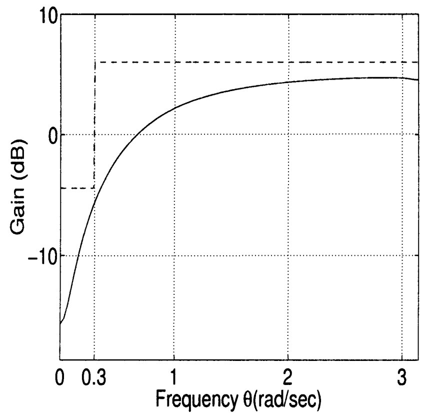

Fig. 4. Frequency response of the sensitivity function (21) (solid line) which meets the specification (dashed line).

We have used the method proposed in [5] to find a satisfactory sensitivity function. One sensitivity function in $\mathcal{S}_D^{\rm new}$ which satisfies the specification (20) was found as

$$S(z) = \frac{z^2 - 1.21}{z^2 + 0.57z - 0.30} \tag{21}$$

and the corresponding controller was computed by

$$C(z) = \frac{1 - S(z)}{P(z)S(z)} = \frac{-0.57z - 0.91}{z}.$$

Fig. 4 shows that the sensitivity function (21) indeed meets the specification (20).

This example suggests that we can use additional interpolation conditions for changing the shape of the infimal curve of the sensitivity gain, and for making given specifications to be achievable.

V. Conclusion

In this note, we have formulated a shaping limitation problem for rational sensitivity functions with a degree constraint as an optimization problem. An analytic solution to this problem was presented in a special case where a plant has some unstable poles, relative degree one and no unstable zero. The result is useful, especially in the approach proposed in [5], for circumventing unnecessary search for appropriate sensitivity functions of low degrees, and for motivating us to utilize the functions with higher degrees.

The shaping limitation problem for general cases (with arbitrary relative degree and arbitrary number of unstable zeros in a plant) amounts to solving a nonconvex optimization problem, as can be seen in (13). To solve the problem, we need to devise an efficient numerical method or we have to be content with some estimate of the optimum. This will be a subject of future research.

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Closed-Loop Shaping Based on Nevanlinna–Pick Interpolation With a Degree Bound

Ryozo Nagamune

***Abstract*—This note presents a novel method for shaping the frequency response of a single-input–single-output closed-loop system, based on the theory of Nevanlinna–Pick interpolation with degree constraint. The method imposes a degree bound on the closed-loop transfer function and searches for a function with a desired frequency response. Numerical examples illustrate the potential of the method in designing controllers with lower degrees than the ones obtained by conventional $\mathcal{H}^\infty$ controller design methods with weighting functions.**

***Index Terms*—Closed-loop shaping, degree bound, $\mathcal{H}^\infty$ control, Nevanlinna–Pick interpolation.**

I. Introduction

The objective of this note is to propose a new method for shaping the closed-loop frequency response in a $\mathcal{H}^\infty$ control framework. The shaping method is based on a recently developed theory of Nevanlinna–Pick interpolation with degree constraint ([1], [2]). The main difference from conventional methods in $\mathcal{H}^\infty$ control (see e.g., [3], [4], [6], and [12]) is that, in shaping frequency responses, we do *not* use weighting functions. The main advantage in our approach is that we typically obtain controllers of degree lower than the controller degree designed via conventional $\mathcal{H}^\infty$ control methods. Moreover, the closed-loop frequency response, which discontinuously depends on the choice of weighting functions in general, smoothly depends on our design parameters, which will facilitate controller design based on trial-and-error.

It is well-known that the suboptimal solution set to a scalar $\mathcal{H}^\infty$ control problem is equivalent to the solution set to the classical Nevanlinna–Pick interpolation problem [3]

$$\mathcal{S}_{\rm NP} := \{T_{cl} \in \mathcal{RH}^\infty : \|T_{cl}\|_\infty < \gamma, T_{cl}(z_j) = w_j,\ j = 0, 1, \ldots, n\}.$$

Here, $\{(z_j, w_j)\}_{j=0}^n$ are given self-conjugate pairs of complex numbers with z_j in an unstable region, γ is a given positive number and $\mathcal{RH}^\infty$ is the set of real rational proper stable functions. The interpolant T_{cl} represents some closed-loop transfer function in control problems. Henceforth, we assume that *the set* $\mathcal{S}_{\rm NP}$ *is nonempty*. The condition of this nonemptyness can actually be expressed by the positivity of a Hermitian matrix, called *the Pick matrix* (see, e.g., [11]).

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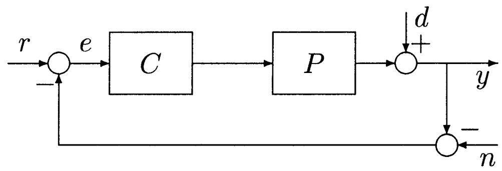


Fig. 1. Standard feedback system.

Remark 1.1: The interpolation constraints may involve derivative constraints, but all the results in this note can be applied even in such cases without any major modification.

A main feature of our approach to shaping of the frequency response of T_{cl} is that we do not let T_{cl} involve weighting functions and search for a desired function from a subset of $\mathcal{S}_{\rm NP}$

$$\mathcal{S}_{\rm NPDC} := \mathcal{S}_{\rm NP} \cap \mathcal{S}_{\rm DC}(n) \tag{1}$$

where the set $\mathcal{S}_{\rm DC}(n)$ is defined as

$$\mathcal{S}_{\rm DC}(n) := \{T_{cl}: \text{ rational and } \deg T_{cl} \leq n\}.$$

Note that the degree bound equals the number of interpolation constraints minus one. The reason for the choice is to guarantee that $\mathcal{S}_{\rm NPDC}$ is nonempty whenever $\mathcal{S}_{\rm NP}$ is. The motivation of degree constraint comes from engineering applications, where simple controllers are preferable in general. Section II will explain how the controller degree is affected by the degree of a closed-loop transfer function, using a standard control structure.

II. Controller Degree Bound

Let us consider the standard feedback system depicted in Fig. 1. Here, P is a given SISO plant, C is a controller to be designed so that the closed-loop system satisfies some prespecified requirements, and d (disturbance), y (output), n (measurement noise), r (reference), and e (error) are scalar signals. The transfer functions S from r to e (also from d to y) and T from r to y (also from n to y) are called the *sensitivity function* and the *complementary sensitivity function*, respectively, and can be expressed by $S := 1/(1 + PC)$ and $T := 1 - S$.

The merit of bounding the degree of a closed-loop transfer function S is attributed to the following fact.

Proposition 2.1: Suppose that the feedback system in Fig. 1 is internally stable. If the sensitivity function S is rational and the plant P is rational and strictly proper, then the controller obtained by

$$C = \frac{1 - S}{PS}$$

is proper and the following degree relationship holds:

$$\deg C \leq \deg P - n_z - n_p + \deg S \tag{2}$$

where n_z and n_p are the numbers of unstable zeros (including infinity) and poles of P, respectively.

Remark 2.2: The proof is straightforward but lengthy, and hence omitted here. Interested readers are referred to [9] for a complete proof.

Remark 2.3: This proposition is valid in both continuous-time and discrete-time cases, and also in cases of multiple and/or boundary unstable plant poles and/or zeros.

Since the plant P is given (and therefore the numbers n_z and n_p are also given), the degree bound of the controller is smaller as the degree of the sensitivity function is. Hence, it is meaningful to bound $\deg\ S$ for obtaining a simple controller.

III. Design Parameters for Closed-Loop Shaping With a Degree Bound

In this section, we shall derive our design parameters for closed-loop shaping. The theory that we use to derive design parameters for shaping is a complete parameterization of the set $\mathcal{S}_{\rm NPDC}$ presented in [2].

Theorem 3.1: [2] There is a diffeomorphism between the set of real polynomial pairs of degree n whose ratios are functions in $\mathcal{S}_{\rm NPDC}$

$$\mathcal{P} := \left\{(a, b): \deg a = \deg b = n, T_{cl} := \frac{b}{a} \in \mathcal{S}_{\rm NPDC}\right\}$$

and the set of real stable polynomials of degree n

$$\Sigma := \{\rho: \deg \rho = n, \rho(z) \neq 0, \qquad \forall\ |z| \geq 1\}. \tag{3}$$

In addition, for each $\rho \in \Sigma$, the corresponding polynomial pair $(a, b) \in \mathcal{P}$ satisfies

$$\gamma^2 a(z)a(z^{-1}) - b(z)b(z^{-1}) = \rho(z)\rho(z^{-1}). \tag{4}$$

Remark 3.2: Although this theorem is stated in discrete-time setting, it is also applicable to continuous-time cases via Möbius transform, as shown in one example later.

Since (4) can also be described as

$$\gamma^2 - T_{cl}(z)T_{cl}(z^{-1}) = \frac{\rho(z)\rho(z^{-1})}{a(z)a(z^{-1})} \tag{5}$$

the n roots of a real polynomial $\rho(z)$, which are self-conjugate and lie in the open unit disc, are called the *spectral zeros of* $\gamma^2 - T_{cl}(z)T_{cl}(z^{-1})$. Since the scaling of ρ by some scalar constant does not affect the function $T_{cl} = b/a$ (see [2, p. 823]), Theorem 3.1 asserts that each element in $\mathcal{S}_{\rm NPDC}$ corresponds to each set of n self-conjugate spectral zeros in a smooth manner. The smoothness is important in controller design involving trial-and-error. Thus, in our shaping approach, *the spectral zeros can be design parameters*. An efficient algorithm to compute T_{cl} from specified n spectral zeros has been developed based on a homotopy continuation method in [10].

It happens that the desired shape cannot be achieved by any function in the set $\mathcal{S}_{\rm NPDC}$ because of the degree restriction. In such cases, we can introduce additional interpolation constraints

$$T_{cl}(\lambda_j) = \alpha_j, \qquad j = 1, \ldots, n_e \tag{6}$$

with $|\lambda_j| > 1$, and redefine the set $\mathcal{S}_{\rm NPDC}$ with these constraints as

$$\mathcal{S}_{\rm NPDC} := \mathcal{S}_{\rm NP} \cap \{T_{cl}: T_{cl} \text{ meets } (6)\} \cap \mathcal{S}_{\rm DC}(n + n_e). \tag{7}$$

Then, we can again use Theorem 3.1 to characterize the set $\mathcal{S}_{\rm NPDC}$ in terms of $n + n_e$ spectral zeros, since the degree bound $n + n_e$ in (7) equals the total number of interpolation constraints minus one. The additional constraints (6) can be chosen freely except with conditions that

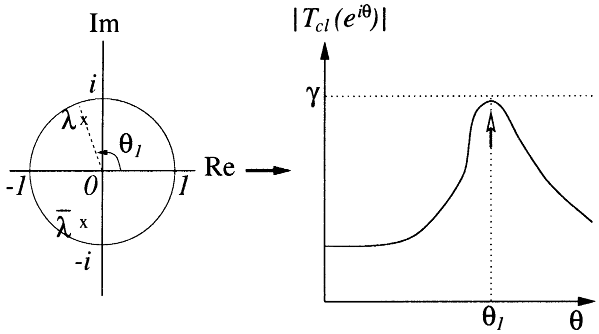


Fig. 2. Influences of a spectral zero $\boldsymbol{\lambda}$ on $|\boldsymbol{T}_{cl}|$.

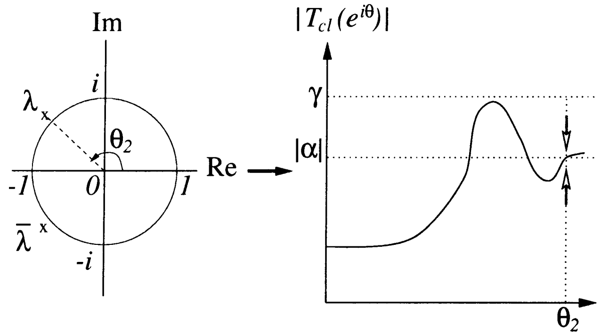


Fig. 3. Influences of an additional interpolation constraint $\boldsymbol{T}_{cl}(\boldsymbol{\lambda}) = \boldsymbol{\alpha}$ on $|\boldsymbol{T}_{cl}|$.

they do not violate the positivity of the corresponding Pick matrix[1] and that the total interpolation data set forms self-conjugate pairs. Consequently, *points and values for the additional constraints can be design parameters for shaping*.

In summary, there are two kinds of design parameters to be tuned for frequency shaping, namely, *spectral zeros* and *additional interpolation constraints*.

IV. Shaping Strategies

In this section, we will propose general strategies to tune the design parameters derived in Section III for the shaping purpose.

A. Tuning of Spectral Zeros

First, the spectral zero λ near the unit circle with the angle θ_1 lifts $|T_{cl}|$ up to the level near γ at the frequency θ_1 (see Fig. 2). This is because of the following relation:

$$\gamma^2 - \left|T_{cl}(e^{i\theta_1})\right|^2 \approx \gamma^2 - T_{cl}(\lambda)T_{cl}(\lambda^{-1}) = 0.$$

Given requirements in the frequency domain, we usually know in advance the frequencies where we *must* have high gain in order to get low gain over some specified frequencies. Thus, spectral zeros near the unit circle are helpful for the achievement of specifications.

B. Tuning of Points for Additional Interpolation Constraints

Next, if a point λ of an additional interpolation constraint is close to the unit circle at the angle θ_2, the additional constraint $T_{cl}(\lambda) = \alpha$ will fix the magnitude of $|T_{cl}|$ close to $|\alpha|$ and the phase of T_{cl} close to that of α at the frequency θ_2 (see Fig. 3). Consequently, additional interpolation constraints can control both gain and phase of T_{cl}, and hence they are quite useful for shaping purpose. However, in view of (7), the introduction of additional constraints increases the degree bound of T_{cl}, which is often undesirable in control. Thus, we should try to use additional constraints as little as possible.

We will mainly utilize these two strategies for shaping of the closed-loop frequency response, instead of using weighting functions. The design parameters assigned away from the unit circle may also be useful (see the first example in Section V), but the influence of those parameters is in general not clear.

[1]Since it is difficult to characterize the set of all $\{(\boldsymbol{\lambda}_j, \boldsymbol{\alpha}_j)\}_{j=1}^{n_e}$ which maintain the positivity of the Pick matrix, we always *check* the positivity after the selection of $\{(\boldsymbol{\lambda}_j, \boldsymbol{\alpha}_j)\}_{j=1}^{n_e}$.

V. Examples Illustrating the New Design Method

In this section, we will present two numerical examples of closed-loop shaping. To illustrate the efficiency of the proposed method, we compare our results with those obtained by some conventional methods. In the examples, we consider the same feedback structure as in Fig. 1 in Section II, and focus on designing a sensitivity function with a desired frequency response.

A. Sensitivity Reduction for a Continuous-Time System

First example deals with the same sensitivity reduction problem as the one presented in [4, p. 77]. Suppose that the continuous-time plant is described by

$$P(s) := \frac{(s-1)(s-2)}{(s+1)(s^2+s+1)}$$

which is stable but has nonminimum phase zeros at $s = \infty, 1$ and 2. The performance specification on the sensitivity function was given in [4] by

$$|S(i\omega)| \le 0.1, \qquad 0 \le \omega < 0.01 \text{ (rad/s)}. \tag{8}$$

The specification (8) is not complete, in the sense that there is no constraint outside the frequency range $[0, 0.01)$ (rad/s) (see [5, p. 25]). Therefore, we add the $\mathcal{H}^\infty$ norm bound constraint

$$\|S\|_\infty < \gamma := 1.3 \tag{9}$$

and try to find a sensitivity function meeting (8) and (9) as well as internal stability of the feedback system.

We shall first construct the set $\mathcal{S}_{\text{NPDC}}$ in (1) by regarding T_{cl} as S. In this case, the interpolation constraints comes from the well-known result on conditions that S must satisfy for internal stability (see, e.g., [12])

$$S(\infty) = S(1) = S(2) = 1.$$

The set $\mathcal{S}_{\text{NPDC}}$ is therefore expressed as

$$\mathcal{S}_{\text{NPDC}} = \{S \in \mathcal{RH}^\infty : \|S\|_\infty < 1.3, \\ S(\infty) = S(1) = S(2) = 1\} \cap \mathcal{S}_{\text{DC}}(2). \tag{10}$$

For consistency with our formulation in Section IV, we use some transformations.

First, by Möbius transform $s = (z-1)/(z+1)$ which conformally maps the right half-plane into the complement of the unit disc, the set corresponding to $\mathcal{S}_{\text{NPDC}}$ is obtained in the discrete-time setting

$$\mathcal{S}_{\text{NPDC}}^d := \{S_d \in \mathcal{RH}^\infty, \|S_d\|_\infty < 1.3, \\ S_d(-1) = S_d(\infty) = S_d(-3) = 1\} \cap \mathcal{S}_{\text{DC}}(2) \tag{11}$$

where the $\mathcal{H}^\infty$ norm is given by $\|S_d\|_\infty := \sup_{\theta\in[-\pi,\pi]} \left|S_d(e^{i\theta})\right|$. The specification (8) is also transformed as

$$|S_d(e^{i\theta})| \le 0.1, \qquad 0 \le \theta < 0.02 \text{ (rad/s)}. \tag{12}$$

Next, note that the condition $S_d(-1) = 1$ in (11) is a boundary interpolation constraint, which makes the Nevanlinna–Pick interpolation problem complicated. We circumvent this difficulty by a variable change and define another function as follows:

$$\tilde{S}_d(z) := S_d\left(\frac{z}{1+\varepsilon}\right)$$

for some small $\varepsilon > 0$. This trick is the discrete-time version of the one presented in [7]. If, for a small ε, we find a function $\tilde{S}_d(z)$ which satisfies (12) from the set

$$\tilde{\mathcal{S}}^d_{\rm NPDC} := \left\{ \tilde{S}_d \in \mathcal{RH}^\infty : \|\tilde{S}_d\|_\infty < 1.3 \right. \\ \left. \tilde{S}_d(-(1+\varepsilon)) = \tilde{S}_d(\infty) = \tilde{S}_d(-3(1+\varepsilon)) = 1 \right\} \cap \mathcal{S}_{\rm DC}(2)$$

then $S_d(z) = \tilde{S}_d((1+\varepsilon)z)$ is in $\mathcal{RH}^\infty$ (i.e., analytic in the complement of the open unit disc), satisfies the interpolation constraints and the degree condition in (11), and meets the specification (12) and the $\mathcal{H}^\infty$ norm condition in (11) approximately. We set the value $\varepsilon = 0.005$ in this example.

It can be shown that the set $\tilde{\mathcal{S}}^d_{\rm NPDC}$ has only one element, namely $\tilde{\mathcal{S}}^d_{\rm NPDC} = \left\{\tilde{S}_d : \tilde{S}_d \equiv 1\right\}$, due to the pole-zero cancellations (see [8] and [9]). Obviously, the function $\tilde{S}_d(z) \equiv 1$ does not satisfy (12) and, hence, it is necessary to add some interpolation constraints for the achievement of the specification. Taking into account the low gain requirement at low frequencies, we introduce one additional interpolation constraint $\tilde{S}_d(1.002) = 0$, and redefine the set $\tilde{\mathcal{S}}^d_{\rm NPDC}$ by

$$\tilde{\mathcal{S}}^d_{\rm NPDC} := \left\{ \tilde{S}_d \in \mathcal{RH}^\infty : \|\tilde{S}_d\|_\infty < 1.3, \tilde{S}_d(1.002) = 0 \right. \\ \left. \tilde{S}_d(-1.005) = \tilde{S}_d(\infty) = \tilde{S}_d(-3.015) = 1 \right\} \cap \mathcal{S}_{\rm DC}(3). \quad (13)$$

Notice that the degree bound in (13) is incremented by one.

Now we have three spectral zeros of the function $1.3^2 - \tilde{S}_d(z)\tilde{S}_d(z^{-1})$ as design parameters; each choice of three spectral zeros determines an element in the set $\tilde{\mathcal{S}}^d_{\rm NPDC}$. When we locate these spectral zeros at $z = 0.25$ and $0.2e^{\pm 2.3562i}$, the corresponding function in the set $\tilde{\mathcal{S}}^d_{\rm NPDC}$ is calculated as

$$\tilde{S}_d(z) = \frac{z^3 + 0.0773z^2 - 0.6047z - 0.4778}{z^3 + 0.1475z^2 - 0.3228z - 0.2653}. \quad (14)$$

Due to the reverse variable changes, the sensitivity function $S(s)$ in the set $\mathcal{S}_{\rm NPDC}$ which corresponds to the function in (14) becomes

$$S(s) = \tilde{S}_d\left(1.005\frac{1+s}{1-s}\right) = \frac{s^3 + 2.6532s^2 + 6.3989s + 0.0096}{s^3 + 3.0042s^2 + 5.3459s + 0.7115}$$

whose frequency response is shown in Fig. 4 with the result in [4]. The controller can be computed by

$$C(s) = \frac{1 - S(s)}{P(s)S(s)} = \frac{0.3510s^3 + 0.7020s^2 + 0.7020s + 0.3510}{s^3 + 2.6532s^2 + 6.3989s + 0.0096}.$$

We can verify that the controller degree meets (2).

As can be seen in Fig. 4, our method has indeed generated a satisfactory sensitivity function by a controller of degree three. On the other hand, the sensitivity function presented in [4] does not quite satisfy the specification, though a controller of degree four is used there.

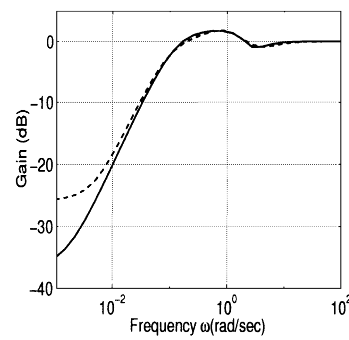


Fig. 4. Frequency response of S obtained by the proposed method (solid line) and that in the book (dashed line).

In this example, the spectral zeros are located far from the unit circle and, hence, it is not clear how these parameters affect the resulting frequency shape. Next, we will give an example in which we can see the influences of the design parameters more clearly.

B. Mixed Sensitivity Reduction for a Discrete-Time System

Here, we consider a more advanced and practical control problem than the sensitivity reduction problem, namely the mixed sensitivity reduction problem. We again assume the feedback structure shown in Fig. 1. Suppose that the plant P is a discrete-time transfer function given by $P(z) = 1/(z - 1.05)$, which has one unstable pole at $z = 1.05$ and one zero at infinity.

The requirements that the closed-loop system must fulfill are assumed to be as follows.

R1) $|S(e^{i\theta})| < 0.1 (= -20 \text{ dB})$ for $\theta \in [0, 0.3]$ (rad/s).

R2) $|T(e^{i\theta})| = |1 - S(e^{i\theta})| < 0.5 (\approx -6.02 \text{ dB})$ for $\theta \in [2.5, \pi]$ (rad/s).

R3) $|S(e^{i\theta})| < 2 (\approx 6.02 \text{ dB})$ for $\theta \in [0, \pi]$ (rad/s).

This type of requirements are typical in practice (see, e.g., [5]). Note that all the previous requirements can be described in terms of only the sensitivity function S. Hence, regarding T_{cl} in (1) as S, we will construct the set $\mathcal{S}_{\rm NPDC}$ as in the first example. Since the uniform bound of $|S|$ should be less than two by the requirement $R3$, we set $\gamma = 2$. Then, the set $\mathcal{S}_{\rm NPDC}$ in (1) becomes

$$\mathcal{S}_{\rm NPDC} = \{S \in \mathcal{RH}^\infty : \|S\|_\infty < 2, S(\infty) = 1, \\ S(1.05) = 0\} \cap \mathcal{S}_{\rm DC}(1). \quad (15)$$

In this example, we can prove that there is no element in the set $\mathcal{S}_{\rm NPDC}$ in (15) which meets all the requirements $R1$, $R2$ and $R3$ simultaneously, because of the tight degree bound $\deg S \leq 1$ (see [8] and [9]). Therefore, we must introduce some additional interpolation constraints to increase the flexibility of the design. When we choose the constraints as $S(1.01e^{\pm 0.3i}) = 0$ for the requirement $R1$, and $S(-1.01) = 1$ for the requirement $R2$, the set $\mathcal{S}_{\rm NPDC}$ is redefined, according to (7), as

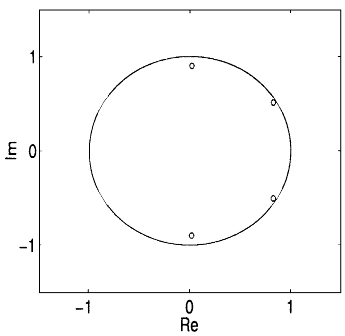
$$\mathcal{S}_{\rm NPDC} := \left\{ S \in \mathcal{RH}^\infty : \|S\|_\infty < 2, S(\infty) = S(-1.01) = 1 \right. \\ \left. S(1.05) = 0, S(1.01e^{\pm 0.3i}) = 0 \right\} \cap \mathcal{S}_{\rm DC}(4). \quad (16)$$

By selecting four spectral zeros of the function $2^2 - S(z)S(z^{-1})$ at $z = 0.97e^{\pm 0.55i}$, $0.9e^{\pm 1.55i}$ [Fig. 5(a)], we can pick up one element from the set $\mathcal{S}_{\rm NPDC}$ in (16) as

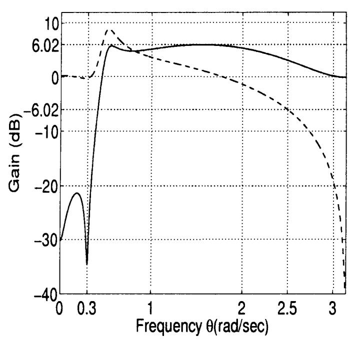
$$S(z) = \frac{z^4 - 2.4048z^3 + 1.3331z^2 + 0.6804z - 0.6158}{z^4 - 1.6159z^3 + 0.8469z^2 - 0.0030z + 0.0028}. \quad (17)$$

$$W_1(z) = \frac{2.9524z^5 - 9.3586z^4 + 11.2760z^3 - 6.0032z^2 + 1.1365z}{3.4242z^5 - 13.1648z^4 + 19.4516z^3 - 13.3012z^2 + 3.7955z - 0.2053}$$
$$W_2(z) = \frac{0.4762z - 0.2381}{z + 0.9}$$

$$S(z) = \frac{z^7 - 4.2044z^6 + 6.3380z^5 - 3.1384z^4 - 1.6153z^3 + 2.3591z^2 - 0.7824z + 0.0435}{z^7 - 3.5584z^6 + 4.5196z^5 - 1.8352z^4 - 0.8501z^3 + 0.9286z^2 - 0.2043z + 0.0000}$$
$$C(z) = \frac{0.6460z^6 - 1.8184z^5 + 1.3032z^4 + 0.7652z^3 - 1.4304z^2 + 0.5781z - 0.0435}{z^6 - 3.1544z^5 + 3.0259z^4 + 0.0388z^3 - 1.5746z^2 + 0.7057z - 0.0414}.$$



(a)



(b)

Fig. 5. Location of the specified spectral zeros and the frequency responses of $\boldsymbol{S}$ and $\boldsymbol{T}$. (a) Location of the spectral zeros (0). (b) Gains of $\boldsymbol{S}$ (solid line) and of $\boldsymbol{T}$ (dashed line).

The frequency response of S in (17) is shown in Fig. 5(b) along with the frequency response of $T = 1 - S$. In the figure, we can verify not only the achievement of all the requirements but also the influences of the spectral zeros and the additional interpolation constraints on the resulting frequency shapes. The corresponding controller is computed as

$$C(z) = \frac{1 - S(z)}{P(z)S(z)} = \frac{0.7889z^3 - 0.4863z^2 - 0.6834z + 0.6186}{z^3 - 1.3548z^2 - 0.0894z + 0.5865}$$

whose degree is consistent with (2).

We tried to solve the same control problem by the conventional weighted $\mathcal{H}^\infty$ technique, that is, to solve the optimization problem:

$$\inf_C \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty \tag{18}$$

subject to internal stability. We have searched appropriate weighting functions W_1 and W_2 in an increasing order of degrees. When we used

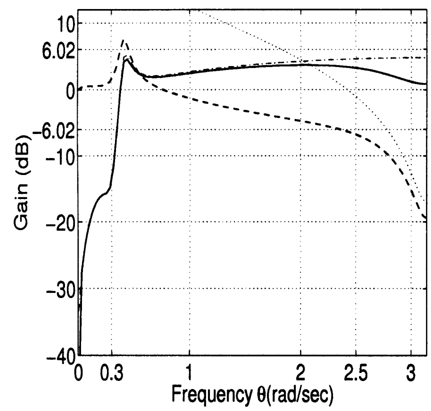


Fig. 6. Frequency responses of $\boldsymbol{S}$ (solid line), $\boldsymbol{T}$ (dashed line), $\boldsymbol{W}_1^{-1}$ (dash-dot line) and $\boldsymbol{W}_2^{-1}$ (dotted line).

weighting functions represented by the first equation shown at the top of the page, we obtained the optimal sensitivity function and controller as shown in the second equation at the top of the page. The frequency responses S and $T = 1 - S$ are shown in Fig. 6.

In Fig. 6, note that the requirement $R1$ is not fulfilled. Indeed, it was difficult in this example to find an acceptable solution by the weighted $\mathcal{H}^\infty$ method. Besides, even if we could find a desired one by further trial and error, the degree of the controller will still be around six, whereas our approach *does* generate a satisfactory solution with a controller whose degree is only three. This demonstrates the efficiency of the proposed shaping technique.

VI. Conclusion

In this note, we proposed a new approach to shaping of the closed-loop frequency response. Spectral zeros of a certain function related to the closed-loop transfer function and additional interpolation constraints was used to shape the frequency response. We gave some rules of thumb to tune these design parameters. The main advantage of the proposed approach is that the degree of the obtained controller becomes relatively low, which was illustrated by some numerical examples.

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Fault Tolerant Control: A Simultaneous Stabilization Result

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***Abstract*—This note discusses the problem of designing fault tolerant compensators that stabilize a given system both in the nominal situation, as well as in the situation where one of the sensors or one of the actuators has failed. It is shown that such compensators always exist, provided that the system is detectable from each output and that it is stabilizable. The proof of this result is constructive, and a worked example shows how to design a fault tolerant compensator for a simple, yet challenging system. A family of second order systems is described that requires fault tolerant compensators of arbitrarily high order.**

***Index Terms*—Controller order, fault tolerant control, sensor faults, simultaneous stabilization.**

I. Introduction

The interest for using fault tolerant controllers is increasing. A number of theoretical results as well as application examples has now been described in the literature; see, e.g., [1]–[9] to mention some of the relevant references in this area.

The approaches to fault tolerant control can be divided into two main classes: *Active* fault tolerant control and *passive* fault tolerant control. In active fault tolerant control, the idea is to introduce a fault detection and isolation block in the control system. Whenever a fault is detected and isolated, a supervisory system takes action, and modifies the structure and/or the parameters of the feedback control system. In contrast, in the passive fault tolerant control approach, a fixed compensator is designed, that will maintain (at least) stability if a fault occurs in the system.

This note will only discuss the passive fault tolerant control approach, also sometimes referred to as *reliable* control. This approach has mainly two motivations. First, designing a fixed compensator can be made in much simpler hardware and software, and might thus be admissible in more applications. Second, classical reliability theory states that the reliability of a system decreases rapidly with the complexity of the system. Hence, although an active fault tolerant control system might in principle accomodate specific faults very efficiently, the added complexity of the overall system by the fault detection system and the supervisory system itself, might in fact sometimes deteriorate plant reliability.

In [10], a fault tolerant control problem has been addressed for systems, where specific sensors could potentially fail such that the corresponding outputs were unavailable for feedback, whereas other outputs were assumed to be available at all times.

In [11, Sec. 5.5], the question of fault tolerant parallel compensation has been discussed, i.e., whether it is possible to design two compensators such that any of them alone or both in parallel will internally stabilize the closed loop system.

The existence results given in [10] and [11] can be considered to be special cases of the main results of this note.

In this note, we shall consider systems for which any sensor (or in the dual case any actuator) might fail, and we wish to determine for which systems such (passive) fault tolerant compensators exist. The main results state that the only precondition for the existence of solutions to this fault tolerant control problem is just stabilizability from each input and detectability of the system from each output.

Throughout this note, $\mathcal{RP}^{p\times m}$ shall denote the set of proper, real-rational functions taking values in $\mathcal{C}^{p\times m}$, and $\mathcal{RSP}^{p\times m}$ shall denote the set of strictly proper, real-rational functions taking values in $\mathcal{C}^{p\times m}$. $\mathcal{RH}_\infty^{p\times m}$ shall denote the set of stable, proper, real-rational functions taking values in $\mathcal{C}^{p\times m}$. The notation $\{s \in \mathcal{R}_{+\infty} : B(s) = 0\}$ will be used as shorthand for zeros of $B(\cdot)$ on the positive real line. The set includes the point at infinity if $\lim_{s\to\infty} B(s) = 0$. For matrices A, B, C, D of compatible dimensions, the expression

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

will be used to denote the transfer function $G(s) = C(sI - A)^{-1}B + D$. Real-rational functions will be indicated by their dependency of a complex variable s (as in $G(s), K(s)$), although the dependency of s will be suppressed in the notation (as in G, K), where no misunderstanding should be possible.

II. Problem Formulation

Consider a system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y_1 &= C_1 x \\ &\vdots \\ y_p &= C_p x \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n, u \in \mathcal{R}^m, y_i \in \mathcal{R}, i = 1\ldots,p$ and $A, B, C_i, i = 1\ldots,p$ are matrices of compatible dimensions. Each of the p measure-

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