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A Shaping Limitation of Rational Sensitivity Functions With a Degree Constraint

Ryozo Nagamune

***Abstract*—This note concerns a certain shaping limitation of sensitivity functions. The focus is on a frequency-wise infimum of gains of rational sensitivity functions with a degree constraint. An explicit infimum is derived for a special case. The result is useful for determining the inability of sensitivity functions of low degrees to achieve a specification in the frequency domain, and thus for motivating the use of higher degree sensitivity functions to fulfill the specification.**

***Index Terms*—Degree constraint, rationality, sensitivity function, shaping limitation.**

I. Introduction

The sensitivity function $S := 1/(1+PC)$, where P is a plant model and C is a controller, plays an important role in representing various performances of feedback systems. Many results on performance limitations have been derived in terms of S; see, e.g., [1], [3], [6], [7], and the references therein. If a given specification contradicts such performance limitations, it is not achievable by any stabilizing feedback controller, which motivates us to change specifications and/or a plant model P. Since any existing controller design method, such as QFT [4] and $\mathcal{H}^\infty$ [2], relies on design iterations based on trial-and-error, it is significant to know what is not achievable in advance for avoiding unnecessary trials.

So far, most of the results on performance limitations of S are concerned with both rational and irrational sensitivity functions. However, in many practical applications, we are interested in *rational* sensitivity functions of *low degrees*, since they typically correspond to simple controllers. It is natural to expect that the rationality and degree restriction may provide tighter and practically more useful limitations than the performance limitations known today. This is the topic of this note.

In this note, by imposing a degree constraint on rational sensitivity functions, we will consider a shaping limitation problem to derive the infimum of the gain $|S|$ at each frequency. The obtained infimum will be useful for determining if we need S of higher degrees to achieve specifications. We will give an explicit frequency-wise infimum of $|S|$ in a certain special case. In general, however, the formulated problem is nonconvex, and thus hard to solve.

This note is organized as follows. In Section II, we will introduce the set of rational sensitivity functions with a degree constraint. For this set, in Section III, we will formulate a shaping limitation problem which will be solved in a special case. In Section IV, we will give a numerical example to illustrate the result of Section III. We will mainly deal with the discrete-time case, but as will be remarked later, the results can be translated into a version for the continuous-time case.

II. Set of Rational Sensitivity Functions With a Degree Constraint

Suppose that a given plant P is a scalar real rational proper transfer function that has relative degree $r > 0$, and has unstable poles p_j of

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The author is with Department of Mechanical Engineering, University of California, Berkeley, CA 94720 USA (e-mail: ryozo@me.berkeley.edu).

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multiplicity ℓ_j, $j=1,\dots,\alpha$, and unstable zeros q_j of multiplicity m_j, $j=1,\dots,\beta$. (Since P is real rational, the unstable poles and zeros are self-conjugate, including multiplicities.) Then, it is known that, for internal stability of the closed-loop system, the sensitivity function S must be stable and satisfy interpolation conditions (see [3] and [6])

$$\begin{cases} \frac{d^k}{dz^k} S(z)\Big|_{z=p_j} = 0, & k=0,\dots,\ell_j-1, \quad j=1,\dots,\alpha \\ \frac{d^k}{dz^k}\left[S(z)-1\right]\Big|_{z=q_j} = 0, & k=0,\dots,m_j-1, \quad j=1,\dots,\beta \\ \frac{d^k}{dz^k}\left[S(z^{-1})-1\right]\Big|_{z=0} = 0, & k=0,\dots,r-1. \end{cases} \tag{1}$$

In addition, if P is rational, then we are often interested in *rational* sensitivity functions, since these will correspond to rational controllers. A rational S should be proper to meet the requirement $S(z^{-1})-1\big|_{z=0}=0$. Therefore, the set of acceptable sensitivity functions when P is real rational is expressed by

$$\mathcal{S} := \{S \in \mathcal{RH}^\infty : S \text{ satisfies } (1)\} \tag{2}$$

where $\mathcal{RH}^\infty$ denotes the set of real rational proper stable transfer functions.

We usually prefer sensitivity functions of low degrees in practice, since such sensitivity functions typically lead to simple controllers. For this reason, in this note, we will focus on a subset of $\mathcal{S}$ which has a degree constraint

$$\mathcal{S}_D := \mathcal{S} \cap \{S : \deg S \le n\}. \tag{3}$$

The degree bound n is chosen to be the total number of interpolation conditions minus one

$$n := \ell + m + r - 1 \quad \ell := \sum_{j=1}^{\alpha} \ell_j \quad m := \sum_{j=1}^{\beta} m_j. \tag{4}$$

The choice of this bound is motivated by the work in [5], where we have proposed a method of searching for a function with a desired shape of the frequency response from a subset of $\mathcal{S}_D$, namely $\hat{\mathcal{S}}_D := \mathcal{S}_D \cap \left\{S \in \mathcal{RH}^\infty : \|S\|_\infty < \gamma\right\}$ for some $\mathcal{H}^\infty$ norm bound $\gamma > 0$. If we know a limitation of $\mathcal{S}_D$, that is also a limitation of $\hat{\mathcal{S}}_D$.

The set $\mathcal{S}_D$ in (3) can be expressed more explicitly. First, since each element S in $\mathcal{S}_D$ is a real rational proper function with $\deg S \le n$, we can parameterize such S as

$$S(z) := \frac{\boldsymbol{b}^T \boldsymbol{z}_n}{[1 \quad \boldsymbol{a}^T]\, \boldsymbol{z}_n} \tag{5}$$

where $\boldsymbol{z}_n := [z^n, z^{n-1}, \dots, 1]^T$, $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^{n+1}$. Secondly, S in $\mathcal{S}_D$ must be stable since $S \in \mathcal{RH}^\infty$. The stability condition is represented as

$$\boldsymbol{a} \in \mathcal{A}_n := \{\boldsymbol{a} \in \mathbb{R}^n : [1 \quad \boldsymbol{a}^T]\, \boldsymbol{z}_n \ne 0,\ \forall\ |z| \ge 1\} \tag{6}$$

where $\mathcal{A}_n$ is the *Schur stability region* in the n-dimensional space. It is well-known that $\mathcal{A}_1 = \{a \in \mathbb{R} : |a| < 1\}$ and $\mathcal{A}_2$ depicted in Fig. 1 are convex sets, while $\mathcal{A}_n$ for $n \ge 3$ are nonconvex.

Lastly, S in $\mathcal{S}_D$ must satisfy the interpolation conditions (1). Using the parameterization in (5), we can write these conditions in terms of coefficient vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as

$$\boldsymbol{b} = Z^{-1} W Z \begin{bmatrix} 1 \\ \boldsymbol{a} \end{bmatrix}. \tag{7}$$

Here, Z is an invertible matrix of size $(n+1) \times (n+1)$ defined by

$$Z := \begin{bmatrix} [A^n b & A^{n-1} b & \dots & b] \\ & I_r & 0_{r \times (n+1-r)} & \end{bmatrix}$$

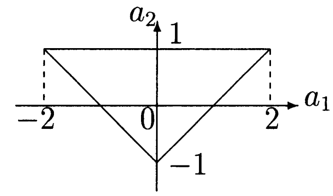


Fig. 1. Schur stability region $\mathcal{A}_2$ (inside the triangle).

which consists of a block diagonal matrix $A := \operatorname{blockdiag}\left\{A_{p_1}^{(\ell_1)}, \dots, A_{p_\alpha}^{(\ell_\alpha)}, A_{z_1}^{(m_1)}, \dots, A_{z_\beta}^{(m_\beta)}\right\}$ and a vector $b := \left[e^{(\ell_1)}, \dots, e^{(\ell_\alpha)}, e^{(m_1)}, \dots, e^{(m_\beta)}\right]^T$ with

$$A_z^{(k)} := \underbrace{\begin{bmatrix} z & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & z \end{bmatrix}}_{k} \qquad e^{(k)} := [\underbrace{1, 0, \dots, 0}_{k}].$$

The matrix W is a $(n+1) \times (n+1)$ matrix

$$W := \begin{bmatrix} 0_\ell & 0 \\ 0 & I_{m+r} \end{bmatrix}. \tag{8}$$

Due to (5)–(7), we have the following.

Proposition 2.1: The set $\mathcal{S}_D$ in (3) can be expressed as

$$\mathcal{S}_D = \left\{S : S(z) = \frac{[1 \quad \boldsymbol{a}^T]\,(Z^{-1} W Z)^T \boldsymbol{z}_n}{[1 \quad \boldsymbol{a}^T]\, \boldsymbol{z}_n}, \boldsymbol{a} \in \mathcal{A}_n\right\}. \tag{9}$$

Before formulating a shaping limitation problem for this set, we will present a case where the set $\mathcal{S}_D$ reduces to a singleton.

Proposition 2.2: Suppose that a plant P has no unstable pole. Then, the set $\mathcal{S}_D$ is a singleton expressed by

$$\mathcal{S}_D = \{S : S \equiv 1\}. \tag{10}$$

Proof: In the case that a plant has no unstable pole, $\ell = 0$ and hence the matrix W in (8) reduces to the identity matrix. From (9), the assertion follows immediately. ■

Since $S \equiv 1$ (or equivalently, $C \equiv 0$) is never satisfactory in applications, this proposition implies that if there is no unstable plant pole ($\ell = 0$), then we have to use sensitivity functions of degree $\deg S \ge n + 1 = r + m$, i.e., at least the relative degree (r) plus the number of unstable plant zeros including multiplicities (m).

III. Sensitivity Shaping Limitation

Now, we will formulate the shaping limitation problem to be considered in this note.

Problem 3.1: For each frequency $\theta \in [0, \pi]$, determine the infimum of the gains of functions in the set $\mathcal{S}_D$. In other words, for each fixed θ, solve the optimization problem

$$\inf_{S \in \mathcal{S}_D} \left|S(e^{i\theta})\right|^2. \tag{11}$$

Remark 3.2: If we require a lower gain than this infimum at some frequencies, then we need to use sensitivity functions of degrees higher than n.

Since the set $\mathcal{S}_D$ is represented in terms of the vector $\boldsymbol{a}$ in (9), we can rewrite the optimization problem (11) as

$$\inf_{\boldsymbol{a} \in \mathcal{A}_n} f_\theta(\boldsymbol{a}) \tag{12}$$

where the cost functional f_θ, defined for each θ, is given by

$$f_\theta(\boldsymbol{a}) := \left|S(e^{i\theta})\right|^2 = \frac{[1 \quad \boldsymbol{a}^T]\,(Z^{-1}WZ)^T \boldsymbol{e}_\theta \boldsymbol{e}_\theta^H Z^{-1}WZ \begin{bmatrix}1\\ \boldsymbol{a}\end{bmatrix}}{[1 \quad \boldsymbol{a}^T]\,\boldsymbol{e}_\theta \boldsymbol{e}_\theta^H \begin{bmatrix}1\\ \boldsymbol{a}\end{bmatrix}} \tag{13}$$

with $\boldsymbol{e}_\theta := [e^{in\theta}, \ldots, e^{i\theta}, 1]^T$. Here, the superscript H means the Hermitian transpose.

Since the functional f_θ is nonconvex and the Schur stability region is a nonconvex set in general, (12) is generally a nonconvex optimization problem that is hard to solve. However, under some assumptions on a plant, we can solve it analytically, and this will be explained next.

Now, we assume the following.

Assumption 3.3: The plant has relative degree one $(r = 1)$, no unstable zero $(m = 0)$ and at least one unstable pole $(\ell \geq 1)$.

Remark 3.4: The case of $\ell = 0$ has already been covered in Proposition 2.2.

We will state the main result in this note.

Theorem 3.5: Under Assumption 3.3, the optimal value of the optimization problem (12) is

$$\inf_{\boldsymbol{a}\in\mathcal{A}_n} f_\theta(\boldsymbol{a}) = \frac{n_\theta}{[2(1+|\cos\theta|)]^n}$$

where n_θ is defined by

$$n_\theta := [1 \quad k_1 \quad \ldots \quad k_n] \times \begin{bmatrix} 1 & \cos\theta & \ldots & \cos n\theta \\ \cos\theta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \cos\theta \\ \cos n\theta & \ldots & \cos\theta & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k_1 \\ \vdots \\ k_n \end{bmatrix} \tag{14}$$

and $\{k_j\}_{j=1}^n$ are determined from given unstable plant poles as

$$\prod_{j=1}^{\alpha}(z-p_j)^{\ell_j} =: z^n + k_1 z^{n-1} + \cdots + k_n.$$

For the proof of this theorem, we need the following.

Lemma 3.6: Denote the denominator of f_θ in (13) by d_θ. Then

$$\sup_{\boldsymbol{a}\in\mathcal{A}_n} d_\theta(\boldsymbol{a}) = [2(1+|\cos\theta|)]^n. \tag{15}$$

Proof: We use a well-known fact that any real polynomial can always be factored into real polynomials of degree at most two. Then, the optimization problem $\sup_{\boldsymbol{a}\in\mathcal{A}_n} d_\theta(\boldsymbol{a})$ is equivalent to another optimization problem as shown in (16) at the bottom of the page, where $\boldsymbol{e}_\theta^{(1)} := [e^{i\theta}, 1]^T$ and $\boldsymbol{e}_\theta^{(2)} := [e^{2i\theta}, e^{i\theta}, 1]^T$. We will show next an analytical solution to each optimization problem in (16).

First, consider the problem $\sup_{a\in\mathcal{A}_1}\left|[1, a]\boldsymbol{e}_\theta^{(1)}\right|^2$. Since $\mathcal{A}_1 = \{a \in \mathbb{R} : |a| < 1\}$ (as was mentioned before), and

$$\left|[1 \quad a]\,\boldsymbol{e}_\theta^{(1)}\right|^2 = [1 \quad a]\begin{bmatrix}1 & \cos\theta \\ \cos\theta & 1\end{bmatrix}\begin{bmatrix}1\\ a\end{bmatrix} = a^2 + 2a\cos\theta + 1$$

it is easy to verify

$$\sup_{a\in\mathcal{A}_1}\left|[1 \quad a]\,\boldsymbol{e}_\theta^{(1)}\right|^2 = 2(1+|\cos\theta|). \tag{17}$$

Next, consider the problem $\sup_{\boldsymbol{a}\in\mathcal{A}_2}\left|[1, \boldsymbol{a}^T]\boldsymbol{e}_\theta^{(2)}\right|^2$. The domain $\mathcal{A}_2$ is a convex polytope (triangle), as shown in Fig. 1. In addition, we have

$$\begin{aligned}\left|[1 \quad \boldsymbol{a}^T]\,\boldsymbol{e}_\theta^{(2)}\right|^2 &= [1 \quad \boldsymbol{a}^T]\begin{bmatrix}1 & \cos\theta & \cos 2\theta \\ \cos\theta & 1 & \cos\theta \\ \cos 2\theta & \cos\theta & 1\end{bmatrix}\begin{bmatrix}1\\ \boldsymbol{a}\end{bmatrix} \\ &= \boldsymbol{a}^T\begin{bmatrix}1 & \cos\theta \\ \cos\theta & 1\end{bmatrix}\boldsymbol{a} \\ &\quad + 2\,[\cos\theta \quad \cos 2\theta]\,\boldsymbol{a} + 1.\end{aligned}$$

Since the coefficient matrix of the quadratic term is nonnegative definite, the supremum of this functional is attained at one of the vertices of $\mathcal{A}_2$. By some calculations, we can obtain the value of the functional at each vertex as

$$\left|[1 \quad \boldsymbol{a}^T]\,\boldsymbol{e}_\theta^{(2)}\right|^2 = \begin{cases} 4(1+\cos\theta)^2, & \boldsymbol{a} = [2, 1]^T \\ 4(1-\cos\theta)^2, & \boldsymbol{a} = [-2, 1]^T \\ 4\sin^2\theta, & \boldsymbol{a} = [0, -1]^T. \end{cases}$$

By taking the maximum of these three values, we obtain

$$\sup_{\boldsymbol{a}\in\mathcal{A}_2}\left|[1 \quad \boldsymbol{a}^T]\,\boldsymbol{e}_\theta\right|^2 = [2(1+|\cos\theta|)]^2. \tag{18}$$

Substitution of (17) and (18) into (16) gives (15). (The result does not distinguish even and odd n.) ■

Proof of Theorem 3.5: Under Assumption 3.3, since $m = 0$ and $r = 1$ [and, thus, $n = \ell$ due to (4)], the matrices W and Z become

$$W = \begin{bmatrix} 0_n & \\ & 1 \end{bmatrix} \quad Z = \begin{bmatrix} [A^n b & \ldots & Ab & b] \\ 1 & 0_{1\times n} & \end{bmatrix}$$

and, hence, the relation between $\boldsymbol{b}$ and $\boldsymbol{a}$ is simplified as

$$\begin{aligned}\boldsymbol{b} &= Z^{-1}WZ\begin{bmatrix}1\\ \boldsymbol{a}\end{bmatrix} \\ &= Z^{-1}\begin{bmatrix}0_{n\times 1}\\ 1\end{bmatrix} = \begin{bmatrix} 1 \\ -[A^{n-1}b \quad \ldots \quad b]^{-1}A^n b\end{bmatrix}.\end{aligned}$$

This means that the numerator of f_θ is independent of the vector $\boldsymbol{a}$. Indeed, the numerator of f_θ can be simplified as in (14). This can be seen from the fact that S in $\mathcal{S}_D$ can be parameterized as

$$S(z) = \frac{\prod_{j=1}^{\alpha}(z-p_j)^{\ell_j}}{z^n + a_1 z^{n-1} + \cdots + a_n}.$$

Therefore, the minimization of f_θ is equivalent to the maximization of d_θ as

$$\inf_{\boldsymbol{a}\in\mathcal{A}_n} f_\theta(\boldsymbol{a}) = \frac{n_\theta}{\sup_{\boldsymbol{a}\in\mathcal{A}_n} d_\theta(\boldsymbol{a})}.$$

Consequently, due to Lemma 3.6, the proof is complete. ■

Remark 3.7: A plant P, that has relative degree zero and one real unstable zero q_1 of multiplicity one, can be transformed into a plant $\hat{P}$ with relative degree one and no unstable zero, via a linear fractional transformation of the variable z:

$$\hat{P}(z) := P\left(\frac{1+q_1 z}{z+q_1}\right). \tag{19}$$

$$\sup_{\boldsymbol{a}\in\mathcal{A}_n} d_\theta(\boldsymbol{a}) = \begin{cases} \left(\sup_{\boldsymbol{a}\in\mathcal{A}_2}\left|[1 \quad \boldsymbol{a}^T]\,\boldsymbol{e}_\theta^{(2)}\right|^2\right)^{n/2}, & \text{if } n \text{ is even} \\ \sup_{a\in\mathcal{A}_1}\left|[1 \quad a]\,\boldsymbol{e}_\theta^{(1)}\right|^2 \cdot \left(\sup_{\boldsymbol{a}\in\mathcal{A}_2}\left|[1 \quad \boldsymbol{a}^T]\,\boldsymbol{e}_\theta^{(2)}\right|^2\right)^{(n-1)/2}, & \text{if } n \text{ is odd} \end{cases} \tag{16}$$

Note that $\hat{P}(\infty) = P(q_1) = 0$. Since the outside of the unit circle is mapped into itself bijectively by this variable change, it does not introduce any extra unstable zero in $\hat{P}$, and unstable poles are mapped into the outside of the unit circle. Therefore, we can apply Theorem 3.5 to $\hat{P}$ and obtain the frequency-wise infimum of $|S|$, denoted by $\hat{l}(\theta)$. By scaling the infimum $\hat{l}(\theta)$ with the inverse transformation of (19), we can compute the frequency-wise infimum $l(\theta)$ for the original plant P as

$$l(\theta) = \hat{l}\left(\frac{1}{i}\log\frac{1+q_1 e^{i\theta}}{e^{i\theta}+q_1}\right).$$

Remark 3.8 (The Continuous-Time Case): A shaping limitation in the continuous-time case can also be derived as follows. First, we reduce the continuous-time plant to the discrete-time plant via a bilinear transformation $z = (1+s)/(1-s)$. Secondly, we apply Theorem 3.5 to the discrete-time plant for computing the gain infimum at each frequency θ. Finally, we scale the discrete-time frequency back to the continuous one ω by

$$e^{i\theta} = \frac{1+i\omega}{1-i\omega}.$$

IV. A Numerical Example

In this section, the result of Section III is illustrated by a numerical example. The plant treated in this example is described by

$$P(z) = \frac{1}{z+1.1}$$

which has relative degree one and one unstable pole at $z = -1.1$. The interpolation conditions (1) in this case are simply $S(\infty) = 1$ and $S(-1.1) = 0$. Suppose that we are given a specification that the sensitivity function must satisfy

$$\left|S(e^{i\theta})\right| < \gamma_u(\theta) := \begin{cases} 0.6, & \theta \in [0, 0.3] \\ 2, & \theta \in [0.3, \pi]. \end{cases} \tag{20}$$

Due to Bode integral formula in the discrete-time case [7], it is necessary to have (see [6, Cor. 3.4.6])

$$\|S\|_\infty \geq \left(\frac{1}{0.6}\right)^{0.3/(\pi-0.3)} |1.1|^{\pi/(\pi-0.3)} \approx 1.1727.$$

This necessary condition does not contradict the uniform bound in (20). However, it does not give any insight how S can be simple to achieve the specification.

Now, consider the set

$$\mathcal{S}_D := \Big\{S \in \mathcal{RH}^\infty : S(\infty) = 1$$
$$S(-1.1) = 0,\ \deg(S) \leq 1\Big\}.$$

The limitation of this set can be obtained by Theorem 3.5 and is shown in Fig. 2, with frequency responses of several elements in $\mathcal{S}_D$. We can see in the figure that all the frequency responses are above the frequency-wise infimum.

In Fig. 3(a), we see that $\gamma_u(\theta)$ is smaller than the infimum for $\mathcal{S}_D$ over low frequencies. Therefore, the specification (20) cannot be met by any sensitivity function of degree one, and hence we have to increase the degree of S to achieve (20).

In increasing the degree of S, we introduce the additional interpolation condition $S(1.1) = 0$, and consider the new set

$$\mathcal{S}_D^{\text{new}} := \Big\{S \in \mathcal{RH}^\infty : S(\infty) = 1$$
$$S(-1.1) = S(1.1) = 0,\ \deg(S) \leq 2\Big\}.$$

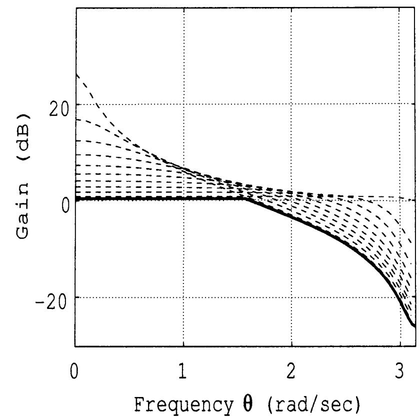
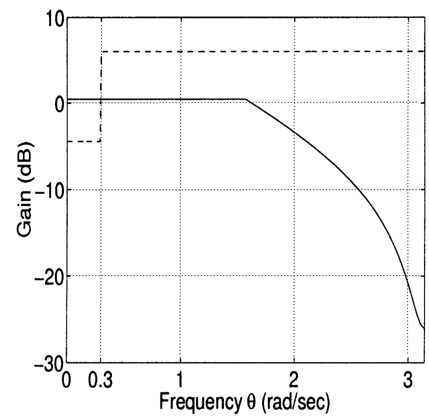
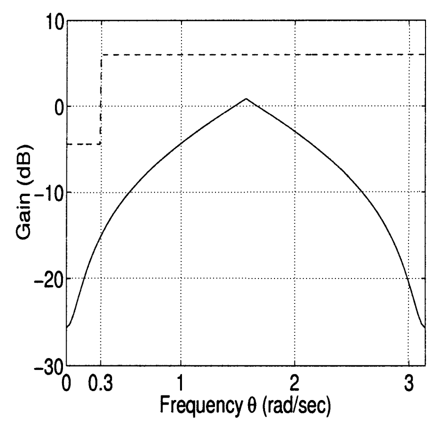


Fig. 2. Frequency-wise infimum of frequency responses of elements in $\mathcal{S}_D$ (solid line) and frequency responses of several elements in $\mathcal{S}_D$ (dashed lines).



(a)



(b)

Fig. 3. Relationship between the infimum $l(\theta)$ (solid lines) and the specification $\gamma_u(\theta)$ (dashed lines).

There are two reasons for the introduction of the additional condition; one is to make the new set $\mathcal{S}_D^{\text{new}}$ in a form that Theorem 3.5 is applicable, and the other is to decrease the infimal curve at low frequencies. Due to Theorem 3.5, the limitation of the set $\mathcal{S}_D^{\text{new}}$ can be computed and is shown in Fig. 3(b). The figure shows that the infimum for $\mathcal{S}_D^{\text{new}}$ is indeed under the specification over all frequencies. Hence, we cannot rule out the possibility that there is an S in $\mathcal{S}_D^{\text{new}}$ which achieves (20). Therefore, we should search for an appropriate S in $\mathcal{S}_D^{\text{new}}$. Note that we cannot guarantee that there does exist an appropriate S in $\mathcal{S}_D^{\text{new}}$ beforehand.

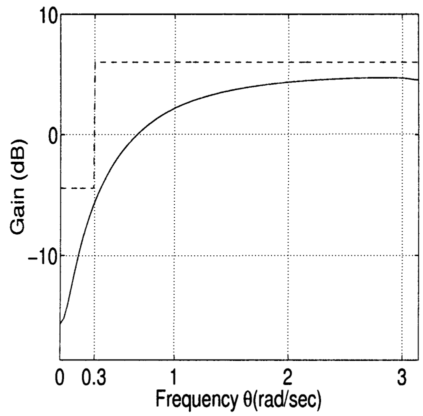


Fig. 4. Frequency response of the sensitivity function (21) (solid line) which meets the specification (dashed line).

We have used the method proposed in [5] to find a satisfactory sensitivity function. One sensitivity function in $\mathcal{S}_D^{\mathrm{new}}$ which satisfies the specification (20) was found as

$$S(z) = \frac{z^2 - 1.21}{z^2 + 0.57z - 0.30} \tag{21}$$

and the corresponding controller was computed by

$$C(z) = \frac{1 - S(z)}{P(z)S(z)} = \frac{-0.57z - 0.91}{z}.$$

Fig. 4 shows that the sensitivity function (21) indeed meets the specification (20).

This example suggests that we can use additional interpolation conditions for changing the shape of the infimal curve of the sensitivity gain, and for making given specifications to be achievable.

V. Conclusion

In this note, we have formulated a shaping limitation problem for rational sensitivity functions with a degree constraint as an optimization problem. An analytic solution to this problem was presented in a special case where a plant has some unstable poles, relative degree one and no unstable zero. The result is useful, especially in the approach proposed in [5], for circumventing unnecessary search for appropriate sensitivity functions of low degrees, and for motivating us to utilize the functions with higher degrees.

The shaping limitation problem for general cases (with arbitrary relative degree and arbitrary number of unstable zeros in a plant) amounts to solving a nonconvex optimization problem, as can be seen in (13). To solve the problem, we need to devise an efficient numerical method or we have to be content with some estimate of the optimum. This will be a subject of future research.

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Closed-Loop Shaping Based on Nevanlinna–Pick Interpolation With a Degree Bound

Ryozo Nagamune

***Abstract*—This note presents a novel method for shaping the frequency response of a single-input–single-output closed-loop system, based on the theory of Nevanlinna–Pick interpolation with degree constraint. The method imposes a degree bound on the closed-loop transfer function and searches for a function with a desired frequency response. Numerical examples illustrate the potential of the method in designing controllers with lower degrees than the ones obtained by conventional $\mathcal{H}^\infty$ controller design methods with weighting functions.**

***Index Terms*—Closed-loop shaping, degree bound, $\mathcal{H}^\infty$ control, Nevanlinna–Pick interpolation.**

I. Introduction

The objective of this note is to propose a new method for shaping the closed-loop frequency response in a $\mathcal{H}^\infty$ control framework. The shaping method is based on a recently developed theory of Nevanlinna–Pick interpolation with degree constraint ([1], [2]). The main difference from conventional methods in $\mathcal{H}^\infty$ control (see e.g., [3], [4], [6], and [12]) is that, in shaping frequency responses, we do *not* use weighting functions. The main advantage in our approach is that we typically obtain controllers of degree lower than the controller degree designed via conventional $\mathcal{H}^\infty$ control methods. Moreover, the closed-loop frequency response, which discontinuously depends on the choice of weighting functions in general, smoothly depends on our design parameters, which will facilitate controller design based on trial-and-error.

It is well-known that the suboptimal solution set to a scalar $\mathcal{H}^\infty$ control problem is equivalent to the solution set to the classical Nevanlinna–Pick interpolation problem [3]

$$\mathcal{S}_{\mathrm{NP}} := \{T_{cl} \in \mathcal{RH}^\infty : \|T_{cl}\|_\infty < \gamma, T_{cl}(z_j) = w_j, \quad j = 0, 1, \dots, n\}.$$

Here, $\{(z_j, w_j)\}_{j=0}^n$ are given self-conjugate pairs of complex numbers with z_j in an unstable region, γ is a given positive number and $\mathcal{RH}^\infty$ is the set of real rational proper stable functions. The interpolant T_{cl} represents some closed-loop transfer function in control problems. Henceforth, we assume that *the set $\mathcal{S}_{\mathrm{NP}}$ is nonempty*. The condition of this nonemptyness can actually be expressed by the positivity of a Hermitian matrix, called *the Pick matrix* (see, e.g., [11]).

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The author is with the Department of Mechanical Engineering, University of California, Berkeley, CA 94720 USA (e-mail: ryozo@me.berkeley.edu).

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