetric) PDE-operators of second order which lead to best possible bounds for associated Greens functions and is therefore well suited to analyse eigenfunctions and eigenvalues in elliptic boundary value problems.

### Work by Beurling and Carleman

Before turning to the topics listed above a passage from my notes on analytic function theory is inserted describing some contributions by Arne Beurling (1905-1987) and Torsten Carleman (1892-1949).

Let us first recall some of Carleman's contributions. His most productive period ranges from 1916 until 1940 covering more than fifty articles of a very high standard. Below we shall not describe his wellknown contributions to quasi-analytic classes which appear at several occasions in these notes. His most valuable result is the spectral resolutions for unbounded self-adjoint operators on a Hilbert space whose complete proof was given in his book *xxx* (Uppsala University 1923). Unbounded self-adjoint operators appear already in his thesis from 1916 devoted to the Neuman boundary value problems with corner points on the boundary. Here that Carleman introduced methods to handle non-symmetric operators which admit suitable factorisations and after reduce the spectral analysis to symmetric kernels. We describe such a case in § 11 in the appendix devoted to functional analysis. Among other contributions by Carleman prior to 1920 one can mention his inequality for the operator norm of resolvents which is a veritable cornerstone in spectral theory of linear operators. Concerning harmonic measures it was Carleman who first recognized its power to study limits and growth of harmonic and analytic functions. The pioneering method appeared in his article *Sur les fonctions inverses des fonctions d'ordre fini* [Arkiv för matematik 1921] which gave a uniform but not sharp bound for the number of asymptotic values. The sharp bound which settled the conjecture posed by Denjoy in 1907 was proved independently by Ahlfors and Beurling in 1929. In §xx we expose Carlemans proof from 1933 which relies upon a differential inequality and has the merit that the methods can be adapted to more general cases. See also page xx in Nevanlinna's text-book [Nevanlinna] for comments about *Carleman's principle for extensions of domains*. A first lesson of this principle is Carleman's elegant proof of a result originally due to Ernst Lindelöf. It asserts that if  $f(z)$  is a bounded analytic function in a half-disc  $D_+ = \{|z| < 1 \cap \Re(z) > 0\}$  and if there exists a Jordan curve  $\gamma$  which except for one end-point at the origin belongs to  $D_+$  such that  $\lim_{z \rightarrow 0} f(z) = 0$  holds along  $\gamma$ , then  $f(z)$  tends uniformly to zero in every Fatou sector  $\{|z| < 1 \cap \{-a < \arg(z) < a\}$  where  $0 < a < \pi/2$ . The beginner should look at the proof in Chapter XX to get a first glimpse of the usefulness to introduce the subharmonic function  $\log |f|$  and employ the harmonic measure. Carleman also used subharmonic functions to extend results of the Phragmén-Lindelöf type. Here is an example. Put  $B = \{0 < \Re z < 1\}$  which is an open and simply connected domain in  $\mathbf{C}$ . Let  $f(z)$  be analytic in  $B$  and for each  $0 < x < 1$  we put

$$m(x) = \max_y |f(x + iy)| \quad \text{and} \quad \phi(x) = \int_{-\infty}^{\infty} |f(x + iy)|^2 dy$$

We assume that these functions are bounded on all intervals  $\delta \leq x \leq 1 - \delta$  when  $\delta > 0$  are small. Phragmén and Lindelöf proved that the function  $\phi(x)$  is convex on  $(0, 1)$ . Carleman proved that  $\phi(x)$  satisfies the differential inequality

$$(*) \quad \phi''(x) \geq \frac{\phi'(x)^2 + m(x)}{\phi(x)}$$

which in particular gives the convexity. In § xx we shall give the proof of (\*) which employs the subharmonic function  $\log |f|$ . Let us also recall Carleman's extension of Liouville's theorem which goes as follows: Consider arcs parametrized by injective and continuous maps  $s \rightarrow \gamma(s) = x(s) + iy(s)$  from the half-open interval  $[0, 1)$  in  $\mathbf{C}$  where  $\gamma(0) = 0$  is the origin and  $|\gamma(s)| \rightarrow +\infty$  as  $s \rightarrow 1$ . Let  $\gamma_1, \gamma_2$  be a pair of such arcs whose sole common point is the origin and together give an unbounded domain  $\Omega$  whose boundary is the union of the two arcs. Suppose that  $f(z)$  is analytic in  $\Omega$  and extends continuous to  $\partial\Omega$  with a finite maximum norm

$$m^* = \sup_{z \in \gamma_1 \cup \gamma_2} |f(z)|$$

Since we can find  $\gamma$ -arcs inside  $\Omega$  which tend to infinity one cannot appeal to the ordinary maximum principle and conclude that the absolute value  $|f(z)|$  stays below  $m^*$  for all  $z \in \Omega$ .

However, if the maximum principle fails then the function

$$\omega_f(\theta) = \max_r |f(re^{i\theta})| \quad : re^{i\theta} \in \Omega$$

cannot stay too small. Lindelöf gave examples of non-constant entire functions  $f$  where the  $\omega_f$ -functions is bounded for all  $\theta$ -angles outside a null-set. Investigating how these Lindelöf-functions increase, Carleman established a result in the article *Sur l'extension d'un théorème de Liouville* which asserts that when the  $\gamma$ -arcs are straight half-lines so that  $\Omega = \{\alpha < \arg z < \beta\}$  is a sector and the maximum principle fails, then one must have a divergent integral:

$$\int_{\alpha}^{\beta} \log \log^+ \omega_f(\theta) d\theta = +\infty$$

Above a double log-function appears so the the divergence forces the  $\omega_f$ -function to be very large in the average. A proof is given in § XX and in § XX we also expose later results due to Beurling where the pair of  $\gamma$ -curves are of a more general type than straight lines.

*Carleman's work on PDE-equations.*

The student in analysis is confronted with several disciplines where analytic function theory and Fourier analysis is one aspect, and PDE-theory another. As expected the interaction is close and many results in PDE-theory use complex methods and Fourier analysis. Let us describe one of Carleman's results whose proof rely upon methods from all these disciplines.

Let  $\Omega$  be a bounded open domain in  $\mathbf{R}^3$  with a  $C^1$ -boundary. We are given a symmetric  $3 \times 3$ -matrix  $A(x)$  whose elements  $\{a_{pq}(x)\}$  are real-valued continuous functions on the closure  $\bar{\Omega}$ . Let  $b_1, b_2, b_3$  and  $c$  be four other real-valued continuous functions on  $\bar{\Omega}$ . We get the differential operator  $L$  acting on  $C^2$ -functions  $u(x)$  by

$$L(u) = \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} a_{pq}(x) \frac{\partial^2 u}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} b_p(x) \frac{\partial u}{\partial x_p} + c(x)u$$

Assume that  $L$  is elliptic which means that there is a constant  $\rho > 0$  such that

$$\min_{\xi} \left| \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} a_{pq}(x) \cdot \xi_p \xi_q \right| \geq \rho$$

where the minimum is taken over all  $x \in \bar{\Omega}$  and every real  $\xi$ -vector of unit length. Fredholm's classical theory about integral equations implies that the eigenvalues  $\lambda$  for which there exists a non-zero function  $u$  such that

$$L(u) + \lambda \cdot u = 0 \quad \text{and} \quad u|_{\partial\Omega} = 0$$

is a discrete set of complex numbers. When  $L$  is symmetric these eigenvalues are real and tend to  $+\infty$ . Results about their asymptotic distribution were established by Weyl and Courant around 1930. In the article [Carleman 1936] their asymptotic expansion was extended to cover non-symmetric elliptic operators. The result from [ibid] goes as follows where we for each  $x \in \Omega$  introduce the non-zero value of the determinant of  $A(x)$  which by the elliptic condition may be taken to be everywhere positive.

**Theorem.** *Let  $|\lambda_n|$  be the absolute value of the  $n$ :th eigenvalue ordered in a non-decreasing sequence. Then*

$$|\lambda_n| \simeq \left[ \frac{1}{6\pi} \int_{\Omega} \frac{dx}{\det A(x)} \right]^{-\frac{2}{3}} \cdot n^{2/3}$$

**Remark.** The  $\simeq$ -sign means that the difference of the terms above are of small ordo  $n^{2/3}$  as  $n \rightarrow +\infty$ . The asymptotic formula does not depend upon the lower order differentials of  $L$ . This follows from the fact that even if eigenvalues can be complex, they are asymptotically real. More precisely Carelan proved that there exist positive constants  $c_1, c_2$  such that  $\Re \lambda \geq -c_1$  and

$$|\Im \lambda| \leq c_2 \cdot \sqrt{\Re \lambda + c_1}$$

hold for all eigenvalues.

*Work by Arne Beurling.*

Many results by Beurling have led to new areas in analytic function theory, harmonic analysis and potential theory. We have not tried to include his deepest results which for example appear in joint articles with Lars Ahlfors and Paul Malliavin. For a more detailed account we refer to [Beurling:Collected Works 1-2] whose foreword contains a very informative text written by Lars Ahlfors and Lennart Carleson. A remarkable feature in Beurling's work is the mixture of geometric considerations and "hard calculus". An example is the construction of extremal distances which Beurling introduced in 1929 and used to estimate harmonic measures. Below follows a description of some major results in his thesis. Personally I find Beurling's proofs in [ibid] utmost instructive and transparent. So in this perspective [ibid] appears as a veritable classic devoted to analytic function theory. The results below speak for themselves. The less experienced reader should consider the subsequent material as a "bird's view" and whenever necessary consult the Chapters in these notes for concepts which are used without hesitation below.

*A. The Milloux problem.*

Consider an analytic function  $f(z)$  in the unit disc where  $|f(z)| < 1$  for all  $z$  and suppose there exist some  $\delta > 0$  such that

$$\min_{\theta} |f(re^{i\theta})| \leq \delta \quad \forall 0 \leq r < 1$$

Then it follows that

$$(*) \quad \max_{\theta} |f(re^{i\theta})| \leq \delta^{\frac{2}{\pi} \cdot \arcsin \frac{1-r}{1+r}} \quad : 0 < r < 1$$

Prior to this result in Beurling's thesis, M.E Schmidt had proved (\*) under the constraint that there exists a Jordan arc from the origin to the unit circle along which  $|f| \leq \delta$  in which case one profits upon a suitable conformal mapping which in a similar context had been used by Koebe. To prove (\*) in the general case Beurling introduced totally new ideas.

*Minimum modulus of analytic functions.*

Chapter XX in [ibid] studies the minimum modulus of an analytic function. When  $f(z)$  is analytic in a disc  $\{|z| < R\}$  we set

$$m_f(r) = \max_{|z|=r} |f(z)| \quad : \quad \mu_f(r) = \min_{|z|=r} |f(z)| \quad : \quad 0 < r < R$$

The following result is proved in [ibid: page 94]:

**Theorem.** *Let  $0 < r_1 < r_2 < R$  and  $\alpha > 0$  satisfies  $0 < \alpha \leq m_f(r_1)$ . Then*

$$(**) \quad \int_{r_1}^{r_2} \chi(\{\mu_f \leq \alpha\}) \cdot \log r \, dr \leq \log 4 + 2 \int_{r_1}^{r_2} \log \log \frac{m_f(r)}{\alpha} \, dr$$

where  $\chi(\{\mu_f \leq \alpha\})$  denotes the characteristic function of the set where  $\mu_f(r)$  takes values  $\leq \alpha$ .

**Remark.** In the article [xxx] from 1933 Nevanlinna found another proof of (\*) which has the merit that it consolidates why (\*) is sharp. Using constructions by Nevanlinna, Beurling could then demonstrate that the inequality (\*\*) gets sharp when  $\alpha$  tends to zero which is the interesting part of the theorem.

*Upper bounds for harmonic measures*

Let  $D$  be a simply connected domain and  $\gamma \subset \partial D$  is a subarc. If  $z \in D$  we denote by  $r(z; \partial D)$ , resp.  $r(z, \gamma)$  the euclidian distance from  $z$  to the boundary and to  $\gamma$  respectively. With these notations Theorem III : page 55 in [ibid] asserts that

$$(***) \quad \mathbf{m}(D, \gamma; z) \leq \frac{4}{\pi} \cdot \arctan \sqrt{\frac{r(z, \partial D)}{r(z, \gamma)}}$$

where the left hand side is the harmonic measure at  $z$  with respect to  $\gamma$ . Examples show that this upper bound is sharp. Next, an inequality, which in a slightly weaker form was established in [ibid] to demonstrate the Denjoy conjecture, was refined later in Beurling's article [xx] from 1940 and goes as follows: Let  $\Omega$  be a Jordan domain and  $\omega \subset \partial\Omega$  a sub-arc of its boundary. Let  $z_0 \in \Omega$  and suppose there exists a positive harmonic function  $\psi$  in  $\Omega$  satisfying:

$$\psi(z_0) = 0 \quad : \quad \min_{z \in \omega} \psi(z) = L$$

$$A = \iint_{\Omega} |\nabla(\psi)|^2 \, dx dy < \infty$$

**Theorem.** *When  $\psi$  exists as above one has the inequality*

$$(****) \quad \mathbf{m}(\Omega; \omega)(z_0) \leq e^{-\pi \cdot \frac{L^2}{A}}$$

**Remark.** This result has a wide range of applications where examples occur in § xx and § xx. A detailed proof of (\*\*\*) appears in section §§ from Special Topics.

*Regular points for the Dirichlet problem.*

Sections in [ibid:page 63-69] are devoted to Dirichlet's problem. Let  $\Omega$  be an open subset of  $\mathbf{C}$ . If  $f$  is a real-valued and continuous function on  $\partial\Omega$  it can be extended to some  $F \in C^0(\bar{\partial}\Omega)$ . Now  $\Omega$  can be exhausted by an increasing sequence of subdomains  $\{\Omega_n\}$  for which the Dirichlet problem is solvable. Such exhaustions were originally considered by Lebesgue and explicit constructions were given by de Vallé Poussin in 1910. To each  $n$  we find the harmonic function  $H_n$  in  $\Omega_n$  which solves the Dirichlet problem with boundary values given by the restriction of  $F$  to  $\partial\Omega_n$ . In Chapter V we prove Wiener's result that the sequence  $\{H_n\}$  converges to a unique harmonic function  $W_f$  in  $\Omega$  which is independent of the chosen exhaustion. One refers to  $W_f$  as Wiener's generalised solution. A boundary point  $z_0$  is called regular if

$$\lim_{z \rightarrow z_0} W_f(z) = f(z_0) \quad : \quad \forall f \in C^0(\partial\Omega)$$

A criterion for a boundary point to be regular was established by Boulignad in 1923. Namely, a boundary point  $z_0$  is regular if and only if there exists a positive harmonic function  $V$  in  $\Omega$  such that

$$\lim_{z \rightarrow z_0} V(z) = 0$$

This condition is rather implicit so one seeks geometric properties to decide if a boundary point is regular or not. In [ibid] Beurling established a sufficient regularity condition which goes as follows: Let  $z_0 \in \partial\Omega$  and for a given  $R > 0$  we consider the circular projection of the closed complement of  $\Omega$  onto the real interval  $0 < r < R$  defined by:

$$E_\Omega(0, R) = \{0 < r < R : \exists z \in \{|z - z_0| = r\} \cap \mathbf{C} \setminus \Omega\}$$

Following Beurling one says that  $z_0$  is logarithmically dense (Point frontière de condensation logarithmique) if the integral

$$(1) \quad \int_{E_\Omega(0, R)} \log r \, dr = +\infty$$

Beurling proved that if one has a divergent integral in (1) then  $z_0$  is a regular boundary point. The proof relies upon an inequality which has independent interest. Here is the situation considered in [ibid: page 64-66]: Let  $\Omega$  be an open set - not necessarily connected and  $R > 0$  where  $\Omega$  is general contains points of absolute value  $\geq R$ . Consider a harmonic function  $U$  in  $\Omega$  with the following properties:

$$(i) \quad \limsup_{z \rightarrow z_*} U(z) \leq 0 \quad : \quad z_* \in \partial\Omega \cap \{|z| \leq R\}$$

$$(ii) \quad U(z) \leq M \quad : \quad z \in \{|z| = R\} \cap D$$

**Theorem.** When (i-ii) hold one has the inequality below for every  $0 < r < R$

$$\max_z U(z) \leq 2M \cdot e^{-\frac{K}{2}}$$

where the maximum in the left hand side is taken over  $\Omega \cap \{r < |z| < R\}$  and

$$K = \int_{E_\Omega(r, R)} \log r \, dr$$

*Extremal distances.*

Above we announced results from [ibid] whose proofs apart from a more standard analytic function theory used new geometric methods. Let us first cite Beurling where he gives the following attribute to original work by Poincaré: *Rappelons que dans la théorie des fonctions analytiques, on a introduit des éléments géométriques non euclidiennes, invariants par rapports a certaines transformations, et cela surtout pour simplifier la théorie dont il s'agit.* In his lecture at the Scandinavian Congress in Copenhagen 1946, Beurling describes in more detail the usefulness of extremal lengths.

*In geometric function theory one often tries to characterize or determine a certain mapping or quantity by an extremal property. The method goes back to Riemann who introduced variational methods in function theory in the form of Dirichlet's principle. When you want to characterize a function by an extremal property, then the class of competing functions is very important. The wider you can make this class the more you can say about the extremal function and the easier it becomes to find good majorants or minorants, as the case may be.*

*The extremal distance in simply connected domains.*

In his thesis Beurling defined an extremal distance between pairs of points  $z_0, z_1$  in simply connected domains  $\Omega$  with a finite area. The construction goes as follows: First the interior distance is defined by

$$\rho(z_0, z_1; \Omega) = \inf_{\gamma} \int_{\gamma} |dz|$$

where the infimum is taken over rectifiable Jordan arcs which stay in  $\Omega$  and join the two points. Set

$$\lambda_*(z_0, z_1; \Omega) = \sqrt{\frac{\pi}{\text{Area}(\Omega)}} \cdot \rho(z_0, z_1, \Omega)$$

One refers to  $\lambda_*$  as the reduced distance. It is not a conformal invariant and to overcome this default Beurling considered the family of *all triples*  $(\Omega^*, z_0^*, z_1^*)$  which are conformally equivalent to the given triple, i.e. there exists a conformal mapping  $f: \Omega \rightarrow \Omega^*$  such that  $f(z_\nu) = z_\nu^*$ . The *extremal distance* is now defined by

$$\lambda(z_0, z_1; \Omega) = \sup \lambda_*(z_0^*, z_1^*; \Omega^*)$$

with the supremum taken over all equivalent triples. By construction  $\lambda$  yields a conformal invariant. Another conformal invariant of a triple  $(z_0, z_1, \Omega)$  is the Green's function  $G(z_0, z_1; \Omega)$ . Recall that it is a symmetric function of the pair  $z_0, z_1$  and keeping  $z_1$  fixed

$$z \mapsto G(z, z_1; \Omega) - \log \frac{1}{z - z_1}$$

is a harmonic function  $\Omega$  which is zero on  $\partial\Omega$ . In [ibid: Théorème 1: page 29] the following fundamental result is proved:

**Theorem.** *For each triple  $(z_0, z_1; \Omega)$  one has the equality*

$$(*) \quad e^{-2G} + e^{-\lambda^2} = 1$$

*About the proof.* By conformal invariance it suffices to prove the equality for a triple  $(0, a, D)$  where  $D$  is the unit disc where the pair of points is the origin and a real point  $0 < a < 1$ . In this case

$$G = \log \frac{1}{a}$$

Hence the theorem amounts to prove the equality

$$e^{-\lambda^2} = 1 - a^2$$

or equivalently that

$$(*) \quad \lambda^2 = \log \frac{1}{1 - a^2}$$

We will show (\*) in § XX. The equation has several applications but at this moment we refrain from discussing more details in this introduction.

#### *General extremal metrics.*

Ten years later Beurling realised the need for a more extensive class of extremal distances which can be applied for domains which are not simply connected. The constructions below (with slightly different notations) appear in the section *Extremal Distance and estimates for Harmonic Measure* [Collected work. Vol 1. page 361-385].

*The numbers  $\lambda(E, K; \Omega)(z_0)$ .* Let  $\Omega$  be a Jordan domain in the complex  $z$ -plane and consider a pair of sets  $E, K$  where  $E \subset \partial\Omega$  and  $K$  is a compact subset of  $\Omega$ . To a point  $z_0 \in \Omega \setminus K$  we introduce the family  $\mathcal{J}(E, K; z_0)$  of rectifiable Jordan arcs  $\gamma$  with the following properties: Apart from its end-points  $\gamma$  stays in  $\Omega \setminus K$  and passes through  $z_0$ . Moreover, the end-points of  $\gamma$  divides the Jordan curve  $\partial\Omega$  into a pair of closed intervals  $\omega_1$  and  $\omega_2$  and the last constraint on  $\gamma$  is that  $E$  is contained in one of these  $\omega$ -intervals. Notice that neither  $K$  or  $E$  are assumed to be connected. Next, let  $\mathcal{A}$  be the family of positive and continuous functions  $\rho$  in  $\Omega$  for which the squared area integral

$$\iint_{\Omega} \rho^2(x, y) dx dy = 1$$

To each such  $\rho$ -function we set

$$(*) \quad L(\rho) = \inf_{\gamma \in \mathcal{J}} \int \rho(z) |dz|$$

where we for simplicity put  $\mathcal{J} = \mathcal{J}(E, K; z_0)$ . The extremal distance between  $E$  and  $z_0$  taken in  $\Omega \setminus K$  is defined by

$$(**) \quad \lambda(E, K; \Omega)(z_0) = \sup_{\rho \in \mathcal{A}} L(\rho)$$

A notable point is that these  $\lambda$ -numbers are conformal invariants. More precisely, let  $(E^*, \Omega^*, K^*)$  be another triple and  $f: \Omega \setminus K \rightarrow \Omega^* \setminus K^*$  is a conformal mapping which extends continuously to the boundary and gives a homeomorphism between the closed Jordan arcs  $\partial\Omega$  and  $\partial\Omega^*$  where  $E^* = f(E)$  and in addition a bijective map between the  $\mathcal{J}$ -families above. Then one has the equality

$$\lambda(E, K; \Omega)(z_0) = \lambda(E^*, K^*; \Omega^*)(f(z_0))$$

Consider a triple  $(E; \Omega, K)$  as above and when  $z_0 \in \Omega \setminus K$  there exists the harmonic measure  $\mathbf{m}(E; \Omega, K)(z_0)$ .

**Theorem.** *For every triple  $E; \Omega, K$  and each  $z_0 \in \Omega \setminus K$  where  $\lambda(E; \Omega, K)(z_0) \geq 2$  one has the inequality*

$$\mathbf{m}(E; \Omega, K)(z_0) \leq 3\pi \cdot e^{-\pi \cdot \lambda(E; \Omega, K)(z_0)}$$

This result is often used to estimate harmonic measures. The point is that in the right hand side we can use  $\rho$ -functions which need not maximize the  $L$ -functional in (\*) and since  $L(\rho) \leq \lambda(E; \Omega, K)(z_0)$  one has in particular

$$\mathbf{m}(E; \Omega, K)(z_0) \leq 3\pi \cdot e^{-\pi \cdot L(\rho)}$$

Examples in § XX demonstrate the usefulness of such majorisations.

**The simply connected case.** Here  $\Omega$  is a Jordan domain and  $K = \emptyset$ . Consider a pair of closed subsets  $E$  and  $F$  in  $\partial\Omega$  which both are a finite unions of closed subintervals. Denote by  $\mathcal{J}(E, F)$  the family of Jordan arcs  $\gamma$  with one end-point in  $E$  and the other in  $F$  while the interior stays in  $\Omega$ . Set

$$\lambda(E; F; \Omega) = \max_{\rho} \min_{\gamma \in \mathcal{J}} \int_{\gamma} \rho \cdot d|z| \quad : \quad \rho \in \mathcal{A}$$

This yields a conformal invariant which can be determined under the extra condition that the sets  $E$  and  $F$  are *separated* which means that there exists a pair of points  $p, q$  in  $\partial\Omega \setminus E \cup F$  which divide  $\partial\Omega$  in two intervals where  $E$  belongs to one and  $F$  to the other. Under this condition one has the equality

$$(1) \quad \lambda(E; F; \Omega) = \mathcal{NS}(E; F; \Omega)$$

where  $\mathcal{NS}(E; F; \Omega)$  is the Neumann-Schwarz number associated to the triple  $(E, F; \Omega)$  and found as follows: Let  $u$  be the harmonic function in  $\Omega$  with boundary values  $u = 1$  on  $E$  and zero on  $F$  while the normal derivative  $\frac{\partial \phi}{\partial n} = 0$  on  $\partial\Omega \setminus E \cup F$ . If  $v$  is the harmonic conjugate then  $v$  is increases on  $E$  and decreases on  $F$  and when  $v$  is normalised so that the range  $v(E)$  is an interval  $[0, h]$  it follows that  $f = u + iv$  is a conformal mapping from  $\Omega$  onto a rectangular slit domain:

$$\{0 < x < 1\} \times \{0 < y < h\} \cup S_{\nu}$$

where  $\{S_{\nu}\}$  are horizontal intervals directed into the rectangle and with end-points on one of the vertical sides  $\{x = 0\}$  or  $\{y = 0\}$ . See § XX for a figure. The Neumann-Schwarz number is given by

$$\mathcal{NS}(E, F; \Omega) = \frac{1}{\sqrt{h}}$$

In § xx we prove that it is a conformal invariant and the proof of (1) is therefore reduced to the case when  $\Omega$  is a rectangle as above with some removed spikes while  $E$  and  $F$  are the opposed vertical lines. If  $\rho^*$  is the constant function  $\frac{1}{\sqrt{h}}$  the fact that horisontal straight lines minimize arc-lengths when we move from  $E$  to  $F$  imply that

$$(2) \quad L(\rho^*) = \frac{1}{\sqrt{h}}$$

On the other hand, let  $\rho \in \mathcal{A}$  be non-constant. Now

$$0 < \iint_{\Omega} \left(\rho - \frac{1}{\sqrt{h}}\right)^2 dx dy = 2 - \frac{2}{\sqrt{h}} \iint_{\Omega} \rho dx dy \implies \iint_{\Omega} \rho dx dy < \sqrt{h}$$

It follows that

$$\min_{0 \leq y \leq h} \int_0^1 \rho(x, y) dx < \frac{1}{\sqrt{h}}$$

In the left hand side appear competing  $\gamma$ -curves and hence

$$L(\rho) < \frac{1}{\sqrt{h}}$$

This proves that  $\rho^*$  maximizes the  $L$ -function and the equality (2) entails (1).

**Another inequality** Let  $\Omega$  be a bounded and connected domain whose boundary consists of a finite family of disjoint closed Jordan curves  $\{\Gamma_\nu\}$ . Consider a pair  $(z_0, \gamma)$  where  $z_0 \in \Omega$  and  $\gamma$  is a Jordan arc starting at  $z_0$  and with an end-point  $p$  on one boundary curve, say  $\Gamma_1$ . Next, let  $K$  be a compact subset of  $\Gamma_1 \setminus \{p\}$ . For each point  $z \in \gamma$  we get the harmonic measure  $\mathbf{m}_K(z)$  which is the value at  $z$  of the harmonic function in  $\Omega$  whose boundary values are one on  $K$  and zero in  $\partial\Omega \setminus K$ . Set

$$h^*(K, \gamma; z_0) = \max_{z \in \gamma} \mathbf{m}_K(z)$$

The interpretation is that one seeks a point  $z \in \gamma$  where the probability for the Brownian motion which starts at  $z$  and escapes at some point in  $K$  before it has reached points in  $\partial\Omega \setminus K$ , is as large as possible. An upper bound of  $h^*$  is found using the notion of extremal length. Namely, let  $\lambda(K; \gamma)$  be the extremal length for the family of curves in  $\Omega$  with one end-point in  $K$  and the other on  $\gamma$ . Then Beurling proved that

$$(*) \quad h^*(K, \gamma; z_0) \leq 5 \cdot e^{-\pi \cdot \lambda(K; \gamma)}$$

**Remark** In random walks under a Brownian motion the inequality above shows that if  $K$  is a large obstacle and  $\gamma$  small, then the  $h^*$ -function can be majorised if some lower bound for the extremal length can be proved. This is of interest in configurations with "narrow channels" when one seeks paths out to the boundary and illustrates a typical application of extremal lengths.

### On quasianalytic series.

Let us finish with a specific result which illustrates Beurling's vigour in "hard analysis". The theorem below was presented in a lecture at the Scandianvan congress in Helsinki (1938:?) A closed subset  $E$  of  $\mathbf{C}$  has positive perimeter if there exists a positive constant  $c$  such that

$$\sum \ell(\gamma_\nu) \geq c$$

hold for every denumerable family of closed and rectifiable Jordan curves  $\{\gamma_\nu\}$  which surround  $E$ , i.e.  $E$  is contained in the union of the associated open Jordan domains. Let  $\{\alpha_k\}$  be a sequences of distinct complex numbers and  $\{A_k\}$  another sequence sequence such that

$$(i) \quad \sum |A_k| < \infty$$

Set

$$(*) \quad F(z) = \sum \frac{A_k}{z - \alpha_k}$$

To each  $k$  we put  $r_k = |A_1| + \dots + |A_k|$  and define  $\mathcal{E}_f$  by

$$\mathcal{E}_f = \left\{ z \in \mathbf{C} : \sum_{k=2}^{\infty} \frac{A_k}{\sqrt{r_{k-1}}} \cdot \frac{1}{|z - \alpha_k|} < \infty \right\}$$

It is easily seen that (i) entails that the complement of  $\mathcal{E}_f$  is a null set in  $\mathbf{C}$ . One says that (\*) is quasi-analytic if  $F(z)$  cannot vanish on a closed subset  $E$  of  $\mathcal{E}_f$  with positive perimeter. The following was proved in [ibid];

**Theorem.** *The series (\*) is quasi-analytic in the sense above if*

$$\liminf_{n \rightarrow \infty} r_n^{\frac{1}{n}} < 1$$

In § XX we expose some steps in the proof where some remarkable inequalities for rational functions occur.

*A final comment*

Carleman and Beurling studied at Uppsala University, where Carleman entered in 1911 and Beurling in 1924. Both had Erik Holmgren and Anders Wiman as teachers and to this one can add that both of them as young researchers met Ernst Lindelöf whose classic text-book *Théorie des Résidues* from 1907 was studied at an early stage by both of them. Beurling presented his Ph.D-thesis on November 4 1933, but its major contents was already written in 1929. At that time Carleman had already served during five years as director at Institute Mittag-Leffler. In the period 1933-1938 Beurling attended Carleman's lectures at the Mittag-Leffler institute and got considerable inspiration from this. One can mention Carleman's lectures about the generalised Fourier transform in 1935 and refined versions of Ikehara's theorem and other Tauberian results and soon after this Beurling went further and proved Tauberian theorems with remainder terms. But there was never a "competitive situation" during the years when Beurling prepared his application for the chair at Uppsala University replacing Homgren after his retirement in 1937. On the contrary, preserved letters between Carleman and Beurling show that they esteemed each others work. For example, in his thesis Beurling gave full credit to Carleman's first attack on the Denjoy conjecture from 1921 and added the following comments upon Carleman's proof from 1933 of the Denjoy conjecture with a sharp bound  $2n$  for entire functions of order  $n$ : *Plus récemment M. Carleman en perfectionnant sa méthode initiale est arrivé au resultat  $\leq 2n$  d'une manière fort élégante.* Beurling's own proof from 1929 has the merit that it can be adapted to other cases than asymptotic values. An example is the following remarkable result from his thesis:

**Theorem** *Let  $f(z)$  be a bounded analytic function in the half space  $\Re z > 0$ . Set*

$$\mu(r, f) = \min_{-\pi/2 < \theta < \pi/2} |f(re^{i\theta})|$$

*Then  $f$  converges uniformly to zero in every sector  $-\pi/2 + \delta \leq \theta \leq \pi/2 - \delta$ . under the condition that*

$$\lim_{r \rightarrow \infty} \mu(r, f) = 0$$

At the end of 1930:s Carleman's health became weak and from this time Beurling took the leading role in Swedish mathematics. His work after 1940 contains a wealth of new results where one may mention his ingenious idea to relate inner functions in the unit disc with closed subspaces of the Hilbert space  $H^2(T)$  which are invariant under multiplication with  $z$ . While reading articles from their collected work I always get fascinated by the transparency and generality while they announce and prove deep results. I can only hope that the sections in these notes which expose material from their work may inspire readers to continue studies of articles by these two eminent mathematicians.

### § 1: Beurlings criterion for the Riemann hypothesis.

Before we study Riemann's  $\zeta$ -function in § 1.B we expose a certain closure theorem.

#### § A.1 A theorem on functions defined by a semi-group

Let  $f(x)$  be a complex-valued function in  $L^2(0, 1)$  which is not identically close to  $x = 0$ , i.e.

$$(*) \quad \int_0^\delta |f(x)| \cdot dx > 0 \quad : \quad \forall \delta > 0$$

For each  $0 < a < 1$  we set

$$f_a(x) = f(ax)$$

We restrict each  $f_a$  to  $(0, 1)$  and denote by  $\mathcal{C}_f$  the linear space generated by  $\{f_a : 0 < a < 1\}$ . Thus, a function in  $\mathcal{C}_f$  is expressed as a finite sum

$$\sum c_k \cdot f_{a_k}(x)$$

where  $\{c_k\}$  are complex numbers and  $0 < a_1 < \dots < a_n < 1$  some finite tuple in  $(0, 1)$ . For every  $1 < p < 2$  we can identify  $\mathcal{C}_f$  with a subspace of  $L^p(0, 1)$  whose closure is denoted by  $\mathcal{C}_f(p)$ .

**A.1.0 The function  $F(s)$ .** It is defined by

$$(1) \quad F(s) = \int_0^1 f(x) \cdot x^{s-1} \cdot dx$$

Then  $F$  is analytic in the half-plane  $\Re(s) > 1/2$  for if  $\sigma = \Re(s) > 1/2$  the Cauchy-Schwarz inequality gives

$$(**) \quad |F(\sigma + it)| \leq \sqrt{\int_0^1 |f(x)|^2 \cdot dx} \cdot \sqrt{\int_0^1 |x|^{2\sigma-2} \cdot dx} = \|f\|_2 \cdot \sqrt{\frac{1}{2\sigma-1}}$$

**A.1.1 Theorem.** *If there exists some  $1 < p < 2$  such that  $\mathcal{C}_f(p)$  is a proper subspace of  $L^p[0, 1]$ , then  $F(s)$  extends to a meromorphic function in the whole complex  $s$ -plane whose poles are confined to the open half-plane  $\Re(s) < 1/2$ . Moreover, for every pole  $\lambda$  the function  $x^{-\lambda}$  belongs to  $\mathcal{C}_f(p)$ .*

*Proof.* Recall that  $L^q(0, 1)$  is the dual of  $L^p(0, 1)$  where  $\frac{1}{q} = 1 - \frac{1}{p}$ . The assumption that  $\mathcal{C}_f(p) \neq L^p(0, 1)$  gives a non-zero  $k(x) \in L^q(0, 1)$  such that

$$(1) \quad \int_0^1 k(x)f(ax) \cdot dx = 0 \quad : \quad 0 < a < 1$$

To the  $k$ -function we associate the transform

$$(2) \quad K(s) = \int_0^1 k(x) \cdot x^{-s} \cdot dx$$

Hölder's inequality implies that  $K(s)$  is analytic in the half-plane  $\Re(s) < \frac{1}{p}$ . Define a function  $g(\xi)$  for every real  $\xi > 1$  by

$$g(\xi) = \int_0^1 k(x) \cdot f(\xi x) \cdot dx$$

Hölder's inequality gives

$$|g(\xi)| \leq \left[ \int_0^1 |k(x)|^q \cdot dx \right]^{\frac{1}{q}} \cdot \left[ \int_0^{\frac{1}{\xi}} |f(\xi x)|^p \cdot dx \right]^{\frac{1}{p}}$$

With  $\xi > 1$  we notice that the last factor after a variable substitution is equal to  $\|f\|_p \cdot |\xi|^{-1/p}$  and hence we have

$$(3) \quad |g(\xi)| \leq \|k\|_q \cdot \|f\|_p \cdot \xi^{-1/p} \quad : \quad \xi > 1$$

Next, put

$$(4) \quad G(s) = \int_1^\infty g(\xi) \cdot \xi^{s-1} \cdot d\xi$$

From (3) it follows that  $G(s)$  is analytic in the half-space  $\Re s < 1/p$ . Consider the strip domain:

$$\square = 1/2 < \Re s < 1/p$$

Variable substitutions of double integrals show that the following holds in  $\square$ :

$$(5) \quad G(s) = F(s) \cdot K(s)$$

*Conclusion.* It follows from (5) that  $F$  extends to a meromorphic function in the whole  $s$ -plane. The inequality (\*\*) shows that no poles appear during the meromorphic continuation across  $\Re s = 1/2$ . Hence  $F$  either is an entire function or else it has a non-empty set of poles where each pole  $\lambda$  has real part  $< 1/2$ . At this stage we are prepared to finish the proof of Theorem A.1.1.

**Existence of at least one pole.** There remains to prove that  $F$  has at least one pole. We argue by contradiction, i.e. suppose that  $F$  is an entire function and consider a real number  $1/2 < \alpha < 1/p$ . The construction of  $F$  shows that its restriction to the half-space  $\Re s \geq \alpha$  is bounded and it is also clear that

$$(i) \quad \lim_{\sigma \rightarrow +\infty} F(\sigma + it) = 0$$

In the half-space  $\Re s \leq \alpha$  we know that  $F = \frac{G}{K}$  where  $G$  and  $K$  both are bounded and at the same time their quotient is analytic in this half-space. Moreover their constructions imply that

$$\lim_{\sigma \rightarrow -\infty} G(\sigma + it) = 0 \quad \text{and} \quad |K(\sigma + it)| = 0$$

Next, a wellknown result by F. and R. Nevanlinna gives some  $M > 0$  and a real number  $c$  such that

$$(iii) \quad |F(\sigma + it)| \leq M \cdot e^{c(\sigma - \alpha)} \quad \text{holds when} \quad \sigma \leq \alpha$$

If  $c \geq 0$  we see that the entire function  $F$  is bounded and (i) implies that  $F = 0$ . But this is impossible since it entails that  $f = 0$ .

**The case  $c < 0$ .** When this holds we set  $a = e^c$  so that  $0 < a < 1$  and define

$$(iv) \quad F_1(s) = \int_0^1 f(ax)x^{s-1} \cdot ds$$

Here a variable substitution gives

$$(v) \quad F_1(s) = a^s (F(s) - \int_a^1 f(x)x^{s-1} \cdot ds)$$

It follows that the entire function  $F_1(s)$  is bounded so by Liouville's theorem it is identically zero. Si by (iv) the the transform of the function  $f_a(x) = f(ax)$  is identically zero. This means precisely that  $f$  vanishes on the interval  $[0, a]$ . But this was excluded by condition (\*) which shows that  $F$  cannot be an entire function.

**The case at pole.** Suppose that  $F$  as a pole at some  $\lambda$  with real part  $< 1/2$ . Since  $G$  is analytic in  $\Re(s) < 1/2$  the equality (5) implies that  $\lambda$  is a zero of  $K$ . Notice also that the presence of the pole of  $F$  at  $\lambda$  is *independent* of the chosen  $L^q$ -function  $k$  which is  $\perp$  to  $\mathcal{C}_f$ . Hence the following implication holds:

$$k \perp \mathcal{C}_f(p) \implies K(\lambda) = \int_0^1 k(x)x^{-\lambda} \cdot dx = 0$$

The Hahn-Banach theorem entails that the  $L^p$ -function  $x^{-\lambda}$  belongs to  $\mathcal{C}_h(p)$  which proves the last claim in Theorem A.1.1.

*B.1 Application to the  $\zeta$ -function.*

Let  $\rho(x)$  denote the 1-periodic function on the positive real  $x$ -line where  $\rho(x) = x$  if  $0 < x < 1$ . So if  $\{x\}$  is the integral part of  $x$  then

$$\rho(x) = x - \{x\}$$

To each  $0 < \theta < 1$  we get the function

$$\rho_\theta(x) = \rho(\theta/x)$$

whose restriction to  $(0, 1)$  gives a non-negative function with jump-discontinuities at the discrete set of  $x$ -values where  $\theta/x$  is an integer. Denote by  $\mathcal{D}$  the linear space of functions on  $(0, 1)$  of the form

$$f(x) = \sum c_\nu \cdot \rho(\theta_\nu/x)$$

where  $0 < \theta_1 < \dots < \theta_N < 1$  is a finite set and  $\{c_\nu\}$  complex numbers such that

$$\sum c_\nu \cdot \theta_\nu = 0$$

**B.1 Theorem.** *The Riemann hypothesis is valid if and only if the identity function 1 belongs to the closure of  $\mathcal{D}$  in  $L^2(0, 1)$ .*

The proof will use the following formula:

**B.2 Proposition.** *For each  $0 < \theta < 1$  one has the equality*

$$\int_0^1 \rho(\theta/x) x^{s-1} \cdot dx = \frac{\theta}{s-1} - \frac{\theta^s \cdot \zeta(s)}{s} \quad \text{when } \Re s > 1$$

*Proof.* The variable substitutions  $x \rightarrow \theta \cdot y$  and  $y \rightarrow 1/u$  identifies the left hand side with

$$\theta^s \cdot \int_0^{1/\theta} \rho(1/y) \cdot y^{s-1} \cdot dy =$$

$$(i) \quad \theta^s \cdot \int_\theta^\infty \rho(u) \cdot u^{-s-1} \cdot du$$

To evaluate (i) we first consider the integral

$$(ii) \quad \int_1^\infty \rho(u) \cdot u^{-s-1} \cdot du = \sum_{n=1}^\infty \int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du$$

where the last equation used the periodicity of  $\rho$ . An integration by parts gives for each  $n \geq 1$ :

$$\int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du = -\frac{1}{s}(n+1)^{-s} + \frac{1}{s} \int_0^1 \frac{du}{(n+u)^s}$$

After a summation over  $n$  we see that (ii) becomes

$$-\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s} \int_1^\infty u^{-s} \cdot du = -\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s(s-1)} = -\frac{\zeta(s)}{s} + \frac{1}{s-1}$$

It follows that the left hand side in (\*) is equal to

$$\begin{aligned} & \theta^s \cdot \left[ \int_\theta^1 u \cdot u^{-s-1} \cdot du - \frac{\zeta(s)}{s} + \frac{1}{s-1} \right] = \\ & \theta^s \cdot \left[ \frac{\theta^{-s+1} - 1}{s-1} - \frac{\zeta(s)}{s} + \frac{1}{s-1} \right] = \frac{\theta}{s-1} - \frac{\theta^s \cdot \zeta(s)}{s} \end{aligned}$$

Now we are prepared to begin the proof of Theorem B.1

2. The case when 1 in the  $L^2$ -closure of  $\mathcal{D}$ .

If  $\epsilon > 0$  this assumption gives some  $f \in \mathcal{D}$  such that the  $L^2$ -norm of  $1 + f$  is  $< \epsilon$ . Since  $\sum c_\nu \cdot \theta_\nu = 0$ , Proposition 7.2 gives:

$$\int_0^1 (1 + f(x)) \cdot x^{s-1} \cdot dx = \frac{1}{s} - \frac{\zeta(s)}{s} \cdot \sum c_\nu \cdot \theta_\nu^s$$

With  $s = \sigma + it$  and  $\sigma > 1/2$  we have  $x^{s-1}$  in  $L^2$  and Cauchy-Schwarz inequality gives:

$$\left| \int_0^1 (1 + f(x)) \cdot x^{s-1} \cdot dx \right| \leq \|f\|_2 \cdot \sqrt{\int_0^1 x^{2\sigma-2} \cdot dx} = \|f\|_2 \cdot \frac{1}{\sqrt{2\sigma-1}}$$

Hence we obtain

$$\left| \frac{1}{s} - \frac{\zeta(s)}{s} \cdot \sum c_\nu \cdot \theta_\nu^s \right| \leq \epsilon \cdot \frac{1}{\sqrt{2\sigma-1}} \quad : \quad \sigma > 1/2$$

If  $\zeta(s_*) = 0$  holds for some  $s_* = \sigma_* + it_*$  with  $\sigma_* > 1/2$ , the left hand side is reduced to  $\frac{1}{|\zeta_*|}$ . Since we can find  $f$  as above for every  $\epsilon > 0$  it would follow that

$$\frac{1}{|\zeta_*|} \leq \epsilon \cdot \frac{1}{\sqrt{2\sigma_*-1}} \quad \text{for every } \epsilon > 0$$

But this is clearly not possible so if 1 belongs to the  $L^2$ -closure of  $\mathcal{D}$  then the Riemann-Hypothesis holds.

3. Proof of necessity.

There remains to show that if 1 is outside the  $L^2$ -closure of  $\mathcal{D}$  then the  $\zeta$ -function has a zero in the half-plane  $\Re s > 1/2$ . To show this we introduce a family of linear operators  $\{T_a\}$  as follows: If  $0 < a < 1$  and  $g(x)$  is a function on  $(0, 1)$  we set

$$T_a(g)(x) = g(x/a) \quad : \quad 0 < x < a$$

while  $T_a(g) = 0$  when  $x \geq a$ .

**Exercise.** Show that each  $T_a$  maps  $\mathcal{D}$  into itself and one has the inequality

$$\|T_a(f)\|_2 \leq \|f\|_2$$

Since 1 is outside the  $L^2$ -closure of  $\mathcal{D}$  its orthogonal complement in the Hilbert space is  $\neq 0$  which gives a non-zero  $g \in L^2(0, 1)$  such that

$$(*) \quad \int_0^1 f(x) \cdot g(x) \cdot dx = 0 \quad : \quad f \in \mathcal{D}$$

Since  $\mathcal{D}$  is invariant under the  $T$ -operators it follows that if  $0 < a < 1$  then we also have

$$(1) \quad 0 = \int_0^a f(x/a) \cdot g(x) \cdot dx = a \cdot \int_0^1 f(x) \cdot g(ax) \cdot dx$$

At this stage we apply Theorem 6.1. To begin with we show that the  $g$ -function satisfies (\*) in Theorem 6.1. For suppose that  $g = 0$  on some interval  $(0, a)$  with  $a > 0$ . Choose some  $b$  where

$$a < b < \min(1, 2a)$$

Now  $\mathcal{D}$  contains the function  $f(x) = b\rho(x/a) - a\rho(x/b)$ . The reader may verify that  $f(x) = 0$  for  $x > b$  and is equal to the constant  $a$  on  $(a, b)$ . With (1) applied to  $f$  we therefore get

$$\int_a^b g(x) \cdot dx = 0$$

This means that the primitive function

$$G(x) = \int_0^x g(u) \cdot du$$

has a vanishing derivative on the interval  $(a, b)$ . The derivative is also zero on  $(0, a)$  where  $g = 0$ . We conclude that  $G = 0$  on the interval  $(0, b)$  so the  $L^2$ -function  $g$  is almost everywhere a fixed constant on this interval. But this constant is zero since  $g = 0$  on  $(0, a)$ . Hence we have shown that  $g = 0$  on the whole interval  $(0, b)$ . We can repeat this with  $a$  replaced by  $b$  and conclude that  $g$  also is zero on the interval

$$0 < x < \min(1, 2b) = \min(1, 4a)$$

After a finite number of steps  $2^m a \geq 1$  and hence  $g$  would be identically zero on  $(0, 1)$  which is not the case. Hence Theorem 6.1 applies to  $g$  and gives some  $\lambda_*$  with  $\Re \lambda_* < 1/2$  such that  $x^{-\lambda_*}$  belongs to  $\mathcal{C}_g(p)$  for every  $p < 2$ . Next, for each  $\theta > 0$  we get the  $\mathcal{D}$ -function

$$x \mapsto \rho(1/x) - \frac{1}{\theta} \cdot \rho(\theta/x)$$

Since (1) holds for all  $0 < a < 1$  it follows that

$$(2) \quad \int_0^1 [\rho(1/x) - \frac{1}{\theta} \cdot \rho(\theta/x)] \cdot x^{-\lambda_*} \cdot dx = 0$$

Put  $s_* = 1 - \lambda_*$ . The formula in Proposition 7.2 shows that the vanishing in (2) gives

$$\frac{\theta^{s_*} - 1}{s_*} \cdot \zeta(s_*) = 0$$

This hold for every  $0 < \theta < 1$  and we can choose  $\theta$  so that  $\theta^{s_*} - 1 \neq 0$  which gives the requested zero  $\zeta(s_*) = 0$  where  $\Re(s_*) = 1 - \Re(\lambda_*) > 1/2$ .

## Lecture 2: Product measures and a Schrödinger equation

**Introduction.** We expose material from the article [Beurling]. First we insert comments from [Beurling] about the significance of the Main Theorem to be announced in § 0.

The article *Théorie relativiste de l'électron et l'interprétation de la mécanique quantique* was published 1932 where Schrödinger raised a new and unorthodox question concerning Brownian motions leading to new mathematical problems of considerable interest. Consider a Brownian motion which takes place in a bounded region  $\Omega$  of the euclidian space  $\mathbf{R}^d$  for some  $d \geq 2$ . At time  $t = 0$  the densities of particles under observation is a non-negative function  $f_0(x)$  defined on  $\Omega$ . Classically the density at a later time  $t > 0$  is equal to a function  $x \mapsto u(x, t)$  where  $u(x, t)$  solves the heat equation

$$\frac{\partial u}{\partial t} = \Delta(u)$$

with boundary conditions

$$(1) \quad u(x, 0) = f_0(x) \quad \text{and} \quad \frac{\partial u}{\partial \mathbf{n}}(x, t) = 0 \quad \text{when} \quad x \in \partial\Omega \quad \text{and} \quad t > 0$$

Schrödinger took into the account the reality of quantum physics where the observed density of particles at a time  $t_1 > 0$  does not coincide with  $u(x, t_1)$ . He posed the problem to find the most probable development during the time interval  $[0, t_1]$  which leads to the state at time  $t_1$  and concluded that the requested density function which substitutes the heat-solution  $u(x, t)$  should belong to a non-linear class of functions formed by products

$$(*) \quad w(x, t) = u_0(x, t) \cdot u_1(x, t)$$

where  $u_0$  is a solution to (1) while  $u_1(x, t)$  is a solution to the adjoint equation

$$(2) \quad \frac{\partial u_1}{\partial t} = -\Delta(u) \quad : \quad \frac{\partial u_1}{\partial \mathbf{n}}(x, t) = 0 \quad \text{on} \quad \partial\Omega$$

defined when  $t < t_1$ . This leads to a new type of Cauchy problem: Given a pair of non-negative functions  $f_0, f_1$  such that

$$\int_{\Omega} f_0 \cdot dx = \int_{\Omega} f_1 \cdot dx$$

one asks if there exists a  $w$ -function satisfying

$$(**) \quad w(x, 0) = f_0(x) \quad : \quad w(x, t_1) = f_1(x)$$

The solvability of this non-linear boundary value problem was left open by Schrödinger. A first account about eventual solutions was presented by I.N. Bernstein in a plenary talk held at the IMU-congress in Zürich 1932. When  $\Omega$  is a bounded set and has a smooth boundary one can use the Poisson-Greens function for the classical equation (\*) and rewrite Schrödinger's equation to a system of non-linear integral equations. Namely, let  $K(x_1, x_2)$  be the Poisson-Greens kernel in  $\Omega \times \Omega$ . Given a pair of functions  $g_0, g_1$  in  $\Omega$  we get solutions  $u_0, u_1$  to the classical heat equation and its adjoint with initial conditions

$$u_0(x, 0) = g_0(x) \quad : \quad u_1(x, t_1) = g_1(x)$$

The product  $w$  satisfies (\*\*) if and only if the following non-linear system is satisfied:

$$(1) \quad g_0(x_1) \cdot \int_{\Omega} K(x_1, x_2) \cdot g_1(x_2) dx_2 = f_0(x_1)$$

$$(2) \quad g_1(x_2) \cdot \int_{\Omega} K(x_1, x_2) \cdot g_0(x_1) dx_1 = f_1(x_2)$$

Let us also recall that  $K$  is a symmetric function, i.e.  $K(x_1, x_2) = K(x_2, x_1)$  and when  $\Omega$  is bounded with a  $C^1$ -boundary there exist positive constants such that  $C_* \leq K(x_1, x_2) \leq C^*$ . Notice also that a necessary condition for the existence of a pair  $g_1, g_2$  is that

$$(3) \quad \int_{\Omega} f_0(x) dx = \int_{\Omega} f_1(x) dx$$

Beurling proved that when (3) holds for a pair of non-negative  $f$ -functions then the system (1-2) has a unique solution  $(g_0, g_1)$ . Indeed, this follows from the Main Theorem below applied to the case  $n = 2$ , the positive  $K$ -function and the product of the two measure spaces given by the Lebesgue measure on  $\Omega$ .

**Remark.** The original problem by Schrödinger asked for solutions when the kernel function is positive but not bounded away from zero. The basic case occurs when one takes the product of the real line with itself and consider the Gaussian kernel. Here a refined version of the Main Theorem is needed. We give comments about this in § 1. Following [Beurling] we shall now study product measures and derive general results which cover the system (1-2). A merit in Beurling's subsequent study is that it discloses the inherent and rather simple nature of Schrödinger's problem.

## II.0 Product measures and the $\mathcal{T}$ -operator

Let  $n \geq 2$  and consider an  $n$ -tuple of sample spaces  $\{X_\nu = (\Omega_\nu, \mathcal{B}_\nu)\}$ . Thus,  $\Omega_\nu$  is a set and  $\mathcal{B}_\nu$  a Boolean  $\sigma$ -algebra of subsets. Then one has the product:

$$Y = \prod X_\nu$$

whose sample space is the set-theoretic product  $\prod \Omega_\nu$  and the Boolean  $\sigma$ -algebra is generated by  $\{\mathcal{B}_\nu\}$ .

**0.1 Product measures.** Let  $\{\gamma_\nu\}$  be an  $n$ -tuple of signed measures on  $X_1, \dots, X_n$ . There exists a unique measure  $\gamma^*$  on  $Y$  such that

$$\gamma^*(E_1 \times \dots \times E_n) = \prod \gamma_\nu(E_\nu)$$

hold for every  $n$ -tuple of  $\{\mathcal{B}_\nu\}$ -measurable sets. We refer to  $\gamma^*$  as the product measure and write

$$\gamma^* = \prod \gamma_\nu$$

**0.2 Remark.** The set of product measures is a proper non-linear subset of the linear space of all signed measures on  $Y$ . This is already seen when  $n = 2$  with two discrete sample spaces, i.e.  $X_1$  and  $X_2$  consists of  $N$  points for some integer  $N$ . Every  $N \times n$ -matrix with non-negative elements  $\{a_{jk}\}$  give a probability measure  $\mu$  on  $X_1 \times X_2$  when the double sum  $\sum \sum a_{jk} = 1$ . Here  $\mu$  is a product measure if and only if there exist  $N$ -tuples  $\{\alpha_j$  and  $\{\beta_k\}$  such that  $\sum \alpha_j = \sum \beta_k = 1$  and  $a_{jk} = \alpha_j \cdot \beta_k$ .

**A normalisation.** If  $\gamma^*$  is a product of non-negative measures its factors  $\{\gamma_\nu\}$  are uniquely determined when their masses are normalised so that

$$\gamma_\nu(\Omega_\nu) = [\gamma^*(\Omega)]^{\frac{1}{n}}$$

From now on such a normalisation holds while we construct non-negative product measures.

**0.3 The space  $\mathcal{A}$ .** If  $g_\nu$  is a function on  $\Omega_\nu$  we get a function  $g_\nu^*$  on the product  $\Omega$  defined by

$$g_\nu^*(x_1, \dots, x_n) = g_\nu(x_\nu)$$

This gives the linear space of functions on  $\Omega$  which can be written as

$$(i) \quad a = g_1^* + \dots + g_n^*$$

Notice that a pair of product measures  $\gamma$  and  $\mu$  on  $Y$  are equal if and only if

$$\int_Y a \cdot d\gamma = \int_Y a \cdot d\mu \quad : \quad \forall a \in \mathcal{A}$$

**0.4 The  $\mathcal{T}_k$ -operators.** Let  $k$  be a real-valued  $\mathcal{B}$ -measurable function such that both  $k$  and  $k^{-1}$  are bounded, i.e. the range is contained in an interval  $[a, b] : 0 < a < b < \infty$ . Consider a product measure  $\gamma^*$ . If  $1 \leq \nu \leq n$  we find a unique measure on  $X_\nu$  denoted by  $(k \cdot \gamma)_\nu$  such that

$$\int_Y g_\nu^* \cdot k \cdot d\gamma^* = \int_{X_\nu} g_\nu \cdot d(k \cdot \gamma)_\nu$$

hold for every bounded  $\mathcal{B}_\nu$ -measurable function  $g_\nu$  on  $X_\nu$ . Now we get the product measure

$$(*) \quad \mathcal{T}_k(\gamma^*) = \prod (k\gamma)_\nu$$

Let  $\mathcal{S}_k^*$  be the family of non-negative product measures  $\gamma^*$  on  $Y$  such that

$$\int_Y k \cdot d\gamma^* = 1$$

Next, denote by  $\mathcal{S}_1^*$  the family of probability measures which in addition are product measures. It is clear that  $\mathcal{T}_k$  gives a map from  $\mathcal{S}_k^*$  into  $\mathcal{S}_1^*$  and using the observation from (0.3.1) one verifies that the map is injective. Less obvious is the surjectivity. More precisely one has:

**Main Theorem.**  $\mathcal{T}_k$  yields a homeomorphism between  $\mathcal{S}_k^*$  and  $\mathcal{S}_1^*$ .

**0.5 Remark.** Above we refer to the norm topology on the space of measure, i.e. if  $\gamma_1$  and  $\gamma_2$  are two measures on  $Y$  then the norm  $\|\gamma_1 - \gamma_2\|$  is the total variation of the signed measure  $\gamma_1 - \gamma_2$ . The reader may verify that  $\mathcal{S}_k^*$  and  $\mathcal{S}_1^*$  both appear as closed subsets in the normed space of signed measures on  $Y$ . Recall also that the space of measures on  $Y$  is complete under this norm which will be used in the subsequent proof. The proof of surjectivity is achieved by solving a variational problem in § 0.7 where we shall need some inequalities which are announced below.

**0.6 Some useful inequalities.** Let  $\gamma_1$  and  $\gamma_2$  be a pair of product measures such that

$$\left| \int_Y g_\nu^* \cdot d\gamma_1 - \int_Y g_\nu^* \cdot d\gamma_2 \right| \leq \epsilon \quad : \quad 1 \leq \nu \leq n$$

hold for some  $\epsilon > 0$  and every function  $g_\nu$  on  $X_\nu$  with maximum norm  $\leq 1$ . Then the norm

$$(0.6.1) \quad \|\gamma_1 - \gamma_2\| \leq n \cdot \epsilon$$

The proof of (0.6.1) is left to the reader. The hint is to make repeated use of Fubini's theorem. More generally, let  $k$  be a bounded measurable function on  $Y$  and  $\gamma, \mu$  is a pair of product measures. Denote by  $\mathcal{A}_*$  the set of  $\mathcal{A}$ -functions  $a$  with maximum norm  $\leq 1$ . Then there exists a constant  $C$  which only depends on  $k$  and  $n$  such that

$$(0.6.2) \quad \|\mathcal{T}_k(\mu) - \gamma\| \leq \max_{a \in \mathcal{A}_*} \left| \int_Y a(kd\mu - d\gamma) \right|$$

Again we leave the proof as an exercise.

### 0.7 A variational problem.

If  $a = \sum g_\nu^*$  is a function in  $\mathcal{A}$  we get the exponential function

$$e^a = \prod e^{g_\nu^*}$$

For every product measure  $\gamma^* = \prod \gamma_\nu$  we obtain a new product measure

$$\prod e^{a_\nu} \cdot \gamma_\nu$$

which is denoted by  $e^a \cdot \gamma^*$ . Next, for every pair  $\gamma \in \mathcal{S}_1^*$  and  $a \in \mathcal{A}$  we set

$$(0.7.1) \quad W(a, \gamma) = \int_Y (e^a k - a) \cdot d\gamma \quad \text{and} \quad W_*(\gamma) = \min_{a \in \mathcal{A}} W(a, \gamma)$$

The following result will be used later on where  $\gamma$  is an arbitrary measure on  $\mathcal{S}_1^*$  and  $k^*$  denotes the maximum of the positive  $k$ -function.

**0.8. Lemma.** Let  $\epsilon > 0$  and  $a \in \mathcal{A}$  be such that  $W(a, \gamma) \leq W_*(\gamma) + \epsilon$ . Then

$$\int e^a \cdot k \cdot \gamma \leq \frac{k^* + \epsilon}{1 - e^{-1}}$$

*Proof.* For every positive real number  $s$  the function  $a - s$  belongs to  $\mathcal{A}$  and by the hypothesis  $W(a - s, \gamma) \geq W(a, \gamma) - \epsilon$ . This entails that

$$\begin{aligned} \int e^a k \cdot d\gamma &\leq \int_Y e^{a-s} \cdot k d\gamma + s \int k \cdot d\gamma + \epsilon \implies \\ &\int (1 - e^{-s}) \cdot e^a \cdot k d\gamma \leq s \cdot k^* + \epsilon \end{aligned}$$

Lemma 0.8 follows if we take  $s = 1$ .

**0.9 Remark.** If  $k_*$  is the minimum of  $k$  we have

$$\int e^a \cdot \gamma \leq \frac{1}{k_*} \cdot \int e^a \cdot k \cdot \gamma$$

So Lemma 0.8 gives the inequality

$$(0.9.1) \quad \int e^a \cdot \gamma \leq \frac{1}{k_*} \cdot \frac{k^* + \epsilon}{1 - e^{-1}}$$

Now we announce a result which gives the requested surjectivity of the operator  $\mathcal{T}_k$ .

**0.10 Proposition.** Let  $\gamma$  be a measure in  $\mathcal{S}_1^*$  and  $\{a_\nu\}$  a sequence in  $\mathcal{A}$  such that

$$\lim W(a_\nu, \gamma) = W_*(\gamma)$$

Then the sequence  $\{e^{a_\nu} \cdot \gamma\}$  converges to a measure  $\mu \in \mathcal{S}_k^*$  such that  $\mathcal{T}_k(\mu) = \gamma$ .

*Proof.* Keeping  $\gamma$  fixed we set  $W(a) = W(a, \gamma)$ . Let  $0 < \epsilon < 1$  and consider a pair  $a, b$  in  $\mathcal{A}$  such that  $W(a)$  and  $W(b)$  both are  $\leq W_*(\gamma) + \epsilon$ . Since  $\frac{1}{2}(a + b)$  belongs to  $\mathcal{A}$  we get

$$(i) \quad 2 \cdot W\left(\frac{1}{2}(a + b)\right) \geq 2 \cdot W_*(\gamma) \geq W(a) + W(b) - 2\epsilon$$

Notice that

$$(ii) \quad W(a) + W(b) - 2 \cdot W\left(\frac{1}{2}(a + b)\right) = \int_Y [e^a + e^b - 2 \cdot e^{\frac{1}{2}(a+b)}] \cdot k d\gamma$$

Next, we have the algebraic identity

$$e^a + e^b - 2 \cdot e^{\frac{1}{2}(a+b)} = (e^{a/2} - e^{b/2})^2$$

It follows from (i-ii) that

$$(iii) \quad \int_Y (e^{a/2} - e^{b/2})^2 \cdot k \cdot d\gamma \leq 2\epsilon$$

The identity  $|e^a - e^b| = (e^{a/2} + e^{b/2}) \cdot |e^{a/2} - e^{b/2}|$  and the Cauchy-Schwarz inequality give:

$$(iv) \quad \left[ \int_Y |e^a - e^b| \cdot k \cdot d\gamma \right]^2 \leq 2\epsilon \cdot \int_Y (e^{a/2} + e^{b/2}) \cdot k \cdot d\gamma$$

By Lemma 0.8 the last factor is bounded by a fixed constant and (iv) gives a constant  $C$  such that

$$(v) \quad \int_Y |e^a - e^b| \cdot k \cdot d\gamma \leq C \cdot \sqrt{\epsilon}$$

Next, let  $k_*$  be the minimum value taken by  $k$  on  $Y$ . Replacing  $C$  by  $C/k_*$  we get

$$(vi) \quad \int_Y |e^a - e^b| \cdot d\gamma \leq C \cdot \sqrt{\epsilon}$$

Now (vi) applies to pairs in the sequence  $\{a_\nu\}$  and shows that  $\{e^{a_\nu} \cdot d\gamma\}$  is a Cauchy sequence with respect to the norm of measures on  $Y$ . So by the completeness of measures there exists a non-negative measure  $\mu$  such that

$$(vii) \quad \lim_{\nu \rightarrow \infty} \|e^{a_\nu} \cdot \gamma - \mu\| = 0$$

The equality  $\mathcal{T}_k(\mu) = \gamma$ . Consider the  $a$ -functions in the minimizing sequence. If  $\rho \in \mathcal{A}$  is arbitrary we have

$$W(a_\nu + \rho) \geq W(a_\nu) - \epsilon_\nu$$

where  $\epsilon_\nu \rightarrow 0$ . This gives

$$(1) \quad \int_Y [k e^{a_\nu} (1 - e^\rho) + \rho] \cdot d\gamma \leq \epsilon_\nu$$

Next, we use the inequality

$$|e^x - 1 - x| \leq x^2 \quad : \quad -\log 2 \leq x \leq \log 2$$

So if the maximum norm  $|\rho|_Y \leq \log 2$  we can write

$$(2) \quad e^\rho = 1 + \rho + \rho_1 \quad \text{where} \quad 0 \leq \rho_1 \leq \rho^2$$

Then we see that (1) gives

$$\int_Y (\rho - k e^{a_\nu} \cdot \rho) \cdot d\gamma \leq \epsilon_\nu + \int_Y \rho_1 \cdot k \cdot e^{a_\nu} \cdot \gamma$$

Lemma 0.8 and the inequality for  $\rho_1$  in (2) above gives a constant  $C$  which is independent of  $\nu$  such that the right hand side is majorised by

$$(3) \quad \epsilon_\nu + C \cdot \|\rho\|_Y^2$$

The same inequality holds with  $\rho$  replaced by  $-\rho$  which entails that

$$\left| \int_Y (k e^{a_\nu} - 1) \cdot \rho \cdot d\gamma \right| \leq \epsilon_\nu + C \cdot \|\rho\|_Y^2$$

Now we apply the inequality (0.6.2) while we use  $\rho$ -functions in  $\mathcal{A}$  with norms  $\sqrt{\epsilon_\nu}$  starting from large  $\nu$  so that  $\sqrt{\epsilon_\nu} \leq \log 2$ . This gives the following inequality for total variations:

$$\|\mathcal{T}_k(e^{a_\nu} \cdot \gamma) - \gamma\| \leq (1 + C) \cdot n\sqrt{\epsilon}$$

Passing to the limit it follows that the equality

$$\mathcal{T}_k(\mu) = \gamma$$

Since  $\gamma \in S_1^*$  was arbitrary we have proved that the  $\mathcal{T}_k$  yields a surjective map from  $S_k^*$  to  $S_1^*$  which finishes the proof of the Main Theorem.

### § 1. The singular case.

Let  $n = 2$  and  $k(x_1, x_2)$  is a bounded and strictly positive continuous function on  $Y = X_1 \times X_2$ . Let  $\gamma \in S_1^*$  satisfy:

$$(1) \quad \int_Y \log k \cdot d\gamma > -\infty$$

**1.1. Theorem.** *There exists a unique non-negative product measure  $\mu$  on  $Y$  such that  $\mathcal{T}_k(\mu) = \gamma$ .*

**1.2 Remark.** In general the measure  $\mu$  need not have finite mass but the proof shows that  $k$  belongs to  $L^1(\mu)$ , i.e.

$$\int_Y k \cdot d\mu < \infty$$

As pointed out by Beurling Theorem 1.1 can be applied to the case  $X_1 = X_2 = \mathbf{R}$  both are copies of the real line and

$$k(x_1, x_2) = g(x_1 - x_2)$$

where  $g$  is the density of a Gaussian distribution which after a normalisation of the variance is taken to be

$$\frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2}$$

So here  $\gamma$  satisfies the the integrability condition

$$\iint (x_1 - x_2)^2 \cdot d\gamma(x_1, x_2) < \infty$$

Theorem 1.1 is proved in [ibid; page 218-220] using similar but technically more involved methods as in the proof of the Main Theorem. Concerning higher dimensional cases, i.e. singular versions of the Main Theorem when  $n \geq 3$ , Beurling gives the following comments at the end of [ibid]:

*The proof of the Main Theorem relies heavily on the condition that  $k \geq a$  for some  $a > 0$ . If this lower bound condition is dropped the individual equation  $\mathcal{K}(\gamma) = \mu$  may still be meaningful, but serious complications will arise concerning the global uniqueness if  $n \geq 3$  and the proof of Theorem 1.1 for the case  $n \geq 3$  cannot be duplicated.*

### III. The Laplace operator and the Helmholtz equation

**Introduction.** Let  $D$  be a domain in the  $(x, y)$ -plane where one has a stationary irrotational flow of an ideal fluid described a stream function  $v(z)$  which is harmonic in  $D$ . The gradient vector  $\nabla(v)$  is the velocity vector of the flow and assumed to be everywhere  $\neq 0$ . In general  $v$  is only defined in a subdomain  $D_*$  of  $D$ , and the part of  $\partial D_*$ . On each component of the full boundary  $\partial D_*$  we assume that  $v$  is constant on every connected component which means that these curves are streamlines of the flow. In 1867 Helmholtz posed the problem to find a pair  $(v, D_*)$  such that the length  $|\nabla(v)|$  is constant on  $\partial D_* \cap D$ . More generally one can impose the condition that the function

$$p \mapsto |\nabla(v)(p)|$$

agrees with a given positive and continuous function  $\Phi$  on  $\partial D_* \cap D$ . This leads to a free boundary value problem which goes as follows:

Let  $D^*$  be the exterior disc  $\{|z| > 1\}$  in  $\mathbf{C}$  and  $\Phi(z)$  is a given continuous and positive function in  $D^*$ . Denote by  $\mathcal{J}^*$  the family of closed Jordan curves  $\gamma$  in  $D^*$  which together with  $\{|z| = 1\}$  borders an annulus. For every such  $\gamma$  there exists the unique harmonic function  $v$  in the annulus bordered by  $\{|z| = 1\}$  and  $\gamma$ , where  $v = 1$  on  $\{|z| = 1\}$  and  $v = 0$  on  $\gamma$ . One seek  $\gamma$  so that

$$(*) \quad |\nabla(v)(p)| = \Phi(p) \quad : p \in \gamma$$

This problem is studied in a series of articles by Beurling. Here we expose results from *On free-boundary problems for the Laplace operator* [Princeton: Seminars analytic functions 1957]. To solve (\*) Beurling proceeds as follows. Let us say that  $\gamma$  is of type  $\mathcal{B}(Q)$  if

$$(1) \quad \limsup_{z \rightarrow p} |\nabla(v)(z)| \leq \Phi(p) \quad : p \in \gamma$$

We consider also the class  $\mathcal{A}(Q)$  where

$$(2) \quad \liminf_{z \rightarrow p} |\nabla(v)(z)| \geq \Phi(p) \quad : p \in \gamma$$

If a curve  $\gamma$  belongs to the intersection  $\mathcal{A}(Q) \cap \mathcal{B}(Q)$  then (\*) holds. There remains to find conditions on  $Q$  in order that this intersection is non-empty and the uniqueness amounts to show that this intersection is reduced to a single curve in  $\mathcal{J}^*$ . Following [ibid] we shall establish some sufficiency results for existence as well as uniqueness. First we impose a growth condition on  $Q$  when one approaches the unit circle:

$$(3) \quad \lim_{|z| \rightarrow 1} \left[ \log \frac{1}{|z|} \right]^{-1} \cdot Q(z) = 0$$

**Remark.** As explained in § XX this condition is invariant under conformal mappings between our chosen annulus and others. From now on (3) is assumed. A first major result in [ibid] is:

**Theorem.** *If  $\mathcal{B}(Q) \neq \emptyset$  then the free boundary value problem has at least one solution.*

Next, we seek conditions in order that  $\mathcal{B}(Q) \neq \emptyset$ . To each  $a < 1$  we denote by  $B_a(Q)$  the family of curves  $\gamma$  such that

$$(i) \quad \limsup_{z \rightarrow p} |\nabla(v)(z)| \leq a \cdot \Phi(p) \quad : p \in \gamma$$

Set

$$\mathcal{B}_*(Q) = \bigcup_{a < 1} B_a(Q)$$

**Theorem.** *Suppose there exists a pair of curves  $\gamma \in \mathcal{A}(Q)$  and  $\gamma_0 \in \mathcal{B}(Q)$  such that  $\gamma$  stays in the annulus bordered by  $\{|z| = 1\}$  and  $\gamma_0$ . Then (\*) has a solution  $\gamma_*$  where  $\gamma_*$  is between  $\gamma$  and  $\gamma_0$ .*

*Uniqueness.*

For each rectifiable and closed Jordan curve  $\gamma$  in the exterior disc with winding number one we set

$$l_\gamma(Q) = \int_\gamma Q \cdot |ds_\gamma|$$

where  $ds_\gamma$  is the arc-length measure. Set

$$(1) \quad l_*(Q) = \inf_\gamma l_\gamma(Q)$$

**Definition.** *The circle  $\{|z| = 1\}$  is said to be convex with respect to  $Q$  if the infimum in (1) can be achieved by a sequence of Jordan curves  $\gamma$  which tend to the circle.*

Thus, convexity means that for each  $\delta > 0$  when we take the infimum over  $\gamma$  curves which stay in the narrow annulus  $1 < |z| < 1 + \delta$  for arbitrary small  $\delta > 0$ , then this infimum is  $\geq l_*(Q)$ .

**Theorem.** *If  $\{|z| = 1\}$  is  $Q$ -convex and  $\log Q$  is subharmonic there cannot exist more than one solution to (\*).*

*About the proofs.*

They rely upon similar methods as in Beurling's conformal mapping theorem to be presented in § III.B. See also chapter VI: vol 2 [Collected work] for further studies related to the free boundary value problem.

### III.B Beurling's conformal mapping theorem.

**Introduction.** Let  $D$  be the open unit disc  $|z| < 1$ . Denote by  $\mathcal{C}$  the family of conformal maps  $w = f(z)$  which map  $D$  onto some simply connected domain  $\Omega_f$  which contains the origin and satisfy:

$$f(0) = 0 \quad \text{and} \quad f'(0) \text{ is real and positive.}$$

Riemann's mapping theorem asserts that for every simply connected subset  $\Omega$  of  $\mathbf{C}$  which is not equal to  $\mathbf{C}$  there exists a unique  $f \in \mathcal{C}$  such that  $\Omega_f = \Omega$ . We are going to construct a subfamily of  $\mathcal{C}$ . Consider a positive and bounded continuous function a function  $\Phi$  defined in the whole complex  $w$ -plane.

**0.1 Definition.** *The set of all  $f \in \mathcal{C}$  such that*

$$(*) \quad \lim_{r \rightarrow 1} \max_{0 \leq \theta \leq 2\pi} [|f'(re^{i\theta})| - \Phi(f(re^{i\theta}))] = 0$$

*is denoted by  $\mathcal{C}_\Phi$ .*

**Remark.** Thus, when  $f \in \mathcal{C}_\Phi$  then the difference of the absolute value  $|f'(z)|$  and  $\Phi(f(z))$  tends uniformly to zero as  $|z| \rightarrow 1$ . Let  $M$  be the upper bound of  $\Phi$ . The maximum principle applied to the complex derivative  $f'(z)$  gives

$$|f'(z)| \leq M \quad : \quad z \in D$$

Hence  $f(z)$  is a continuous function in the open disc  $D$  whose Lipschitz norm is uniformly bounded by  $M$ . This implies that  $f$  extends to a continuous function in the closed disc, i.e.  $f$  belongs to the disc algebra  $A(D)$ . Notice also that (\*) implies that the function  $z \mapsto |f'(z)|$  extends to a continuous function on  $\bar{D}$ .

**1. Theorem.** *Assume that  $\text{Log} \frac{1}{\Phi(w)}$  is subharmonic. Then  $\mathcal{C}_\Phi$  contains a unique function  $f^*$ .*

**Remark.** In the special case when  $\Phi(w) = \Phi(|w|)$  is a radial function we notice that for every  $\rho > 0$  such that  $\Phi(\rho) = \rho$  it follows that the function  $f(z) = \rho \cdot z$  belongs to  $\mathcal{C}_\Phi$ . So for a radial  $\Phi$ -function where different  $\rho$ -numbers exist one does not have uniqueness. The reader may verify that a radial function  $\Phi$  for which  $\Phi(\rho) = \rho$  has several solutions cannot satisfy the condition in Theorem 1. Next, let us give examples of  $\Phi$ -functions which satisfy the condition in Theorem 1. Consider an arbitrary real-valued and non-negative  $L^1$ -function  $\rho(t, s)$  which has compact support. Set

$$\Phi(w) = \exp \left[ \int \log \frac{1}{|w - t - is|} \cdot \rho(t, s) \cdot dt ds \right]$$

Here  $\log \frac{1}{\Phi}$  is subharmonic and Theorem 1 asserts that there exists a unique simply connected domain  $\Omega$  which contains the origin such that the normalised conformal mapping function  $f: D \rightarrow \Omega$  satisfies

$$|f'(e^{i\theta})| = \Phi(f(e^{i\theta})) \quad : \quad 0 \leq \theta \leq 2\pi.$$

The proof of Theorem 1 relies upon some results where we only assume that the  $\Phi$ -function is continuous and positive.

**The family  $\mathcal{A}_\Phi$ .** A conformal map  $f(z)$  in  $\mathcal{C}$  belongs to  $\mathcal{A}_\Phi$  if

$$(1) \quad \limsup_{|z| \rightarrow 1} |f'(z)| - \Phi(f(z)) \leq 0$$

**Remark.** By the definition of limes superior this means that for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$|f'(z)| \leq \Phi(f(z)) + \epsilon \quad : \quad \text{for all } 1 - \delta < |z| < 1.$$

**The maximal region**  $\Omega^*(\Phi)$ . With  $\Phi$  given we define a bounded open subset in the  $w$ -plane as follows:

$$\Omega^*(\Phi) = \cup f(D) \quad : \quad \text{union taken over all } f \in \mathcal{A}_\Phi.$$

With these notations we have

**2. Theorem.** *The maximal region  $\Omega^*(\Phi)$  is simply connected. Moreover, there exists a unique conformal map with  $f^*(D) = \Omega^*(\Phi)$  which in addition belongs to  $\mathcal{C}_\Phi$ .*

**The family**  $\mathcal{B}_\Phi$ . It consists of all  $f \in \mathcal{C}$  such that

$$\text{Lim.inf.}_{|z| \rightarrow 1} |f'(z)| - \Phi(f(z)) \geq 0$$

To this family we assign minimal region

$$\Omega_*(\Phi) = \cap f(D) \quad \text{The intersection taken over all } f \in \mathcal{B}_\Phi$$

**3. Theorem.** *The set  $\Omega_*(\Phi)$  is simply connected and the unique  $f_* \in \mathcal{C}$  for which  $f_*(D) = \Omega_*(\Phi)$  belongs to  $\mathcal{C}_\Phi$ .*

The constructions of the maximal and the minimal region give

$$(1) \quad \Omega_*(\Phi) \subset \Omega^*(\Phi)$$

In general this inclusion it is strict as seen by the example when  $\Phi$  is radial. But when  $\text{Log } \frac{1}{|\Phi|}$  is subharmonic the uniqueness in Theorem 1 asserts that one has the equality  $\Omega_*(\Phi) = \Omega^*(\Phi)$ .

**Remark about the proofs.** Following Beurling's article [Beur] we shall give the details of the proof of Theorem 2. Concerning Theorem 3 it is substantially harder and for this part of the proof we refer to [Beur: p. 127-130] for details, Before we enter the proof of Theorem 2 we show how Theorem 2 and 3 together give the uniqueness in Theorem 1.

### A. Proof of Theorem 1.

Let  $\Phi$  be as in Theorem 1. Admitting Theorem 2 and 3 we get the two simply connected domains  $\Omega^*(\Phi)$  and  $\Omega_*(\Phi)$ . Keeping  $\Phi$  fixed we set  $\Omega^* = \Omega^*(\Phi)$  and  $\Omega_* = \Omega_*(\Phi)$ . Since  $\Omega_* \subset \Omega^*$  Riemann's mapping theorem gives an inequality for the first order derivative at  $z = 0$ :

$$(i) \quad f'_*(0) \leq (f^*)'(0)$$

Next, we can write

$$\Phi(w) = e^{U(w)}$$

where  $U(w)$  by assumption is super-harmonic. We can solve the Dirichlet problem with respect to the domain  $\Omega^*$ . This gives the harmonic function  $U^*$  in  $\Omega^*$  where

$$U^*(w) = U(w) \quad w \in \partial\Omega^*.$$

Similarly we find the harmonic function  $U_*$  in  $\Omega_*$  such that

$$(*) \quad U_*(w) = U(w) \quad w \in \partial\Omega_*.$$

Next, since  $f^* \in \mathcal{C}_\Phi$  we have the equality

$$(ii) \quad \log |(f^*)'(z)| = U(f(z)) \quad |z| = 1$$

Now  $\log |(f^*)'(z)|$  and  $U^*(f(z))$  are harmonic in  $D$  and (ii) gives:

$$\log (f^*)'(0) = U^*(0)$$

In a similar way we find that

$$\log f'_*(0) = U_*(0)$$

Since  $U$  is super-harmonic in  $\Omega^*$  and  $\partial\Omega_*$  is a closed subset of  $\bar{\Omega}^*$  we get:

$$U(w) \geq U^*(w) \quad w \in \partial\Omega_*$$

From (\*) it therefore follows that  $U_* \geq U^*$  holds in  $\Omega_*$ . So in particular

$$\log f'_*(0) = U_*(0) \geq U^*(0) = \log (f^*)'(0)$$

Together with (i) we conclude that  $f_*(0) = (f^*)'(0)$ . Finally, the uniqueness in Riemann's mapping theorem gives  $\Omega_* = \Omega^*$  and hence that  $f_* = f^*$  which proves Theorem 1.

## B. Proof of Theorem 2.

The first step in the proof is to construct a certain "union map" defined by a finite family  $f_1, \dots, f_n$  of functions  $\mathcal{A}_\Phi$ . Set

$$(*) \quad S_\nu = f_\nu(D) \quad \text{and} \quad S_* = \cup S_\nu$$

So above  $S_*$  is a union of Jordan domains which in general can intersect each other in a rather arbitrary fashion.

**B.1 Definition.** *The extended union denoted by  $EU(S_*)$  is defined as follows: A point  $w$  belongs to the extended union if there exists some closed Jordan curve  $\gamma$  which contains  $w$  in its interior domain while  $\gamma \subset S_*$ .*

**Exercise.** Verify that the extended union is *simply connected*.

**B.2 Lemma** *Let  $f_*$  be the unique normalised conformal map from  $D$  onto the extended union above. Then  $f_* \in \mathcal{A}_\Phi$ .*

*Proof.* First we reduce the proof to the case when all the functions  $f_1, \dots, f_n$  extend to be analytic in a neighborhood of the closed disc  $\bar{D}$ . In fact, with  $r < 1$  we set  $f_\nu^r(z) = f_\nu(rz)$  and get the image domains  $S_\nu[r] = f_\nu^r(D) = f_\nu(D_r)$ . Put  $S_*[r] = \cup S_\nu[r]$  and construct its extended union which we denote by  $S_{**}[r]$ . Next, let  $\epsilon > 0$  and consider the new function  $\Psi(w) = \Phi(w) + \epsilon$ . Let  $f_*[r]$  be the conformal map from  $D$  onto  $S_{**}[r]$ . If Lemma B.2 has been proved for the  $n$ -tuple  $\{f_\nu^r\}$  it follows by continuity that  $f_*[r]$  belongs  $\mathcal{A}_\Psi$  if  $r$  is sufficiently close to one. Passing to the limit we see that  $f_* = \lim_{r \rightarrow 1} f_*[r]$  and we get  $f_* \in \mathcal{A}_\Psi$ . Since  $\epsilon > 0$  is arbitrary we get  $f_* \in \mathcal{A}_\Phi$  as required.

After this preliminary reduction we consider the case when each  $f$ -function extends analytically to a neighborhood of the closed disc  $|z| \leq 1$ . Then each  $S_\nu$  is a closed real analytic Jordan curve and the boundary of  $S_*$  is a finite union of real analytic arcs and some corner points. In particular we find the outer boundary which is a piecewise analytic and closed Jordan curve  $\Gamma$  and the extended union is the Jordan domain bordered by  $\Gamma$ . It is also clear that  $\Gamma$  is the union of some connected arcs  $\gamma_1, \dots, \gamma_N$  and a finite set of corner points and for each  $1 \leq k \leq N$  there exists  $1 \leq \nu(k) \leq n$  such that

$$\gamma_k \subset \partial S_{\nu(k)}$$

Denote by  $\{F_\nu = f_\nu^{-1}\}$  and  $F = f_*^{-1}$  the inverse functions and put:

$$G = \text{Log} \frac{1}{|F|} \quad : \quad G_\nu = \text{Log} \frac{1}{|F_\nu|} \quad : \quad 1 \leq \nu \leq n.$$

With  $1 \leq \nu \leq n$  kept fixed we notice that  $G_\nu$  and  $G$  are super-harmonic functions in  $S_\nu$  and the difference

$$H = G - G_\nu$$

is superharmonic in  $S_\nu$ . Next, consider a point  $p \in \partial S_\nu$ . Then  $|F_\nu(p)| = 1$  and hence  $G_\nu(p) = 0$ . At the same time  $p$  belongs to  $\partial S_*$  or the interior of  $S_*$  so  $|F(p)| \leq 1$  and hence  $G(p) \geq 0$ . This shows that  $H \geq 0$  on  $\partial S_\nu$  and by the minimum principle for harmonic functions we obtain:

$$(i) \quad H(q) \geq 0 \quad \text{for all } q \in S_\nu$$

Let us then consider some boundary arc  $\gamma_k$  where  $\gamma \subset \partial S_\nu$ , i.e. here  $\nu = \nu(k)$ . Now  $H = 0$  on  $\gamma_k$  and since (i) holds it follows that the *outer normal derivative*:

$$(ii) \quad \frac{\partial H}{\partial n}(p) \leq 0 \quad p \in \gamma_k$$

Since  $|F| = |F_\nu| = 1$  holds on  $\gamma_k$  and the gradient of  $H$  is parallel to the normal we also get:

$$\frac{\partial G}{\partial n}(p) = -|F'(w)| \quad \text{and} \quad \frac{\partial G_\nu}{\partial n}(p) = -|F'_\nu(w)| \quad : w \in \gamma_k$$

Hence (ii) above gives

$$(iii) \quad |F'(w)| \geq |F'_\nu(w)| \quad \text{when} \quad w \in \gamma_k$$

Next, since  $f_\nu \in \mathcal{A}_\Phi$  we have

$$(iv) \quad |f'_\nu(F_\nu(w))| \leq \Phi(w)$$

and since  $F_\nu$  is the inverse of  $f_\nu$  we get

$$1 = f'_\nu(F_\nu(w)) \cdot F'_\nu(w)$$

Hence (iv) entails

$$(v) \quad |F'_\nu(w)| \geq \frac{1}{\Phi(w)}$$

We conclude from (iii) that

$$(vi) \quad |F'(w)| \geq \frac{1}{\Phi(w)} \quad : w \in \gamma_k$$

This holds for all the sub-arcs  $\gamma_1, \dots, \gamma_n$  and hence we have proved the inequality

$$(*) \quad |F'(w)| \geq \frac{1}{\Phi(w)} \quad \text{for all} \quad w \in \Gamma$$

except at a finite number of corner points. To settle the situation at corner points we notice that Poisson's formula applied to the harmonic function  $\log |f'_*(z)|$  in the unit disc gives

$$(vii) \quad \log |f'_*(z)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot \log |f'(e^{i\theta})| \cdot d\theta.$$

Next, since  $F$  is the inverse of  $f_*$  we have

$$|f'_*(z)| \cdot |F'(f(z))| = 1 \quad \text{for all} \quad |z| = 1.$$

Hence (vi) gives

$$|f'(z)| \leq \Phi(f_*(z)) \quad \text{for all} \quad |z| = 1.$$

With  $\Phi = e^U$  we therefore get

$$\log |f'_*(z)| \leq U(f(z)) \quad \text{for all} \quad |z| = 1.$$

From the Poisson integral (vii) it follows that

$$\log |f'_*(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot U(f_*(e^{i\theta})) \cdot d\theta. \quad z \in D$$

**A passage to the limit.** In addition to the obvious equi-continuity the passage to the limit requires some care which is exposed in [Beurling: Lemma 1, page 122]. Passing to the limit as  $|z| \rightarrow 1$  the continuity of  $\Phi$  implies that  $f_*$  belongs to  $\mathcal{A}_\Phi$  which proves Lemma B.2.

### B.3 The construction of $f^*$

By the uniform bound for Lipschitz norms the family  $\mathcal{A}_\Phi$  is equi-continuous. We can therefore find a denumerable dense subset  $\{h_\nu\}$ . It means that to every  $f \in \mathcal{A}_\Phi$  and every  $\epsilon > 0$  there exists some  $h_\nu$  such that the maximum norm  $|f - h_\nu|_D < \epsilon$ . It follows that

$$\Omega^* = \cup h_\nu(D)$$

Next, to every  $n \geq 2$  we have the  $n$ -tuple  $h_1, \dots, h_n$  and by Lemma B. 2 we construct the function  $f_n$  where we have the inclusions

$$h_\nu(D) \subset f_n(D) \quad : \quad 1 \leq \nu \leq n$$

Moreover, the image domains  $\{f_n(D)\}$  increase with  $n$ . So (i) above gives

$$(*) \quad \Omega^* = \cup f_n(D)$$

Next,  $\{f_n\}$  is a normal family of analytic functions and since their image domains increase it follows there exists the limit function  $f^*$  which belongs to  $\mathcal{C}$  and (\*) above gives the equality  $f^*(D) = \Omega^*$ . There remains to prove that  $f^*$  also belongs to  $\mathcal{C}_\Phi$ . To get the inclusion

$$f^* \in \mathcal{C}_\Phi$$

we establish a relation between  $\Phi$  and the maximal domain  $\Omega^*(\Phi)$ .

**B.4 Proposition.** *Let  $\Psi$  be a positive continuous function which is equal to  $\Phi$  outside  $\Omega^*(\Phi)$  while its restriction to  $\Omega^*(\Phi)$  is arbitrary. Then one has the equality*

$$\Omega^*(\Phi) = \Omega^*(\Psi)$$

*Proof.* The assumption gives

$$(i) \quad \Psi(w) = \Phi(w) \quad \text{for all } w \in \partial\Omega^*(\Phi)$$

It follows that  $f^* \in \mathcal{A}_\Psi$ . Hence the equality  $f^*(D) = \Omega^*(\Phi)$  and the construction of  $\Omega^*(\Psi)$  give the inclusion

$$(ii) \quad \Omega^*(\Phi) \subset \Omega^*(\Psi)$$

Next, let  $h^*$  be the mapping function associated to  $\Psi$ . By the construction of the maximal region  $\Omega^*(\Phi)$  we get:

$$(iv) \quad \Omega^*(\Psi) = h^*(D) \subset \Omega^*(\Phi)$$

Hence (iii) and (iv) give the requested equality

$$(v) \quad \Omega^*(\Psi) = \Omega^*(\Phi)$$

### B.5 A special choice of $\Psi$

Keeping  $\Phi$  fixed we put  $\Omega^*(\Phi) = \Omega^*$  to simplify the notations. We have the  $U$ -function such that

$$(vi) \quad \Phi(w) = e^{-U(w)}$$

Now  $U(w)$  is a continuous function on  $\partial\Omega^*$  and solving the Dirichlet problem we obtain the function  $U_*(w)$  where  $U_* = U$  outside  $\Omega^*$ , and in  $\Omega^*$  the function  $U_*$  is the harmonic extension of the boundary function  $U$  restricted to  $\partial\Omega^*$ . Set

$$\Psi(w) = e^{U_*(w)}$$

Proposition B.4 gives

$$(i) \quad f^* \in \mathcal{A}_\Psi$$

Next, consider the function in  $D$  defined by:

$$V(z) = \log \left| \frac{df^*(z)}{dz} \right| - U_*(f(z))$$

From (i) it follows that  $V(z)$  either is identically zero in  $D$  or everywhere  $< 0$ . It is also clear that if  $V = 0$  then  $f^* \in \mathcal{C}_\Phi$  as required. So there remains only to prove:

**B.6 Lemma.** *The function  $V(z)$  is identically zero in  $D$ .*

*Proof.* Assume the contrary. So now

$$(i) \quad \left| \frac{df^*(z)}{dz} \right| < e^{U_*(z)} \quad \text{for all } z \in D.$$

Let  $F(w)$  be the inverse of  $f$  so that :

$$F'(f^*(z)) \cdot \frac{df^*(z)}{dz} = 1 \quad z \in D$$

Then (i) gives:

$$(ii) \quad |F'(w)| > e^{-U_*(w)} \quad w \in \Omega^*$$

Next, let  $V(w)$  be the harmonic conjugate of  $U_*(w)$  normalised so that  $V(0) = 0$  and set

$$H(w) = \int_0^w e^{-U_*(\zeta) + iV(\zeta)} \cdot d\zeta.$$

Then (ii) gives

$$(iii) \quad |H'(w)| < |F'(w)| \quad w \in \Omega^*$$

Since  $H(w(0)) = F(w(0)) = 0$  it follows that

$$(iv) \quad \inf_{w \in \Omega^*} \frac{|H(w)|}{|F(w)|} = r_0 < 1$$

Since  $|F(w)| < 1$  in  $\Omega^*$  while  $|F(w)| \rightarrow 1$  as  $w$  approaches  $\partial\Omega^*$  we see that (iv) entails the domain

$$(v) \quad R_0 = \{w \in \Omega^* : |H(w)| < r_0\}$$

has at least one boundary point  $w_*$  which also belongs to  $\partial\Omega^*$ . Next, the function  $H(w)$  is analytic in  $R_0$  and its derivative is everywhere  $\neq 0$  while  $|H(w)| = 1$  on  $\partial R_0$ . It follows that  $H$  gives a conformal map from  $R_0$  onto the disc  $|z| < r_0$ . Let  $h(z)$  be the inverse of this conformal mapping. Now we get the analytic function in  $D$  defined by

$$g(z) = h(r_0 z)$$

Next, let  $|z| = 1$  and put  $w = h(r_0 z)$ . Then

$$(v) \quad |g'(z)| = r_0 \cdot |h'(r_0 z)| = r_0 \cdot \frac{1}{|H'(g(z))|} = r_0 \cdot e^{U_*(g(z))} = r_0 \cdot \Psi(g(z)) < \Psi(g(z))$$

At the same time we have a common boundary point

$$w_* \in \partial\Omega^* \cap \partial g(D)$$

Since the  $g$ -function extends to a continuous function on  $|z| \leq 1$  there exists a point  $e^{i\theta}$  such that

$$g(e^{i\theta}) = w_*$$

Now we use that  $r_0 < 1$  above. The continuity of the  $\Psi$  gives some  $\epsilon > 0$  such that for any complex number  $a$  which belongs to the disc  $|a - 1| < \epsilon$ , it follows that the function

$$z \mapsto a \cdot g(z)$$

belongs to  $\mathcal{A}_\Psi$ . Finally, by Proposition B.4 the maximal region for the  $\Psi$ -function is equal to  $\Omega^*$  and we conclude that

$$ag(e^{i\theta}) = aw_* \in \Omega^* \quad |a - 1| < \epsilon$$

This would mean that  $w_*$  is an *interior point* of  $\Omega^*$  which contradicts that  $w_* \in \partial\Omega^*$ . Hence Lemma B.6 is proved.

### C. Proof of Theorem 3.

First we have a companion to Lemma B.2. Namely, let  $g_1, \dots, g_n$  be a finite set in  $\mathcal{B}_\Phi$ . Set  $S_\nu = g_\nu(D)$ . Following [Beur: page 123] we give

**C.1 Definition.** *The reduced intersection of the family  $\{S_\nu\}$  is defined as the set of these points  $w$  which can be joined with the origin by a Jordan arc  $\gamma$  contained in the intersection  $\cap S_\nu$ . The resulting domain is denoted by  $RI\{S_\nu\}$ .*

**C.2 Proposition.** *The domain  $RI\{S_\nu\}$  is simply connected and if  $g \in \mathcal{C}$  is the normalised conformal mapping onto this domain, then  $g \in \mathcal{B}_\Phi$ .*

The proof of this result can be carried out in a similar way as in the proof of Lemma B.2 so we leave out the details. Next, starting from a dense sequence  $\{g_\nu\}$  in  $\mathcal{B}_\Phi$  we find for each  $n$  the function  $f_n \in \mathcal{B}_\Phi$  where

$$f_n(D) = RI\{S_\nu\} \quad : \quad S_\nu = g_\nu(D) \quad : \quad 1 \leq \nu \leq n.$$

Here the simply connected domains  $\{f_n(D)\}$  decrease and there exists the limit function  $f_* \in \mathcal{C}$  where

$$f_*(D) = \Omega_*$$

There remains to prove

**C.3 Proposition.** *One has  $f_* \in \mathcal{C}_\Phi$ .*

**Remark.** Proposition C.3 requires a quite involved proof and is given in [Beur: page 127-130]. We shall not try to present all the details and just sketch the strategy in the proof. Put

$$m = \inf_{g \in \mathcal{B}_\Phi} g'(0)$$

Next,  $f_*$  belongs to  $\mathcal{B}_\Phi$  because we have the trivial inclusion  $\mathcal{C}_\Phi \subset \mathcal{B}_\Phi$  and using this Beurling proved that

$$m \geq \min_{w \in \Omega_*} \Phi(w)$$

Next, starting with Proposition C.2 above, Beurling introduces a normal family and proves that

$$f'_*(0) = m$$

Thus,  $f_*$  is a solution to an extremal problem. This is used in the final part of Beurling's proof to establish the inclusion  $f_* \in \mathcal{C}_\Phi$ . Let us remark that this part of the proof relies upon some subtle set-theoretic constructions where the family of regions of the *Schoenflies' type* are introduced in [Beur: page 121]. The whole analysis involves topological investigations of independent interest.

### Lecture IV: Eigenvalues for the Laplace operator.

Let  $\Omega$  be a connected bounded domain in the complex  $z$ -plane of class  $\mathcal{D}(C^1)$  with the Green function  $G(p, q)$ . On the Hilbert space  $L^2(\Omega)$  of square integrable functions we define the linear operator  $\mathcal{G}$  by:

$$\mathcal{G}(u)(p) = \frac{1}{2\pi} \cdot \iint_{\Omega} G(p, q) \cdot u(q) \, dx dy$$

Since  $G(p, q)$  is square integrable over  $\Omega \times \Omega$  this yields a Hilbert-Schmidt operator. Suppose that  $\lambda$  be a non-zero complex number and  $u$  is some  $L^2$ -function in  $\Omega$  satisfying the integral equation

$$(0.1) \quad u(p) + \lambda \cdot \iint_{\Omega} G(p, q)u(q) = 0 \quad : \quad p \in \Omega$$

The properties of Green's function entails that  $u$  is harmonic in  $\Omega$  and satisfies the equation

$$(0.2) \quad \Delta(u) + \lambda \cdot u = 0$$

Moreover, since  $G(p, q) = 0$  when  $p \in \partial\Omega$  and  $q \in \Omega$  it follows that  $u = 0$  on the boundary.

**Exercise.** Show conversely that if  $u$  is a  $C^2$ -function in  $\Omega$  which satisfies (0.2) and in addition extends to a continuous function which is zero on  $\partial\Omega$  then  $u$  solves the integral equation (0.1).

Next, the integral formula from § XX shows that if  $u$  satisfies (0.2) then

$$\iint_{\Omega} [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2] \, dx dy = \lambda \cdot \iint_{\Omega} u^2 \, dx dy$$

Hence non-zero  $u$ -solutions can only exist when  $\lambda$  are real and positive. The result below is a special case from the Fredholm-Hilbert theory:

**0.1 Theorem.** *There exists a non-decreasing sequence  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  of positive real numbers and to each  $\lambda_n$  one has a real-valued function  $\phi_n$  where  $\phi_n = 0$  on  $\partial\Omega$  and*

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0$$

*holds in  $\Omega$ . Here  $\{\phi_n\}$  is an orthonormal set in the Hilbert space  $L^2(\Omega)$  and Green's function satisfies the equation*

$$G(p, q) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n}$$

*where the right hand side converges when  $p \neq q$ .*

**Remark.** As usual eigenvalues are repeated when the corresponding finite dimensional eigenspace has dimension  $> 1$ .

#### 1. Asymptotic formulas.

Theorem 0.1 and Ikehara's theorem lead to asymptotic formulas. The first result is due to Weyl:

**1.1 Theorem.** *One has the limit formula*

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \frac{4\pi}{\text{area}(\Omega)}$$

Concerning the eigenfunctions the following holds:

**1.2 Theorem.** *For every  $p \in \Omega$  one has*

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi}$$

Passing to partial derivatives of the  $\phi$ -functions similar asymptotic formulas hold. The result for first order partial derivatives is:

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{1}{\lambda_n^2} \cdot \sum_{k=1}^{k=n} \left( \frac{\partial \phi_k}{\partial x}(p) \right)^2 = \frac{1}{16\pi}$$

and similarly for the partial  $y$ -derivative.

**Remark.** The proofs below are taken from Carleman's lecture at the Scandinavian Congress in Copenhagen 1934. The strategy is to employ Green resolvents. For every complex number  $\lambda$  outside the discrete set  $\{\lambda_n\}$  we find the function which for each fixed  $q \in \Lambda$  satisfies

$$\Delta G(p, q; \lambda) + \lambda \cdot G(p, q; \lambda) = 0 \quad : \quad p \in \Omega \setminus \{q\}$$

and at  $p = q$  it has the same singularity as  $G(p, q)$ , i.e.  $\log \frac{1}{|p-q|}$ . Finally,  $G(p, q; \lambda) = 0$  when  $p \in \partial\Omega$ . Theorem 0.1 gives the equation:

**1.3 Theorem** *One has the equality*

$$G(p, q; \lambda) - G(p, q) = \frac{1}{2\pi\lambda} \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda - \lambda_n)}$$

Next we introduce a function which will be used to prove the asymptotic formulas.

**1.4 The function  $\Phi(p, s)$ .** For each  $p \in \Omega$  we define a function of the complex variable  $s$ :

$$\Phi(p, s) = \sum_{n=1}^{\infty} \frac{\phi_n^2(p)}{\lambda_n^s}$$

This is an analytic function of the complex variable  $s$  defined in some half-plane  $\Re s > K$  provided that  $K$  is sufficiently large. It turns out that  $\Phi(p, s)$  has a meromorphic extension to the whole complex  $s$ -plane.

**1.5 Theorem** *The function  $\Phi(p, s)$  extends to a meromorphic in the whole complex  $s$ -plane with a simple pole at  $s = 1$  whose residue is  $\frac{1}{4\pi}$  and zeros at  $0, -1, -2, \dots$*

Theorem 1.5 is proved in § 2 and in § 3 we show how to derive the asymptotic formulas using Ikehara's theorem applied to the  $\Phi$ -function.

## 2. Proof of Theorem 1.5.

Consider the smallest eigenvalue  $\lambda_1$  and with  $0 < a < \lambda_1$  we first establish some integral formulas.

**2.1 Lemma** *For every pair  $0 < a < b$  of real numbers one has*

$$b^{-s} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\lambda}{b(\lambda - b)\lambda^s} \cdot d\lambda$$

*Proof.* Set  $\lambda = bz$  and  $\alpha = \frac{a}{b}$  so that  $0 < \alpha < 1$  and the right hand side becomes

$$b^{-s} \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{z}{(1-z)z^s} \cdot dz$$

When  $\Re z > 1$  the reader can verify that the integral in the right hand side is 1 and deduce Lemma 2.1. The next result is left as an exercise.

**2.2 Lemma** *For every pair  $0 < a < b$  of real numbers and  $\Re s > 1$  the integral in the right hand side of Lemma 2.1 is equal to*

$$\frac{a^{s-1}}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i(1-s)\theta}}{b(b - a^{i\theta})} d\theta + \frac{\sin \pi s}{\pi} \cdot \int_a^{\infty} \frac{1}{b(b + \lambda)\lambda^s} d\lambda$$

### 2.3 The function $F(p, \lambda)$

Since  $G(p, q; \lambda)$  and  $G(p, q)$  have the same singularity  $\log \frac{1}{|p-q|}$  along the diagonal it follows that

$$(i) \quad G(p, p; \lambda) - G(p, p) = \frac{1}{2\pi\lambda} \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)}$$

when  $p \in \Omega$  where the right hand side is a meromorphic function of  $\lambda$  with poles confined to the set  $\{\lambda_n\}$ . Set

$$F(p, \lambda) = G(p, p; \lambda) - G(p, p)$$

Keeping  $p \in \Omega$  fixed we apply Lemma 2.1 and 2.2 with  $b = \lambda_n$  for every  $n \geq 1$ . Then a summation over  $n$  gives

**Lemma 2.4** *One has the equality*

$$(*) \quad \Phi(p, s) = \frac{a^{s-1}}{4\pi^2} \int_{-\pi}^{\pi} e^{i(1-s)\theta} \cdot F(p, ae^{i\theta}) d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} \frac{F(p, -\lambda)}{\lambda^s} d\lambda$$

The first term in (\*) is an entire function of  $s$  since  $0 < a < \lambda_1$  is a fixed real number and  $F(p, \lambda)$  is analytic in the open disc of radius  $|\lambda_1|$  centered at the origin. So  $\Phi(p, s)$  extends to a meromorphic function with a simple pole at  $s = 1$  if the same is true for the function

$$(2.5) \quad F_*(p, s) = \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} \frac{F(p, -\lambda)}{\lambda^s} d\lambda$$

where we in addition should verify that the residue at  $s = 1$  is  $\frac{1}{4\pi}$ . To attain this we introduce another family of functions.

### 2.6 The functions $H(p, q; \kappa)$ .

Define the analytic function  $K(z)$  in the half-plane  $\Re z > 0$  by

$$K(z) = \int_1^{\infty} \frac{e^{-zt}}{\sqrt{t^2 - 1}} dt$$

With  $\kappa$  kept fixed we get the function

$$K_{\kappa}(p, q) = K(\kappa|p - q|)$$

where  $|p - q|$  is the distance between a pair of points  $p, q$  in  $\Omega$ . For each  $\lambda > 0$  we set  $\kappa = \sqrt{\lambda}$ . By §-XX there exist unique functions  $\{H(p, q; \kappa)\}$  which for each fixed  $q \in \Omega$  gives:

$$(i) \quad \Delta H(p, q; \kappa) - \kappa^2 \cdot H(p, q; \kappa) = 0 : p \in \Omega \quad \text{and} \quad H(p, q; \kappa) = K_{\kappa}(p, q) : p \in \partial\Omega$$

From (i) the reader may verify the equality:

$$(ii) \quad G(p, q; -\lambda) = K_{\kappa}(p, q) - H(p, q; \kappa) \quad \text{hold for each} \quad \lambda > 0$$

Next we perform a limit as  $q \rightarrow p$ . The construction of the  $K$ -function gives:

$$(iii) \quad \lim_{q \rightarrow p} K_{\kappa}(p, q) - \log \frac{1}{|p - q|} = -\log \kappa + \log 2 - \log \gamma$$

Using the function  $H_*(p)$  from § 0.4 to get rid of  $\log \frac{1}{|p-q|}$  we therefore get the equation:

$$(iv) \quad F(p, -\lambda) = -\log \kappa + \log 2 - \log \gamma + H_*(p) - H(p, p; \kappa)$$

Above the last term depends upon  $\lambda$  via the equality  $\kappa = \sqrt{\lambda}$ . We are concerned with the integral in the right hand side of (2.5) and first we study the behaviour of

$$(v) \quad s \mapsto \int_a^{\infty} \frac{H(p, p; \sqrt{\lambda})}{\lambda^s} d\lambda$$

It turns out that this yields an entire function of  $s$ . To see this we use the PDE-equation satisfied by  $H$  in (i) and establish some estimates. For each  $p \in \Omega$  we set  $\ell(p) = \text{dist}(p, \partial\Omega)$ .

**2.7 Lemma** *There exists a constant  $A$  such that the inequalities below hold for each  $p \in \Omega$  and every  $\kappa > 0$ :*

$$0 \leq H(p, p; \kappa) \leq K(\kappa \cdot \ell(p)) \quad \text{and} \quad H(p, p; \kappa) \leq A \cdot e^{-\alpha\kappa}$$

### 2.8 The meromorphic extension of $F_*(p, s)$

The last inequality in Lemma 2.7 yields an exponential decay and it follows that (v) yields an entire function of  $s$  which we denote by  $g_p(s)$  and now (iv) gives

$$F_*(p, s) = \frac{\sin \pi s}{2\pi^2} \cdot \left[ - \int_a^\infty \frac{\log \sqrt{\lambda}}{\lambda^s} d\lambda + (\log 2 - \gamma + H_*(p, p)) \cdot \int_a^\infty \frac{1}{\lambda^s} d\lambda + g_p(s) \right]$$

A computation gives:

$$(2.7.i) \quad - \int_a^\infty \frac{\log \sqrt{\lambda}}{\lambda^s} d\lambda = -\frac{1}{2} \cdot \frac{a^{s-1} \log a}{s-1} + \frac{1}{2} \cdot \frac{a^{s-1}}{(s-1)^2}$$

At the same time, with the constant  $A = \log 2 - \gamma + H_*(p, p)$  the function

$$s \mapsto A \cdot \int_a^\infty \frac{1}{\lambda^s} d\lambda$$

has a simple pole at  $s = 1$  which is compensated by the zero of the sine-function  $\frac{\sin \pi s}{2\pi^2}$  and the reader can check that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot A \cdot \int_a^\infty \frac{1}{\lambda^s} d\lambda$$

is an entire function of  $s$ . In (2.7.i) we have a double pole at  $s = 1$  which after multiplication with the sine-function gives a simple pole and the reader can verify that the residue is  $\frac{1}{4\pi}$  which finishes the proof of Theorem 1.5.

### 3. Proofs of the asymptotic formulas.

Theorem 1.5 and Ikehara's theorem from § xX entail that

$$(3.1) \quad \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi}$$

Hence Theorem 1.2 is proved and to get Theorem 1.3 we perform an integration over  $\Omega$  so that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \iint_{\Omega} \Phi(p, s) dx dy$$

where we simply have used that each  $\phi$ -function has a squared integral equal to one over  $\Omega$ .

**3.1 Exercise.** Use the equations from the proof of theorem 1.5 to show that after an integration over  $\Omega$  one has

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \frac{\text{Area}(\Omega)}{4\pi} \cdot \frac{1}{s-1} + J(s)$$

where  $J(s)$  is analytic in the closed half-plane  $\Re s \geq 1$  with continuous boundary values on  $\Re s = 1$ . Finally, conclude from this that Ikehara's Theorem gives Theorem 1.2.

**3.2. The limit formula (1.3)** The proof of (1.3) uses similar methods as above but this time some extra technicalities appear where estimates for partial derivatives of the Green's function are needed. The reader may consult [Carleman: page 38-40] for the details which give (1-4) or try to carry out the proof. Actually one has a general limit formula for higher order mixed partial derivatives of the  $\phi$ -functions. More precisely, the following is proved in [ibid]:

**3.3. Theorem.** *For every pair of non-negative integers  $j, m$  and each  $p \in \Omega$  one has the limit formula*

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^{j+m+1}} \cdot \sum_{k=1}^{k=n} \left( \frac{\partial^{j+m} \phi_k}{\partial x^j \partial y^j} \right)^2(p) = \frac{1}{\pi \cdot 2^{2m+2j+2}} \cdot \frac{(2m)! \cdot (2j)!}{m! \cdot j! \cdot (m+j+1)!}$$

#### 4. Vibrating planes.

Let  $D$  be a membrane with constant density of mass  $m$  and tension  $k > 0$ . The boundary is fixed by a plane curve  $C$  placed in the horizontal  $(x, y)$ -plane and the function  $u = u(x, y, t)$  is the deviation in the vertical direction while the membrane is in motion. Here  $t$  is a time variable and by Hooke's law the  $y$ -function satisfies the wave equation

$$(*) \quad \frac{d^2 u}{dt^2} = \frac{k}{m} \cdot \Delta u$$

where the boundary condition is that  $u(p, t) = 0$  for each  $p \in C$ . The time dependent kinetic energy becomes

$$T(t) = \frac{m}{2} \iint_{\Omega} \left( \frac{du}{dt} \right)^2 dx dy$$

The potential energy becomes

$$V(t) = \frac{k}{2} \iint_{\Omega} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dx dy$$

The general solution to (\*) is

$$(**) \quad u(p, t) = 2 \cdot \sum_{\nu=1}^{\infty} c_{\nu} \cos(\kappa_{\nu} t) \phi_{\nu}(p)$$

where  $\{c_{\nu}\}$  is a sequence of real numbers. Define the mean kinetic energy at individual points  $p \in D$  by

$$L(p) = \frac{m}{2} \cdot \lim_{\tau \rightarrow \infty} \int_0^{\tau} \left( \frac{du}{dt} \right)^2(p) \cdot d\tau$$

**Exercise.** Show that (\*\*) entails that

$$L(p) = k \cdot \sum |c_{\nu}|^2 \lambda_{\nu} \phi_{\nu}(p)^2$$

**Remark.** The  $c$ -numbers decay in a physically realistic solution so that the series above converges. For each positive number  $w$  we can consider the contribution from high frequencies and set

$$L_w(p) = k \cdot \sum_{\lambda_{\nu} > w} |c_{\nu}|^2 \lambda_{\nu} \cdot \phi_{\nu}(p)^2$$

Next, the mean kinetic energy at a point  $p \in D$  is defined by:

$$W(p) = \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \int_0^{\tau}$$

A computation gives

$$V(p) = k \cdot \sum |c_{\nu}|^2 \cdot \left[ \frac{\partial \phi_{\nu}}{\partial x}(p)^2 + \frac{\partial \phi_{\nu}}{\partial y}(p)^2 \right]$$

The mean potential energy which comes from high frequencies is defined by

$$V_w(p) = k \cdot \sum_{\lambda\nu > w} |c_\nu|^2 \cdot \left[ \frac{\partial \phi_\nu}{\partial x}(p)^2 + \frac{\partial \phi_\nu}{\partial y}(p)^2 \right]$$

When the regularity of  $\partial\Omega$  is relaxed, for example if  $\partial\Omega$  is a union of planar parts where pairs intersect at lines and "ugly corner points" appear when more than two planar parts meet, then the kernel function  $K_h$  is unbounded and may even fail to be square integrable, i.e. it can occur that

$$\iint_{\partial\Omega \times \partial\Omega} |K(p, q)|^2 d\sigma(p) d\sigma(q) = +\infty$$

In this situation the analysis becomes more involved and leads to the spectral theory of unbounded linear operators. One can also go further and allow  $u$ -solutions to the equation  $u = \lambda \cdot \mathcal{K}_h(u)$  which are measurable functions. In other words, the domain of definition for the integral operator  $\mathcal{K}_h$  is extended. Then it turns out that the spectrum of  $\mathcal{K}_h$  may contain non-discrete parts outside the real line. We treat this case for planar domains in § XX where a specific case occurs if  $\Omega$  is a bounded open subset of  $\mathbf{R}^2$  bordered by a finite family of disjoint piecewise linear Jordan curves, i.e. by polygons. When  $h$  is a positive function on  $\partial\Omega$  the planar kernel is given by

$$K_h(p, q) = \frac{1}{\pi} \cdot \frac{\langle p - q, \mathbf{n}_*(q) \rangle}{|p - q|^2}$$

Let  $\{\alpha_\nu\}$  be the family of interior angles at the corner points from the union of the polygons above. So here  $0 < \alpha_\nu < \pi$  for each  $\nu$  and put:

$$R = \min_{\nu} \frac{\pi}{\pi - \alpha_\nu}$$

In his thesis *Über das Neumann-Poincarésche Problem für ein gebiet mit Ecken* from 1916, Carleman proved that  $\mathcal{K}_h(\lambda)$  extends to a meromorphic function in the open disc  $|\lambda| < R$  where a finite set of real and simple poles can occur. But in contrast to the smooth case the continuation beyond this disc is in general quite complicated. More precisely, when the domain of  $\mathcal{K}_h$  is extended to measurable functions  $u$  with finite logarithmic energy:

$$\iint_{\partial\Omega \times \partial\Omega} \left| \log \frac{1}{|p - q|} \right| \cdot |u(p)| \cdot |u(q)| d\sigma(p) d\sigma(q) < \infty$$

there appears in general a non-real spectrum outside the disc of radius  $R$  which need not consist of discrete points. We remark that Carleman's study of the Neumann-Poincaré operators for non-smooth domains led to the theory about unbounded self-adjoint operators on Hilbert spaces. Carleman's book *Sur les équations singuliers à noyau réel et symétrique* from 1923 proves the spectral theorem for unbounded operators and constitutes one of his major contributions in mathematics.

## Lecture V. Elliptic operators and their fundamental solutions

### Contents.

*Introduction.*

*0.A Elliptic operators*

*0.B Boundary value problems*

*1. The construction of  $\Phi(x, \xi; \kappa)$*

*2. Green's functions*

*3. The almost reality of eigenvalues.*

### Introduction.

The material below is taken from Carleman's lectures at Institute Mittag Leffler in 1935. Points in  $\mathbf{R}^3$  are denoted by  $x = (x_1, x_2, x_3)$  and we shall consider second order differential operators:

$$P(x, \partial_x) = \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} a_{pq}(x) \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=01}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + b(x)$$

The functions  $\{a_{pq}\}$  are of class  $C^2$  while  $\{a_p\}$  are of class  $C^1$  and  $b$  is continuous. In addition the functions are real-valued and the leading coefficients satisfy  $a_{pq} = a_{qp}$  which for every  $x$  gives the symmetric  $3 \times 3$ -matrix  $A(x) = \{a_{pq}(x)\}$ .

**0.A Elliptic operators.** If the matrix  $A(x)$  is positive for every  $x$ , i.e. if the eigenvalues are positive, one says that  $P$  is an elliptic differential operator. From now on we restrict the attention to such PDE-operators and to ensure convergence of volume integrals taken over the whole of  $\mathbf{R}^3$  we add the conditions that

$$\lim_{|x| \rightarrow \infty} a_{pp}(x) = 1 \quad : \quad 1 \leq p \leq 3$$

while  $\{a_{pq}\}$  for  $p \neq q$  and  $a_1, a_2, a_3, b$  tend to zero as  $|x| \rightarrow +\infty$ . This means that  $P$  approaches the Laplace operator when  $|x|$  is large. Recall the notion of a fundamental solution which classically is called *Eine Grundlösung*. First the regularity of the coefficients of a PDE-operator  $P$  enable us to construct the adjoint operator:

$$P^*(x, \partial_x) = P - 2 \cdot \left( \sum_{p=1}^{p=3} \left( \sum_{q=1}^{q=3} \frac{\partial a_{pq}}{\partial x_q} \right) \cdot \frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3} \frac{\partial a_p}{\partial x_p} \right) + 2 \cdot \sum \sum \frac{\partial^2 a_{pq}}{\partial x_p \partial x_q}$$

Partial integration gives the equation below for every pair of  $C^2$ -functions  $\phi, \psi$  in  $\mathbf{R}^3$  with compact support:

$$\int P(\phi) \cdot \psi \, dx = \int \phi \cdot P^*(\psi) \, dx$$

where the volume integrals are taken over  $\mathbf{R}^3$ . A locally integrable function  $\Phi(x)$  in  $\mathbf{R}^3$  is a fundamental solution to  $P(x, \partial_x)$  if

$$\psi(0) = \int \Phi \cdot P^*(\psi) \, dx$$

hold for every  $C^2$ -function  $\psi$  with compact support. Next, to each positive number  $\kappa$  we get the PDE-operator  $P - \kappa^2$  and a function  $\Phi(x; \kappa)$  is a fundamental solution to  $P - \kappa^2$  if

$$(1) \quad \psi(0) = \int \Phi(x; \kappa) \cdot (P^* - \kappa^2)(\psi(x)) \, dx$$

hold for compactly supported  $C^2$ -functions  $\psi$ . Above  $\kappa$  appears as an index of  $\Phi$ , i.e. for each fixed  $\kappa$  we have the locally integrable function  $x \mapsto \Phi(x; \kappa)$ . Next, the origin can be replaced by a variable point  $\xi$  in  $\mathbf{R}^3$  and then one seeks a function  $\Phi^*(x, \xi; \kappa)$  with the property that

$$(*) \quad \psi(\xi) = \int \Phi(x, \xi; \kappa) \cdot (P^* - \kappa^2)(\psi(x)) dx$$

hold for all  $\xi \in \mathbf{R}^3$  and every  $C^2$ -function  $\psi$  with compact support. Keeping  $\kappa$  fixed this means that  $\Phi(x, \xi; \kappa)$  is a function of six variables defined in  $\mathbf{R}^3 \times \mathbf{R}^3$ . Fundamental solutions are in general not unique. But in § 1 we give a canonical construction of fundamental solutions  $\Phi(x, \xi; \kappa)$  for all sufficiently large  $\kappa$ .

**0.B Boundary value problems.** The operator  $P$  can be restricted to bounded open domains  $\Omega$  where boundary value problems appear. A basic problem is to seek pairs  $(u, \lambda)$  where  $u(x)$  is a complex-valued function which satisfies the PDE-equation

$$(*) \quad P(x, \partial_x)u(x) + \lambda \cdot u(x) = 0$$

in  $\Omega$  and  $u = 0$  on  $\partial\Omega$ . Set

$$\Delta(x) = \det(\{a_{pq}(x)\})$$

It turns out that the set of eigenvalues is a discrete subset of  $\mathbf{C}$ . Admitting this for a while we let  $\{\lambda_n\}$  be the eigenvalues to (\*) arranged so that their absolute values are non-decreasing and a single eigenvalue is repeated according to the dimension of the corresponding space of eigenfunctions. Then the following asymptotic formula holds:

**0.B.1 Theorem.** *For every elliptic operator  $P$  as above and each bounded domain  $\Omega$  one has the limit formula*

$$(*) \quad \lim_{n \rightarrow \infty} \frac{|\lambda_n|}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_{\Omega} \frac{dx}{\sqrt{\Delta(x)}}$$

**0.B.2 Remark.** The formula above is due to Courant and Weyl when  $P$  is symmetric. The extension to non-symmetric operators was achieved in the article *Über die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen* which actually is a resumé of Carleman's lectures at Institute Mittag-Leffler in 1935. The proof of Theorem 0.B.1 relies upon the construction of fundamental solutions in § 1. After this has been achieved, the asymptotic formula (\*) is derived via Tauberian theorems for Dirichlet series where a crucial result goes as follows: Let  $\{a_\nu\}$  and  $\{\lambda_\nu\}$  be two sequences of positive numbers where  $\lambda_\nu \rightarrow +\infty$  and the series

$$f(x) = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\lambda_\nu + x}$$

converges when  $x > x_*$  for some positive number  $x_*$ . Next, for every  $x > 0$  we define the function

$$\mathcal{A}(x) = \sum_{\{\lambda_\nu < x\}} a_\nu$$

In other words, with  $x > 0$  we find the largest integer  $\nu(x)$  such that  $\lambda_{\nu(x)} < x$  and then  $\mathcal{A}(x)$  is the sum over the  $a$ -numbers up to this index.

**0.B.3 Tauberian theorem.** *Suppose there exists a constant  $A > 0$  and some  $0 < \alpha < 1$  such that*

$$(*) \quad \lim_{x \rightarrow \infty} x^\alpha \cdot f(x) = A$$

*Then it follows that*

$$\lim_{x \rightarrow \infty} \mathcal{A}(x) = \frac{A}{\pi} \cdot \frac{\sin \pi \alpha}{1 - \alpha} \cdot x^{1-\alpha}$$

### 1. The construction of $\Phi(x, \xi; \kappa)$ .

1.1 *The case when  $P$  has constant coefficients.* Here the fundamental solution was already given by Newton. Of course, he did not use the today's vocabulary but the formula below can be read off from *Principia*. We have the positive and symmetric  $3 \times 3$ -matrix  $A = \{a_{pq}\}$ . Let  $\{b_{pq}\}$  be the elements of the inverse matrix and recall that they are found via Cramér's rule:

$$b_{pq} = \frac{A_{pq}}{\Delta}$$

where  $\Delta = \det(A)$  and  $\{A_{pq}\}$  are the cofactor minors of the  $A$ -matrix. Put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq} a_p a_q - b}$$

with  $\kappa$  so large that the term under the square-root is  $> 0$ . Consider the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p a_q x_q$$

With these notations Newton's fundamental solution at  $x = 0$  becomes

$$(*) \quad H(x; \kappa) = \frac{1}{\sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha \sqrt{B(x)} - \frac{1}{2} \sum_{p,q} b_{pq} a_p \cdot x_q}$$

**Exercise.** Verify by Stokes formula that  $H(x; \kappa)$  indeed yields a fundamental solution to the PDE-operator  $P(\partial_x) - \kappa^2$ .

#### 1.2 *The case with variable coefficients.*

Choose  $\kappa_0 > 0$  such that

$$\kappa_0^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi) > 0 \quad \text{hold for all } \xi \in \mathbf{R}^3$$

For every  $\kappa \geq \kappa_0$  we set

$$(i) \quad \alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi)}$$

$$(ii) \quad H(x, \xi; \kappa) = \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi) \sqrt{B(x)} - \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) \cdot x_q}$$

With  $\xi$  kept fixed  $x \mapsto H(x, \xi; \kappa)$  is real analytic outside the origin and the singularity at  $x = 0$  is of Newton's type. In particular  $x \rightarrow H(x, \xi; \kappa)$  is locally integrable as a function of  $x$  in a neighborhood of the origin. For every fixed  $\xi$  we define the differential operator in the  $x$ -space:

$$L_*(x, \partial_x, \xi; \kappa) = \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi))) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi)) \frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

Keeping  $\xi$  fixed we apply  $L_*$  to the function  $x \mapsto H(x - \xi, \xi; \kappa)$  and put:

$$(iii) \quad F(x, \xi; \kappa) = \frac{1}{4\pi} \cdot L_*(x, \partial_x, \xi; \kappa)(H(x - \xi, \xi, \kappa))$$

**1.3 Two estimates.** The hypothesis that  $\{a_{pq}(x)\}$  are of class  $C^2$  and  $\{a_p(x)\}$  of class  $C^1$ , together with the limit conditions (\*) in § XX give the existence of positive constants  $C, C_1$  and  $k$  such that the following hold when  $\kappa \geq \kappa_0$ :

$$(1.3.1) \quad |H(x - \xi, \xi; \kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x - \xi|} \quad : \quad |F(x, \xi; \kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|^2}}{|x - \xi|^2}$$

**Exercise.** Verify the two inequalities above.

**1.4 An integral equation.** We seek  $\Phi(x, \xi; \kappa)$  which solves the equation:

$$(1) \quad \Phi(x, \xi; \kappa) = \iiint F(x, y; \kappa) \cdot \Phi(y, \xi; \kappa) dy + F(x, \xi; \kappa)$$

where the integral is taken over  $\mathbf{R}^3$ . To solve (1) we construct the Neumann series of  $F$ . Thus, starting with  $F^{(1)} = F$  we set

$$(1.4.1) \quad F^{(n)}(x, \xi; \kappa) = \int_{\mathbf{R}^3} F(x, y; \kappa) \cdot F^{(n-1)}(y, \xi; \kappa) dy \quad : \quad n \geq 2$$

Then (1.3.1) gives the inequality

$$(i) \quad |F^{(2)}(x, \xi; \kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|^2}}{|x-y|^2 \cdot |\xi-y|^2} \cdot dy$$

To estimate (i) we notice that the triple integral after the substitution  $y - \xi \rightarrow u$  becomes

$$(ii) \quad C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2 \cdot |u|^2} \cdot du$$

In (ii) the volume integral is integrated in polar coordinates and becomes

$$(iii) \quad C_1^2 \cdot \int_0^\infty \int_{S^2} \frac{e^{-k\kappa r^2}}{|x-r \cdot w-\xi|^2} \cdot dw dr$$

where  $S^2$  is the unit sphere and  $dw$  the area measure on  $S^2$ . Hence (iii) is equal to

$$(iv) \quad 2\pi C_1^2 \cdot \int_0^\infty \int_0^\pi \frac{e^{-k\kappa r^2}}{(x-\xi)^2 + r^2 - 2r \cdot |x-\xi| \cdot \sin \theta} \cdot d\theta dr =$$

$$\frac{2\pi C_1^2}{|x-\xi|} \cdot \int_0^\infty e^{-k\kappa|x-\xi|t} \cdot \log \left| \frac{1+t}{1-t} \right| \cdot \frac{dt}{t}$$

where the last equality follows by a straightforward computation.

**1.5 Exercise.** Show that (iv) gives the estimate

$$|F^{(2)}(x, \xi; \kappa)| \leq \frac{2\pi C_1^2 \cdot C_1^*}{\kappa \cdot |x-\xi|^2}$$

where  $C_1^*$  is a fixed positive constant which is independent of  $x$  and  $\xi$  and use an induction over  $n$  to get:

$$(*) \quad |F^{(n)}(x, \xi; \kappa)| \leq \frac{C_1}{|x-\xi|^2} \cdot \left[ \frac{2\pi C_1^2 \cdot C_1^*}{\kappa} \right]^{n-1} \quad n \geq 2$$

**1.6 Conclusion.** With  $\kappa$  so large that  $2\pi \cdot C_1^2 \cdot C_1^* < \kappa$  it follows from (\*) that the series

$$\sum_{n=1}^{\infty} F^{(n)}(x, \xi; \kappa)$$

converges when  $x \neq \xi$  and gives the requested solution  $\Phi(x, \xi; \kappa)$ . Moreover,  $\Phi(x, \xi; \kappa)$  satisfies a similar estimate as in (1.3.1) above with another constant than  $C_2$  instead of  $C_1$ . Using Green's formula one easily verifies that  $\Phi(x, \xi; \kappa)$  gives a fundamental solution of the PDE-operator  $P(x, \partial_x) - \kappa^2$  with a pole at  $\xi$ . The constructions above show that the functions

$$x \mapsto \Phi(x, \xi; \kappa) \quad \text{and} \quad x \mapsto H(x - \xi, \xi; \kappa)$$

have the same singularities at  $x = \xi$ . Consider the difference

$$(1.6.1) \quad \Psi(x, \xi; \kappa) = \Phi(x, \xi; \kappa) - H(x - \xi, \xi; \kappa)$$

**1.6.2 Exercise.** Use the previous constructions to show that for every  $0 < \gamma \leq 2$  there is a constant  $C_\gamma$  such that

$$|\Psi(x, \xi; \kappa)| \leq \frac{C_\gamma}{(\kappa|x - \xi|)^\gamma}$$

hold for every pair  $(x, \xi)$  and every  $\kappa \geq \kappa_0$ . Together with the estimate (1.3.1) for the  $H$ -function this gives an estimate for the fundamental solution  $\Phi$ .

## 2. Green's functions.

Let  $\Omega$  be a bounded domain in  $\mathbf{R}^3$ . A Green's function  $G(x, y; \kappa)$  attached to  $\Omega$  and the PDE-operator  $P(x, \partial_x; \kappa)$  is a function which for fixed  $\kappa$  is defined in  $\Omega \times \Omega$  and has the following properties:

$$(*) \quad G(x, y; \kappa) = 0 \quad \text{when } x \in \partial\Omega \quad \text{and } y \in \Omega$$

$$(**) \quad \psi(y) = \int_{\Omega} (P^*(x, \partial_x) - \kappa^2)(\psi(x)) \cdot G(x, y; \kappa) dx \quad : \quad y \in \Omega$$

hold for all  $C^2$ -functions  $\psi$  with compact support in  $\Omega$ . To find  $G$  we solve Dirichlet problems. With  $\xi \in \Omega$  kept fixed one has the continuous function on  $\partial\Omega$ :

$$x \mapsto \Phi^*(x, \xi; \kappa)$$

Solving Dirichlet's problem gives a unique  $C^2$ -function  $w(x)$  which satisfies:

$$P(x, \partial_x)(w) + \kappa^2 \cdot w = 0 \quad \text{holds in } \Omega \quad \text{and} \quad w(x) = \Phi(x, \xi; \kappa) \quad : \quad x \in \partial\Omega = 0$$

From the above it is clear that this gives the requested  $G$ -function, i.e. one has:

**2.1 Proposition.** *The the function*

$$G(x, \xi; \kappa) = \Phi(x, \xi; \kappa) - w(x) \quad \text{satisfies} \quad (* - **)$$

Using the estimates for the  $\phi$ -function from § 1 we get estimates for the  $G$ -function above. Start with a sufficiently large  $\kappa_0$  so that  $\Phi^*(x, \xi; \kappa_0)$  is a positive function of  $(x, \xi)$ . Then the following hold:

**2.2 Theorem.** *One has*

$$G(x, \xi; \kappa_0) = \frac{1}{\sqrt{\Delta(x)} \cdot \sqrt{\Phi(x, \xi; \kappa_0)}} + R(x, \xi)$$

where the remainder function satisfies the following for all pairs  $(x, \xi)$  in  $\Omega$ :

$$|R(x, \xi)| \leq C \cdot |x - \xi|^{-\frac{1}{4}}$$

and the constant  $C$  only depends on the domain  $\Omega$  and the PDE-operator  $P$ .

**Remark.** Above the negative power of  $|x - \xi|$  is a fourth-root which means that the remainder term  $R$  is more regular compared to the first term which behaves like  $|x - \xi|^{-1}$  on the diagonal  $x = \xi$ .

**2.3 Exercise.** Prove Theorem 2.3 If necessary, consult [Carleman: page xx-xx9 for details.

**2.4 The integral operator  $\mathcal{J}$ .** Consider the integral operator which sends a function  $u$  in  $\Omega$  to

$$\mathcal{J}_u(x) = \int_{\Omega} G(x, \xi; \kappa_0) \cdot u(\xi) d\xi$$

The construction of the Green's function gives:

$$(2.4.1) \quad (P - \kappa_0^2)(\mathcal{J}_u)(x) = u(x) \quad : \quad x \in \Omega$$

In other words, if  $E$  denotes the identity we have the operator equality

$$(2.4.2) \quad P(x, \partial_x) \circ \mathcal{J} = \kappa_0^2 \cdot \mathcal{J} + E$$

Consider pairs  $(u, \gamma)$  such that

$$(2.4.3) \quad u(x) + \gamma \cdot \mathcal{J}_u(x) = 0 \quad : \quad x \in \Omega$$

The vanishing from (\*) in § 2 for the  $G$ -function implies that  $\mathcal{J}_u(x) = 0$  on  $\partial\Omega$ . Hence every  $u$ -function which satisfies in (2.4.3) for some constant  $\gamma$  vanishes on  $\partial\Omega$ . Next, applying  $P$  to (2.4.3) the operator formula (2.4.2) gives

$$0 = P(u) + \gamma \kappa_0^2 \cdot \mathcal{J}_u + \gamma \cdot u \implies P(u) + (\gamma - \kappa_0^2)u = 0$$

**2.4.4 Conclusion.** The boundary value problem (\*) from 0.B is equivalent to find eigenfunctions of  $\mathcal{J}$  via (2.4.3) above.

### 3. Almost reality of eigenvalues.

Consider the set of eigenvalues  $\lambda$  to (\*) in (0.B). Then we have:

**3.1 Proposition.** *There exist positive constants  $C_*$  and  $c_*$  such that every eigenvalue  $\lambda$  to (\*) in (0.B) satisfies*

$$|\Im \lambda|^2 \leq C_* (\Re \lambda) + c_*$$

*Proof.* Let  $u$  be an eigenfunction where  $P(u) + \lambda \cdot u = 0$ . Stokes theorem and the vanishing of  $u$  on  $\partial\Omega$  give:

$$\begin{aligned} 0 &= \int_{\Omega} \bar{u} \cdot (P + \lambda)(u) dx = - \int_{\Omega} \sum_{p,q} a_{pq}(x) \cdot \frac{\partial u}{\partial x_p} \frac{\partial \bar{u}}{\partial x_q} dx + \\ &\int_{\Omega} \bar{u} \cdot \left( \sum a_p(x) \frac{\partial u}{\partial x_p} \right) dx + \int_{\Omega} |u(x)|^2 \cdot b(x) dx + \lambda \cdot \int_{\Omega} |u(x)|^2 dx \end{aligned}$$

Write  $\lambda = \xi + i\eta$ . Separating real and imaginary parts we find the two equations:

$$(i) \quad \xi \int |u|^2 dx = \int \sum_{p,q} a_{p,q}(x) \frac{\partial u}{\partial x_p} \cdot \frac{\partial \bar{u}}{\partial x_q} dx + \int \left( \frac{1}{2} \cdot \sum \frac{\partial a_p}{\partial x_p} - b \right) \cdot |u|^2 dx$$

$$(ii) \quad \eta \int |u|^2 dx = \frac{1}{2i} \int \sum a_p \left( u \frac{\partial \bar{u}}{\partial x_p} - \bar{u} \frac{\partial u}{\partial x_p} \right) dx$$

Set

$$A = \int |u|^2 dx \quad : \quad B = \int |\nabla(u)|^2 dx$$

Since  $P$  is elliptic there exists a positive constant  $k$  such that

$$\sum_{p,q} a_{p,q}(x) \frac{\partial u}{\partial x_p} > k \cdot |\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants  $c_1, c_2, c_3$  such that

$$(iii) \quad A\xi > c_1 B - c_2 B \quad : \quad A|\eta| < c_3 \cdot \sqrt{AB}$$

Here (iii) implies that  $\xi > -c_2$  and the reader can also confirm that

$$(iv) \quad B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A \cdot c_2 \cdot \sqrt{\frac{\xi + c_2}{c_1}} \quad : \quad |\eta| < c_3 \cdot \sqrt{\frac{\xi + c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 3.1.

#### 4. Asymptotic formulas.

Using the results above where we have found a good control of the integral operator  $\mathcal{J}$  and the conclusion in 2.4.4 identifies eigenvalues to  $\mathcal{J}$  with those from (\*) in (0.B). Now Tauberian theorems give the asymptotic formula in Theorem 0.B.1. For details the reader may consult [Carleman: Collected work page xx-xx].

**Final Remark.** As pointed out by Carleman similar asymptotic formulas as in Theorem 0.B.1 hold for elliptic operators of arbitrary even order  $2m$  where  $m$  is a positive integer. The procedure is to start with a canonically defined fundamental solution for operators with constant coefficients where one can start with Froitz John's fundamental solution instead of Newton's. After the construction of fundamental solutions" for elliptic and positive PDE-operators with variable coefficients is carried out by similar methods as above. The restriction to the 3-dimensional case is of course not essential, i.e. proofs are verbatim the same in every dimension  $n \geq 4$ .

## VI. A Neumann problem with non-linear boundary values.

**Introduction.** We expose Carleman's article *Über eine nichtlineare Randwertaufgabe bei der Gleichung  $\Delta u = 0$*  (Mathematisches Zeitschrift vol. 9 (1921). Here is the equation to be considered: Let  $\Omega$  be a bounded domain in  $\mathbf{R}^3$  with  $C^1$ -boundary and  $\mathbf{R}^+$  the non-negative real line where  $u$  is the coordinate. Let  $F(u, p)$  be a real-valued and continuous function defined on  $\mathbf{R}^+ \times \partial\Omega$ . Assume that

$$(0,1) \quad u \mapsto F(u, p)$$

is strictly increasing for every  $p \in \partial\Omega$  and that  $F(0, p) \geq 0$ . Moreover,

$$(0,2) \quad \lim_{u \rightarrow \infty} F(u, p) = +\infty$$

holds uniformly with respect to  $p$ . For a given point  $Q_* \in \Omega$  we seek a function  $u(x)$  which is harmonic in  $\Omega \setminus \{Q_*\}$  and at  $Q_*$  it is locally  $\frac{1}{|x-Q_*|}$  plus a harmonic function and on  $\partial\Omega$  the inner normal derivative  $\partial u / \partial n$  satisfies the equation

$$(*) \quad \frac{\partial u}{\partial n}(p) = F(u(p), p) \quad : p \in \partial\Omega$$

Finally it is also assumed that  $u$  extends to a continuous function on  $\partial\Omega$ .

**Theorem.** *For each  $F$  as above the boundary value problem has a unique solution.*

**Remark.** The strategy in Carleman's proof is to consider the family of boundary value problems where we for each  $0 \leq h \leq 1$  seek  $u_h$  to satisfy

$$(*) \quad \frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p), p) \quad : p \in \partial\Omega$$

where  $u_h$  has the same pole as  $u$  above. Starting with  $h = 0$  one has a classical linear Neumann problem where a unique solution exists. To proceed from  $h = 0$  to  $h = 1$  the idea is to use a "homotopy argument" where one first easily reduces the proof to the case when  $F$  is a real-analytic function of  $u$ . Then the subsequent proof will show that if we have found a solution  $u_{h_0}$  for some  $0 \leq h_0 < 1$ , we obtain solutions  $u_h$  when  $h_0 < h < h_0 + \epsilon$  for sufficiently small  $\epsilon$  by solving an infinite system of linear boundary value problems. It goes without saying that this method is restricted to favorable cases as above where the uniqueness and robust properties of these solutions are present as we show below. But it is instructive to see how one can employ analytic series to handle such cases. Now we turn to the proof of the theorem and start with the reduction to the case when  $F$  is real-analytic.

**A.0. Proof of uniqueness.** Suppose that  $u_1$  and  $u_2$  are two solutions and notice that  $u_2 - u_1$  is harmonic in  $\Omega$ . If  $u_1 \neq u_2$  we may assume that the maximum of  $u_2 - u_1$  is  $> 0$  and attained at some  $p_* \in \partial\Omega$ . The strict maximum principle for harmonic functions gives:

$$(i) \quad u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*)$$

for all  $x \in \Omega$ . With  $v = u_2 - u_1$  we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p), p) - F(u_1(p), p)$$

Here (0.1) entails that  $\frac{\partial v}{\partial n}(p_*) > 0$  and since we have an inner normal derivative this violates (i) which proves the uniqueness.

**A.1 Monotonic properties.** Let  $F_1$  and  $F_2$  be two functions satisfying (0.1) and (0.2) where

$$F_1(u, p) \leq F_2(u, p)$$

hold for all  $(u, p) \in \mathbf{R}^+ \times \partial\Omega$ . If  $u_1$ , respectively  $u_2$  solve (\*) for  $F_1$  and  $F_2$  it follows that  $u_2(q) \leq u_1(q)$  for all  $q \in \Omega$ . To see this we set  $v = u_2 - u_1$  which is harmonic in  $\Omega$ . If  $p \in \partial\Omega$  we get

$$(i) \quad \frac{\partial v}{\partial n}(p) = F_2(u_2(p), p) - F_1(u_1(p), p) \geq 0$$

Suppose that the maximum of  $v$  is  $> 0$  and let the maximum be attained at some point  $p_*$ . Since (i) is an inner normal it follows that we must have  $0 = \frac{\partial v}{\partial n}(p)$  which would entail that

$$F_2(u_2(p_*)p_*) > F_2(u_1(p_*), p_*) \geq F_1(u_1(p_*), p_*) \implies$$

and this contradicts the strict inequality  $u_2(p_*) > u_1(p_*)$  since we have an increasing function in (0.1).

**A.2. A bound for the maximum norm.** Let  $u$  be a solution to (\*) and let  $M_u$  be the maximum norm of its restriction to  $\partial\Omega$ . Choose  $p^* \in \partial\Omega$  such that

$$(1) \quad u(p^*) = M_u$$

Let  $G$  be the Green's function which has a pole at  $Q_*$  while  $G = 0$  on  $\partial\Omega$ . Now

$$h = u - M_u - G$$

is a harmonic function in  $\Omega$ . On the boundary we have  $h \leq 0$  and  $h(p^*) = 0$ . So  $p^*$  is a maximum point for this harmonic function in the whole closed domain  $\bar{\Omega}$ . It follows that

$$\begin{aligned} \frac{\partial h}{\partial n}(p^*) &\leq 0 \implies \\ F(u(p^*), p^*) &= \frac{\partial u}{\partial n}(p^*) \leq \frac{\partial G}{\partial n}(p^*) \end{aligned}$$

Set

$$A^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

Then we have

$$(*) \quad F(M_u, p^*) \leq A^*$$

Hence the assumption (0.2) for  $F$  this gives a robust estimate for the maximum norm  $M_u$ . Next, let  $m_u$  be the minimum of  $u$  on  $\partial\Omega$  and consider the harmonic function

$$h = u - m_u - G$$

This time  $h \geq 0$  on  $\partial\Omega$  and if  $u(p_*) = m_u$  we have  $h(p_*) = 0$  so here  $p_*$  is a minimum for  $h$ . It follows that

$$\frac{\partial h}{\partial n}(p_*) \geq 0 \implies F(u(p_*), p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$A_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

$$(**) \quad F(m_u, p^*) \geq A_*$$

**Remark.** Above  $0 < A_* < A^*$  are constants which are independent of  $F$ . Hence the maximum norms of solutions  $u = u_F$  are controlled if the  $F$ -functions stay in a family where (0.2) holds uniformly.

## B. The linear equation.

Let  $f(p)$  and  $W(p)$  be a pair of continuous functions on the boundary  $\partial\Omega$  where  $W$  is positive, i.e.  $W(p) > 0$  for every boundary point. The classical Neumann theorem asserts that there exists a unique function  $U$  which is harmonic in  $\Omega$ , extends to a continuous function on the closed domain and its inner normal derivative satisfies:

$$(1) \quad \partial U / \partial n(p) = W(p) \cdot U(p) + f(p) \quad p \in \partial \Omega$$

The uniqueness is a consequence of Green's formula. For suppose that  $U_1$  and  $U_2$  are two solutions to (1) and set  $v = U_1 - U_2$ . Since  $v$  is harmonic in  $\Omega$  it follows that:

$$\iiint_{\Omega} |\nabla(v)|^2 dx dy dz + \iint_{\partial \Omega} v \cdot \partial v / \partial n \cdot dS = 0$$

Here  $\partial v / \partial n = W(p)v$  and since  $W(p) > 0$  holds on  $\partial \Omega$  we conclude that  $v$  must be identically zero. For the unique solution to (1) some estimates hold. Namely, set

$$M_U = \max_p U(p) \quad \text{and} \quad m_U = \min_p U(p)$$

Since  $U$  is harmonic in  $\Omega$  the the maximum and the minimum are taken on the boundary. If  $U(p^*) = M_U$  for some  $p^* \in \partial \Omega$  we have  $\partial U / \partial n(p^*) \leq 0$ . Set

$$W_* = \min_p W(p)$$

By assumption  $W_* > 0$  and we get

$$M_U \cdot W(p^*) + f(p^*) = \partial U / \partial n(p^*) \leq 0 \implies M_U \leq \frac{|f|_{\partial \Omega}}{W_*}$$

where  $|f|_{\partial \Omega}$  is the maximum norm of  $f$  on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial \Omega}}{W_*}$$

Hence the following inequality holds for the the maximum norm  $|U|_{\partial \Omega}$  :

$$(*) \quad |U|_{\partial \Omega} \leq \frac{|f|_{\partial \Omega}}{W_*}$$

**B.1 Estimates for first order derivatives.** Let  $p \in \partial \Omega$  and denote by  $N$  the inner normal at  $p$ . Since  $\partial \Omega$  is of class  $C^1$  a sufficiently small line segment from  $p$  along  $N$  stays in  $\Omega$ . So at points  $q = p + \ell \cdot N$  we can take the directional derivative of  $U$  along  $N_p$ . This gives a function

$$\ell \mapsto \partial U / \partial N(p + \ell \cdot N)$$

Since the boundary is  $C^1$  these functions are defined on a fixed interval  $0 \leq \ell \leq \ell^*$  for all  $p$ . With these notations there exists a constant  $B$  such that

$$(**) \quad \left| \partial U / \partial N(p + \ell \cdot N) \right| \leq B \cdot \|\partial U / \partial n\|_{\partial \Omega} \quad : \quad p \in \partial \Omega \quad : \quad 0 \leq \ell \leq \ell^*$$

where the size of  $B$  is controlled by the maximum norm of  $f$  on  $\partial \Omega$  and the positive constant  $W_*$  above.

### C. Proof of Theorem

Armed with the results above we can begin the proof of the Theorem. To begin with it suffices to prove the theorem when  $F(u, p)$  is an analytic function with respect to  $u$ . For if we then take an arbitrary  $F$ -function satisfying (0.1) and (0.2), then  $F$  is uniformly approximated by a sequence  $\{F_n\}$  of analytic functions and if  $\{u_n\}$  are the unique solutions to  $\{F_n\}$  then the estimates in (B) show that there exists a limit function  $\lim_{n \rightarrow \infty} u_n = u$  where  $u$  solves (\*) for the given  $F$ -function. So let us now assume that  $u \mapsto F(u, p)$  is a real-analytic function on the positive real axis for each  $p \in \partial \Omega$  where local power series converge uniformly with respect to  $p$ . In this situation there remains to prove the *existence* of a solution  $u$  to the PDE in (\*) above Theorem 1. To attain this we proceed as follows.

**C.1 The succesive solutions  $\{u_h\}$ .** To each real number  $0 \leq h \leq 1$  we seek a solution  $u_h$  where

$$(1) \quad \frac{\partial u_h}{\partial n}(p) = h \cdot F(u_h, p) + (1 - h) \cdot u_h(p)$$

With  $h = 0$  we have a linear equation

$$(2) \quad \frac{\partial u}{\partial n}(p) = u(p)$$

which is solved by the Green's function with a pole at  $Q_*$ . Next, suppose that  $0 \leq h_0 < 1$  and that we have found the solution  $u_{h_0}$  in (1) above. Set  $u_0 = u_{h_0}$  and with  $h = h_0 + \alpha$  for some small  $\alpha > 0$  we shall find  $u_h$  by a series

$$(3) \quad u_h = u_0 + \sum_{\nu=1}^{\infty} \alpha^\nu \cdot u_\nu$$

The pole at  $p_*$  occurs already in  $u_0$ . So  $u_1, u_2, \dots$  will be a sequence of harmonic functions in  $\Omega$ . There remains to find this sequence so that  $u_h$  yields a solution to (1). We will show that this can be achieved when  $h - h_0 = \alpha$  is sufficiently small. To begin with the results from (B) give positive constants  $0 < c_1 < c_2$  such that

$$(4) \quad 0 < c_1 \leq u_0(p) \leq c_2 \quad : p \in \partial\Omega$$

Now we use the analyticity of  $F$  with respect to  $u$  which enable us to write:

$$(5) \quad F(u_h(p), p) = F(u_0(p) + \sum_{k=1}^{\infty} c_k(p) \cdot [\sum_{\nu=1}^{\infty} \alpha^\nu u_\nu(p)]^k$$

where  $\{c_k(p)\}$  are continuous functions on  $\partial\Omega$  which appear in an expansion

$$(6) \quad F(u_0(p) + \xi, p) = F(u_0(p), p) + \sum_{k=1}^{\infty} c_k(p) \cdot \xi^k$$

In the last series expansion we notice that (4) and the hypothesis on  $F$  entail that the radius of convergence has a uniform bound below, i.e. there exists  $\rho > 0$  which is independent of  $p \in \partial\Omega$  and a constant  $K$  such that

$$(7) \quad \sum_{k=1}^{\infty} |c_k(p)| \cdot \rho^k \leq K$$

hold for all  $p \in \partial\Omega$ . Now the equation (1) is solved via a system of equations for the harmonic functions  $\{u_\nu\}$  which are determined inductively while  $\alpha$ -powers are identified. The linear  $\alpha$ -term gives the equation

$$(i) \quad \frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1 - h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p)$$

For  $u_2$  we find that

$$(ii) \quad \frac{\partial u_2}{\partial n} = (1 - h_0)u_2 - u_1 + h_0 c_1(p)u_2 + c_1(p)u_1 + c_2(p)u_1^2$$

In general, for  $\nu \geq 3$  one has

$$(iii) \quad \frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \dots, u_{\nu-1}, p)$$

where  $\{R_\nu\}$  are polynomials in the preceding  $u$ -functions whose coefficients are continuous functions derived via the  $c$ -functions above. Here the function  $c_1(p)$  is given by

$$c_1(p) = \frac{\partial F(u_0(p), p)}{\partial u}$$

which by the hypothesis on  $F$  is positive on  $\partial\Omega$ . Next, since  $u_0$  is a solution we also have a pair of positive constants  $0 < c_* < c^*$  such that

$$(iv) \quad c_* < c_1(p) \leq c^* \quad : p \in \partial\Omega$$

Hence the function

$$W(p) = (1 - h_0) + h_0 \cdot c_1(p)$$

is positive on  $\partial\Omega$ . Now estimates for the linear inhomogeneous equations in (B) can be applied where the  $f$ -functions are the  $R$ -polynomials. Then (7) and a majorising positive series expressing maximum norms show that if  $\alpha$  is sufficiently small then the series (3) converges and gives the requested solution for (1). Moreover, the positive  $\alpha$  can be taken *independently* of  $h_0$ . So together with the uniqueness of solutions  $u_h$  whenever they exist, it follows that we can move from  $h = 0$  until  $h = 1$  where we get the requested solution  $u$  to the PDE in Theorem 1.

**Remark.** The reader may consult page 106 in [Carleman] where the existence of a uniform constant  $\alpha > 0$  for which the series (3) converge for every  $h$  is demonstrated by an explicit majorant series.