SINGULAR OSCILLATORY INTEGRALS ON $\mathbb{R}^n$

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ABSTRACT. Let $\mathcal{P}_{d,n}$ denote the space of all real polynomials of degree at most d on $\mathbb{R}^n$. We prove a new estimate for the logarithmic measure of the sublevel set of a polynomial $P \in \mathcal{P}_{d,1}$. Using this estimate, we prove that

$$\sup_{P\in\mathcal{P}_{d,n}} \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} \frac{\Omega(x/|x|)}{|x|^n} dx \right| \leq c \log d \, (\|\Omega\|_{L\log L(S^{n-1})} + 1),$$

for some absolute positive constant c and every function Ω with zero mean value on the unit sphere S^{n-1}. This improves a result of Stein from [3].

1. INTRODUCTION

We denote by $\mathcal{P}_{d,n}$ the vector space of all real polynomials of degree at most d in $\mathbb{R}^n$. Let K be a $-n$ homogeneous function on $\mathbb{R}^n$, that is,

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n}, \tag{1.1}$$

where Ω is some function on the unit sphere S^{n-1}. Consider the principal value integral

$$I_n(P) = \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right|.$$

Stein has proved in [3] that if Ω has zero mean value on the unit sphere, then

$$|I_n(P)| \leq c_d \|\Omega\|_{L^\infty(S^{n-1})}, \tag{1.2}$$

for some constant c_d depending only on d. We wish to obtain sharp estimates of the form (1.2). The one dimensional analogue, namely the estimate

$$\left| p.v. \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c \log d, \tag{1.3}$$

which was proved in [2], suggests that the constant c_d in (1.2) could be replaced by $c \log d$ for some absolute positive constant c. The fact that this is indeed the case is the content of the following theorem.

Theorem 1.1. *Suppose that $K(x) = \Omega(x/|x|)/|x|^n$ where Ω has zero mean value on the unit sphere S^{n-1}. There exists an absolute positive constant c such that*

$$\sup_{P\in\mathcal{P}_{d,n}} \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \log d \; (\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

2000 *Mathematics Subject Classification.* Primary 42B20; Secondary 26D05.

Remark 1.2. Suppose that $K(x) = \Omega(x/|x|)/|x|^n$ where the function Ω is odd on the unit sphere. It is an immediate consequence of the one-dimensional result that

$$\sup_{P\in\mathcal{P}_{d,n}} \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x)dx \right| \le c \log d \; \|\Omega\|_{L^1(S^{n-1})}$$

for some absolute positive constant c.

The main ingredient of the proof of Theorem 1.1 is an estimate for the logarithmic measure of the sublevel set of a real polynomial in one dimension. This is a lemma of independent interest which we now state.

Lemma 1.3 (The logarithmic measure lemma). *Let $P(x) = \sum_{k=0}^d b_k x^k$ be a real valued polynomial of degree at most d, $\alpha > 0$ and $M = \max\{|b_k| : \frac{d}{2} < k \le d\}$. If $E = \{x \ge 1 : |P(x)| \le \alpha\}$, then*

$$\int_E \frac{dx}{x} \le c \min\left(\left(\frac{\alpha}{M}\right)^{\frac{1}{d}}, 1 + \frac{1}{d} \log \frac{\alpha}{M} \right),$$

where c is an absolute positive constant.

Notation. We will use the letter c to denote an absolute positive constant which might change even in the same line of text.

2. Preliminary Results

As is usually the case when one deals with oscillatory integrals, a key Lemma is the classical van der Corput Lemma.

Lemma 2.1 (van der Corput). *Let $\phi : [a, b] \to \mathbb{R}$ be a C^1 function and suppose that $|\phi'(t)| \ge 1$ for all $t \in [a, b]$ and ϕ' changes monotonicity N times in $[a, b]$. Then, for every $\lambda \in \mathbb{R}$,*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \le \frac{cN}{|\lambda|}$$

where c is an absolute constant independent of a,b and ϕ.

The proof of Lemma 2.1 is a simple integration by parts.

We will also need a precise estimate for the Lebesgue measure of the sublevel set of a polynomial on $\mathbb{R}^n$.

Theorem 2.2 (Carbery,Wright). *Suppose that $K \subset \mathbb{R}^n$ is a convex body of volume 1 and $P \in \mathcal{P}_{d,n}$. Let $1 \le q \le \infty$. Then,*

$$|\{x \in K : |P(x)| \le \alpha\}| \le c \min(qd, n) \alpha^{\frac{1}{d}} \|P\|_{L^q(K)}^{-\frac{1}{d}}.$$

This is a consequence of a more general Theorem of Carbery and Wright and can be found in [1].

Corollary 2.3. *Let P be a real homogeneous polynomial of degree k on $\mathbb{R}^n$. Then*

$$\int_{S^{n-1}} \frac{\|P\|_{L^\infty(S^{n-1})}^{\frac{1}{2k}}}{|P(x')|^{\frac{1}{2k}}} d\sigma_{n-1}(x') \le c. \tag{2.1}$$

Proof of Corollary 2.3. Let $B = B(0,\rho)$ be the ball of volume 1 on $\mathbb{R}^n$. For $\epsilon < \frac{1}{k}$ and some $\lambda > 0$ to be defined later, we have

$$\begin{aligned}\int_B |P(x)|^{-\epsilon}dx &= \int_0^\infty |\{x \in B : |P(x)|^{-\epsilon} \geq \alpha\}|d\alpha \\ &\leq \lambda + \int_\lambda^\infty |\{x \in B : |P(x)| < \alpha^{-\frac{1}{\epsilon}}\}|d\alpha \\ &\leq \lambda + cn\|P\|_{L^\infty(B)}^{-\frac{1}{k}} \frac{\lambda^{-\frac{1}{k\epsilon}+1}}{\frac{1}{k\epsilon}-1},\end{aligned}$$

using Theorem 2.2. Optimizing in λ we get

$$\int_B |P(x)|^{-\epsilon}dx \leq \left(cn\frac{k\epsilon}{1-k\epsilon}\right)^{k\epsilon} \|P\|_{L^\infty(B)}^{-\epsilon}.$$

Using polar coordinates and setting $\epsilon = \frac{1}{2k} < \frac{1}{k}$, we then get

$$\begin{aligned}\|P\|_{L^\infty(S^{n-1})}^{\frac{1}{2k}} \int_{S^{n-1}} |P(x')|^{-\frac{1}{2k}} d\sigma_{n-1}(x') &\leq c\frac{n^{\frac{3}{2}}}{\rho^n} = c\frac{n^{\frac{3}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \\ &\leq c\frac{n^{\frac{3}{2}}(e\pi)^{\frac{n}{2}}}{(\frac{n}{2}+1)^{\frac{n+1}{2}}} \leq c,\end{aligned}$$

which completes the proof. □

3. The logarithmic measure lemma

The proof of Lemma 1.3 is motivated by an argument of Vinogradov from [4], used to estimate the *Lebesgue* measure of the sublevel set of a polynomial in a bounded interval. We fix a polynomial $P(x) = \sum_{k=0}^d b_k x^k$ and look at the set $E = \{x \geq 1 : |P(x)| \leq \alpha\}$. Note that by replacing α with αM in the statement of the lemma, it is enough to consider the case $M = 1$. Since E is a closed set we can find points $x_0, x_1, \ldots, x_d \in E$ such that $x_0 < x_1 < \cdots < x_d$ and

$$\frac{1}{d}\int_E \frac{dx}{x} = \int_{E\cap[x_j,x_{j+1}]} \frac{dx}{x} \leq \log\frac{x_{j+1}}{x_j}, \qquad 0 \leq j \leq d-1.$$

We set $\mu = \int_E \frac{dx}{x}$ and $t = e^{\frac{\mu}{d}} > 1$ and we have that $x_{j+1} \geq tx_j$, $0 \leq j \leq d-1$. The Lagrange interpolation formula is

$$P(x) = \sum_{j=0}^d P(x_j)\frac{(x-x_0)\cdots\widehat{(x-x_j)}\cdots(x-x_d)}{(x_j-x_0)\cdots\widehat{(x_j-x_j)}\cdots(x_j-x_d)},\ x \in \mathbb{R},$$

where $\hat{u}$ means that u is omitted. Thus,

$$b_k = \sum_{j=0}^d P(x_j)(-1)^{d-k}\frac{\sigma_{d-k}(x_0,\ldots,\widehat{x_j},\ldots,x_d)}{(x_j-x_0)\cdots\widehat{(x_j-x_j)}\cdots(x_j-x_d)},$$

where σ_l is the l-th elementary symmetric function of its variables. Therefore

$$\begin{aligned}
|b_k| &\leq \alpha \sum_{j=0}^{d} \frac{\sigma_{d-k}(x_0,\ldots,\widehat{x_j},\ldots,x_d)}{|x_j-x_0|\cdots|\widehat{x_j-x_j}|\cdots|x_j-x_d|} \\
&= \alpha \sum_{j=0}^{d} \frac{\sigma_k(\frac{1}{x_0},\ldots,\widehat{\frac{1}{x_j}},\ldots,\frac{1}{x_d})}{(\frac{x_j}{x_0}-1)\cdots(\frac{x_j}{x_{j-1}}-1)(1-\frac{x_j}{x_{j+1}})\cdots(1-\frac{x_j}{x_d})} \\
&\leq \alpha \sum_{j=0}^{d} \frac{\sigma_k(1,\ldots,\widehat{\frac{1}{t}},\ldots,\frac{1}{t^d})}{(t^j-1)\cdots(t-1)(1-\frac{1}{t})\cdots(1-\frac{1}{t^{d-j}})}.
\end{aligned}$$

It is easy to see that there exists precisely one j, $0 \leq j \leq \frac{d-1}{2} < d$, for which

$$t^{j-1} < \frac{2t^d}{t^{d+1}+1} \leq t^j. \tag{3.1}$$

It is exactly for this j that $(t^j-1)\cdots(t-1)(1-\frac{1}{t})\cdots(1-\frac{1}{t^{d-j}})$ takes its minimum value as j runs from 0 to d. On the other hand we have

$$\sum_{j=0}^{d} \sigma_k\left(1,\ldots,\widehat{\frac{1}{t_j}},\ldots,\frac{1}{t^k}\right) = (d+1-k)\sigma_k\left(1,\ldots,\frac{1}{t^d}\right)$$

and, hence

$$\begin{aligned}
|b_k| &\leq \alpha\ (d+1-k)\sigma_k\left(1,\ldots,\frac{1}{t^d}\right)\frac{1}{(t^j-1)\cdots(t-1)(1-\frac{1}{t})\cdots(1-\frac{1}{t^{d-j}})} \\
&\leq \frac{\alpha\ (d+1-k)\binom{d+1}{k}}{1\cdot t\cdots t^k}\frac{1}{(t^j-1)\cdots(t-1)(1-\frac{1}{t})\cdots(1-\frac{1}{t^{d-j}})}. \qquad (3.2)
\end{aligned}$$

From (3) we easily see that $t^j < 2$ and, since $\frac{\log(x-1)}{x}$ is increasing in the interval $(1,2)$, we find

$$\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1) = \\
&= \frac{t}{t-1}\left(\frac{\log(t-1)}{t}(t-1)+\cdots+\frac{\log(t^j-1)}{t^j}(t^j-t^{j-1})\right) \\
&\geq \frac{t}{t-1}\int_1^{t^j}\frac{\log(x-1)}{x}dx = \frac{t}{t-1}\int_0^{t^j-1}\frac{\log x}{1+x}dx. \qquad (3.3)
\end{aligned}$$

Similarly, since $\frac{\log(1-x)}{x}$ is decreasing in the interval $(0,1)$ we get

$$\begin{aligned}
&\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right) = \\
&= \frac{1}{t-1}\left(\frac{\log(1-\frac{1}{t^{d-j}})}{\frac{1}{t^{d-j}}}\left(\frac{1}{t^{d-j-1}}-\frac{1}{t^{d-j}}\right)+\cdots+\frac{\log(1-\frac{1}{t})}{\frac{1}{t}}\left(1-\frac{1}{t}\right)\right) \\
&\geq \frac{1}{t-1}\int_{\frac{1}{t^{d-j}}}^{1}\frac{\log(1-x)}{x}dx = \frac{1}{t-1}\int_0^{1-\frac{1}{t^{d-j}}}\frac{\log x}{1-x}dx. \qquad (3.4)
\end{aligned}$$

We let

$$A = \frac{t^d-1}{t^d+1}, \quad B = t^j-1, \quad \Gamma = 1-\frac{1}{t^{d-j}},$$

and, obviously, $0 < A, B, \Gamma < 1$. From (3.3) and (3.4) we have

$$
\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1)+\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right) \geq \\
\geq\ & \frac{t}{t-1}\int_0^{t^j-1}\frac{\log x}{1+x}dx+\frac{1}{t-1}\int_0^{1-\frac{1}{t^{d-j}}}\frac{\log x}{1-x}dx \\
=\ & \frac{t}{t-1}\int_0^{B}\frac{\log x}{1+x}dx+\frac{1}{t-1}\int_0^{\Gamma}\frac{\log x}{1-x}dx \\
=\ & -\frac{t}{t-1}B\log\frac{1}{B}-\frac{1}{t-1}\Gamma\log\frac{1}{\Gamma}-O\Big(\frac{t}{t-1}B\Big)-O\Big(\frac{1}{t-1}\Gamma\Big).
\end{aligned}
$$

From (3) we get $B, \Gamma \leq \frac{t^{d+1}-1}{t^{d+1}+1}$ and, since $\frac{t+1}{t-1}\frac{t^{d+1}-1}{t^{d+1}+1}$ is decreasing in $t \in (1, +\infty)$, we find

$$\frac{t}{t-1}B \leq \frac{t+1}{t-1}\frac{t^{d+1}-1}{t^{d+1}+1} \leq d+1$$

and, similarly,

$$\frac{1}{t-1}\Gamma \leq \frac{t+1}{t-1}\frac{t^{d+1}-1}{t^{d+1}+1} \leq d+1.$$

Therefore

$$
\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1)+\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right) \geq \\
\geq\ & -\frac{t}{t-1}B\log\frac{1}{B}-\frac{1}{t-1}\Gamma\log\frac{1}{\Gamma}-cd \\
\geq\ & -2\frac{2}{t-1}A\log\frac{1}{A}-\frac{1}{t-1}\left(B\log\frac{1}{B}+\Gamma\log\frac{1}{\Gamma}-2A\log\frac{1}{A}\right)-cd.
\end{aligned}
$$

Now

$$
\begin{aligned}
B\log\frac{1}{B}+\Gamma\log\frac{1}{\Gamma}-2A\log\frac{1}{A} &= (B+\Gamma-2A)\log\frac{1}{A}+A\frac{B}{A}\log\frac{A}{B}+A\frac{\Gamma}{A}\log\frac{A}{\Gamma} \\
&\leq \left(\frac{B+\Gamma}{A}-2\right)A\log\frac{1}{A}+cA.
\end{aligned}
$$

Using (3)

$$\frac{B+\Gamma}{A}-1 \leq \frac{2(t-1)}{t^{d+1}+1}$$

and we conclude that

$$
\begin{aligned}
\frac{1}{t-1}\left(B\log\frac{1}{B}+\Gamma\log\frac{1}{\Gamma}-2A\log\frac{1}{A}\right) &\leq \frac{2}{t^{d+1}+1}A\log\frac{1}{A}+\frac{c}{t-1}A \\
&\leq c+c\frac{t+1}{t-1}\frac{t^d-1}{t^d+1} \leq cd.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1)+\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right) \geq \\
\geq\ & -\frac{2}{t-1}A\log 1A-cd
\end{aligned}
$$

and, finally, (3.2) implies that for some $k > \frac{d}{2}$

$$1 \leq \frac{c_o^d \alpha}{t^{\frac{k(k-1)}{2}}} \left(\frac{1}{A}\right)^{\frac{2A}{t-1}},$$

where c_o is an absolute positive constant.

case 1: $c_o \alpha^{\frac{1}{d}} < \frac{1}{2}$. Then, since $\frac{2A}{t-1} \leq \frac{t+1}{t-1} A \leq d$, we get

$$A^d \leq A^{\frac{2A}{t-1}} \leq c_o \alpha$$

which implies

$$\frac{t^d - 1}{t^d + 1} \leq A \leq c_o\ \alpha^{\frac{1}{d}}$$

and, finally,

$$\mu \leq e^{\mu} - 1 = t^d - 1 \leq 4 c_o \alpha^{\frac{1}{d}}.$$

case 2: $c_o \alpha^{\frac{1}{d}} \geq \frac{1}{2}$, $t^d < 2$. Then

$$e^{\mu} = t^d < 4 c_o \alpha^{\frac{1}{d}}$$

which shows that

$$\mu < \log(4 c_o) + \frac{\log \alpha}{d}.$$

case 3: $c_o \alpha^{\frac{1}{d}} \geq \frac{1}{2}$, $t^d \geq 2$. Then $A \geq \frac{1}{3}$ and $\frac{2A}{t-1} \leq \frac{t+1}{t-1} A \leq d$ and, hence,

$$\frac{1}{3^d} t^{\frac{k(k-1)}{2}} \leq c_o^d \alpha.$$

We conclude that

$$\mu \leq \frac{2d^2}{k(k-1)} \left(\log(3 c_o) + \frac{\log \alpha}{d} \right) \leq c \left(1 + \log \frac{\alpha}{d} \right)$$

since $k > \frac{d}{2}$.

4. Proof of Theorem 1.1

Let Ω be a function with zero mean value on the unit sphere S^{n-1} belonging to the class $L \log L(S^{n-1})$, that is

$$\|\Omega\|_{L \log L(S^{n-1})} = \int_{S^{n-1}} |\Omega(x')|(1 + \log^+ |\Omega(x')|) d\sigma_{n-1}(x') < \infty.$$

Set $K(x) = \Omega(x/|x|)/|x|^n$ and let $P \in \mathcal{P}_{d,n}$. We will show the theorem for $d = 2^m$, for some $m \geq 0$. The general case is then an immediate consequence.

We set

$$C_d = \sup_{0<\epsilon<R} \left| \int_{\epsilon \leq |x| \leq R} e^{iP(x)} K(x) dx \right|,$$

where C_d is a constant depending on d, Ω and n. For $0 < \epsilon < R$ we write,

$$I_{\epsilon,R}(P) = \int_{\epsilon \leq |x| \leq R} e^{iP(x)} K(x) dx = \int_{S^{n-1}} \int_{\epsilon}^{R} e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x').$$

For $x' \in S^{n-1}$, we have that $P(rx') = \sum_{j=1}^{d} P_j(x') r^j$ where P_j is a homogeneous polynomial of degree j. Observe that we can omit the constant term, without loss of generality. Set also $m_j = \|P_j\|_{L^\infty(S^{n-1})}$. Since ϵ and R are arbitrary positive numbers, by a dilation in r we can assume that $\max_{\frac{d}{2}<j\leq d} m_j = 1$ and, in particular,

that $m_{j_o} = 1$ for some $\frac{d}{2} < j_o \le d$. We also write $Q(x) = \sum_{j=1}^{\frac{d}{2}} P_j(x)$. We split the integral in two parts as follows

$$\begin{aligned} |I_{\epsilon,R}(P)| &\le \left| \int_{S^{n-1}} \int_\epsilon^1 e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| \\ &+ \left| \int_{S^{n-1}} \int_1^R e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| = I_1 + I_2. \end{aligned}$$

For I_1 we have that

$$\begin{aligned} I_1 &\le \int_{S^{n-1}} \int_0^1 \left| e^{iP(rx')} - e^{iQ(rx')} \right| \frac{dr}{r} |\Omega(x')| d\sigma_{n-1}(x') \\ &+ \left| \int_{S^{n-1}} \int_\epsilon^1 e^{iQ(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| \\ &\le \sum_{\frac{d}{2} < j \le d} \frac{m_j}{j} \|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}} \le c\|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}}. \end{aligned}$$

For I_2 we write

$$\begin{aligned} I_2 &\le \int_{S^{n-1}} \left| \int_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|>d\}} e^{iP(rx')} \frac{dr}{r} \right| |\Omega(x')| d\sigma_{n-1}(x') \\ &+ \int_{S^{n-1}} \int_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|\le d\}} \frac{dr}{r} |\Omega(x')| d\sigma_{n-1}(x'). \end{aligned}$$

Since $\{r \in [1, R] : |\frac{\partial P(rx')}{\partial r}| > d\}$ consists of at most $O(d)$ intervals where $\frac{\partial P(rx')}{\partial r}$ is monotonic, van der Corput's lemma gives the bound

$$\int_{S^{n-1}} \left| \int_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|>d\}} e^{iP(rx')} \frac{dr}{r} \right| |\Omega(x')| d\sigma_{n-1}(x') \le c\|\Omega\|_{L^1(S^{n-1})}.$$

On the other hand, the logarithmic measure lemma implies that

$$\begin{aligned} &\int_{S^{n-1}} \int_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|\le d\}} \frac{dr}{r} |\Omega(x')| d\sigma_{n-1}(x') \le \\ &\le c\|\Omega\|_{L^1(S^{n-1})} + c\frac{1}{d} \int_{S^{n-1}} \log \frac{d}{\max_{\frac{d}{2}<j\le d}\{j|P_j(x')|\}} |\Omega(x')| d\sigma_{n-1}(x'). \end{aligned}$$

Combining the estimates we get

$$C_d \le c\|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}} + c\frac{2j_o}{d} \int_{S^{n-1}} \log \frac{\|P_{j_o}\|_{L^\infty(S^{n-1})}^{\frac{1}{2j_o}}}{|P_{j_o}(x')|^{\frac{1}{2j_o}}} |\Omega(x')| d\sigma_{n-1}(x')$$

and, from Young's inequality,

$$\begin{aligned} C_d \le c\|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}} &+ c\int_{S^{n-1}} \frac{\|P_{j_o}\|_{L^\infty(S^{n-1})}^{\frac{1}{2j_o}}}{|P_{j_o}(x')|^{\frac{1}{2j_o}}} d\sigma_{n-1}(x') + \\ &+ c\int_{S^{n-1}} |\Omega(x')|(1 + \log^+ |\Omega(x')|) d\sigma_{n-1}(x'). \end{aligned}$$

Now, using corollary 2.3 we get

$$C_d \le C_{\frac{d}{2}} + c(\|\Omega\|_{L \log L(S^{n-1})} + 1).$$

Since $d = 2^m$, this means that

$$C_{2^m} \leq C_{2^{m-1}} + c(\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

Using induction on m we get that $C_{2^m} \leq C_1 + cm(\|\Omega\|_{L\log L(S^{n-1})} + 1)$. Observe that C_1 corresponds to some polynomial $P(x) = b_1x_1 + \cdots + b_nx_n$. We write

$$\left|\int_{\epsilon<|x|<R} e^{iP(x)}K(x)dx\right| =$$

$$= \left|\int_{S^{n-1}}\int_{\epsilon}^{R}\{e^{irP(x')} - e^{ir\|P\|_{L^\infty(S^{n-1})}}\}\frac{dr}{r}\Omega(x')d\sigma_{n-1}(x')\right|.$$

Using the simple estimate

$$\left|\int_{\epsilon}^{R}\{e^{iar} - e^{ibr}\}\frac{dr}{r}\right| \leq c + c\left|\log\left|\frac{b}{a}\right|\right|$$

we get

$$\begin{aligned}\left|\int_{\epsilon<|x|<R} e^{iP(x)}K(x)dx\right| &\leq c\|\Omega\|_{L^1(S^{n-1})} + \\ &+c\int_{S^{n-1}}\log\frac{\|P\|^{\frac{1}{2}}_{L^\infty(S^{n-1})}}{|P(x')|^{\frac{1}{2}}}|\Omega(x')|d\sigma_{n-1}(x').\end{aligned}$$

Hence, $C_1 \leq c\|\Omega\|_{L^1(S^{n-1})} + c + \|\Omega\|_{L\log L(S^{n-1})}$ and

$$C_{2^m} \leq cm(\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

The case of general d is now trivial. If $2^{m-1} < d \leq 2^m$ then

$$C_d \leq C_{2^m} \leq cm(\|\Omega\|_{L\log L(S^{n-1})} + 1) \leq c\log d(\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

5. The one dimensional case revisited

We will attempt to give a short proof of the one dimensional analogue of theorem 1.1. This is a slight simplification of the proof in [2], with the aid of the logarithmic measure lemma.

So, fix a real polynomial $P(x) = b_0 + b_1x + \cdots + b_dx^d$ and consider the quantity

$$C_d = \sup_{0<\epsilon<R}\left|\int_{\epsilon<|x|<R} e^{iP(x)}\frac{dx}{x}\right|.$$

By the same considerations as in the $n-$dimensional case, we can assume that P has no constant term and that it can be decomposed in the form

$$P(x) = \sum_{0<j\leq\frac{d}{2}} b_jx^j + \sum_{\frac{d}{2}<j\leq d} b_jx^j = Q(x) + R(x),$$

where $|b_j| \leq 1$ for every $\frac{d}{2} < j \leq d$. As a result

$$\begin{aligned}\left|\int_{\epsilon<|x|<R} e^{iP(x)}\frac{dx}{x}\right| &\leq C_{\frac{d}{2}} + \int_{0<|x|<1}\frac{|R(x)|}{x}dx + \left|\int_{1<|x|<R} e^{iP(x)}\frac{dx}{x}\right| \\ &\leq C_{\frac{d}{2}} + c + I.\end{aligned}$$

We split I as follows

$$I \leq \left| \int_{\{x\in[1,R):|P'(x)|>d\}} e^{iP(x)} \frac{dx}{x} \right| + \int_{\{x\geq 1:|P'(x)|\leq d\}} \frac{dx}{x}.$$

Now, using Proposition 2.1 for the first summand in the above estimate and the logarithmic measure lemma to estimate the second summand, we get that $I \leq c$. But this means that $C_d \leq C_{\frac{d}{2}} + c$ which completes the proof by considering first the case $d = 2^m$ for some m, as in the $n-$dimensional case.

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