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Brief paper

Sensitivity shaping with degree constraint by nonlinear least-squares optimization☆

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Abstract

This paper presents a new approach to shaping of the frequency response of the sensitivity function. In this approach, a desired frequency response is assumed to be specified at a finite number of frequency points. A sensitivity shaping problem is formulated as an approximation problem to the desired frequency response with a function in a class of sensitivity functions with a degree bound. The sensitivity shaping problem is reduced to a finite-dimensional constrained nonlinear least-squares optimization problem. To solve the optimization problem numerically, standard algorithms for an unconstrained version of nonlinear least-squares problems are modified to incorporate the constraint. Numerical examples illustrate how these design parameters are tuned in an intuitive manner, as well as how the design proceeds in actual control problems.

Keywords: Sensitivity function; Shaping; Nonlinear least-squares optimization; Degree bound; Schur polynomial

1. Introduction

It is well-known that the *sensitivity function*, denoted by S, is one of the essential factors in determining performances of feedback systems, such as robust stability and tracking. It has been recognized since the classical control era that sensible control design can be accomplished by designing S appropriately. Thus, it is significant to develop systematic design tools for S.

Much effort has been made for such development, e.g., classical control methodologies such as PID-based control and lead–lag compensations (Horowitz, 1992), both open-loop (McFarlane & Glover, 1992) and closed-loop shaping techniques in H^∞ control (e.g., Doyle, Francis, & Tannenbaum, 1992), an approach based on positive polynomials (Henrion, Šebek, & Kučera, 2003; Henrion, 2003), to name a few. However, these previous tools heavily require designers' engineering experience, knowledge and intuition in manual selection of design parameters such as weighting functions. Even for experienced designers, the manual selection involves trial and error, which is by no means an easy task.

In Byrnes, Georgiou, and Lindquist (2001), a new paradigm is suggested for sensitivity shaping without weighting functions in an H^∞ control framework, and it is further developed in Nagamune and Lindquist (2001) and Nagamune (2004a). The paradigm is based on analytic interpolation theory with degree constraint initiated in Georgiou (1983, 1987a) and carried to completion in Byrnes, Lindquist, Gusev, and Matveev (1995), Byrnes, Gusev, and Lindquist (1998), and Byrnes et al. (2001). In this paradigm, design parameters are *spectral zeros* (or equivalently, *Schur polynomials*) and additional interpolation conditions. We have illustrated through numerical examples that the approach in Nagamune (2004a) often generates controllers of lower degrees than conventional

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H^∞ controller design does. (See also Blomqvist & Nagamune, 2005; Blomqvist, Lindquist, & Nagamune, 2003 for such examples.) However, only guidelines have been provided for the tuning of spectral zeros in Nagamune (2004a), and it would be convenient to have a method for determining these parameters in a certain optimal sense. This is the motivation of this paper.

In this paper, for scalar systems, we shall propose a new method to design S in the frequency domain. We will formulate a sensitivity shaping problem as an approximation problem, for a function in a class of S with a bounded degree, relative to a desired frequency response given at a finite number of frequency points. The problem can be reduced to a finite-dimensional constrained nonlinear least-squares (NLS) optimization problem. To solve the NLS problem numerically, we will use algorithms which are modifications of standard algorithms originally developed for unconstrained NLS optimization. Since the optimization problem is nonconvex, sensible selection of the initial point for the algorithms is crucial. Some rules of thumb for such selection are suggested. Although trial-and-error process is necessary for choosing appropriate design parameters even in our approach, we believe that the way of selecting and tuning design parameters is more intuitive than that in previous approaches. This point will be illustrated through control design examples.

In addition to the advantage of intuitive design, another important advantage of our approach over the conventional H^∞ methodology, including the LMI-based approach (Iwasaki & Skelton, 1994; Gahinet & Apkarian, 1994), is as follows. To shape the frequency response, we will not rely on weighting functions which typically cause the increase of controller degrees. In fact, although we will introduce some "weights" which play a similar role to weighting functions, the weights do *not* affect controller degrees. Also, the weights in our approach do not assume any rationality, while the weighting functions should be rational in most cases. The lack of rationality requirement increases the design flexibility.

2. A sensitivity shaping problem

Consider the feedback system depicted in Fig. 1. Here, P is a given scalar real rational discrete-time plant[1] and C is a controller to be designed to fulfill both internal stability of the feedback system and some given performance specifications. In this paper, we consider only such specifications

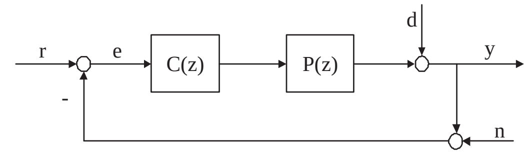

Fig. 1. The feedback system.

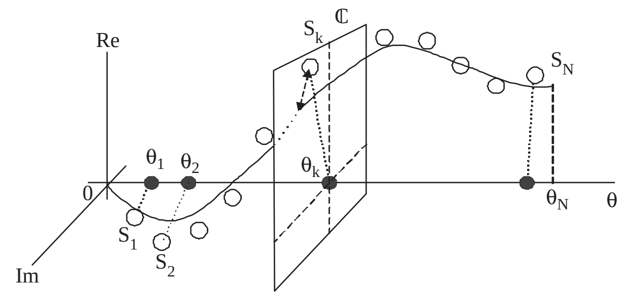

Fig. 2. The frequency response of a "best-approximate" sensitivity function S (solid curve) to data s_k (circles) at frequencies θ_k (black dots on θ-axis).

that can be expressed in terms of the *sensitivity function*

$$S(z) := (1 + P(z)C(z))^{-1} \tag{1}$$

in the frequency domain. (Note that frequency domain specifications on $T := 1 - S$, CS and PS can be transformed into that on S; see Helton & Marino (1998).) More precisely, we assume that, at a given finite number N of frequencies $\boldsymbol{\theta} := \{\theta_k\}_{k=1}^N \subset [0, \pi]$, a "desired" frequency response $\boldsymbol{s} := \{s_k\}_{k=1}^N \subset \mathbb{C}$ of S is given, and we try to find a "best-approximate" sensitivity function S from a class of "allowable" sensitivity functions (see Fig. 2[2]). Next, what we mean by "best-approximate" and "allowable" will be explained.

To clarify the meaning of "best-approximation," we need to introduce a discrepancy between the desired frequency response data $(\boldsymbol{\theta}, \boldsymbol{s})$ and a sensitivity function S. In this paper, we use the weighted squares sum[3]:

$$d_{\boldsymbol{w}}((\boldsymbol{\theta}, \boldsymbol{s}), S) := \frac{1}{2}\sum_{k=1}^{N} \frac{w_k}{|s_k|^2}|S(e^{i\theta_k}) - s_k|^2, \tag{2}$$

where the weights $\boldsymbol{w} := \{w_k\}_{k=1}^N$ are positive scalars to be chosen by the designer; if one wants a better approximation at the frequency θ_k, one can choose a large w_k relative to weights at other frequencies. We remark that any specification of the form $\sum_k w_k|H(e^{i\theta_k}) - h_k|^2$ can be expressed as (2), where w_k and h_k are fixed weights and fixed desired frequency responses given at frequency grid points, and H can be equal to S, CS, PS, or PCS. In (2), the term $|S(e^{i\theta_k}) - s_k|$ is the distance of two complex numbers $S(e^{i\theta_k})$ and s_k in

[1] To be consistent with the mathematical setting in Byrnes et al. (2001); Georgiou (1987b, 1999), we deal with only scalar discrete-time systems in this paper. However, as will be shown in Section 5, our method is applicable even to continuous-time systems with bilinear transformations.

[2] The 3-D plot in Fig. 2 can be interpreted as a combination of the gain plot and the phase plot in the Bode diagram.

[3] Division by $|s_k|^2$ is for normalization. We assume $s_k \neq 0$.

the complex plane; see the dashed arrow in Fig. 2. A "best-approximate" sensitivity function S is the one which minimizes this discrepancy for given $(\boldsymbol{w}, \boldsymbol{\theta}, \boldsymbol{s})$.

In this paper, we call a sensitivity function S "allowable" if it satisfies the following four conditions:

(C1) the internal stability condition,
(C2) n_{e} conditions $S(\lambda_j) = \eta_j,\ j = 1, \ldots, n_{\mathrm{e}}$, which are specified at points $\lambda_j \in \mathbb{C}$ outside the unit disc,
(C3) the H^∞ norm bound condition $\|S\|_\infty < \gamma$, where for a stable rational function $S, \|S\|_\infty := \max_{\theta\in[-\pi,\pi]}|S(\mathrm{e}^{\mathrm{i}\theta})|$, and γ is chosen to be large enough so that there exists an S which satisfies (C1), (C2), and $\|S\|_\infty < \gamma$, and
(C4) rationality and a degree condition, i.e., S must be real rational and $\deg S \leqslant n := n_{\mathrm{p}} + n_{\mathrm{z}} + n_{\mathrm{e}} - 1$, where n_{p} and n_{z} are the number of unstable poles and zeros of the plant P, respectively.

The motivations for these conditions are as follows. (C1) is a standard requirement for any practical feedback system. (C2)–(C4) are motivated by the work in Byrnes et al. (2001) and Nagamune (2004a). (C2) increases the flexibility of the shaping design. (See Nagamune (2004a), where we call these conditions *additional interpolation constraints*.) We may not need this condition for achieving required performance, in which case, we just set $n_{\mathrm{e}} = 0$. As for (C3), there are motivations from both control viewpoint and optimization viewpoint. From control viewpoint, the constraint (C3) is called the *gain-phase margin constraint* (see Helton & Marino (1998), p. 20), and (C3) is important to avoid a large peak gain of S for a large stability margin. From optimization viewpoint, (C3) is useful to avoid choosing an initial point far from the solution in nonconvex optimization that we need to solve; see Section 4. (C4) restricts a class to a degree constrained one, which eventually leads to a restriction on the controller degree; see Nagamune (2004a), Proposition 2.1.

With definitions of the discrepancy $d_{\boldsymbol{w}}$ in (2) and the class of allowable sensitivity functions

$$\mathscr{S} := \{S : S \text{ satisfies (C1)–(C4)}\}, \tag{3}$$

the *sensitivity shaping problem* to be considered in this paper is, for given weights $\boldsymbol{w}$ and data $(\boldsymbol{\theta}, \boldsymbol{s})$, to solve an optimization problem:

$$\inf_{S\in\mathscr{S}} d_{\boldsymbol{w}}((\boldsymbol{\theta}, \boldsymbol{s}), S). \tag{4}$$

Remark 2.1. Condition (C4) is the main difference of $\mathscr{S}$ from the suboptimal solution set to the standard H^∞ control problem. In conventional reduced-order H^∞ controller design, we will have a rank condition (Gahinet & Apkarian, 1994). Such condition will be difficult to exploit as a constraint in optimization, since the feasible set becomes a "thin" set; perturbation of optimization parameters easily violate the feasibility. To the contrary, by bounding $\deg S$ as (C4) instead of $\deg C$, we can formulate a "nicer" optimization problem, in the sense that the feasible set becomes an open connected set. See Eq. (7).

3. A finite-dimensional constrained nonlinear least-squares problem

In this section, we will show that the sensitivity shaping problem (4) can be reduced to a finite-dimensional constrained NLS problem.

Suppose that S is a feasible point of the optimization problem (4), i.e., $S \in \mathscr{S}$. Then, since S satisfies (C4), it can be factored as $S(z) = b(z)/a(z)$, where $a(z) := \boldsymbol{z}^{\mathrm{T}}\boldsymbol{\alpha}$, $b(z) := \boldsymbol{z}^{\mathrm{T}}\boldsymbol{\beta}$, $\boldsymbol{\alpha} \in \mathbb{R}^{n+1}$, $\boldsymbol{\beta} \in \mathbb{R}^{n+1}$ and $\boldsymbol{z} := [z^n, \ldots, z, 1]^{\mathrm{T}}$. In addition, since S satisfies (C1) and (C2), S needs to fulfill $n_{\mathrm{p}} + n_{\mathrm{z}} + n_{\mathrm{e}} (= n + 1)$ interpolation/derivative conditions at unstable poles and zeros (including infinite zeros) of the plant, as well as at points specified by (C2). Due to these $(n + 1)$ conditions, we can derive a linear relation between $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ as $\boldsymbol{\beta} = K\boldsymbol{\alpha}$, for a uniquely determined real matrix K. See Nagamune and Blomqvist (2004) for the detail of the construction of K. Besides, since S satisfies (C3), S must be stable and meet the norm condition $\|S\|_\infty < \gamma$. The stability condition can be stated that the denominator vector $\boldsymbol{\alpha}$ needs to be in the Schur stability region:

$$\mathfrak{S} := \left\{ \begin{array}{c} \boldsymbol{\alpha} := [\alpha_0, \ldots, \alpha_n]^{\mathrm{T}} \in \mathbb{R}^{n+1} : \alpha_0 > 0 \\ \boldsymbol{z}^{\mathrm{T}}\boldsymbol{\alpha} \neq 0, \forall z \in \mathbb{D}^c \end{array} \right\}, \tag{5}$$

with notation $\mathbb{D}^c := \{z \in \mathbb{C} : |z| \geqslant 1\}$ and $\boldsymbol{z}$ defined above. The norm condition can be expressed as $\gamma^2|a(\mathrm{e}^{\mathrm{i}\theta})|^2 - |b(\mathrm{e}^{\mathrm{i}\theta})|^2 > 0$, $\forall\theta \in \mathbb{R}$, which leads to spectral factorization

$$\gamma^2 a(z)a(z^{-1}) - b(z)b(z^{-1}) = \rho(z)\rho(z^{-1}), \tag{6}$$

for a unique[4] spectral factor $\rho(z) := \boldsymbol{z}^{\mathrm{T}}\boldsymbol{\rho}$ with $\boldsymbol{\rho} \in \mathfrak{S}$.

So far, we have explained that each $S \in \mathscr{S}$ corresponds to some $\boldsymbol{\alpha} \in \mathfrak{A}$, where $\mathfrak{A}$ is an open set in $\mathbb{R}^{n+1}$ defined by

$$\mathfrak{A} := \{\boldsymbol{\alpha} \in \mathfrak{S} : \gamma^2|\boldsymbol{e}(\theta)^{\mathrm{T}}\boldsymbol{\alpha}|^2 - |\boldsymbol{e}(\theta)^{\mathrm{T}}K\boldsymbol{\alpha}|^2 > 0, \forall\theta \in \mathbb{R}\},$$

with $\boldsymbol{e}(\theta) := [\mathrm{e}^{\mathrm{i}n\theta}, \mathrm{e}^{\mathrm{i}(n-1)\theta}, \ldots, 1]^{\mathrm{T}}$. The converse is trivial; for each $\boldsymbol{\alpha} \in \mathfrak{A}$, the function $S := (\boldsymbol{z}^{\mathrm{T}}K\boldsymbol{\alpha})/(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{\alpha})$ is in $\mathscr{S}$. We have also explained that, for each $\boldsymbol{\alpha} \in \mathfrak{A}$, there is a unique $\boldsymbol{\rho} \in \mathfrak{S}$. Actually, a much stronger assertion holds for the map between $\mathfrak{A}$ and $\mathfrak{S}$, as stated in the following theorem taken from Byrnes et al. (1995) and Byrnes and Lindquist (2000).

Theorem 3.1. *To each* $\boldsymbol{\rho} \in \mathfrak{S}$, *there exists a unique* $\boldsymbol{\alpha} \in \mathfrak{A}$ *such that* $S(z) = b(z)/a(z)$ *satisfies* (6) *and* $\boldsymbol{\beta} = K\boldsymbol{\alpha}$ *with the uniquely determined K above. The map* $\boldsymbol{h} : \mathfrak{S}$ to $\mathfrak{A}$ *sending* $\boldsymbol{\rho}$ *to* $\boldsymbol{\alpha}$ *is a diffeomorphism.*

[4] Without the positivity condition $\alpha_0 > 0$ in (5), the spectral factor ρ would be determined uniquely up to sign.

The proof of Theorem 3.1 is highly nontrivial. To each $\boldsymbol{\rho} \in \mathfrak{S}$, the existence of $\boldsymbol{\alpha} \in \mathfrak{A}$ in the theorem was proven in Georgiou (1983, 1987a,b). He also conjectured the uniqueness of such $\boldsymbol{\alpha}$. The conjecture was shown to be true in Byrnes et al. (1995) in the context of rational covariance extensions, and later in Georgiou (1999) and Byrnes and Lindquist (2000) for Nevanlinna–Pick interpolation. It was also established in Byrnes et al. (1995) and Byrnes and Lindquist (2000) that the map $\boldsymbol{h}$ is a diffeomorphism, providing a complete parameterization of the set $\mathscr{S}$ in terms of $\boldsymbol{\rho} \in \mathfrak{S}$:

$$\mathscr{S} = \left\{ S(z) = \frac{\boldsymbol{z}^{\mathrm{T}} K \boldsymbol{h}(\boldsymbol{\rho})}{\boldsymbol{z}^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{\rho})} : \boldsymbol{\rho} \in \mathfrak{S} \right\}.$$

Due to this parameterization of $\mathscr{S}$, we can reduce the sensitivity shaping problem (4) to the following finite-dimensional constrained NLS problem:

$$\inf_{\boldsymbol{\rho}\in\mathfrak{S}} \frac{1}{2} \sum_{k=1}^{N} \frac{w_k}{|s_k|^2} \left| \frac{\boldsymbol{e}_k^{\mathrm{T}} K \boldsymbol{h}(\boldsymbol{\rho})}{\boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{\rho})} - s_k \right|^2, \tag{7}$$

where $\boldsymbol{e}_k := \boldsymbol{e}(\theta_k)$, $k = 1, \ldots, N$. See Nagamune and Blomqvist (2004) for the explicit form of the map $\boldsymbol{h}$.

4. Solving the nonlinear least-squares problems

In order to solve the sensitivity shaping problem formulated in Section 2, we need a reliable and numerically robust algorithm to solve the optimization problem in (7). The precise meaning of "solving" will become clear in Section 4.1. The problem can be written as

$$\inf_{\boldsymbol{\rho}\in\mathfrak{S}} \frac{1}{2} \boldsymbol{F}(\boldsymbol{\rho})^{\mathrm{T}} \boldsymbol{F}(\boldsymbol{\rho}), \tag{8}$$

where $\boldsymbol{F} : \mathfrak{S} \mapsto \mathbb{R}^{2N}$ is the vector-valued residual map

$$\boldsymbol{F}(\boldsymbol{\rho}) := [\mathrm{Re}\{f_1(\boldsymbol{\rho})\}, \ldots, \mathrm{Re}\{f_N(\boldsymbol{\rho})\}, \mathrm{Im}\{f_1(\boldsymbol{\rho})\}, \ldots, \mathrm{Im}\{f_N(\boldsymbol{\rho})\}]^{\mathrm{T}},$$

$$f_k(\boldsymbol{\rho}) := \frac{\sqrt{w_k}}{|s_k|} \left(\frac{\boldsymbol{e}_k^{\mathrm{T}} K \boldsymbol{h}(\boldsymbol{\rho})}{\boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{\rho})} - s_k \right), \quad k = 1, \ldots, N. \tag{9}$$

4.1. Properties of the optimization problem

Since the domain $\mathfrak{S}$ of problem (8) is open, there is no guarantee that there exists a minimizer in $\mathfrak{S}$. In addition, since the cost functional in (8) is nonconvex and the domain $\mathfrak{S}$ in general is a nonconvex set, a global minimizer may not be unique, and there may even be several local minima. Therefore, by "solving" (8), we mean either finding a *local minimizer* in $\mathfrak{S}$ or an *approximation in* $\mathfrak{S}$ *of a local infimizer* within a certain tolerance.

A major advantage with the formulated NLS problem is the smoothness of the cost functional in (8). This smoothness is due to the continuous differentiability of the residual vector $\boldsymbol{F}$ with respect to $\boldsymbol{\rho}$; see Nagamune and Blomqvist (2004) for derivative expressions. This enables local search algorithms based on derivative information, which will be proposed in Section 4.2. For derivative-based algorithms, nonconvexity means that it will not converge to a global minimizer unless algorithms are initialized properly. This makes the problem of finding good initial points important. Some guideline to select proper initial points will be given in Section 4.3.

4.2. Two modified algorithms

The formulation as a NLS problem also has the advantage that the problem class is well-studied and that there are several efficient and numerically robust algorithms for solving the problem available; see e.g. Nash and Sofer (1996). Especially, two popular algorithms are the *Gauss–Newton* and the *Levenberg–Marquardt* methods, which were originally developed for unconstrained NLS problems. Here, we will modify these two algorithms in order to incorporate the constraint $\boldsymbol{\rho} \in \mathfrak{S}$. We will treat the constraint implicitly; more precisely, we will enforce a bound on the step length so that an updated point stays in $\mathfrak{S}$. As stopping criteria, we will either require the gradient to be close to zero, or that the norm of the step is small for detecting $\boldsymbol{\rho}$ getting close to the boundary of the feasible region. Detailed descriptions are given in Nagamune and Blomqvist (2004).

In the algorithms proposed above, we need to check feasibility ($\boldsymbol{\rho} \in \mathfrak{S}$) and to compute the residual vector $\boldsymbol{F}$ and its Jacobian $\nabla\boldsymbol{F}$. To check whether $\boldsymbol{\rho} \in \mathfrak{S}$, we can, e.g., recursively compute the corresponding partial reflection coefficients and check that they are less or equal to one in modulus, since ρ is a real polynomial. Computing $\boldsymbol{F}$ and $\nabla\boldsymbol{F}$ for a given point $\boldsymbol{\rho} \in \mathfrak{S}$ involves the computation of $\boldsymbol{h}(\boldsymbol{\rho})$ as shown in (9). This computation can be done by the continuation method developed in Blomqvist, Fanizza, and Nagamune (2003), which however requires some computational effort.

4.3. Determining a good initial point

The initialization of the algorithm is most important since the problem in general is nonconvex. If we have a controller design to be improved incrementally we can initialize with that solution. Otherwise we propose to use what we might call the *approximate peak solution*. This also serves as the default initial point in the MATLAB implementation (Blomqvist & Nagamune, 2004).

The approximate peak solution is motivated by the tuning rules of Nagamune (2004a). The most effective tuning rule is to place a complex conjugate pair of roots of ρ close to the unit circle at the frequency corresponding to a desired peak gain of the sensitivity function. Approximately knowing a desired peak location we place a pair of roots correspondingly and the rest in origin. Starting at the maximum entropy (ME) solution, we can use the continuation method of Blomqvist et al. (2003) to determine the approximate peak solution. The ME solutions can be computed using the formula (Georgiou & Lindquist, 2003, Eq. (6.2), p. 2915) for the positive real setting, with bilinear transformations.

5. Design procedure and examples

Next, through a couple of examples from the control literature, we shall explain how to select and tune design parameters ($\boldsymbol{\theta}$, $\boldsymbol{s}$, $\boldsymbol{w}$, γ and (λ_j, η_j)) to satisfy given design specifications. These problems assume the feedback structure depicted in Fig. 1. To focus on the presentation of the selection and tuning strategies, we will skip the exposition of the physical meanings in each problem, and present it just as a mathematical problem. Readers interested in detailed problem settings are referred to each book from which each problem is taken. In this section, "NLSsolver" stands for the nonlinear least-squares optimization solver, which realizes the theory in Sections 3 and 4.

5.1. Flexible beam control

Here, we will deal with a control problem in Doyle et al. (1992, Sections 10 & 12) where a desired sensitivity function is naturally available from the specification.

5.1.1. Problem setting

The continuous-time plant P is given as

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.77}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)}. \tag{10}$$

Our goal in this problem is to design a strictly proper controller C which satisfies, for a step reference r,

- the settling time is less than 8 s,
- the overshoot is less than 10%, and
- the control input fulfills $|u(t)| \leqslant 0.5$ for all t.

In Doyle et al. (1992), the first two requirements in the time domain have been approximated by a requirement in the frequency domain as a desired sensitivity function $S_d(s) := s(s+1.2)/(s^2+1.2s+1)$. We also aim at designing a sensitivity function similar to S_d, with extra consideration of control input constraint.

5.1.2. Initial selection of design parameters

Using S_d, we extract our desired frequency response at a finite number of frequencies. We take 100 points in the frequency $[10^{-3}, 10^3]$ (rad/s), equally distanced in the logarithmic scale, as $\boldsymbol{\omega} := \{\omega_k\}_{k=1}^{100}$. With these points, we set our desired frequency response $(\boldsymbol{\theta}, \boldsymbol{s})$ in the discrete-time setting as

$$\boldsymbol{\theta} := \{\theta_k : \mathrm{e}^{\mathrm{i}\theta_k} = (1 + \mathrm{i}\omega_k)(1 - \mathrm{i}\omega_k)^{-1}, \omega_k \in \boldsymbol{\omega}\}, \tag{11}$$

$$\boldsymbol{s} := \{s_k := S_d(\mathrm{i}\omega_k), \omega_k \in \boldsymbol{\omega}\}. \tag{12}$$

Since we have initially no information on frequency emphasis, weights are set as $\boldsymbol{w} := \{w_k := 1,\ k = 1, \ldots, 100\}$. The uniform upper bound of the sensitivity gain is chosen as $\gamma := 1.5$. We do not use any additional interpolation condition in this problem. From the gain plot of S_d, we would like to have a peak gain around 1 rad/s. Therefore, we always set the initial point for optimization to a $\boldsymbol{\rho}$ in $\mathfrak{S}$ that has its roots at $\pm 0.95\mathrm{i}$, which corresponds to an approximate peak solution having its peak close to 1 rad/s in the continuous-time setting.

5.1.3. Controller design

With the initial selection of design parameters, NLSsolver outputs a controller C_0 and a sensitivity function S_0. Several frequency and time responses are plotted in Fig. 3. The uppermost figure shows the Bode plot of S_0 with the desired frequency response $(\boldsymbol{\theta}, \boldsymbol{s})$. As can be seen, NLSsolver indeed generates S_0 approximating $(\boldsymbol{\theta}, \boldsymbol{s})$.

Now, we check the original time domain specifications. The lower figures in Fig. 3 show the step response and the input signal. Although the step response meets the specification, the input signal is too large to fulfill the specification $|u(t)| \leqslant 0.5$. Therefore, we need to update some of our design parameters, and redesign a controller.

To see the cause of large input signal, we draw the Bode plot of the controller C_0 in the middle of Fig. 3. From the figure, we see that there is a sharp gain peak around 20 rad/s. In fact, this frequency coincides with the frequency of the input oscillation. Therefore, one natural way to suppress the input is to lower the gain peak of C.

Now, we update the design parameters. Since $C = (1 - S)/PS$, we need to make S close to one to decrease the gain of C. Desired frequency response s_k is almost one around frequency 20 rad/s, and thus, we increase the weight w_k around the frequency to fit S closer to s_k. (We do not change other design parameters in this example.) After some trial and error, we have chosen weights $\boldsymbol{w}$ as in Fig. 4, that results in the following controller and sensitivity function:

$$C(s) = \frac{2.706s^3 + 1.931s^2 + 75.51s + 0.05028}{s^4 + 7.698s^3 + 33.59s^2 + 126.8s + 143}, \tag{13}$$

$$S(s) = \frac{s^4 + 2.789s^3 + 19.9s^2 + 29.13s}{s^4 + 2.789s^3 + 19.9s^2 + 25.62s + 19.38}. \tag{14}$$

The resulting Bode plots and response signals are shown in Fig. 5, with response signals in Doyle et al. (1992). The figures show that the sharp peak disappeared in the gain of C, which has been done at the price of degradation of sensitivity fitting, and that the original time domain specifications are

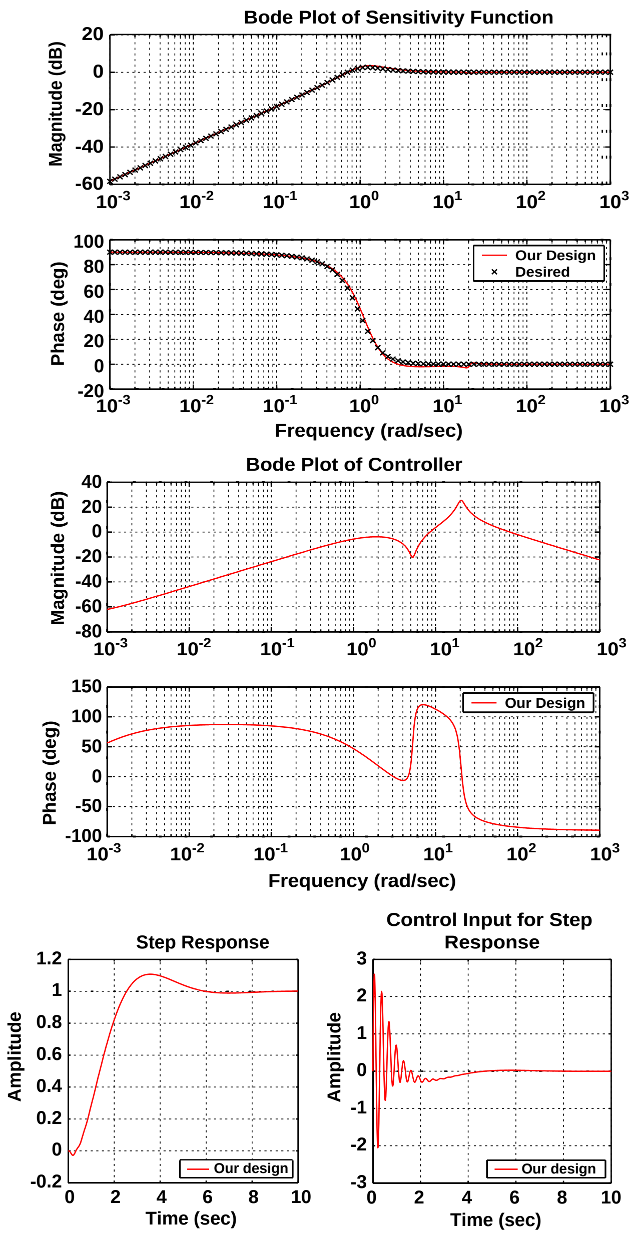

Fig. 3. Bode plots and response signals by initial design.

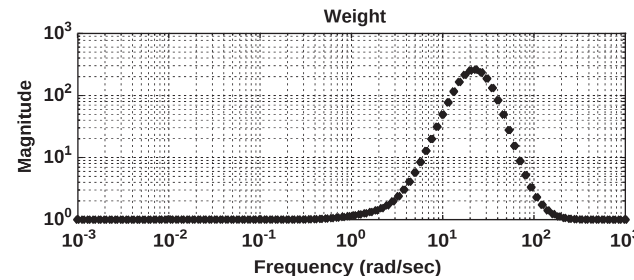

Fig. 4. Weight w.

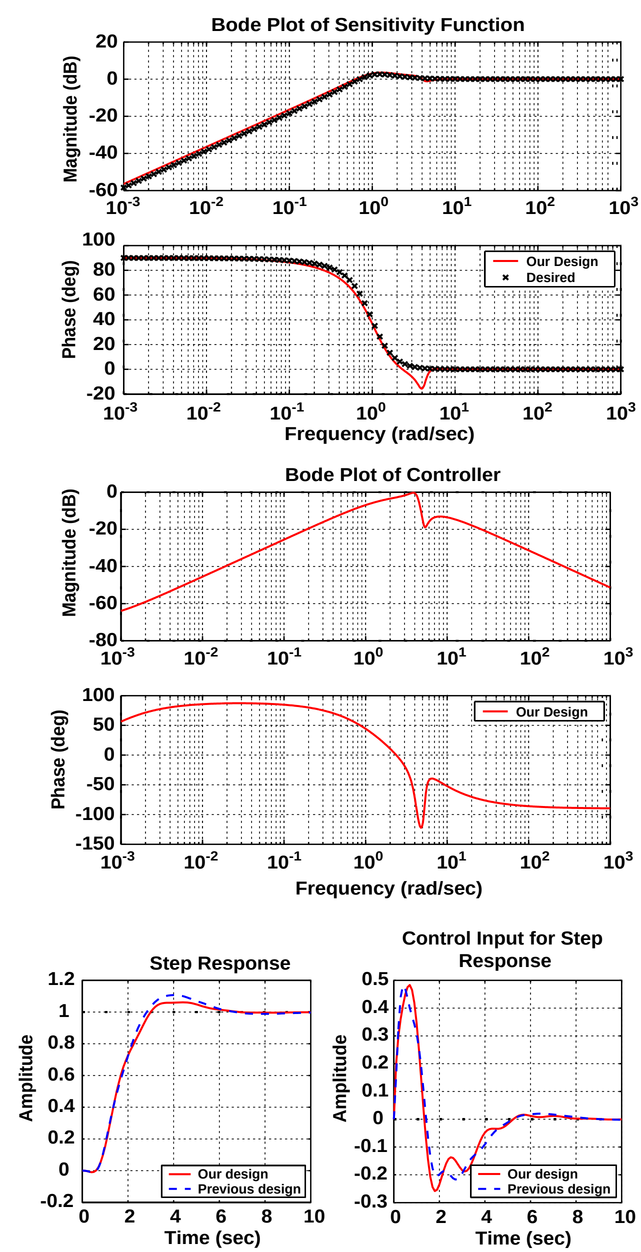

Fig. 5. Bode plots and response signals by the new design.

indeed satisfied. Also, one can see that we have obtained a similar performance to that in Doyle et al. (1992). We stress that the degree of the controller (13) is half of the one obtained in Doyle et al. (1992).

5.2. Slide drive control

Here, we will deal with a slide drive control problem in the book by Helton and Marino (1998, Chapter 6.2). In this control problem, in contrast to the first example, a desired sensitivity function is not available at the outset. We will explain how to solve such problem with our approach.

5.2.1. Problem setting

The plant P is given by

$$P(s) := \frac{2s^2 + 10s + 100}{s^4 + 7.01s^3 + 110.47s^2 + 452.6s + 521},$$

which is stable and minimum-phase. The performance specifications for the continuous-time sensitivity function are given as

$$\begin{array}{ll} |S(i\omega)| < -20\,\text{dB}, & \omega \in [0, 0.1], \\ |S(i\omega)| < -10\,\text{dB}, & \omega \in [0.1, 1.0], \\ |S(i\omega)| < 6\,\text{dB}, & \omega \in [1.0, 5.0], \\ |1 - S(i\omega)| < -20\,\text{dB}, & \omega \in [5.0, 10.0], \\ |1 - S(i\omega)| < -40\,\text{dB}, & \omega \in [10.0, \infty]. \end{array} \tag{15}$$

5.2.2. Initial selection of design parameters

First of all, since the plant is stable and minimum-phase, we can show that our allowable set $\mathscr{S}$ would be a singleton $\mathscr{S} = \{S : S \equiv 1\}$ without additional constraints $S(\lambda_j) = \eta_j$; see Proposition II.2 in Nagamune (2004b). The case of $S \equiv 1$ (i.e., $C \equiv 0$) is obviously unsatisfactory, and thus we need to introduce at least one additional constraint. Here, we will initially use two constraints as $S(\pm 0.01i) = 0.1$ ($= -20\,\text{dB}$) in the continuous-time setting to take into account the specification over low frequencies.

Next, we need to construct a desired frequency response from specification (15). We take 50 points in [0.01, 1] (rad/s) and 50 points in [5, 100] (rad/s), equally distanced in the logarithmic scale, denoted by $\boldsymbol{\omega} := \{\omega_k\}_{k=1}^{100}$. With these points, we set our desired frequency response $(\boldsymbol{\theta}, \boldsymbol{s})$ in the discrete-time setting to $\boldsymbol{\theta}$ in (11) and $\boldsymbol{s}$ as shown in Fig. 6, for the specifications over [0, 1] (rad/s) and $[5, \infty]$ (rad/s). On the other hand, the specification over the intermediate frequencies [1, 5] (rad/s) is taken care of by setting the uniform upper bound to $\gamma := 2$ ($\approx 6\,\text{dB}$). The weights are set as $\boldsymbol{w} := \{w_k := 1,\ k = 1, \ldots, 100\}$.

Remark 5.1. As we have no information about how to select desired phases, we set phases to zero. Although this selection may not be a best one, it will be one natural selection. After the first design, we will obtain an idea how the phase should look like; see the design below.

5.2.3. Controller design

Using the initial selections of design parameters and by choosing the initial $\boldsymbol{\rho}$ whose roots locate at $0.99(1 \pm 3i)/(1 \mp 3i)$ (i.e., gain peak around 3 rad/s in the continuous-time setting), the NLSsolver returns the controller C_0 and the sensitivity function S_0. Bode plots of the sensitivity function and complementary sensitivity function are shown in Fig. 6, in which we can see that some specifications in (15) are not satisfied.

Now, we will utilize S_0 to generate new desired frequency response data $(\boldsymbol{\theta}, \boldsymbol{s})$. The vector $\boldsymbol{\omega}$ is taken at 100 frequencies $\boldsymbol{\omega} := \{\omega_k\}_{k=1}^{100}$, equally distanced in the logarithmic scale over [0.01, 100]. Then, $\boldsymbol{\theta}$ is obtained by (11), and a vector $\boldsymbol{s}$ is given by $\boldsymbol{s} := \{s_k := S_0(i\omega_k),\ k = 1, \ldots, 100\}$.

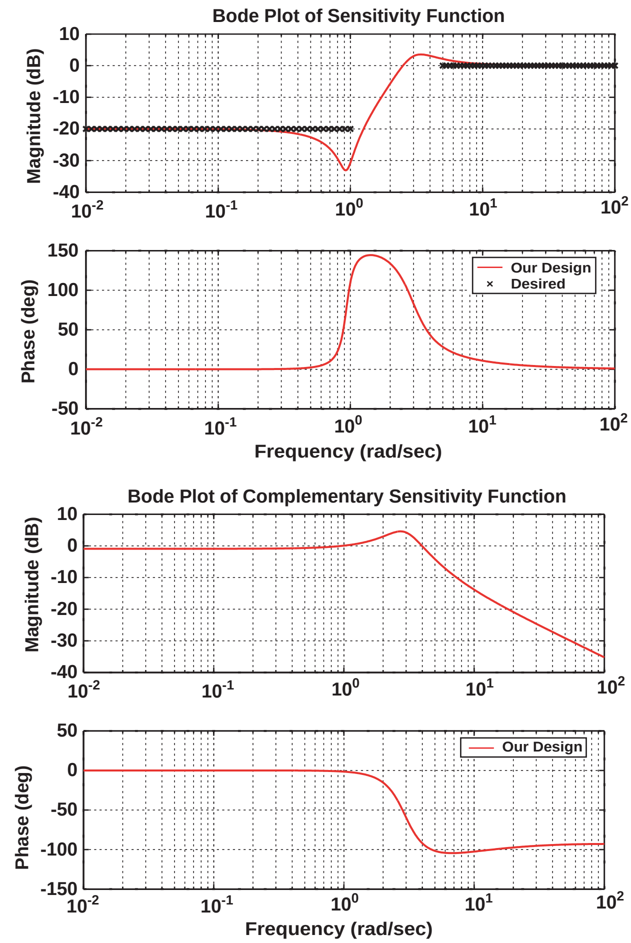

Fig. 6. Bode plots of S_0 and $T_0 := 1 - S_0$.

Our strategy here is to modify this $\boldsymbol{s}$, as well as to modify/add additional constraints if necessary, after every design iteration so that the specifications in (15) are fulfilled. More concretely, one design iteration consists of (i) gradually changing, toward the achievement of specifications, $\boldsymbol{s}$ and/or current additional conditions $S(\lambda_k) = \eta_k$, (ii) introducing new additional conditions if necessary, (iii) adopting an initial point $\boldsymbol{\rho}$ for NLSsolver as the minimizer of the previous design if the design is not bad, and (iv) designing a new controller and checking the performance in the Bode plot. After a number of design iterations, with four additional conditions (see the circles in Fig. 7), we have obtained a controller and a sensitivity function as

$$C(s) = \frac{0.2238s^7 + 8.875s^6 + 82.58s^5 + 965.5s^4 + 4231s^3 + 7977s^2 + 8233s + 5493}{s^7 + 9.281s^6 + 85.02s^5 + 291.9s^4 + 741.3s^3 + 551.1s^2 + 594.1s + 96.21},$$

$$S(s) = \frac{s^5 + 4.281s^4 + 13.61s^3 + 9.814s^2 + 11.69s + 1.924}{s^5 + 4.281s^4 + 14.06s^3 + 24.43s^2 + 24.97s + 23.01}.$$

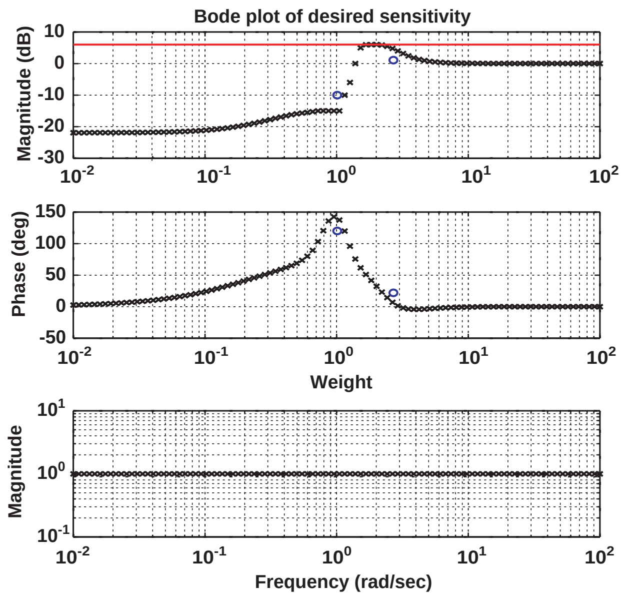

Fig. 7. Design parameters in the final design. The horizontal line in the uppermost figure is the level $\gamma=2$, and the circles correspond to additional conditions.

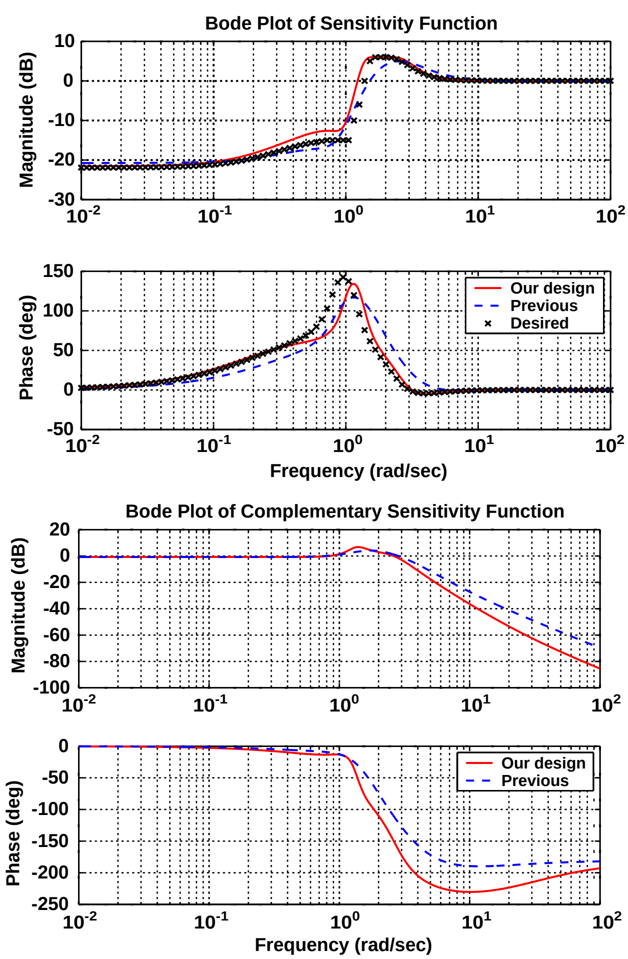

Fig. 8. Bode plots of S and $T := 1 - S$.

The corresponding Bode plots of S and $T := 1 - S$ are shown in Fig. 8, with the design result in Helton and Marino (1998, Chapter 6.2). Although the complementary sensitivity slightly violates the requirements over high frequencies, we have obtained much lower gain in those frequencies than that designed in the book by Helton and Marino (1998). Note that the degree of controller is seven, comparable to the controller degree in Helton and Marino (1998, p. 80), which was eight.

Remark 5.2. At this point, it is quite heuristic to select (λ_k, η_k) for additional constraints, even though we have some guidelines for the selections as was presented in Nagamune (2004a). How to select these design parameters in a certain optimal sense is still an open question.

6. Conclusions

In this paper, we have proposed a new approach to design the sensitivity function in the frequency domain. We have formulated a sensitivity shaping problem, and reduced it to a finite-dimensional constrained nonlinear least-squares optimization problem. To solve this problem, we have modified the Gauss–Newton and the Levenberg–Marquardt methods to incorporate the constraint. Numerical examples from the control literature have demonstrated the usefulness of the proposed method in designing relatively low degree controllers. We have developed a user-friendly software for the sensitivity shaping based on the developed theory (Nagamune & Blomqvist, 2004). A multivariable extension of the proposed sensitivity shaping method is currently under investigation. In addition, a numerical comparison with convex optimization approaches to sensitivity shaping, such as Iwasaki and Hara (2003) and Grassi et al. (2001), will be an interesting subject.

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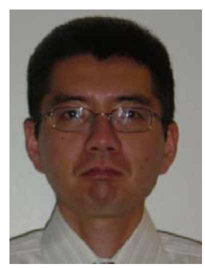

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