A SHARP BOUND FOR THE STEIN-WAINGER OSCILLATORY INTEGRAL

IOANNIS R. PARISSIS

Abstract. Let $\mathcal{P}_d$ denote the space of all real polynomials of degree at most d. It is an old result of Stein and Wainger [4] that

$$\sup_{P\in\mathcal{P}_d}\left|p.v.\int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t}\right| \leq C_d$$

for some constant C_d depending only on d. On the other hand, Carbery, Wainger and Wright in [2] claim that the true order of magnitude of the above principal value integral is $\log d$. We prove that

$$\sup_{P\in\mathcal{P}_d}\left|p.v.\int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t}\right| \sim \log d.$$

1. Introduction

Let $\mathcal{P}_d$ be the vector space of all real polynomials of degree at most d in $\mathbb{R}$. For $P \in \mathcal{P}_d$ we consider the principal value integral

$$I(P) = \left|p.v.\int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t}\right|.$$

We wish to estimate the quantity $I(P)$ by a constant $C(d)$ depending only on the degree of the polynomial d. This amounts to estimating the integral

$$I_{(\epsilon,R)}(P) = \left|\int_{\epsilon\leq|t|\leq R} e^{iP(t)}\frac{dt}{t}\right|$$

by some constant $C(d)$ independent of ϵ, R and P.

This problem is quite old and in fact has been answered some thirty years ago by Stein and Wainger in [4] and [6]. They showed that the quantity $I(P)$ is bounded by a constant C_d depending only on d. Their proof is very simple and uses a combination of induction and Van der Corput's lemma. Let us recall the latter since we'll also be using it in what follows.

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Proposition 1.1 (van der Corput)**.** *Let $\phi : [a,b] \to \mathbb{R}$ be a C^k function and suppose that $|\phi^{(k)}(t)| \geq 1$ for some $k \geq 1$ and all $t \in [a,b]$. If $k = 1$ suppose in addition that ϕ' is monotonic. Then, for every $\lambda \in \mathbb{R}$,*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq \frac{Ck}{|\lambda|^{\frac{1}{k}}}$$

where C is an absolute constant independent of a,b,k and ϕ.

For a proof of this very well known result with Ck replaced by C_k see for example [3]. A proof that the constant C_k can be taken to be linear in k can be found in [1].

On the other hand, Carbery, Wainger and Wright have conjectured in [2] that the true order of magnitude of the principal value integral is $\log d$. The main result of this paper is the proof of this conjecture. This is the content of:

Theorem. *There exist two absolute positive constants c_1 and c_2 such that*

$$c_1 \log d \leq \sup_{P \in \mathcal{P}_d} \left| p.v. \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c_2 \log d.$$

Remark 1.2. Suppose that K is a $-n$ homogeneous function on $\mathbb{R}^n$, odd and integrable on the unit sphere. Then, by the one-dimensional result, we trivially get that there is an absolute positive constant c, such that:

$$\left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \|K\|_{L^1(S^{n-1})} \log d,$$

for every polynomial P on $\mathbb{R}^n$, of degree at most d.

Notation. We will use the letter c to denote an absolute positive constant which might change even in the same line of text. Also, the notation $A \sim B$ means that there exist absolute positive constants c_1 and c_2 such that $c_1 B \leq A \leq c_2 B$.

2. Aknowledgements

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3. The lower bound in the Theorem

In this section we will construct a real polynomial P of degree at most d such that the inequality

$$I(P) = \left| p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \geq c \log d \tag{3.1}$$

holds. The general plan of the construction is as follows. We will first construct a *function* f (which will not be a polynomial) such that $I(f) \geq c \log n$. We will then construct a polynomial P of degree $d = 2n^2 - 1$ that approximates the function f in a way that $|I(f) - I(P)|$ is small (small means $o(\log n)$ here). Since $\log n \sim \log d$ this will yield our result.

Lemma 3.1. *For n a large positive integer, let $f(t)$ be the continuous function which is equal to 1 for $\frac{1}{n} \le t \le 1-\frac{1}{n}$, equal to -1 for $-1+\frac{1}{n} \le t \le -\frac{1}{n}$, equal to 0 for $|t| \ge 1$ and linear in each interval $[-1,-1+\frac{1}{n}]$, $[-\frac{1}{n},\frac{1}{n}]$ and $[1-\frac{1}{n},1]$. Then,*

$$I(f) = \left| p.v. \int_{\mathbb{R}} e^{if(t)} \frac{dt}{t} \right| \ge c \log n. \tag{3.2}$$

Proof. The proof is more or less straightforward.

$$\begin{aligned} I(f) &= 2\left|\int_0^1 \frac{\sin f(t)}{t} dt\right| \\ &\ge 2\left|\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\sin f(t)}{t} dt\right| - 2\left|\int_0^{\frac{1}{n}} \frac{\sin f(t)}{t} dt\right| - 2\left|\int_{1-\frac{1}{n}}^{1} \frac{\sin f(t)}{t} dt\right| \\ &\ge 2\sin 1 \log(n-1) - 2\int_0^{\frac{1}{n}} \frac{f(t)}{t} dt - 2\int_{1-\frac{1}{n}}^{1} \frac{f(t)}{t} dt \\ &= 2\sin 1 \log(n-1) - 2 - 2n\log\frac{n}{n-1} + 2 \\ &\ge 2\sin 1 \log(n-1) - 4 \ \ge \ c\log n. \end{aligned}$$

□

We now want to construct a polynomial which approximates the function f. We will do so by convolving the function f with a "polynomial approximation to the identity". To be more specific, for $k \in \mathbb{N}$ and $x \in \mathbb{R}$ define the function

$$\phi_k(x) = c_k \left(1 - \frac{x^2}{4}\right)^{k^2} \tag{3.3}$$

where the constant c_k is defined by means of the normalization

$$\int_{-2}^{2} \phi_k(x) dx = 1. \tag{3.4}$$

Observe that

$$1 = c_k \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right)^{k^2} dx = 4c_k \int_0^1 (1-x^2)^{k^2} dx = 2c_k B\left(\frac{1}{2}, k^2+1\right),$$

where $B(\cdot,\cdot)$ is the beta function. Using standard estimates for the beta function we see that $c_k \sim k$.

Define, next, the functions P_k in $\mathbb{R}$ as

$$P_k(t) = \int_{-1}^{1} f(x)\phi_k(t-x) dx, \tag{3.5}$$

where f is the function of Lemma 3.1. It is clear that the functions P_k are *polynomials* of degree at most $2k^2$. The following lemma deals with some technical issues concerning the polynomials P_k.

Lemma 3.2. *Let P_k be defined as in (3.5) above.*
(i) P_k is an odd polynomial of degree $2k^2-1$ with leading coefficient

$$a_k = (-1)^{k^2+1}\frac{2c_k k^2}{4^{k^2}}\Big(1-\frac{1}{n}\Big).$$

That is

$$P_k(t) = a_k t^{2k^2-1} + \cdots \ .$$

(ii) As a consequence of (i) we have for all t

$$|P_k^{(2k^2-1)}(t)| \geq c(2k^2-1)!\frac{k^3}{4^{k^2}}.$$

(iii) For $t\in[-1,1]$ we have

$$P_k(t) = \int_0^2 \big(f(t+x)+f(t-x)\big)\phi_k(x)dx.$$

Proof. (i) Using (3.5) we have

$$\begin{aligned} P_k(-t) &= \int_{-1}^1 f(x)\phi_k(-t-x)dx = \int_{-1}^1 f(x)\phi_k(t+x)dx \\ &= \int_{-1}^1 f(-x)\phi_k(t-x)dx = -P_k(t). \end{aligned}$$

Next, from (3.5) we have that

$$\begin{aligned} P_k(t) &= c_k\int_{-1}^1 f(x)\sum_{m=0}^{k^2}\binom{k^2}{m}\Big(-\frac{(t-x)^2}{4}\Big)^m dx \\ &= c_k\sum_{m=0}^{k^2}\binom{k^2}{m}\frac{(-1)^m}{4^m}\int_{-1}^1 f(x)(t-x)^{2m}dx \\ &= c_k\frac{(-1)^{k^2}}{4^{k^2}}\int_{-1}^1 f(x)(x-t)^{2k^2}dx \\ &+ c_k\sum_{m=0}^{k^2-1}\binom{k^2}{m}\frac{(-1)^m}{4^m}\int_{-1}^1 f(x)(t-x)^{2m}dx. \end{aligned}$$

It is now easy to see that the two highest order terms come from the first summand in the above formula. Therefore,

$$\begin{aligned} P_k(t) &= c_k\frac{(-1)^{3k^2}}{4^{k^2}}\int_{-1}^1 f(x)dx\ t^{2k^2} - c_k\frac{(-1)^{k^2}2k^2}{4^{k^2}}\int_{-1}^1 f(x)xdx\ t^{2k^2-1} + \cdots \\ &= (-1)^{k^2+1}\frac{2c_k k^2}{4^{k^2}}\Big(1-\frac{1}{n}\Big)t^{2k^2-1} + \cdots \ . \end{aligned}$$

(ii) We just use the result of (i) and that $c_k \sim k$.

(iii) Fix a $t \in [-1,1]$. Then,

$$\begin{aligned}\int_{-2}^{2} f(t-x)\phi_k(x)dx &= \int_{\mathbb{R}} f(t-x)\phi_k(x)\chi_{[-2,2]}(x)dx \\ &= \int_{-1}^{1} f(x)\phi_k(t-x)\chi_{[-2,2]}(t-x)dx \\ &= \int_{-1}^{1} f(x)\phi_k(t-x)dx \\ &= P_k(t).\end{aligned}$$

However, since ϕ_k is even,

$$P_k(t) = \int_{-2}^{2} f(t-x)\phi_k(x)dx = \int_{0}^{2} \big(f(t+x)+f(t-x)\big)\phi_k(x)dx.$$

□

We are now ready to prove the lower bound for $I(P)$.

Proposition 3.3. *Let P_n be the polynomial defined in (3.5) where n is the large positive integer used to define the function f in Lemma 3.1. Then P_n is a polynomial of degree $d = 2n^2 - 1$ and*

$$I(P_n) = \left| p.v. \int_{\mathbb{R}} e^{iP_n(t)} \frac{dt}{t} \right| \geq c \log d.$$

Proof. Since P_n is odd,

$$I(P_n) = 2\left|\int_0^{+\infty} \frac{\sin P_n(t)}{t} dt\right|,$$

and it suffices to show that for all $R \geq 1$

$$\left|\int_0^{R} \frac{\sin P_n(t)}{t} dt\right| \geq c\log d \sim c\log n. \tag{3.6}$$

By part (ii) of Lemma 3.2 and a standard application of Proposition 1.1 (Van der Corput) we see that

$$\left|\int_1^{R} \frac{\sin P_n(t)}{t} dt\right| \leq c$$

for all $R \geq 1$. As a result, the proof will be complete if we show that

$$I_1(P_n) = \left|\int_0^{1} \frac{\sin P_n(t)}{t} dt\right| \geq c\log n. \tag{3.7}$$

Using Lemma 3.1 and the triangle inequality we get

$$I_1(P_n) \geq c\log n - |I_1(P_n) - I(f)| \tag{3.8}$$

and, in order to show (3.7), it suffices to show that

$$|I_1(P_n) - I(f)| = o(\log n). \tag{3.9}$$

We have that

$$\begin{aligned} |I_1(P_n) - I(f)| &= \left| \int_0^1 \frac{\sin P_n(t) - \sin f(t)}{t} dt \right| \\ &\leq \int_0^1 \frac{|P_n(t) - f(t)|}{t} dt. \end{aligned}$$

Using part (iii) of Lemma 3.2 and (3.4), we get

$$|P_n(t) - f(t)| \leq \int_0^2 |f(t+x) + f(t-x) - 2f(t)| \phi_n(x) dx$$

for $0 \leq t \leq 1$. Hence

$$|I_1(P_n) - I(f)| \leq \int_0^2 \int_0^1 \frac{|f(t+x) + f(t-x) - 2f(t)|}{t} \, dt \; \phi_n(x) dx.$$

Now, the desired result, condition (3.9), is the content of the following lemma. □

Lemma 3.4. *Let* $A(x,t) = |f(t+x) + f(t-x) - 2f(t)|$. *Then,*

$$\int_0^2 \int_0^1 \frac{A(x,t)}{t} \, dt \; \phi_n(x) dx = o(\log n).$$

Proof. Firstly, it is not difficult to establish that

$$A(x,t) \leq 4 \min(nx, nt, 1) \tag{3.10}$$

$$A(x,t) = 0, \quad \text{when } \frac{1}{n} \leq t - x \leq t + x \leq 1 - \frac{1}{n} \,. \tag{3.11}$$

Indeed,

$$\begin{aligned} A(x,t) &\leq |f(t+x) - f(t)| + |f(t-x) - f(t)| \\ &\leq nx + nx \leq 2nx. \end{aligned}$$

On the other hand,

$$\begin{aligned} A(x,t) &= |f(t+x) - f(x) + f(t-x) - f(-x) - 2f(t)| \\ &\leq |f(t+x) - f(x)| + |f(t-x) - f(-x)| + 2|f(t)| \\ &\leq nt + nt + 2nt = 4nt. \end{aligned}$$

Inequality (3.10) now follows by the fact that $|f|$ is bounded by 1 and (3.11) is trivial to prove.

We split the integral $\int_0^2 \int_0^1 \cdots dt dx$ into seven integrals:

$$\int_0^2 \int_{\frac{1}{2}}^1 \cdots dt dx + \int_0^{\frac{1}{n}} \int_0^x \cdots dt dx + \int_{\frac{1}{n}}^2 \int_0^{\frac{1}{n}} \cdots dt dx + \int_0^{\frac{1}{n}} \int_x^{x+\frac{1}{n}} \cdots dt dx$$
$$+ \int_0^{\frac{1}{2}-\frac{1}{n}} \int_{x+\frac{1}{n}}^{\frac{1}{2}} \cdots dt dx + \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}} \int_{\frac{1}{n}}^{x+\frac{1}{n}} \cdots dt dx + \int_{\frac{1}{2}-\frac{1}{n}}^2 \int_{\frac{1}{n}}^{\frac{1}{2}} \cdots dt dx.$$

We estimate each of the seven integrals separately.

$$\int_0^2 \int_{\frac{1}{2}}^1 \frac{A(x,t)}{t} dt \phi_n(x) dx \leq 4 \log 2 \int_0^2 \phi_n(x) dx = 2 \log 2.$$

$$\begin{aligned}\int_0^{\frac{1}{n}}\int_0^x \frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_0^{\frac{1}{n}}\int_0^x \frac{4nt}{t}dt\phi_n(x)dx\\ &= \int_0^{\frac{1}{n}} 4nx\phi_n(x)dx \;\leq\; 2.\end{aligned}$$

$$\begin{aligned}\int_{\frac{1}{n}}^2\int_0^{\frac{1}{n}} \frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_{\frac{1}{n}}^2\int_0^{\frac{1}{n}} \frac{4nt}{t}dt\phi_n(x)dx\\ &= \int_0^{\frac{1}{n}} 4\phi_n(x)dx \;\leq\; 2.\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{1}{n}}\int_x^{x+\frac{1}{n}} \frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_0^{\frac{1}{n}}\int_x^{x+\frac{1}{n}} \frac{4nx}{t}dt\phi_n(x)dx\\ &= \int_0^{\frac{1}{n}} 4nx\log\left(1+\frac{1}{nx}\right)\phi_n(x)dx \;\leq\; 2.\end{aligned}$$

For $\int_0^{\frac{1}{2}-\frac{1}{n}}\int_{x+\frac{1}{n}}^{\frac{1}{2}}$ we have $\frac{1}{n}\leq t-x\leq t+x\leq 1-\frac{1}{n}$ and, by (3.11), $A(x,t)=0$. Hence

$$\int_0^{\frac{1}{2}-\frac{1}{n}}\int_{x+\frac{1}{n}}^{\frac{1}{2}} \frac{A(x,t)}{t}dt\phi_n(x)dx = 0.$$

Next

$$\begin{aligned}\int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}}\int_{\frac{1}{n}}^{x+\frac{1}{n}} \frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}}\int_{\frac{1}{n}}^{x+\frac{1}{n}} \frac{4}{t}dt\phi_n(x)dx\\ &\leq 4\int_{\frac{1}{n}}^1 \log(nx+1)\phi_n(x)dx.\end{aligned}$$

Now, fix some $\alpha\in(0,1)$. Write

$$\begin{aligned}\int_{\frac{1}{n}}^1 \log(nx+1)\phi_n(x)dx &= \int_{\frac{1}{n}}^{\frac{1}{n^\alpha}}\cdots\, dx + \int_{\frac{1}{n^\alpha}}^1\cdots\, dx\\ &\leq \frac{\log(n^{1-\alpha}+1)}{2} + c_n\log(n+1)\int_{\frac{1}{n^\alpha}}^1\left(1-\frac{x^2}{4}\right)^{n^2}dx\\ &\leq \frac{\log(n^{1-\alpha}+1)}{2} + cn\log(n+1)\; e^{-\frac{1}{4}n^{2(1-\alpha)}}.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}\frac{\int_{\frac{1}{n}}^1\log(nx+1)\phi_n(x)dx}{\log n}\leq\frac{1-\alpha}{2}$$

and, since α is arbitrary in $(0,1)$,

$$\int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}}\int_{\frac{1}{n}}^{x+\frac{1}{n}} \frac{A(x,t)}{t}dt\phi_n(x)dx = o(\log n).$$

Finally,

$$\begin{aligned}\int_{\frac{1}{2}-\frac{1}{n}}^{2}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_{\frac{1}{2}-\frac{1}{n}}^{2}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{4}{t}dt\phi_n(x)dx\\ &\leq 4\log\frac{n}{2}\;c_n\int_{\frac{1}{2}-\frac{1}{n}}^{2}\left(1-\frac{x^2}{4}\right)^{n^2}dx\\ &\leq cn\log ne^{-\frac{1}{16}n^2} \;=\; o(1).\end{aligned}$$

□

4. The upper bound in the Theorem

We set

$$K_d=\sup_{P\in\mathcal{P}_d,\epsilon,R}\left|\int_{\epsilon\leq|t|\leq R}e^{iP(t)}\frac{dt}{t}\right|. \tag{4.1}$$

We take any polynomial P, of degree at most d, which we can assume has no constant term, that is, $P(0)=0$. We set $k=[\frac{d}{2}]$ and we write

$$\begin{aligned}P(t) &= a_1t+a_2t^2+\cdots+a_kt^k+a_{k+1}t^{k+1}+\cdots+a_dt^d\\ &= Q(t)+R(t),\end{aligned}$$

where $Q(t)=a_1t+a_2t^2+\cdots+a_kt^k$ and $R(t)=a_{k+1}t^{k+1}+\cdots+a_dt^d$. Let $|a_l|=max_{k+1\leq j\leq d}|a_j|$ for some $k+1\leq l\leq d$. By a change of variables in the integral in (4.1) we can assume that $|a_l|=1$ and thus that $|a_j|\leq 1$ for every $k+1\leq j\leq d$. Now split the integral in (4.1) in two parts as follows

$$\begin{aligned}\left|\int_{\epsilon\leq|t|\leq R}e^{iP(t)}\frac{dt}{t}\right| &\leq \left|\int_{\epsilon\leq|t|\leq 1}e^{iP(t)}\frac{dt}{t}\right|+\left|\int_{1\leq|t|\leq R}e^{iP(t)}\frac{dt}{t}\right|\\ &= I_1+I_2.\end{aligned} \tag{4.2}$$

For I_1 we have that

$$\begin{aligned}I_1 &\leq \left|\int_{\epsilon\leq|t|\leq 1}\left[e^{iP(t)}-e^{iQ(t)}\right]\frac{dt}{t}\right|+\left|\int_{\epsilon\leq|t|\leq 1}e^{iQ(t)}\frac{dt}{t}\right|\\ &\leq \int_{\epsilon\leq|t|\leq 1}\left|e^{iP(t)}-e^{iQ(t)}\right|\frac{dt}{t}+K_{[\frac{d}{2}]}\\ &\leq \int_{0\leq|t|\leq 1}\frac{|R(t)|}{t}dt+K_{[\frac{d}{2}]}\\ &\leq 2\sum_{j=k+1}^{d}\frac{|a_j|}{j}+K_{[\frac{d}{2}]}\;\leq\;\sum_{j=k+1}^{d}\frac{1}{j}+K_{[\frac{d}{2}]}\;\leq\;c+K_{[\frac{d}{2}]}.\end{aligned}$$

For the second integral in (4.2) we have that

$$I_2\leq\left|\int_{1\leq t\leq R}e^{iP(t)}\frac{dt}{t}\right|+\left|\int_{-R\leq t\leq -1}e^{iP(t)}\frac{dt}{t}\right|=I_2^{+}+I_2^{-}.$$

For some $\alpha>0$ to be defined later split I_2^{+} into two parts as follows:

$$I_2^{+}\leq\int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}}\frac{dt}{t}+\left|\int_{\{t\in[1,R]:|P'(t)|>\alpha\}}e^{iP(t)}\frac{dt}{t}\right|.$$

Since $\{t \in [1, R] : |P'(t)| > \alpha\}$ consists of at most $O(d)$ intervals where P' is monotonic, using Proposition 1 we get the bound

$$\left| \int_{\{t\in[1,R]:|P'(t)|>\alpha\}} e^{iP(t)} \frac{dt}{t} \right| \leq c\frac{d}{\alpha}.$$

For the logarithmic measure of the set $\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}$, observe that

$$\begin{aligned} \int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}} \frac{dt}{t} &\leq \sum_{m=0}^{\infty} \int_{\{t\in[2^m,2^{m+1}]:|P'(t)|\leq\alpha\}} \frac{dt}{t} \\ &= \sum_{m=0}^{\infty} \int_{\{2^m t\in[2^m,2^{m+1}]:|P'(2^m t)|\leq\alpha\}} \frac{dt}{t} \\ &= \sum_{m=0}^{\infty} \int_{2^m\{t\in[1,2]:|P'(2^m t)|\leq\alpha\}} \frac{dt}{t} \\ &= \sum_{m=0}^{\infty} \int_{\{t\in[1,2]:|P'(2^m t)|\leq\alpha\}} \frac{dt}{t}. \end{aligned}$$

We have thus showed that

$$\int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}} \frac{dt}{t} \leq \sum_{m=0}^{\infty} |\{t \in [1, 2] : |P'(2^m t)| \leq \alpha\}|. \tag{4.3}$$

In order to finish the proof we need a suitable estimate for the sublevel set of a polynomial. This is the content of the following lemma.

Lemma 4.1 (Vinogradov)**.** *Let $h(t) = b_0 + b_1 t + \cdots + b_n t^n$ be a real polynomial of degree n. Then,*

$$|\{t \in [1, 2] : |h(t)| \leq \alpha\}| \leq c\left(\frac{\alpha}{\max_{0\leq k\leq n} |b_k|}\right)^{\frac{1}{n}}.$$

This Lemma is due to Vinogradov [5]. We postpone the proof of Lemma 4.1 until after the end of the proof of the upper bound.

Consider the polynomial $P'(2^m t)$ with coefficients $ja_j 2^{m(j-1)}$, $1 \leq j \leq d$. Clearly, $\max_{1\leq j\leq d} |ja_j 2^{m(j-1)}| \geq |la_l 2^{m(l-1)}| \geq ([\frac{d}{2}] + 1)2^{m[\frac{d}{2}]}$. Using Lemma 4 and (4.3), we get

$$\int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}} \frac{dt}{t} \leq c\alpha^{\frac{1}{d-1}} \sum_{m=0}^{\infty} \left(\frac{1}{([\frac{d}{2}] + 1)2^{m[\frac{d}{2}]}}\right)^{\frac{1}{d-1}} \leq c\alpha^{\frac{1}{d-1}}.$$

Obviously, a similar estimate holds for I_2^-. Summing up the estimates we get

$$\left| \int_{\epsilon\leq|t|\leq R} e^{iP(t)} \frac{dt}{t} \right| \leq c + c\frac{d}{\alpha} + c\alpha^{\frac{1}{d-1}} + K_{[\frac{d}{2}]}.$$

Optimizing in α we get that

$$\left| \int_{\epsilon\leq|t|\leq R} e^{iP(t)} \frac{dt}{t} \right| \leq c + K_{[\frac{d}{2}]} \tag{4.4}$$

and hence

$$K_d \leq c + K_{[\frac{d}{2}]}.$$

In particular we have

$$K_{2^n} \le c + K_{2^{n-1}}.$$

Using induction on n we get that $K_{2^n} \le cn$. It is now trivial to show the inequality for general d. Indeed, if $2^{n-1} < d \le 2^n$ then $K_d \le K_{2^n} \le cn \le c \log d$.

For the sake of completeness we give the proof of Lemma 4.1.

Proof of Lemma 4.1. The set $E_\alpha = \{t \in [1,2] : |h(t)| \le \alpha\}$ is a union of intervals. We slide them together to form a single interval I of length $|E_\alpha|$ and pick $n+1$ equally spaced points in I. If we slide the intervals back to their original position we end up with $n+1$ points $x_0, x_1, x_2, \dots, x_n \in E_\alpha$ which satisfy

$$|x_j - x_k| \ge |E_\alpha| \frac{|j-k|}{n}. \tag{4.5}$$

The Lagrange polynomial which interpolates the values $h(x_0)$, $h(x_1)$,..., $h(x_n)$ coincides with $h(x)$:

$$h(x) = \sum_{j=0}^{n} h(x_j) \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}.$$

Therefore we get for the coefficients of h that

$$b_k = \sum_{j=0}^{n} h(x_j) \frac{(-1)^{n-k}\sigma_{n-k}(x_0,\dots,\hat{x_j},\dots,x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$$

for $k = 0, 1, \dots, n$. In the above formula $\sigma_{n-k}(x_0,\dots,\hat{x_j},\dots,x_n)$ is the $(n-k)$-th elementary symmetric function of $x_0,\dots,\hat{x_j},\dots,x_n$ where x_j is omitted. Using the estimate $\sigma_{n-k}(x_0,\dots,\hat{x_j},\dots,x_n) \le \binom{n}{n-k} 2^{n-k}$ together with (4.5) we get that, for every $k = 0, 1, \dots, n$,

$$\begin{aligned} |b_k| &\le \binom{n}{n-k} 2^{n-k} n^n \frac{\alpha}{|E_\alpha|^n} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \\ &= \binom{n}{n-k} 2^{2n-k} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n} \quad \le \quad c \frac{8^n}{\sqrt{n}} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n}, \end{aligned}$$

where we used the estimate $\binom{n}{n-k} \le \binom{n}{[\frac{n}{2}]} \le c \frac{2^n}{\sqrt{n}}$. Hence

$$\max_{0 \le k \le n} |b_k| \le c \frac{8^n}{\sqrt{n}} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n}$$

and solving with respect to $|E_\alpha|$ we get

$$|E_\alpha| \le c \left(\frac{\alpha}{\max_{0 \le k \le n} |b_k|} \right)^{\frac{1}{n}}.$$

□

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Department of Mathematics, University of Crete, Knossos Avenue 71409, Iraklio–Crete, Greece

E-mail address: ypar@math.uoc.gr