B. Proof of Lemma 2.3

We can assume that all components of x, y, u, v are nonzero. To see why this is so, suppose that the result was proven for this case and we were given four arbitrary nonzero vectors x, y, u, and v. We could transform them via a single nonsingular transformation T such that each component of Tx, Ty, Tu, Tv was nonzero (Lemma 2.2). Then for all Hermitian matrices P we would have $(Tx)^T P(Ty) = x^T(T^T PT)y$, and hence, that $(Tx)^T P(Ty) = -k(Tu)^T P(Tv)$. Then, $Tx = \alpha Tu$ and, thus, $x = \alpha u$ or $Tx = \beta Tv$ and $x = \beta v$. So, we shall assume that all components of x, y, u, v are nonzero. Suppose that x is not a scalar multiple of u to begin with. Then, for any index i with $1 \leq i \leq n$, there is some other index j and two nonzero real numbers c_i, c_j such that

$$x_i = c_i u_i \quad x_j = c_j u_j, \qquad c_i \neq c_j. \tag{8}$$

Choose one such pair of indexes i, j. Equating the coefficients of p_{ii}, p_{jj} and p_{ij}, respectively, in the identity $x^T Py = -ku^T Pv$ yields the following equations:

$$x_i y_i = -k u_i v_i \tag{9}$$

$$x_j y_j = -k u_j v_j \tag{10}$$

$$(x_i y_j + x_j y_i) = -k(u_i v_j + u_j v_i). \tag{11}$$

If we combine (8) with (9) and (10), we find

$$y_i = -\frac{k}{c_i} v_i \tag{12}$$

$$y_j = -\frac{k}{c_j} v_j. \tag{13}$$

Using (9)–(13), we find $c_i u_i y_j + c_j u_j y_i = -k(u_i v_j + u_j v_i)$. Hence, $u_i v_j (c_j - c_i/c_j) = u_j v_i (c_j - c_i/c_i)$. Recall that $c_i \neq c_j$, so we can divide by $c_j - c_i$ and rearrange the terms to get

$$\frac{c_i}{c_j} = \left(\frac{v_i}{v_j}\right)\left(\frac{u_j}{u_i}\right). \tag{14}$$

However, using (8), we find

$$\frac{c_i}{c_j} = \left(\frac{x_i}{x_j}\right)\left(\frac{u_j}{u_i}\right). \tag{15}$$

Combining (14) and (15) yields

$$\frac{v_i}{v_j} = \frac{x_i}{x_j}. \tag{16}$$

Thus, $x_i = cv_i$, $x_j = cv_j$ for some constant c. Now, if we select any other index k with $1 \leq k \leq n$, and write $x_k = c_k u_k$ then c_k must be different to at least one of c_i, c_j. Without loss of generality, we may take it that $c_k \neq c_i$. Then, the aforementioned argument can be repeated with the indexes i and k in place of i and j to yield

$$x_i = cv_i \quad x_k = cv_k. \tag{17}$$

However, this can be done for any index k so we conclude that $x = cv$ for a scalar c. So, we have shown that if x is not a scalar multiple of u, then it is a scalar multiple of v. To complete the proof, note that if $x = \beta v$ for a scalar β then by (9), $\beta v_i y_i = -k u_i v_i$ for all i. Thus $y = -(k/\beta)u$ as claimed. The same argument will show that if $x = \alpha u$ for a scalar α, then $y = -(k/\alpha)v$. Q.E.D

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A Robust Solver Using a Continuation Method for Nevanlinna–Pick Interpolation With Degree Constraint

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***Abstract*—This note modifies a previous algorithm for solving a certain convex optimization problem, introduced by Byrnes, Georgiou, and Lindquist, to determine any Nevanlinna–Pick interpolant satisfying degree constraint. The modified algorithm is based on a continuation method with predictor-corrector steps and it turns out to be quite efficient and numerically robust.**

***Index Terms*—Continuation method, degree constraint, Nevanlinna–Pick interpolation, predictor–corrector step.**

I. INTRODUCTION

This note proposes a new solver for computing interpolants for the Nevanlinna–Pick interpolation problem with degree constraint (NPDC), formulated as follows.

NPDC: Suppose that a set $\mathcal{D} := \left\{(z_j, w_j) \in \mathbb{C}^2\right\}_{j=0}^{n}$, with distinct $\{z_j\}$ and $|z_j| > 1$, is given under the following assumptions.

A1) The Pick matrix P is positive definite, where

$$P := \left[\frac{w_i + \bar{w}_j}{1 - z_i^{-1}\bar{z}_j^{-1}}\right]_{i,j=0}^{n}. \tag{1}$$

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A2) $(z_j, w_j) \in \mathcal{D}$ whenever $(\bar{z}_j, \bar{w}_j) \in \mathcal{D}$.

A3) $z_0 = \infty$ and w_0 is real.

For the set $\mathcal{D}$, the problem is to determine all the functions f satisfying: i) interpolation constraints: $f(z_j) = w_j, j = 0, 1, \ldots, n$, ii) strictly positive realness: f is analytic and Re $f(z) > 0$ for all $|z| \geq 1$, and iii) degree constraint: f is rational of McMillan degree at most n.

Assumption A1) guarantees that the solution set of NPDC is nonempty, A2) leads to the restriction of the solutions f to rational functions with real coefficients (see [3, Corollary 4.6]), which are especially relevant to applications, and A3) is for mathematical convenience.

The classical Nevanlinna–Pick interpolation problem (see, e.g., [16]) requires only conditions i) and ii), but here, we have an additional condition iii) and this completely alters the mathematical problem. The theory for NPDC has been developed in [2], [3], [7], and [8], and its usefulness in applications has been recognized in [2] and [12]. The theory completely parameterizes interpolants for NPDC in terms of Schur polynomials.

To compute each interpolant for a specified Schur polynomial ρ of degree n, we need to solve an optimization problem [2], [3]

$$\min_{q \in \mathcal{Q}_+} J_\rho(q),\ J_\rho(q) := \langle q + q^*, w + w^* \rangle - \left\langle \log\left(q + q^*\right), \frac{\rho\rho^*}{\tau\tau^*} \right\rangle. \quad (2)$$

Here, τ is a fixed real polynomial of degree n which depends on the interpolation points as $\tau(z) := \prod_{j=1}^{n}(z - z_j^{-1})$, the domain $\mathcal{Q}_+$ is defined by

$$\mathcal{Q}_+ := \left\{ q := \frac{\pi}{\tau},\ \begin{array}{c} \pi : \text{real polynomial of degree } n \\ q\left(e^{i\theta}\right) + q\left(e^{-i\theta}\right) > 0,\ \forall \theta \in [-\pi, \pi] \end{array} \right\} \quad (3)$$

$q^*(z) := q(z^{-1})$, w is any real function that is analytic in $|z| \geq 1$ and satisfies the interpolation constraints $w(z_j) = w_j, j = 0, 1, \ldots, n$ and the inner product $\langle a, b \rangle$ is defined for two real $\mathcal{L}z$ functions a and b by $\langle a, b \rangle := \int_{-\pi}^{\pi} a(e^{i\theta}) b(e^{-i\theta}) d\theta / 2\pi$. With the optimal solution q, the interpolant f is obtained by spectral factorization

$$q(z) + q\left(z^{-1}\right) = \frac{\alpha(z)\alpha\left(z^{-1}\right)}{\tau(z)\tau\left(z^{-1}\right)} \quad (4)$$

by solving $\alpha\beta^* + \alpha^*\beta = \rho\rho^*$ and by setting $f = \beta/\alpha$.

It was shown in [3] that (2) is a convex optimization problem with a unique interior minimum, i.e., stationary point. To find the stationary point numerically, an algorithm based on Newton's method was proposed in [2] and [3]. This algorithm involves a system of linear equations. The system of linear equations in this algorithm can be ill-conditioned and the solution may be numerically inaccurate if the condition number of the Hessian H is large. This occurs when q is close to the boundary of the region $\mathcal{Q}_+$, since it is known (see [3]) that the function $J_\rho(q)$ has an infinite gradient at the boundary. In addition, the spectral factorization in (4) is often numerically hard to solve when roots of the polynomial α lie near the unit circle, in which case, q is near the boundary of $\mathcal{Q}_+$. These disadvantages are crucial since engineering applications often require such q, generating a sharp peak of frequency responses (see [2] and [12]). Thus, a new solver which overcomes the drawbacks needs to be developed. This is the subject of this note.

We employ coefficients of α in (4) as variables in the optimization to avoid the spectral factorization. Although this yields a nonconvex objective function in a nonconvex region, it turns out that the function has a unique stationary point in the region and that it is locally convex around the stationary point. To find the stationary point, we apply the Euler–Newton continuation method [1]. This idea was inspired by [6], where the solver for the rational covariance extension problem with degree constraint was developed with the continuation method. However, as will be seen later, the objective function we treat is slightly different from the one in [6] and, hence, one needs to use different treatments at some parts.

The note is organized as follows. In Section II, we transform J_ρ in (2) into a new function by means of a variable change. Section III discusses attractive properties of the new function from the optimization viewpoint. Based on the properties, a continuation method is applied to solve the minimization problem of the objective function in Section IV. Section V gives one numerical example to illustrate the efficiency of the proposed solver. A preliminary version of this note has appeared in [13], [14]. The proposed solver will be useful in many important engineering problems which are reducible to the Nevanlinna–Pick interpolation problems, e.g., the problems presented in [2], [4], [5], [9], [10], and [17].

II. Formulation in Terms of the Coefficients of α

In [2, Appendix B], it was shown that the first term of J_ρ in (2) does not depend on a particular choice of the function w and that it can be represented in terms of P in (1) as

$$\langle q + q^*, w + w^* \rangle = \boldsymbol{\gamma}^H P \boldsymbol{\gamma} \quad (5)$$

if the unique (modulo ± 1) minimum-phase a, derived by

$$q(z) + q\left(z^{-1}\right) = a(z)a\left(z^{-1}\right) \quad (6)$$

for some $q \in \mathcal{Q}_+$, is expressed in terms of Cauchy kernels as $a(z) = \sum_{j=0}^{n} \gamma_j z/(z - \bar{z}_j^{-1})$ and if we define the coefficient vector as $\boldsymbol{\gamma} := [\gamma_0, \gamma_1, \ldots, \gamma_n]^T \in \mathbb{R} \times \mathbb{C}^n$.

We will further transform J_ρ to express it in terms of coefficients of α in (4). By comparing (4) and (6), we can express a as

$$a(z) = \frac{\alpha(z)}{\tau(z)} =: \frac{\alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n}{z^n + \tau_1 z^{n-1} + \cdots + \tau_n}. \quad (7)$$

Then, it can be shown that the coefficient vector $\boldsymbol{\alpha} := [\alpha_0, \alpha_1, \ldots, \alpha_n]^T \in \mathbb{R}^{n+1}$ is expressed by

$$\boldsymbol{\alpha} = L_n V \boldsymbol{\gamma} \quad (8)$$

where the invertible matrices V and L_n are defined by

$$V := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \bar{z}_0^{-1} & \bar{z}_1^{-1} & \cdots & \bar{z}_n^{-1} \\ \vdots & \vdots & & \vdots \\ \bar{z}_0^{-n} & \bar{z}_1^{-n} & \cdots & \bar{z}_n^{-n} \end{bmatrix}$$

$$L_n := \begin{bmatrix} 1 & & & \\ \tau_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ \tau_n & \cdots & \tau_1 & 1 \end{bmatrix}.$$

Due to (5)–(8), the function J_ρ in (2) becomes

$$J_\rho(q) = \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\langle \log|\alpha|, \Psi \rangle + 2\langle \log|\tau|, \Psi \rangle \quad (9)$$

where $\Psi := \rho\rho^*/\tau\tau^*$ and the positive–definite matrix

$$K := L_n^{-T}\left(V^{-1}\right)^H P V^{-1} L_n^{-1} \in \mathbb{R}^{(n+1)\times(n+1)} \quad (10)$$

is completely determined by the interpolation data. Since the last term in (9) is constant, it does not affect the minimization problem. Noting $\langle \log|\alpha|, \Psi \rangle = \langle \log \alpha, \Psi \rangle$ for any real polynomial α, we define

$$g_\rho(\boldsymbol{\alpha}) := \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\langle \log \alpha, \Psi \rangle. \quad (11)$$

We also define the Schur stability region by

$$\mathcal{S}_n := \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{n+1} :\ \begin{array}{l} \alpha(z) := \sum_{k=0}^{n} \alpha_k z^{n-k} \neq 0, \forall |z| \geq 1, \\ \alpha_0 > 0 \end{array} \right\}.$$

The Schur stability region is open and nonconvex for $n \geq 3$. Since spectral factorization (4) defines a one-one correspondence between points in $\mathcal{S}_n$ and $\mathcal{Q}_+$, the problems of finding the minimizer $q \in \mathcal{Q}_+$ of $J_\rho(q)$ in (2) and of finding the minimizer $\boldsymbol{\alpha} \in \mathcal{S}_n$ of $g_\rho(\boldsymbol{\alpha})$ in (11)

are equivalent. Thus, instead of (2), the rest of this note focuses on the problem

$$\min_{\boldsymbol{\alpha}\in\mathcal{S}_n} g_\rho(\boldsymbol{\alpha}), \quad \text{where} \quad g_\rho(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\langle \log \alpha, \Psi\rangle. \tag{12}$$

Remark II.1: The optimization problem (12) is nonconvex, which is generally considered to be more difficult than the convex one. However, there are advantages here to solve the nonconvex problem. Namely, we can avoid numerical difficulties caused by spectral factorization and ill-conditioning of a system of linear equations. Besides, we can still solve the nonconvex problem (12) in an efficient way, as explained in Section IV.

Remark II.2: This objective function differs from the one dealt with in [6] in that Ψ in (11) is a ratio of pseudo polynomials, not just a pseudo polynomial. Because of this difference, the approach in [6] cannot be adopted directly here.

III. Properties of the Function g_ρ

In this section, we derive some attractive properties of g_ρ in (11), which are relevant to the optimization problem (12).

A. Unique Stationary Point

We first state a lemma which implies that *the search for the global optimizer of g_ρ is equivalent to finding the unique stationary point.*

Lemma III.1: The function g_ρ has a unique stationary point $\hat{\boldsymbol{\alpha}}$ in $\mathcal{S}_n$. In addition, the point $\hat{\boldsymbol{\alpha}}$ corresponds to the minimizer $\hat{q} \in \mathcal{Q}_+$ of $J_\rho(q)$ via spectral factorization (4).

Proof: First, for a vector $\boldsymbol{v} = [v_0, v_1, \ldots, v_n]^T$, introduce the operator

$$T_{\boldsymbol{v}} := \begin{bmatrix} v_0 & \cdots & v_n \\ \vdots & \iddots & \\ v_n & & \end{bmatrix} + \begin{bmatrix} v_0 & \cdots & v_n \\ & \ddots & \vdots \\ & & v_0 \end{bmatrix}$$

which is known to be invertible whenever $\boldsymbol{v}$ is in $\mathcal{S}_n$. For an $\boldsymbol{\alpha} \in \mathcal{S}_n$, there exists a unique strictly positive real function $q \in \mathcal{Q}_+$ such that (4) holds. Define a vector $\boldsymbol{q}$ which consists of the Markov parameters of the unique q as $\boldsymbol{q} = [q_0, q_1, \ldots, q_n]^T$, where $q(z) = q_0 + q_1 z^{-1} + \cdots + q_n z^{-n} + \cdots$. Then, the vector $\boldsymbol{q}$ can be explicitly written in terms of $\boldsymbol{\alpha}$ (by using [2, p. 3193, eq. (3.12)]) as

$$\boldsymbol{q} = \frac{L_n^{-1} T_{\boldsymbol{\tau}}^{-1} T_{\boldsymbol{\alpha}} \boldsymbol{\alpha}}{2} \tag{13}$$

where $\boldsymbol{\tau} := [1, \tau_1, \ldots, \tau_n]^T$. It is straight-forward to verify that $\partial \boldsymbol{q}/\partial \boldsymbol{\alpha} = L_n^{-1} T_{\boldsymbol{\tau}}^{-1} T_{\boldsymbol{\alpha}}$. Then, due to the chain rule, we have

$$\frac{\partial g_\rho}{\partial \boldsymbol{\alpha}} = \left(L_n^{-1} T_{\boldsymbol{\tau}}^{-1} T_{\boldsymbol{\alpha}}\right)^T \frac{\partial g_\rho}{\partial \boldsymbol{q}}.$$

We know from [3] that the function J_ρ has a unique stationary point in $\mathcal{Q}_+$. Suppose that the Markov parameter vector of the unique stationary point is denoted by $\hat{\boldsymbol{q}}$. Then, it satisfies $\partial g_\rho/\partial \boldsymbol{q}|_{\boldsymbol{q}=\hat{\boldsymbol{q}}} = 0$ and hence $\partial g_\rho/\partial \boldsymbol{\alpha}|_{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}} = 0$, where $\hat{\boldsymbol{\alpha}}$ is a unique point in $\mathcal{S}_n$ which corresponds via (4) to $\hat{\boldsymbol{q}}$. On the other hand, for each point $q \in \mathcal{Q}_+$ other than $\hat{\boldsymbol{q}}$, let us denote a unique point in $\mathcal{S}_n$ corresponding to the q via (4) as $\bar{\boldsymbol{\alpha}}$ and define $\bar{\boldsymbol{q}}$ by (13) with $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$. Then, we can conclude $\partial g_\rho/\partial \boldsymbol{\alpha}|_{\boldsymbol{\alpha}=\bar{\boldsymbol{\alpha}}} \neq 0$, since $L_n^{-1} T_{\boldsymbol{\tau}}^{-1} T_{\bar{\boldsymbol{\alpha}}}$ is nonsingular and $\partial g_\rho/\partial \boldsymbol{q}|_{\boldsymbol{q}=\bar{\boldsymbol{q}}} \neq 0$. ■

B. Nonconvexity of g_ρ

The function J_ρ is strictly convex on $\mathcal{Q}_+$ (see [3]), but the global convexity is lost for the new objective function g_ρ. Here, one example is given to illustrate this statement.

Consider the interpolation data $(z_0, w_0) = (\infty, 3)$ and $(z_1, w_1) = (2, 1.5)$. When we set $\rho(z) = z - 0.9$, the cost function g_ρ is represented for an appropriate matrix K as

$$g_\rho(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\left\langle \log \alpha, \frac{(z-0.9)\left(z^{-1}-0.9\right)}{(z-0.5)\left(z^{-1}-0.5\right)} \right\rangle.$$

We will show that the Hessian is not positive definite at some particular point. Take $\tilde{\boldsymbol{\alpha}} = [0.1, 0.05]^T$, which is obviously in $\mathcal{S}_1$. Then, the smallest eigenvalue of the Hessian of g_ρ at $\tilde{\boldsymbol{\alpha}}$ becomes negative. (The detail of the calculation is presented in [14]). Hence, *the function g_ρ is not globally convex in general on the region $\mathcal{S}_n$.*

C. Local Convexity of g_ρ Around the Optimum

We next give another lemma which motivates us to resort to the procedure based on a continuation method to solve (12).

Lemma III.2: The function g_ρ is strictly convex in a neighborhood of the global minimum.

Proof: With a lengthy calculation, we can show that the Hessian $H_{\boldsymbol{\alpha}}$ is expressed by

$$H_{\boldsymbol{\alpha}} = \left(\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\alpha}}\right)^T H_{\boldsymbol{q}} \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\alpha}} + \left[\left(\frac{\partial g_\rho}{\partial \boldsymbol{q}}\right)^T \frac{\partial^2 \boldsymbol{q}}{\partial \alpha_l \partial \alpha_k}\right]_{k,l=0}^{n} \tag{14}$$

where the second term denotes the $(n+1)\times(n+1)$ matrix whose $(k+1, l+1)$ entry appears in the bracket. Since $H_{\boldsymbol{q}}$ is positive definite and $\partial g_\rho/\partial \boldsymbol{q}$ vanishes at the optimal point (see [3]) and also since $(\partial \boldsymbol{q}/\partial \boldsymbol{\alpha}) = L_n^{-1} T_{\boldsymbol{\tau}}^{-1} T_{\boldsymbol{\alpha}}$ is invertible at the optimum, $H_{\boldsymbol{\alpha}}$ is positive definite at the optimum. By continuity of the Hessian $H_{\boldsymbol{\alpha}}$, $H_{\boldsymbol{\alpha}}$ is positive definite in a neighborhood of the optimum. ■

IV. Continuation Method

Recall that our goal is to find, for a given positive–definite matrix K and a given $\Psi = \rho\rho^*/\tau\tau^*$, the global minimizer of the optimization problem (12). Owing to Lemma III.1, this is equivalent to finding the unique stationary point $\boldsymbol{\alpha}$ in $\mathcal{S}_n$. This point is characterized as a solution of the system of nonlinear equations

$$\nabla_{\boldsymbol{\alpha}} g_\rho(\boldsymbol{\alpha}) := \frac{\partial g_\rho(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = 2K\boldsymbol{\alpha} - 2\left\langle \frac{1}{\alpha}\boldsymbol{z}, \Psi \right\rangle = 0 \tag{15}$$

where $\boldsymbol{z} := [z^n, z^{n-1}, \ldots, 1]^T$ and, with abuse of notation, the inner product means component-wise inner products, that is

$$\left\langle \frac{1}{\alpha}\boldsymbol{z}, \Psi \right\rangle := \begin{bmatrix} \left\langle \frac{z^n}{\alpha}, \Psi \right\rangle \\ \left\langle \frac{z^{n-1}}{\alpha}, \Psi \right\rangle \\ \vdots \\ \left\langle \frac{1}{\alpha}, \Psi \right\rangle \end{bmatrix}.$$

For an arbitrarily specified Schur polynomial ρ, the solution $\boldsymbol{\alpha}$ of (15) must be determined in an iterative way. Because of the lack of global convexity of g_ρ, the initial point of the iteration must be chosen sufficiently close to the (unknown) solution, which is generally hard. On the other hand, if ρ equals τ, then $\Psi \equiv 1$ and the solution of the system

$$2K\boldsymbol{\alpha} - 2\left\langle \frac{1}{\alpha}\boldsymbol{z}, 1 \right\rangle = 0 \tag{16}$$

is well-known as the so-called *central solution*, which is easy to obtain. (Indeed, the solution of (15) in this case can be obtained by solving a system of linear equations; see [2]). Therefore, by starting with the central solution, we shall apply a continuation method [1] to solve (15). The method is tailored so that each point of iterations from the central solution stays in a region of convexity of a family of optimization problems tending to (12). This removes the problem caused by the lack of global convexity of the function g_ρ.

A. Problem Normalization

Before explaining the continuation method, for technical reasons, we normalize Ψ by defining a new function

$$\Psi_1(z) := \frac{\Psi(z)}{\langle \Psi, 1\rangle}$$

so that $\langle \Psi_1, 1\rangle = 1$. This normalization can be interpreted as a scaling of ρ by $1/\sqrt{\langle \Psi, 1\rangle}$. Here, we should note that $\langle \Psi, 1\rangle > 0$. It can be verified that a solution $\boldsymbol{\alpha}$ of the system

$$2K\boldsymbol{\alpha} - 2\left\langle \frac{1}{\alpha}\boldsymbol{z}, \Psi_1 \right\rangle = 0 \tag{17}$$

is a multiple of the solution of (15) by the same scaling $1/\sqrt{\langle \Psi, 1\rangle}$ and so is β obtained via $\alpha\beta^* + \alpha^*\beta = \rho\rho^*$. Hence, this scaling does *not* affect the solution $f = \beta/\alpha$ of NPDC. Therefore, we can shift our attention from (15) to the normalized problem (17).

B. Continuation Method

The continuation method that we apply to solve (17) uses a homotopy

$$\boldsymbol{h}(\boldsymbol{\alpha}, \cdot) : [0, 1] \to \mathbb{R}^{n+1}$$

such that $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = 0$ and $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = 0$ coincide respectively with (16) and (17). Recall that we want to solve $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = 0$, whereas $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = 0$ is easy to solve. The idea of the continuation method is sketched as follows. First, we solve a system of linear equations $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = 0$. Next, we consider a slightly different system $\boldsymbol{h}(\boldsymbol{\alpha}, \nu) = 0$ with a small-positive ν. This is a nonlinear system and, hence, an iterative method is necessary to obtain the solution. We shall apply Newton's method here. The initial point of the iteration is determined based on the solution of the previous system $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = 0$ and this step is called the *predictor step*, while the step of solving the new system $\boldsymbol{h}(\boldsymbol{\alpha}, \nu) = 0$ is called the *corrector step*. After solving $\boldsymbol{h}(\boldsymbol{\alpha}, \nu) = 0$, we increase the value of ν and go through the predictor and corrector steps again. We repeat this procedure until ν becomes one, that is, until we obtain the solution of $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = 0$. In the following, we first construct the homotopy $\boldsymbol{h}(\boldsymbol{\alpha}, \cdot)$ and explain the predictor and corrector steps later.

Now, let us define a class of functions

$$\Phi(z, \nu) := 1 + \nu\left(\Psi_1(z) - 1\right),\ \nu \in [0, 1].$$

Note that $\Phi(z, 0) = 1$, $\Phi(z, 1) = \Psi_1(z)$ and due to the problem normalization, $\langle 1, \Phi(z, \nu)\rangle = 1,\ \forall \nu \in [0, 1]$. Since $\Phi(e^{i\theta}, 0) \equiv 1 > 0$ and $\Phi(e^{i\theta}, 1) = \Psi_1(e^{i\theta}) > 0$ for all $\theta \in [-\pi, \pi]$, it is easy to verify that $\Phi(e^{i\theta}, \nu) > 0$ for all $\theta \in [-\pi, \pi]$ and $\nu \in [0, 1]$. Hence, for each ν, there exists a unique Schur polynomial ρ_ν satisfying $\Phi(z, \nu) = \rho_\nu \rho_\nu^*/\tau\tau^*$. (In fact, we do not need to determine ρ_ν in our procedure; see [14]). Due to Lemma III.1, for each $\nu \in [0, 1]$, the optimization problem

$$\min_{\boldsymbol{\alpha} \in \mathcal{S}_n} g_{\rho_\nu}(\boldsymbol{\alpha}) \quad g_{\rho_\nu}(\boldsymbol{\alpha}) := \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\langle \log \alpha, \Phi(z, \nu)\rangle$$

has a unique minimizer in $\mathcal{S}_n$, characterized by

$$\boldsymbol{h}(\boldsymbol{\alpha}, \nu) := \nabla_{\boldsymbol{\alpha}} g_{\rho_\nu}(\boldsymbol{\alpha}) = 2K\boldsymbol{\alpha} - 2\left\langle \frac{1}{\alpha}\boldsymbol{z}, \Phi(z, \nu) \right\rangle = 0. \tag{18}$$

Since $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = 0$ and $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = 0$ are exactly (16) and (17), respectively, we have constructed the desired homotopy $\boldsymbol{h}(\boldsymbol{\alpha}, \nu)$ in (18).

For each $\nu \in [0, 1]$, denote the unique solution of (18) in $\mathcal{S}_n$ by $\hat{\boldsymbol{\alpha}}(\nu)$. The class of vectors $\{\hat{\boldsymbol{\alpha}}(\nu) : \nu \in [0, 1]\}$ forms a trajectory in the $(n + 1)$-dimensional Euclidean space and the role of the predictor and corrector steps is to follow the trajectory from $\nu = 0$ to $\nu = 1$.

Before describing the predictor and corrector steps, we state one important fact for trajectory following, which is the direct consequence of the implicit function theorem [15]. Note that the Hessian of $g_{\rho_\nu}(\boldsymbol{\alpha})$, $\nabla^2_{\boldsymbol{\alpha}} g_{\rho_\nu}(\boldsymbol{\alpha}) = \partial \boldsymbol{h}/\partial \boldsymbol{\alpha}(\boldsymbol{\alpha}, \nu)$ is positive definite (and hence invertible) on the trajectory $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^1$ due to Lemma III.2.

Lemma IV.1: The function $\hat{\boldsymbol{\alpha}}(\nu)$ is continuously differentiable with respect to ν over the interval [0, 1]. Besides, the derivative is given by

$$\frac{d\hat{\boldsymbol{\alpha}}}{d\nu}(\nu) = -\left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\alpha}}(\boldsymbol{\alpha}, \nu)\right)^{-1}\left(\frac{\partial \boldsymbol{h}}{\partial \nu}(\boldsymbol{\alpha}, \nu)\right)\Bigg|_{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}(\nu)}.$$

C. Predictor Step

We will apply *Euler's method* in this step. To be more specific, for a given point $\hat{\boldsymbol{\alpha}}(\nu)$ and a step size $\delta\nu$, we move the point $\hat{\boldsymbol{\alpha}}(\nu)$ in the direction

$$\boldsymbol{d}\left(\hat{\boldsymbol{\alpha}}(\nu)\right) := -\left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\alpha}}(\boldsymbol{\alpha}, \nu)\right)^{-1}\left(\frac{\partial \boldsymbol{h}}{\partial \nu}(\boldsymbol{\alpha}, \nu)\right)\Bigg|_{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}(\nu)} \tag{19}$$

by the step size $\delta\nu$, that is

$$\boldsymbol{\alpha}(\nu + \delta\nu) = \hat{\boldsymbol{\alpha}}(\nu) + \boldsymbol{d}\left(\hat{\boldsymbol{\alpha}}(\nu)\right)\delta\nu. \tag{20}$$

This predictor step approximates the trajectory curve by a straight line. Note that the notation $\boldsymbol{\alpha}$ is used in the left-hand side of (20) instead of $\hat{\boldsymbol{\alpha}}$, since the point after the predictor step is not guaranteed to lie on the trajectory. To obtain $\boldsymbol{\alpha}(\nu + \delta\nu)$, we need to calculate $\boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu))$ and to choose step size $\delta\nu$.

First, the computation of the vector $\boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu))$ in (19) is explained. The first part is the inverse of the Hessian of g_{ρ_ν} at $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}(\nu)$ and can be computed according to the procedure presented in [14]. The second part becomes as follows:

$$\frac{\partial \boldsymbol{h}}{\partial \nu}(\boldsymbol{\alpha}, \nu)\Bigg|_{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}(\nu)} = \nabla_{\boldsymbol{\alpha}}\tilde{g}\left(\hat{\boldsymbol{\alpha}}(\nu)\right) - 2K\hat{\boldsymbol{\alpha}}(\nu) \tag{21}$$

where the function $\tilde{g}$ is defined by

$$\tilde{g}(\boldsymbol{\alpha}) := \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\left\langle \log \alpha, \frac{\eta\tau^* + \tau\eta^*}{\tau\tau^*} \right\rangle$$

where $\eta(z) := [z^{n-1}, z^{n-2}, \ldots, 1]L_{n-1}\boldsymbol{\psi}$. The gradient term in (21) can be determined by the same calculation as in [14].

Next, a reasonable method is proposed to determine the step length $\delta\nu$ such that the updated point $\boldsymbol{\alpha}(\nu + \delta\nu)$ does not deviate from the (unknown) point $\hat{\boldsymbol{\alpha}}(\nu + \delta\nu)$ on the trajectory too much. A small deviation from the trajectory is necessary, due to the lack of global convexity and Lemma III.2, for convergence of the Newton's iteration in the corrector step which follows this predictor step. Since, for any ν in [0, 1], the point $\hat{\boldsymbol{\alpha}}(\nu)$ satisfies the identity

$$\hat{\boldsymbol{\alpha}}(\nu)^T \nabla_{\boldsymbol{\alpha}} g_{\rho_\nu}\left(\hat{\boldsymbol{\alpha}}(\nu)\right) = 2\hat{\boldsymbol{\alpha}}(\nu)^T K \hat{\boldsymbol{\alpha}}(\nu) - 2\langle 1, \Phi(z, \nu)\rangle \equiv 0$$

due to (18) and since the inner product term equals one due to the normalization, the trajectory $\hat{\boldsymbol{\alpha}}(\nu)$ lies on a hyper-ellipsoid in the $(n + 1)$-dimensional Euclidean space, namely

$$\hat{\boldsymbol{\alpha}}(\nu)^T K \hat{\boldsymbol{\alpha}}(\nu) \equiv 1,\ \forall \nu \in [0, 1]. \tag{22}$$

Hence, the direction vector $\boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu))$ is orthogonal to the normal vector $K\hat{\boldsymbol{\alpha}}(\nu)$ of the hyper-ellipsoid, that is, $\boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu))^T K \hat{\boldsymbol{\alpha}}(\nu) = 0$. From this and the relation (22), we obtain the following:

$$\boldsymbol{\alpha}(\nu + \delta\nu)^T K \boldsymbol{\alpha}(\nu + \delta\nu) = 1 + \boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu))^T K \boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu))(\delta\nu)^2. \tag{23}$$

Since K is positive definite, the point $\boldsymbol{\alpha}(\nu + \delta\nu)$ lies on an enlarged version of the hyper-ellipsoid (22) unless $\boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu)) = 0$. Keeping the last term in (23) small will be helpful to get the small deviation from the original hyper-ellipsoid (22) and hence, from the trajectory. Therefore, for a prespecified small constant $\varepsilon > 0$, we set $\delta\nu$ as

$$\delta\nu = \left(\frac{\varepsilon}{\boldsymbol{d}(\hat{\boldsymbol{\alpha}}(\nu))^T K \boldsymbol{d}\left(\hat{\boldsymbol{\alpha}}(\nu)\right)}\right)^{1/2}. \tag{24}$$

Remark IV.2: We would like to maintain the positivity of the Hessian of $g_{\rho_{\nu+\delta\nu}}$ at the point $\boldsymbol{\alpha}(\nu + \delta\nu)$ generated in the predictor step, for convergence of Newton's method in the corrector step. Therefore, in the case where the Hessian is not positive at $\boldsymbol{\alpha}(\nu + \delta\nu)$, we decrease $\delta\nu$ from (24) until it becomes positive.

D. Corrector Step

Using the point $\boldsymbol{\alpha}(\nu + \delta\nu)$ in (20) as an initial point and fixing ν at $\nu + \delta\nu$, we apply *Newton's method* to pull $\boldsymbol{\alpha}(\nu + \delta\nu)$ back to the

trajectory and to obtain $\hat{\boldsymbol{\alpha}}(\nu+\delta\nu)$. Since the function $g_{\rho_{\nu+\delta\nu}}(\boldsymbol{\alpha})$ is strictly convex around the trajectory due to Lemma III.2, it is expected that the Newton's iteration converges as long as the initial point is not far from the trajectory, that is, the value ε in (24) is chosen appropriately small.

In Newton's method, we need the values of the gradient and the Hessian of $g_{\rho_{\nu+\delta\nu}}(\boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$. The details of the calculations are given in [14]. To increase the robustness of the algorithm, we perform an inaccurate line search by means of the Wolfe test (see [11, p. 214] for a detailed exposition). See [13] and [14] for the whole procedure of the corrector step.

V. Numerical Example

In this section, we will give one example which illustrates the efficiency and the robustness of the proposed solver. We tackle a Nevanlinna–Pick interpolation problem with degree constraint that will be difficult to solve by the previous solver.

We assume the self-conjugate interpolation data

$$\begin{aligned}\mathcal{D}=\Big\{&(\infty,0.6499),(1.1,1)\\&(0.8709\mp 0.8967i,1.0363\pm 0.3338i)\\&(0.3344\mp 1.2044i,0.7085\pm 0.4738i)\\&(-0.6474\pm 0.8893i,1\pm i)\Big\}.\end{aligned}$$

We can verify that the corresponding Pick matrix is positive definite.

Since the number of interpolation constraints is eight, our goal is to find any positive-real interpolant f that is real rational of degree at most seven, by specifying seven spectral zeros in the open unit disc. When the spectral zeros of f (in other words, roots of ρ) are selected at locations $z=0.95e^{\pm 1.22i},\ 0.95e^{\pm 2.3i}, -0.99, \pm 0.99i$, the proposed solver finds a solution $f(z)=\beta(z)/\alpha(z)$, where

$$\begin{aligned}\beta(z):=&1.3852z^7-1.8896z^6+1.5410z^5-0.5285z^4\\&-0.6206z^3+1.5499z^2-1.9570z+1.2167\\\alpha(z):=&2.1315z^7-3.7752z^6+3.8690z^5-2.5704z^4\\&+2.7296z^3-3.8681z^2+3.7799z-1.8708.\end{aligned}$$

We can check that the interpolation constraints are satisfied with precision $\sum_{j=0}^{7}|w_j-f(z_j)|^2<10^{-6}$. The poles of f are located at $z=-0.6798\pm 0.7334i, 0.3607\pm 0.9327i, 0.9003, 0.7545\pm 0.6369i$, some of which are almost on the unit circle. Such Nevanlinna–Pick interpolation problems with degree constraint are considered to be hard to solve numerically. In particular, it is almost impossible for such problems to be solved accurately by the previous solver. Indeed, the condition numbers[1] of the Hessian $H_{\boldsymbol{\alpha}}$ of $g_\rho(\boldsymbol{\alpha})$ and $H_{\boldsymbol{q}}$ of $J_\rho(\boldsymbol{q})$ at the optimum are calculated as

$$\operatorname{cond}(H_{\boldsymbol{\alpha}})=86.95\ll\operatorname{cond}(H_{\boldsymbol{q}})=2.38\times 10^{10}.$$

Additionally, our continuation procedure does not converge without the Wolfe test in this example. On the other hand, the proposed solver with the Wolfe test has an ability to solve such hard problems as this example, which enables us to fully exploit the freedom of the solution set of NPDC in applications.

VI. Conclusion

A new solver for Nevanlinna–Pick interpolation with degree constraint has been presented. The solver relies on a continuation method consisting of Euler–Newton predictor-corrector steps. A change of variables has been introduced in order to remove numerical difficulties caused by the inaccuracy of spectral factorization and the ill-conditioning of a system of linear equations in the previous algorithm. Even though estimates of the condition number of the Hessian and a convergence proof for the algorithm are missing, numerical experiments show that the proposed solver performs in an efficient and numerically robust manner.

[1]We calculated the 2-norm condition number, i.e., the ratio of the largest singular value of a matrix to the smallest one.

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