CLOSED-LOOP SHAPING BY ANALYTIC FUNCTIONS WITH A BOUNDED DEGREE

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Abstract

This paper proposes a new technique for frequency shaping of the closed-loop system. The technique does not involve any weighting function for shaping, which is often used in the conventional H^∞ control theory, but selects a desirable analytic function from a set of H^∞ suboptimal solutions with a bounded degree. The Nevanlinna-Pick interpolation theory with a degree bound plays a central role in deriving shaping strategies of the closed-loop frequency responses. One example illustrates the efficiency of the proposed method by comparing it with a conventional method in H^∞ control.

1 Introduction

H^∞ control theory is a systematic way to design a desirable closed-loop system in the frequency domain. It is well-known that the standard H^∞ control problem can be reduced to the model-matching problem (see [4, 5]). In the scalar case, the model-matching problem is formulated as follows: for given $T_1 \in RH^\infty$ and $T_2 \in RH^\infty$, where RH^∞ denotes the set of real, rational proper functions which are analytic in the unstable region, that is, either the closed right half-plane in continuous-time cases or the complement of the open unit disc in discrete-time cases, solve the optimization problem:

$$\gamma_{opt} = \inf \left\{ \|T_{cl}\|_\infty, \ T_{cl} = T_1 - T_2 Q, \ Q \in RH^\infty \right\}. \quad (1)$$

Throughout this paper, we consider only a single-input single-output (SISO) closed-loop system with an SISO plant. The optimal solution T_{cl} of this minimization problem is known to become a positive scalar multiple of an all-pass function, and therefore the shape of its frequency response is flat over all frequencies.

However, in cases where it is impossible for T_{cl} to attain sufficiently low gain over all frequencies due to inherent performance limitations (see e.g. [11]), the flat optimal solution is often not acceptable, and we need to shape the frequency response of T_{cl} in order to take some tradeoff between frequencies. The conventional shaping method in H^∞ control theory multiplies T_{cl} by a weighting function W and solve (1). The main disadvantage of this approach is that W in general increases the degree of the controller, typically by $\deg W$.

In this paper, we shall take an alternative approach. Instead of finding the optimal solution (1) with W, let us consider the set of H^∞ suboptimal solutions:

$$\mathcal{S} := \left\{ T_{cl} : \|T_{cl}\|_\infty < \gamma, \ T_{cl} = T_1 - T_2 Q, \ Q \in RH^\infty \right\}, \quad (2)$$

where T_1 and T_2 do not include any W and γ is a given positive real number larger than γ_{opt} in (1). Since $\gamma > \gamma_{opt}$, this set has infinitely many functions T_{cl} with various shapes of frequency responses whose gains are bounded by γ, and thus we may choose one function with a desired frequency shape from the set $\mathcal{S}$. Note that the LMI-based method [7] also considers the H^∞ suboptimal solutions and obtain low degree controllers directly even with weighting functions W for frequency shaping. However, to find an appropriate W which indeed solves an original control problem is sometimes a hard task.

It is known that the set $\mathcal{S}$ can also be expressed in terms of interpolation constraints of T_{cl} at unstable zeros (including zeros at infinity) of T_2, as is explained next. For a simple exposition, we assume that the unstable zeros of T_2 are distinct and not on the boundary of the stable region. Then, denoting these zeros by $z_i, i = 0, 1, \cdots, n$, the set $\mathcal{S}$ can be rewritten as

$$\mathcal{S} = \left\{ T_{cl} : \begin{array}{l} \|T_{cl}\|_\infty < \gamma, \ T_{cl} \in RH^\infty, \\ T_{cl}(z_i) = T_1(z_i), i = 0, 1, \cdots, n \end{array} \right\}. \quad (3)$$

Here, we have used the fact that $Q \in RH^\infty$ if and only if $T_{cl} = T_1 - T_2 Q \in RH^\infty$ and $T_{cl}(z_i) = T_1(z_i), i = 0, 1, \cdots, n$. This set can be characterized by means of the Nevanlinna-Pick interpolation theory [12]. The condition for nonempty-ness of $\mathcal{S}$ (which is equivalent to the condition $\gamma > \gamma_{opt}$) is described by the positivity of the Pick matrix:

$$P_c := \left[\frac{\gamma^2 - T_1(z_i)\overline{T_1(z_j)}}{z_i + \bar{z}_j} \right]_{i,j=0}^n, \text{ (continuous-time cases)},$$

$$P_d := \left[\frac{\gamma^2 - T_1(z_i)\overline{T_1(z_j)}}{1 - z_i^{-1}\bar{z}_j^{-1}} \right]_{i,j=0}^n, \text{ (discrete-time cases)},$$

where $\bar{x}$ denotes the complex conjugate of a complex number x and $[m_{i,j}]_{i,j=0}^n$ means the $(n+1) \times (n+1)$ matrix whose (i,j)-entry is $m_{i-1,j-1}$. In the case that the Pick matrix is positive definite, all the elements in $\mathcal{S}$ are obtained by Schur-Nevanlinna algorithm (see [12] for details).

On the other hand, low degree rational closed-loop transfer functions are often preferable to high degree ones. This is because of the fact that the low degree closed-loop system often leads to simple controllers (see [9] for one such example). In addition, it is sometimes advantageous that the closed-loop system works as a low degree system, since such system will be easy to handle for operators. From these reasons, we will impose degree constraint on the functions T_{cl} in the set $\mathcal{S}$. It is known (see [2, 3]) that even when the Pick matrix is positive definite, $\mathcal{S}$ with degree constraint $\deg T_{cl} \leq n-1$ could be empty, depending on the interpolation data, while $\mathcal{S}$ with $\deg T_{cl} \leq n$ is nonempty whenever the Pick matrix is positive definite. (Note that the number of interpolation constraints is $n+1$.) Therefore, we search for a desirable function from the nonempty set:

$$\mathcal{S}_D := \mathcal{S} \cap \{T_{cl} : \text{ rational and } \deg T_{cl} \leq n\}. \tag{4}$$

Now, the following questions arise naturally.

- How can we characterize the set $\mathcal{S}_D$?
- How can we find a desirable function from the set $\mathcal{S}_D$ in an efficient way?
- If we cannot find a desirable function in $\mathcal{S}_D$, what can we do next?

To answer these questions, we shall make use of the Nevanlinna-Pick interpolation theory with degree constraint presented in [2, 3]. In the next section, according to the theory, we will derive the design parameters and some tuning strategies for frequency shaping.

2 Application of the Nevanlinna-Pick interpolation with a degree bound

The set $\mathcal{S}_D$ is completely parameterized due to an analytic interpolation theory, proposed by Byrnes, Georgiou and Lindquist in [2, 3], as follows. In this section, we regard the complement of the open unit disc as the unstable region, but by means of the bilinear transformation, the similar argument holds in the case where the unstable region is the closed right half-plane.

Theorem 2.1 ([2, 3]) *There is a bijective correspondence between the set of pairs of real polynomials with degree n which correspond to functions in $\mathcal{S}_D$:*

$$\mathcal{A} := \{(b,a) : \deg a = \deg b = n,\ T_{cl} := b/a \in \mathcal{S}_D\}, \tag{5}$$

and the set of stable real polynomials with degree n:

$$\Sigma := \{\sigma : \deg \sigma = n,\ \sigma(z) \neq 0, \forall |z| \geq 1\}. \tag{6}$$

In addition, for each $\sigma \in \Sigma$, the corresponding polynomial pair $(b,a) \in \mathcal{A}$ satisfies

$$\gamma^2 a(z)a(z^{-1}) - b(z)b(z^{-1}) = \sigma(z)\sigma(z^{-1}). \tag{7}$$

Note that the polynomials a and b in (5) may have common factors and hence $\deg T_{cl}$ can be less than n. In [2, 3], for each $\sigma \in \Sigma$, the corresponding $(b,a) \in \mathcal{A}$ was shown to be obtained by a solution of a generalized entropy maximization problem. They also proposed the method of finding (b,a) numerically from a prespecified σ, by reducing the maximization problem to a dual finite-dimensional convex optimization problem. See [2, 3] for details.

The equation (7) can also be described as

$$\gamma^2 - T_{cl}(z)T_{cl}(z^{-1}) = \frac{\sigma(z)\sigma(z^{-1})}{a(z)a(z^{-1})}. \tag{8}$$

Thus, the n roots of a real polynomial $\sigma(z)$, that are self-conjugate and lie in the open unit disc, are called *the spectral zeros of* $\gamma^2 - T_{cl}(z)T_{cl}(z^{-1})$. Due to the bijective property in Theorem 2.1, each element in $\mathcal{S}_D$ can be picked up by selecting a set of n self-conjugate spectral zeros. Therefore, the spectral zeros can be design parameters.

It happens that the desired shape cannot be attained by any function in the set $\mathcal{S}_D$ because of the degree restriction. In such cases, we introduce additional interpolation constraints

$$T_{cl}(\lambda_i) = \alpha_i,\ |\lambda_i| > 1,\ |\alpha_i| < \gamma,\ i = 1, \cdots, n_e, \tag{9}$$

and redefine the set $\mathcal{S}_D$ with these constraints as

$$\begin{aligned} \mathcal{S}_D = \mathcal{S} &\cap \{T_{cl} : T_{cl} \text{ meets (9)}\} \\ &\cap \{T_{cl} : \text{ rational and } \deg T_{cl} \leq n + n_e\}. \end{aligned} \tag{10}$$

Then we can again use Theorem 2.1 to characterize the set $\mathcal{S}_D$ in terms of $n+n_e$ spectral zeros, since the degree bound $n+n_e$ equals the number of interpolation constraints minus one. The additional constraints can be chosen freely except the condition that they do not violate the positivity of the corresponding Pick matrix.

In summary, the design parameters to be tuned are

- $n + n_e$ self-conjugate spectral zeros, in the open unit disc
- pairs of points $(\lambda_i, \alpha_i),\ i = 1, \cdots, n_e$ for additional interpolation conditions.

In the following, we will investigate the influences of these parameters on the resulting closed-loop frequency shape.

First, the spectral zero λ near the unit circle with angle θ_1 lifts $|T_{cl}|$ up to the level near γ at the frequency θ_1 (see Figure 1). This is because of the following relation:

$$\gamma^2 - \left|T_{cl}(e^{i\theta_1})\right|^2 \approx \gamma^2 - T_{cl}(\lambda)T_{cl}(\lambda^{-1}) = 0. \tag{11}$$

Given requirements in the frequency domain, we usually know in advance the frequency range where we *must* have high gain in order to get low gain over some specified frequencies. Thus, the spectral zeros near the unit circle are helpful for achievement of specifications.

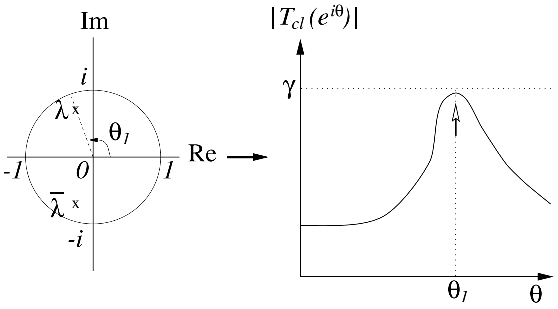


Figure 1: The influences of a spectral zero λ on the frequency response of T_{cl}

Next, if the point of an additional interpolation constraint λ is close to the unit circle at frequency θ_2, the additional constraint $T_{cl}(\lambda) = \alpha$ will fix the magnitude of $|T_{cl}|$ around $|\alpha|$ and the phase of T_{cl} around that of α at the frequency θ_2 (see Figure 2). Consequently, additional interpolation constraints can control both gain and phase of T_{cl}, and hence they are quite useful for shaping purpose.

However, in view of (10), the introduction of additional constraints increases the degree bound of T_{cl}, which is often undesirable by the reasons mentioned before. Thus, we should try to use additional constraints as less as possible.

In the next section, using the tuning strategies of design parameters presented above, we will solve one fundamental control problem, that is, the mixed sensitivity reduction problem.

Remark 1 In the conventional H^∞ control, we can check numerically if the resulting closed-loop frequency response is below the frequency response of the inverse of the weighting function. Our approach does not have such a straightforward way, and we have to check the performance achievement graphically after the controller design.

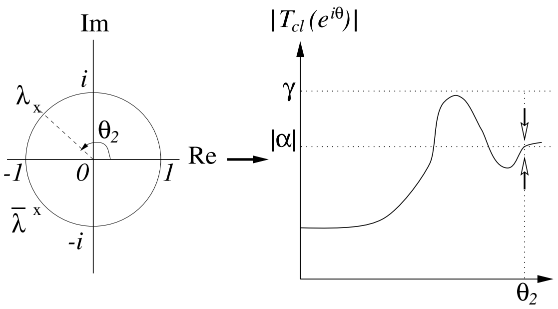


Figure 2: The influences of an additional interpolation constraint $T_{cl}(\lambda) = \alpha$ on the frequency response of T_{cl}

3 One example: mixed sensitivity reduction

Consider the feedback system depicted in Figure 3. Here, $P(z)$ is a discrete-time transfer function given by

$$P(z) = \frac{1}{z - 1.05}, \tag{12}$$

$C(z)$ is a controller to be designed, and d (disturbance), y (output), n (measurement noise), r (reference) and e (error) are scalar signals. The transfer functions S from r to e (also from d to y) and T from r to y (also from n to y) are called *the sensitivity function* and *the complementary sensitivity function*, respectively, and expressed by

$$S(z) = \frac{1}{1 + P(z)C(z)}, \tag{13}$$

$$T(z) = \frac{P(z)C(z)}{1 + P(z)C(z)} = 1 - S(z). \tag{14}$$

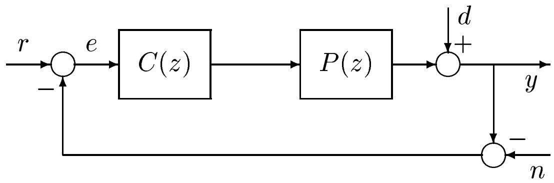


Figure 3: The feedback system

Suppose that specifications that the closed-loop system must fulfill are given as follows.

1. $|S(e^{i\theta})| < 0.1(= -20\text{ dB})$ for $\theta \in [0, 0.3]$ (rad/sec).

   This requirement is for good reference tracking and disturbance attenuation. Since typical reference signal and disturbance have low frequencies, it is common in practice to require low gain of S at low frequencies.

2. $|T(e^{i\theta})| = |1 - S(e^{i\theta})| < 0.5(\approx -6.02\text{ dB})$ for $\theta \in [2.5, \pi]$ (rad/sec).

   This requirement is to attenuate the noise effect on the output. Since noise is usually focused on high frequencies, the low gain of T at the spectrum is also a common requirement.

3. $|S(e^{i\theta})| < 2(\approx 6.02\text{ dB})$ for $\theta \in [0, \pi]$ (rad/sec).

   A small uniform bound of $|S|$ guarantees the large stability margin due to the Nyquist stability criterion. See [6] for more detail explanations on specifications.

The solvability of the problem of finding such controllers that attain all the requirements is unknown in advance by any existing theory at this point, since the specification is not just the

H^∞-norm bound of one function. Hence we anyway have to search for a controller by trial and error in any existing method. (For example, with H^∞ methods, we have to search for an appropriate weighting function by trial and error.)

Note that all the requirements above can be described in terms of only the sensitivity function S. Hence, regarding T_{cl} in (1) as S, we first construct the set $\mathcal{S}_D$. Since the uniform bound of $|S|$ should be less than two by the requirement 3, we set $\gamma = 2$. The plant (12) has one unstable zero at infinity and one unstable pole at $z = 1.05$. In this case, $n = 1$ (the number of interpolation constraints minus one) in (4), and the set $\mathcal{S}_D$ in (4) becomes

$$\mathcal{S}_D = \left\{ S \in RH^\infty : \begin{array}{l} S(\infty) = 1,\ S(1.05) = 0, \\ \|S\|_\infty < 2,\ \deg S \le 1 \end{array} \right\}. \quad (15)$$

Although the set $\mathcal{S}_D$ can be derived by obtaining T_1 and T_2 via the doubly coprime factorization of P (see [5]), it also comes directly from the well-known conditions that S must satisfy for internal stability of the feedback system (see [13]).

In this example, we can prove that there is no element in the set $\mathcal{S}_D$ in (15) which meets all the requirements 1, 2 and 3 simultaneously, because of the strict degree bound $\deg S \le 1$.

Remark 2 The existence range of frequency response of elements in the set

$$\tilde{\mathcal{S}}_D := \left\{ S \in RH^\infty : \begin{array}{l} S(\infty) = 1,\ S(1.05) = 0, \\ \deg S \le 1 \end{array} \right\} \quad (16)$$

can be obtained explicitly. Obviously, $\mathcal{S}_D \subset \tilde{\mathcal{S}}_D$. By showing that there is no function in $\tilde{\mathcal{S}}_D$ fulfilling the requirements, we can conclude that the specifications are not attainable by any sensitivity function in $\mathcal{S}_D$. See details in [10].

Therefore, we must introduce some additional interpolation constraints to increase the flexibility of the design. When we choose the constraints as $S(1.01e^{\pm 0.3i}) = 0$ for the requirement 1, and $S(-1.01) = 1$ for the requirement 2, the set $\mathcal{S}_D$ is redefined, according to (10), as

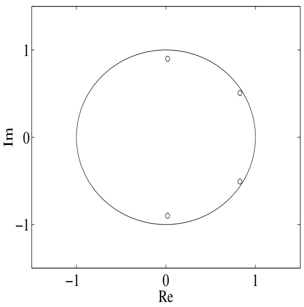
$$\mathcal{S}_D = \left\{ S \in RH^\infty : \begin{array}{l} \|S\|_\infty < 2,\ \deg S \le 4 \\ S(\infty) = S(-1.01) = 1, \\ S(1.05) = 0,\ S(1.01e^{\pm 0.3i}) = 0 \end{array} \right\}. \quad (17)$$

Selection of four spectral zeros at $z = 0.97e^{\pm 0.55i},\ 0.9e^{\pm 1.55i}$ (Figure 4(a)) resulted in high gains around 0.55 rad/sec and 1.55 rad/sec. The corresponding element from the set $\mathcal{S}_D$ in (17) was computed as

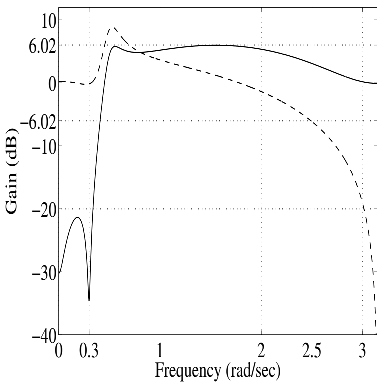
$$S(z) = \frac{z^4 - 2.4048z^3 + 1.3331z^2 + 0.6804z - 0.6158}{z^4 - 1.6159z^3 + 0.8469z^2 - 0.0030z + 0.0028}, \quad (18)$$

whose frequency response is shown in Figure 4(b) with the frequency response of $T = 1 - S$. In the figure, we can verify not only the achievement of all the requirements but also the influences of the spectral zeros and the additional interpolation constraints on the resulting frequency shapes. The corresponding controller is computed by $C = (1 - S)/PS$ as

$$C(z) = \frac{0.7889z^3 - 0.4863z^2 - 0.6834z + 0.6186}{z^3 - 1.3548z^2 - 0.0894z + 0.5865}. \quad (19)$$



(a) The location of the spectral zeros (o)



(b) The gains of S (solid line) and of T (dashed line)

Figure 4: The location of the specified spectral zeros and the frequency responses of $S(z)$ and $T(z) = 1 - S(z)$

We tried to solve the same control problem by the conventional weighted H^∞ technique, that is, to solve the optimization problem:

$$\inf \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty \quad (20)$$

subject to the internal stability condition. We tried to find weighting functions solving the control problem in an increasing order of degree. When we used weighting functions:

$$W_1(z) = \frac{2.95z^5 - 9.36z^4 + 11.28z^3 - 6.00z^2 + 1.14z}{3.42z^5 - 13.16z^4 + 19.45z^3 - 13.30z^2 + 3.80z - 0.21},$$

$$W_2(z) = \frac{0.48z - 0.24}{z + 0.90},$$

we obtained the optimal solution with the frequency responses as shown in Figure 5. In this case, the degree of the controller was six.

Remark 3 This coincides with the fact (see [8]) that the degree bound of the optimal controller becomes the degree of the generalized plant minus one, since in this example,

$$\deg P + \deg W_1 + \deg W_2 = 7.$$

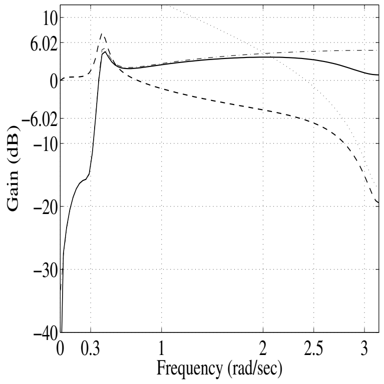


Figure 5: The frequency responses of S (solid line), T (dashed line), W_1^{-1} (dash-dot line) and W_2^{-1} (dotted line)

In Figure 5, note that the requirement 1 is not attained. Indeed, it was difficult in this example to find an acceptable solution by the conventional H^∞ method. Besides, even if we could find one by further trial and error, the degree of the controller will be still around six, whereas our approach does generate a satisfactory solution with a controller whose degree is only three. This shows the efficiency of the proposed shaping techniques.

4 Conclusions

In this paper, we have proposed a new method for shaping of the closed-loop frequency response. The method is based on the analytic interpolation theory with a degree bound. Instead of weighting functions, our design parameters are the spectral zeros and points and values of additional interpolation constraints. The efficiency of our approach was illustrated by one numerical example.

There are two main advantages in the proposed method. First, by bounding the degree of the closed-loop system, the controller is likely to become simple. Secondly, the proposed approach has the potential to easily solve such problems that are difficult to solve by conventional weighted H^∞ method.

Our approach is applicable only for an SISO closed-loop system with an SISO plant at present. However, the extension of the approach to multi-input multi-output cases is also important for many control applications. To this end, we first have to construct the theory of the tangential (or directional) Nevanlinna-Pick interpolation theory (see [1]) with a degree bound, and this will be a future research issue.

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